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Abstract Using Malliavin calculus techniques, we derive an analytical formula for the price of European options, for any model including local volatility and jump Poisson process. We show that the accuracy of the formula depends on the smoothness of the payoff. Our approach relies on an asymptotic expansion related to small diffusion and small jump frequency. As a consequence, the calibration of such model becomes very fast.

Keywords asymptotic expansion · Malliavin calculus · volatility skew and smile · small diffusion process · small jump frequency

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1 Introduction

The standard Black-Scholes formula (1973) has been derived under the assumption of lognormal diffusion with constant volatility to price calls and puts. However those hypotheses are unrealistic under real market conditions because we need to use different volatilities to equate different option strikes K and maturities T. Besides this, the market data shows that the shape of the implied volatilities looks like a smile or a skew.

In order to fit the smile or the skew, Dupire (1994) and Rubinstein (1994) use a local volatility $\sigma_{loc}(t,f)$ depending on time t and state f to fit the market. This hypothesis is interesting for hedging because it maintains the completeness of the

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market. Only in few cases, [ACGL01], one has closed formulas. In the case of homogeneous volatility, singular perturbation techniques have been used to obtain asymptotic expression for the price of vanilla options (call, put) in [HD99]. Other cases had been derived using an asymptotic expansion of the heat kernel for short maturity in [Lab05].

But Andersen and Andreasen in [AA00] show that this sole assumption of local volatility is not compatible with empirical evidences (for instance, the post crash of implied volatility for the S&P500 index). Hence, they derived a model with local volatility plus a jump process to fit the smile (we write it AA model). They calibrate this model by solving the equivalent forward PIDE. This sort of problem could be handled numerically using: an ADI-FFT scheme in [AA00], Finite Element Method in [MvPS03], an explicit implicit PIDE-FFT method for general Lévy processes in [CV05], predictor corrector methods to improve the accuracy of the PIDE in [BM06]. But in the best case, all these methods lead to a time of calibration of the order of one minute (see [AA00]). Can we reduce this computational time? Is it possible to reach a time of calibration as short as the computational time of a closed formula of Merton's type [Mer76]? This is the objective reached in this work.

In order to handle even more general situations, we consider for the one dimensional underlying state process the solution of the stochastic differential equation (SDE):

$$dX_t = \sigma(t, X_{t^-})dW_t + \mu(t, X_{t^-})dt + dJ_t, X_0 = x_0. \tag{1.1}$$

One may think of $(X_t)_t$ as the log asset price for instance. Here $(W_t)_{0\le t\le T}$ is a standard real Brownian motion on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\le t\le T}, \mathbb{P})$ with the usual assumption on the filtration $(\mathscr{F}_t)_{0\le t\le T}$ and $(J_t)_{0\le t\le T}$ is a compound Poisson process defined by: $J_t = \sum_{i=1}^{N_t} Y_i$ where N_t is a counting Poisson process with constant jump intensity λ and $(Y_i)_{i\in\mathbb{N}^*}$ are i.i.d. normal variables with mean η_J and volatility γ_J.

Actually, our further analysis relies on a suitable parameterization w.r.t. $\varepsilon \in [0,1]$:

$$dX_t^\varepsilon = \varepsilon(\sigma(t, X_{t^-}^\varepsilon)dW_t + \mu(t, X_{t^-}^\varepsilon)dt + dJ_t), X_0^\varepsilon = x_0, \tag{1.2}$$

so that $X_t^1 = X_t$. Our first aim is to give an accurate analytic approximation (in some sense) of the expected payoff (fair price of this option)

$$g(\varepsilon) = \mathbb{E}(h(X_T^\varepsilon)) \tag{1.3}$$

for a given terminal function h and for a fixed maturity T, at the value of $\varepsilon = 1$.

The approximation can be applied to the following models:

Example 1.1 AA model on the log-asset.
In that case, $(X_t^1)_t$ is the logarithm of the underlying, σ its volatility, $\mu = \lambda(1 - e^{\eta_J + \frac{\gamma_J^2}{2}}) - \frac{\sigma^2}{2}$ in order to guaranty the martingale property for $(e^{X_t^1})_t$. For a call exercised at maturity T, $h(x) = e^{-\int_0^T r(u)du}(e^{\int_0^T (r(u)-q(u))du}e^x - k)^+$ where r is the risk free rate term and q is the dividend term. This model has been derived in [AA00]. In the sequel we mainly focus our discussion on this model.

Example 1.2 Jump diffusion model on the asset.
$(X_t^1)_t$ is the forward with maturity T, σ its volatility, $\mu = -\lambda\eta_J$. For a call exercised at maturity T, $h(x) = e^{-\int_0^T r(u)du}(x-k)^+$. This model aims at modeling the presence of price jumps in a market with slow movements, where the implied normal volatility is traded instead the implied Black Scholes volatility (Japanese markets in [HKLD02]).

Our approach consists in expanding the price (1.3) with respect to ε. But the accuracy of the expansion is not related to ε because the value of interest $\varepsilon = 1$ is not small (and this is a significant difference with singular perturbation techniques). It is just a tool to derive convenient representations of the derivatives. Actually, using an asymptotic expansion in the context of small diffusions and small jump frequency, we can exhibit estimates of the derivatives, which allows us to make the contribution at any order explicit and to control the error. This is achieved using the infinite dimensional analysis of Malliavin calculus. The key feature of our approach is that we provide explicit formulas for the terms at any order, explicit upper bounds of the errors for general forms of μ, σ. However, the more the parameters μ, σ, λ are small or the more the maturity T is small, the more the expansion is accurate. On real data, the accuracy is indeed very good (less than **2bp** for various strikes and maturities). As a result of these expansions, we prove that the price (1.3) in our general model (1.1) equals to the price in the Merton model plus a combination of Greeks (still in Merton's model). Hence, all these terms are straightforward to numerically evaluate, with a computational cost as cheap as the closed Merton formula. The residual terms (that is the error) is also estimated and their amplitudes depend on the smoothness of the payoff. We distinguish three cases: smooth, vanilla (call, put) and binary payoffs. This is our main contribution. Also, we observe from the approximation price that one may get a volatility smile for short maturities (since we use Merton's model as a proxy) and a volatility skew for long maturities (due to local volatility function).

Comparison with the literature. We refer in particular to Hagan and al in [HKLD02] for the SABR model, to Fouque and al in [FPS00] for stochastic volatility models, or to Antonelli-Scarletti in [AS07]. In all these works, as a difference with our approach, a perturbation analysis w.r.t. the volatility, the mean reversion parameters or the correlation is performed and this leads to write the price as a main term (essentially a Black-Scholes price) plus an integral of Greeks over maturities. In the time homogeneous case, the authors succeed on computing or approximating this integral, which strongly relies on PDE arguments. In our case, we do not approximate the underlying PDE (or the related operator) but owing to Malliavin calculus, we directly focus on the law of the random variable X_T^1 given $X_0 = x_0$ (and not necessarily on the process for any initial condition). Thus, we are able to handle time inhomogeneous coefficients and jumps as well, without extra efforts.

Outline of the paper. In the following, we give some notations and assumptions that will be used throughout the paper. The section 2 is aimed at presenting in an heuristic way our methodology to approximate the expected cost. Rigorous results are proved in Section 5 and 6. In Section 3, we derive financial modeling consequences from these formulas. These observations lead to justify simplified choices of the local volatility, to predict the form of all attainable smiles with their dynamics. In Section 4, we firstly give a methodology to implement the approached formula. Secondly,

we show how to efficiently use our formula for calibration items using a relevant algorithm. Finally, we detail numerical applications in calibration for real market data using our simplified form of local volatility. In Section 5, we give a rigorous sense to our approach of infinitesimal perturbation; the analysis depends on the kinds of payoff (smooth payoff in Theorem 5.1, vanilla options in Theorem 5.4, binary options in Theorem 5.7) and we discuss the impact of the payoff on the accuracy. The proofs of Theorems 5.1-5.4-5.7 are postponed to Section 6. In Section 7, we bring together useful results to make our expansion explicit.

Notations.

Notation 1.1 ***Differentiation.***
If these derivatives have a meaning, we write:

- $\psi^{(i,j)} = \frac{\partial^{i+j}\psi}{\partial x^i \partial y^j}$ *for every function* ψ *of two variables.*
- $X_{i,t} = \frac{\partial^i X_t^\varepsilon}{\partial \varepsilon^i}|_{\varepsilon=0}$ *. These processes play a crucial role in the sequel.*
- $X_{c,1,t}$ *for the continuous part of* $X_{1,t}$.
- $\sigma_t = \sigma(t,x_0), \mu_t = \mu(t,x_0), \sigma_t^{(i)} = \sigma^{(0,i)}(t,x_0), \mu_t^{(i)} = \mu^{(0,i)}(t,x_0)$.

Definitions. The following sets of functions are used to distinguish the payoff functions h.

Definition 1.3 As usual, we define $\mathscr{C}_0^\infty(\mathbb{R})$ as the space of real infinitely differentiable functions h with compact support (smooth payoffs). A function h belongs to $\mathscr{H}$ if $|h(x)| \leq c_1 e^{c_2|x|}$ for any x, for two constants c_1 and c_2. In other words, h has at most an exponential growth. In practice, the case of vanilla options (call-put) is associated to $h \in \mathscr{H}$ and $h^{(1)} \in \mathscr{H}$. For binary options, we only require h to be in $\mathscr{H}$.

In order to make the approximation explicit, we introduce the following family of operators indexed by maturity T.

Definition 1.4 Integral Operator.
The Integral operator $\omega(T)$ is defined by:
for any integrable function l, $\omega(T)_t^l = \int_t^T l_u du$, $\forall t \in [0,T]$.
Its n-times iteration is given by:
for any integrable functions $(l_1,\cdots,l_n)$, $\omega(T)_t^{l_1,\cdots,l_n} = \omega(T)_t^{l_1 \omega(T)^{l_2,\cdots,l_n}}$, $\forall t \in [0,T]$.

The following notations are useful to give an interpretation to the correction terms.

Definition 1.5 Greeks.
Let Z be a random variable. Given a payoff function h, we define the i^{th} Greek for the variable Z by the quantity (if it has a meaning) :

$$\text{Greek}_i^h(Z) = \frac{\partial^i \mathbb{E}[h(Z+x)]}{\partial x^i}|_{x=0}.$$

Under appropriate smoothness assumptions on h, one also has

$$\text{Greek}_i^h(Z) = \mathbb{E}[h^{(i)}(Z)].$$

Assumptions. In order to get accurate approximations, we may assume that coefficients σ, μ are smooth enough. In what follows, N is an integer greater than 4 and sometimes 5.

Assumption *(R_N). The function σ, μ are bounded and of class C^N w.r.t x and the derivatives until order N are bounded.*

This assumption may be restrictive because σ and μ have to be bounded as well their derivatives. Actually, this statement is made only to simplify a bit our analysis, but we can prove that our approximation remains valid if some boundedness requirements are partially relaxed.

To perform the infinitesimal analysis, we rely on smoothness properties which are not given by the payoff functions, but the law of the underlying stochastic models (this is related to Malliavin calculus). The next assumption on the volatility combined to (R_N) guaranties these smoothness properties.

Assumption *(E). σ satisfies an uniform ellipticity condition*

$$\forall (t,x) \in [0,T] \times \mathbb{R}, \qquad \sigma(t,x) \geq I_\sigma$$

for a real positive number I_σ.

2 Smart Taylor Development

Our perturbation approach lies in the framework of small diffusion and small jump frequency that is $\sigma(t,x) = \Delta s(t,x)$, $\mu(t,x) = \Delta m(t,x)$ and $\lambda \leq \Delta$, with a small positive number Δ. All the next results would remain valid if the jump size were small ($|\eta_J|$ and γ_J small) but it is more natural in the AA model to assume few jumps of arbitrary size than arbitrary number of jumps with small size.

Under assumption (R_5), for any t, X_t^ε is almost surely C^4 w.r.t ε (this is an easy extension of [Kun84]). If we put $X_{i,t}^\varepsilon = \frac{\partial^i X_t^\varepsilon}{\partial \varepsilon^i}$, we get

$$\begin{aligned} dX_{1,t}^\varepsilon =& \sigma(t, X_{t^-}^\varepsilon) dW_t + \mu(t, X_{t^-}^\varepsilon) dt + dJ_t \\ &+ \varepsilon X_{1,t^-}^\varepsilon (\sigma^{(0,1)}(t, X_{t^-}^\varepsilon) dW_t + \mu^{(0,1)}(t, X_{t^-}^\varepsilon) dt), X_{1,0}^\varepsilon = 0. \end{aligned} \tag{2.1}$$

From the definitions, $X_{i,t} \equiv \frac{\partial^i X_t^\varepsilon}{\partial \varepsilon^i}|_{\varepsilon=0}$, $\sigma_t^{(i)} \equiv \sigma^{(0,i)}(t,x_0)$ and $\mu_t^{(i)} \equiv \mu^{(0,i)}(t,x_0)$, we easily get

$$\begin{aligned} dX_{1,t} =& \sigma_t dW_t + \mu_t dt + dJ_t, X_{1,0} = 0, \\ dX_{2,t} =& 2X_{1,t^-}(\sigma_t^{(1)} dW_t + \mu_t^{(1)} dt), X_{2,0} = 0. \end{aligned}$$

Now we use the Taylor formula twice: firstly, for X_T^1 at the second order w.r.t ε around x_0, secondly for smooth function h at the first order w.r.t x around $x_0 + X_{1,T}$. One gets:

$$\begin{aligned} \mathbb{E}[h(X_T^1)] =& \mathbb{E}[h(x_0 + X_{1,T} + \frac{X_{2,T}}{2} + ...)] \\ =& \mathbb{E}[h(x_0 + X_{1,T})] + \mathbb{E}[h^{(1)}(x_0 + X_{1,T}) \frac{X_{2,T}}{2}] + ... \end{aligned}$$

Because of $X_{1,T} = \int_0^T (\sigma_t dW_t + \mu_t dt) + J_T$ is a drifted Brownian motion plus a compound Poisson process, the cost can be approximated by a summation of two terms :

- $\mathbb{E}[h(x_0 + X_{1,T})]$: The leading order which corresponds to the Merton price (BS price when $\lambda = 0$) for call or put option.
- $\mathbb{E}[h^{(1)}(x_0 + X_{1,T})\frac{X_{2,T}}{2}]$: The correction term which will be made explicit using the next proposition.

Theorem 2.1 ***(Main approximation price formula).***
Under assumptions $(R_5), (E)$*, one has:*

$$\mathbb{E}[h(X_T)] \approx \mathbb{E}[h(x_0 + X_{1,T})] + \sum_{i=1}^{3} \alpha_{i,T} \mathrm{Greek}_i^h(x_0 + X_{1,T}) + \sum_{i=1}^{3} \beta_{i,T} \mathrm{Greek}_i^h(x_0 + X_{1,T} + Y'), \qquad (2.2)$$

where

$$\begin{aligned}
\alpha_{1,T} &= \int_0^T \mu_t \omega(T)_t^{\mu^{(1)}} dt, \\
\alpha_{2,T} &= \int_0^T (\sigma_t^2 \omega(T)_t^{\mu^{(1)}} + \mu_t \omega(T)_t^{\sigma\sigma^{(1)}}) dt, \\
\alpha_{3,T} &= \int_0^T \sigma_t^2 \omega(T)_t^{\sigma\sigma^{(1)}} dt, \\
\beta_{1,T} &= \lambda \eta_J \int_0^T t \mu_t^{(1)} dt, \\
\beta_{2,T} &= \lambda \int_0^T t(\gamma_J \mu_t^{(1)} + \eta_J \sigma_t \sigma_t^{(1)}) dt, \\
\beta_{3,T} &= \lambda \gamma_J \int_0^T t \sigma_t \sigma_t^{(1)} dt,
\end{aligned}$$

and Y' *is an independent copy of the variables* $(Y_i)_{i \in \mathbb{N}^*}$.

The accuracy of the approached formula is stated in Theorems 5.1-5.4-5.7.

To prove Theorem 2.1, it remains to show that $\mathbb{E}[h^{(1)}(x_0 + X_{1,T})\frac{X_{2,T}}{2}]$ is equal to the last two terms of (2.2). The reader familiar with Malliavin calculus for the computations of Greeks (see [FLLL01], [Gob03], ...) may recognize in the expansion of $\mathbb{E}[h^{(1)}(x_0 + X_{1,T})\frac{X_{2,T}}{2}]$ the generic form of some derivatives (or Greeks) of $\mathbb{E}[h^{(1)}(x_0 + X_{1,T})]$, derivative which is written as the expectation of $h^{(1)}(x_0 + X_{1,T})$ multiplied by a random weight. This is indeed our methodology to explicitly compute the correction terms in the formula (2.2). It relies on the following key lemmas, which are stated for any smooth function l with compact support and for any square predictable (resp. deterministic) process (u_t) (resp. $(v_t), (v_t)$).

Lemma 2.2 *One has:*

$$\mathbb{E}[(\int_0^T u_t dW_t) l(\int_0^T v_t dW_t)] = \mathbb{E}[(\int_0^T v_t u_t dt) l^{(1)}(\int_0^T v_t dW_t)].$$

In the case of deterministic u, it is equal to $\int_0^T v_t u_t dt \, \mathrm{Greek}_1^l(\int_0^T v_t dW_t)$.

Lemma 2.3 *One has :*

$$\begin{aligned}\mathbb{E}[(\int_0^T v_t X_{c,1,t} dt) l(\int_0^T v_t dW_t)] = &\int_0^T v_t \sigma_t \omega(T)_t^v dt \, \mathrm{Greek}_1^l(\int_0^T v_t dW_t) \\ &+ \int_0^T \mu_t \omega(T)_t^v dt \, \mathrm{Greek}_0^l(\int_0^T v_t dW_t).\end{aligned}$$

Lemma 2.4 *One has:*

$$\begin{aligned}\mathbb{E}[(\int_0^T v_t J_t dt) l(J_T)] = \lambda(&\eta_J \int_0^T t v_t dt \, \mathrm{Greek}_0^l(J_T + Y') \\ &+ \gamma_J \int_0^T t v_t dt \, \mathrm{Greek}_1^l(J_T + Y')),\end{aligned}$$

such that Y' *is an independent copy of the variables* $(Y_i)_{i\in\mathbb{N}}$.

The proofs of these technical lemmas are postponed to section 7. We now complete the proof of Theorem 2.1.

Proof We define the new function of two variables : $G(x,y) = h(x+y+x_0+\int_0^T \mu_t dt)$. Using that the set $\{t \in [0,T], X_{1,t} \neq X_{1,t^-}\}$ is of measure zero (see [Sat99]), one has:

$$\begin{aligned}\mathbb{E}[\frac{X_{2,T}}{2} h^{(1)}(x_0 + X_{1,T})] &= \mathbb{E}[\frac{X_{2,T}}{2} G^{(1,0)}(\int_0^T \sigma_t dW_t, J_T)] \\ &= \mathbb{E}[(\int_0^T X_{1,t}(\sigma_t^{(1)} dW_t + \mu_t^{(1)} dt)) G^{(1,0)}(\int_0^T \sigma_t dW_t, J_T)].\end{aligned}$$

Because J_T is independent of $(W_t)_{t\in[0,T]}$ and by Lemma 2.2, one has:

$$\begin{aligned}\mathbb{E}[\frac{X_{2,T}}{2} h^{(1)}(x_0 + X_{1,T})] = &\mathbb{E}[(\int_0^T \sigma_t \sigma_t^{(1)} X_{c,1,t} dt) G^{(2,0)}(\int_0^T \sigma_t dW_t, J_T)] \\ &+ \mathbb{E}[(\int_0^T \mu_t^{(1)} X_{c,1,t} dt) G^{(1,0)}(\int_0^T \sigma_t dW_t, J_T)] \\ &+ \mathbb{E}[(\int_0^T \sigma_t \sigma_t^{(1)} J_t dt) G^{(2,0)}(\int_0^T \sigma_t dW_t, J_T)] \\ &+ \mathbb{E}[(\int_0^T \mu_t^{(1)} J_t dt) G^{(1,0)}(\int_0^T \sigma_t dW_t, J_T)].\end{aligned}$$

Apply lemmas 2.3 and 2.4 and use Definition 1.5 of Greeks to get the result. □

Remark 2.5 The above approximation of the price is a summation of three terms:

1. $\mathbb{E}[h(x_0+X_{1,T})]$: The leading order which corresponds to the price when the parameters σ,μ are deterministic. We know that in this case, there is a closed formula : Merton closed formula for call(put), or by FFT tools for any other payoff because the characteristic function of $X_{1,T}$ is explicit. For instance, the formula for a call in the Merton model (see [Mer76]) on the log asset is:

$$\sum_{i=0}^{\infty}\frac{(\lambda T)^i}{i!}e^{-\lambda T-\int_0^T r(u)du}Call_{BS}\Bigg(e^{x_0+\int_0^T(r(u)-q(u))du+\lambda(1-e^{\eta_J+\frac{\gamma_J^2}{2}})T+i(\eta_J+\frac{\gamma_J^2}{2})},K,T,\sqrt{\int_0^T\frac{\sigma^2(t,x_0)}{T}dt+i\gamma_J^2}\Bigg),$$

   where $Call_{BS}(S,K,T,v)$ is the Black-Scholes price for a call on an underlying S_t with initial condition $S_0=S$, with strike K, volatility v and exercised at maturity T, where the risk free rate and the dividend yield are set to 0%.
2. $\sum_{i=1}^3\alpha_{i,T}\text{Greek}_i^h(x_0+X_{1,T})$: The volatility and drift correction term which depends on the first derivatives of μ and σ. This term can be computed as easily as the main term.
3. $\sum_{i=1}^3\beta_{i,T}\text{Greek}_i^h(x_0+X_{1,T}+Y')$: The jump correction term which depends on the first derivatives of μ, σ and on the jump parameters. Since Y' is also Gaussian and independent of $X_{1,T}$, the computation of theses Greeks are similar to the previous ones, by adding to the mean $\int_0^T\mu_t dt$ and variance $\int_0^T\sigma_t^2dt$ the quantities η_J and γ_J.

Remark 2.6 In the AA model on the log-asset, one has:

$$\begin{aligned}
\alpha_{1,T}=&\frac{1}{2}\int_0^T\sigma_t^2\omega(T)_t^{\sigma\sigma^{(1)}}dt+\lambda(e^{\eta_J+\frac{\gamma_J^2}{2}}-1)\int_0^T t\sigma_t\sigma_t^{(1)}dt,\\
\alpha_{2,T}=&-\frac{3}{2}\int_0^T\sigma_t^2\omega(T)_t^{\sigma\sigma^{(1)}}dt-\lambda(e^{\eta_J+\frac{\gamma_J^2}{2}}-1)\int_0^T t\sigma_t\sigma_t^{(1)}dt,\\
\alpha_{3,T}=&\int_0^T\sigma_t^2\omega(T)_t^{\sigma\sigma^{(1)}}dt,\\
\beta_{1,T}=&-\lambda\eta_J\int_0^T t\sigma\sigma_t^{(1)}dt,\\
\beta_{2,T}=&\lambda(\eta_J-\gamma_J)\int_0^T t\sigma_t\sigma_t^{(1)}dt,\\
\beta_{3,T}=&\lambda\gamma_J\int_0^T t\sigma_t\sigma_t^{(1)}dt.
\end{aligned}$$

Thus, the computation of these constants simply reduces to that of $\int_0^T t\sigma_t\sigma_t^{(1)}dt$ and $\int_0^T\sigma_t^2\omega(T)_t^{\sigma\sigma^{(1)}}dt$.

We mention that we could perform higher order approximation formulas, still explicit. The only difference is that the number of random variables used as argument for the Greeks will increase at each order, and it is in the set $(X_{1,T}+Y_1'+\cdots+Y_i')_{i\in\mathbb{N}}$. We refer to [Miri] for higher order terms.

3 Financial Modeling Consequences

For simplicity, we consider the AA model on the log-asset (an analogous statement would be available for the jump diffusion model on the asset).
The standard Gaussian framework as developed by Black-Scholes (1973), Merton (1976) is realized by making the volatility function σ constant (the result is still available for a function dependent on time only). In order to manage on fixed income (without jump) Andersen and Andreasen [AA02] take a parametric form for σ, Piterbarg[1] [Pit05] also uses the same shape for Power Reverse Dual Currency swaps in order to handle the skew for the FX:

$$\sigma(t,x) = \nu(t)e^{(\beta(t)-1)x}, \tag{3.1}$$

where $\nu(t)$ the relative volatility function, $\beta(t)$ is a time-dependent constant elasticity of variance.

Because of $\mu = \lambda(1-e^{\eta_J+\frac{\gamma_J^2}{2}}) - \frac{\sigma^2}{2}$, the approximation formula (2.2) depends only on $\sigma(t,x_0), \sigma^{(0,1)}(t,x_0), \lambda, \eta_J, \gamma_J$. The volatility given in equation (3.1) may generate all possible values of the following time-dependent functions $\sigma(t,x_0) = \nu(t)e^{(\beta(t)-1)x_0}$, $\sigma^{(0,1)}(t,x_0) = (\beta(t)-1)\nu(t)e^{(\beta(t)-1)x_0}$, because it has two degrees of freedom $\nu(t)$, $\beta(t)$. So this kind of volatility may create all attainable prices in this class of models, and thus all attainable Black Scholes smiles. This justifies the interest for the volatility (3.1).

Attainable Black Scholes smiles of the model. Can we predict what is the general form of the smiles generated by this model?

- For short maturity: according to our approach, the model is close to the Merton model related to $x_0 + X_{1,T}$. So the shape of implied volatilities forms a smile centered in a point close to the money, which is at the left when $\eta_J + \frac{\gamma_J^2}{2} > 0$ (at the right when $\eta_J + \frac{\gamma_J^2}{2} < 0$).
  Formal Proof: Using the approximation formula, the correction terms are $O(\sqrt{T})$. So when T decreases to zero, the price converges to the Merton price. The second statement is easy to check. One can follow the approach done in [Gat02,Mat00] using characteristic functions, or can prove it directly using some derivations of the Merton formula [Mer76].
- For long maturity: the smile becomes a skew which is due to the local volatility function (because the smile for the Merton model flattens for long maturity).

Smile's Dynamics. According to [HKLD02], the smile created by a given model would be compatible with real market if it had the same dynamics as the forward. Is this property fulfilled in this context? The model has the Merton model as a good proxy. The implied volatilities for the Merton model are increasing and depend only on the ratio between the forward and the strike. Therefore, the smile should move in the same direction as the forward.

[1] If σ_{Pit} is the local volatility used in [Pit05] and $L(t) = e^{\int_0^t (r(u)-q(u))du}$, one has $\sigma(t,x) = \sigma_{Pit}(t, L_t e^x)$.

4 Numerical Experiments

In this section, we give details of the implementation for the approximation (2.2). After that, a generic bootstrap algorithm for calibration purpose is derived. Finally, a numerical application of this algorithm is applied to market data (Index Option).

4.1 Numerical Implementation

The case of homogeneous parameters $\sigma(t,x_0),\sigma^{(0,1)}(t,x_0),\mu(t,x_0),\mu^{(0,1)}(t,x_0)$ gives us the coefficients α,β exactly. This result is still valid for higher orders.
In addition, when these parameters are time-dependent, there are two cases.
Either the data are smooth. In which case, we use a Gauss-Legendre curvature formula (see [PTVF]) for the calculus of the terms α,β.
Or the data are piecewise constant. In this case, we can give explicit expressions of α,β as function of the piecewise constant data. Let $T_0=0\leq T_1\leq\cdots,\leq T_n=T$ such that $\sigma(t,x_0),\sigma^{(0,1)}(t,x_0),\mu(t,x_0),\mu^{(0,1)}(t,x_0)$ are constant at each interval $]T_i,T_{i+1}]$ and are equal respectively to $\sigma_{T_{i+1}},\sigma^{(1)}_{T_{i+1}},\mu_{T_{i+1}},\mu^{(1)}_{T_{i+1}}$. Before giving the recurrence relation, we need to introduce the following functions: $\tilde{\omega}_{1,t}=\omega(t)_0^{\sigma^2},\tilde{\omega}_{2,t}=\omega(t)_0^{\mu}$.

Proposition 4.1 ***Recurrence calculus.***
For piecewise constant coefficients, one has:

$$
\begin{aligned}
\alpha_{1,T_{i+1}} =&\alpha_{1,T_i}+(T_{i+1}-T_i)\mu^{(1)}_{T_{i+1}}\tilde{\omega}_{2,T_i}+\frac{(T_{i+1}-T_i)^2}{2}\mu_{T_{i+1}}\mu^{(1)}_{T_{i+1}},\\
\alpha_{2,T_{i+1}} =&\alpha_{2,T_i}+(T_{i+1}-T_i)(\mu^{(1)}_{T_{i+1}}\tilde{\omega}_{1,T_i}+\sigma_{T_{i+1}}\sigma^{(1)}_{T_{i+1}}\tilde{\omega}_{2,T_i})\\
&+\frac{(T_{i+1}-T_i)^2}{2}(\sigma^2_{T_{i+1}}\mu^{(1)}_{T_{i+1}}+\mu_{T_{i+1}}\sigma_{T_{i+1}}\sigma^{(1)}_{T_{i+1}}),\\
\alpha_{3,T_{i+1}} =&\alpha_{3,T_i}+(T_{i+1}-T_i)\sigma_{T_{i+1}}\sigma^{(1)}_{T_{i+1}}\tilde{\omega}_{1,T_i}+\frac{(T_{i+1}-T_i)^2}{2}\sigma^3_{T_{i+1}}\sigma^{(1)}_{T_{i+1}},\\
\beta_{1,T_{i+1}} =&\beta_{1,T_i}+\lambda\,\eta_J\frac{(T^2_{i+1}-T^2_i)}{2}\mu^{(1)}_{T_{i+1}},\\
\beta_{2,T_{i+1}} =&\beta_{2,T_i}+\lambda\frac{(T^2_{i+1}-T^2_i)}{2}(\gamma_J\mu^{(1)}_{T_{i+1}}+\eta_J\sigma_{T_{i+1}}\sigma^{(1)}_{T_{i+1}}),\\
\beta_{3,T_{i+1}} =&\beta_{3,T_i}+\lambda\,\gamma_J\frac{(T^2_{i+1}-T^2_i)}{2}\sigma_{T_{i+1}}\sigma^{(1)}_{T_{i+1}},\\
\tilde{\omega}_{1,T_{i+1}} =&\tilde{\omega}_{1,T_i}+(T_{i+1}-T_i)\sigma^2_{T_{i+1}},\\
\tilde{\omega}_{2,T_{i+1}} =&\tilde{\omega}_{2,T_i}+(T_{i+1}-T_i)\mu_{T_{i+1}}.
\end{aligned}
$$

Proof According to Theorem 2.1, one has:

$$\begin{aligned}
\alpha_{1,T_{i+1}} =& \int_0^{T_i} \mu_t \omega(T_{i+1})_t^{\mu^{(1)}} dt + \int_{T_i}^{T_{i+1}} \mu_t \omega(T_{i+1})_t^{\mu^{(1)}} dt \\
=& \alpha_{1,T_i} + \int_0^{T_i} \mu_t \omega(T_{i+1})_{T_i}^{\mu^{(1)}} dt + \int_{T_i}^{T_{i+1}} \mu_t \omega(T_{i+1})_t^{\mu^{(1)}} dt \\
=& \alpha_{1,T_i} + \omega(T_{i+1})_{T_i}^{\mu^{(1)}} \int_0^{T_i} \mu_t dt + \int_{T_i}^{T_{i+1}} \mu_t \omega(T_{i+1})_t^{\mu^{(1)}} dt \\
=& \alpha_{1,T_i} + (T_{i+1} - T_i)\mu_{T_{i+1}}^{(1)} \tilde{\omega}_{2,T_i} + \frac{(T_{i+1} - T_i)^2}{2} \mu_{T_{i+1}} \mu_{T_{i+1}}^{(1)}.
\end{aligned}$$

The other terms are calculated analogously. □

4.2 Algorithm of Calibration

For this kind of model (AA model on the log asset or on the asset itself), calibration is still challenging as this model has no analytical formula. We can still do a numerical calibration using the forward PIDE as explained in [AA00], but the time of calibration remains quite long (about one minute). With our approach, we can shorten the duration of calibration to be less than one second. We can do that by a simple bootstrapping algorithm using the path dependent formula.

Bootstrap algorithm for piecewise data . Suppose that we want to fit option prices for n maturities $T_0 = 0 \leq T_1 \leq \cdots \leq T_n$ and m strikes $K_1, \cdots, K_m$. First, we search the best fitted parameters $\lambda, \eta_J, \gamma_J$. At each interval $]T_{i-1}, T_i]$, the data σ, $\sigma^{(0,1)}$, μ, $\mu_{(0,1)}$ are constant, equal respectively to σ_{T_i}, $\sigma_{T_i}^{(1)}$, μ_{T_i}, $\mu_{T_i}^{(1)}$ and depending on the vector $\chi_i = (\nu(T_i), \beta(T_i))$ (see formula 3.1). Starting at $i = 1$, we express the coefficient α and β as a function of χ_i using Proposition (4.1). We apply a minimization algorithm (for instance, the Levenberg-Marquardt as described in [PTVF]) in order to fit the implied volatilities for all strikes $K_1, \cdots, K_m$ at maturity T_i using our approximation (2.2). Once the vector χ_i is found, we go to the next step $i+1$, update α and β and compute χ_{i+1}.

4.3 Numerical application

4.3.1 Accuracy of the approximation

Here, we give a short example of the performance of our method. The jump parameters have been set to: $\lambda = 30\%, \eta_J = -8\%, \gamma_J = 35\%$. These parameters are not small ones, especially for the intensity of the Poisson process λ, and the volatility of the jump γ_J. The piecewise constant functions ν and β are equal respectively at each interval of the form $[\frac{i}{20}, \frac{i+1}{20}]$ to $25\% - i \times 0.11\%, 1 - i \times 0,75\%$. The spot, the risk free rate and the dividend yield are set respectively to $100, 4\%, 0\%$.
We observe in the table below that the errors of implied Black Scholes volatilities

between our approximation and the price calculated using a PIDE method do not exceed **2 bp** for a large range of strikes and maturities. The computational time of our formula is less than 4 milliseconds on a $2,6$ GHz Pentium PC. The accuracy of our formula turns to be excellent.

Table 4.1 Error in implied Black Scholes volatilities (in bp) between the approximation formula and the PIDE method expressed as function of maturities in fraction of years and relatives strikes.

T/K	70%	85%	100%	120%	150%
3M	0.02	-0.03	-0.92	-0.07	-0.12
1Y	0.04	0.06	0.15	-0.11	0.01
3Y	0.22	-0.23	0.11	0.41	0.31
5Y	1.39	1.06	-0.01	1.85	1.76

4.3.2 Calibration issues

Here, we calibrate the EURO STOXX 50 Index. The surface of implied B-S volatility is given in the table 4.2.

Table 4.2 Implied B-S volatilities for the EURO STOXX Index expressed as function of maturities in fraction of years and relatives strikes. The risk free rate is equal to 4.08%.

T/K	95%	97.5%	100%	102.5%	105%
3M	20.88%	19.47%	18.13%	16.91%	15.85%
6M	21.12%	20.07%	19.26%	18.55%	17.70%
9M	21.30%	20.47%	19.86%	19.33%	18.65%
1Y	21.39%	20.67%	20.16%	19.71%	19.11%
1.5Y	21.46%	20.90%	20.61%	20.40%	19.92%
2Y	21.89%	21.41%	21.18%	21.02%	20.61%

The jump parameters for the calibrated model are $\lambda = 28.52\%$, $\eta_J = -31.32\%$ and $\gamma_J = 5.11\%$. The diffusion parameters v and β for the calibrated model are given in the table 4.3. These values are realistic. The errors between the implied volatilities generated by the calibrated model and the market data are given in the table 4.4.

Table 4.3 Calibrated values of the piecewise constant functions v and β.

T	v	β
3M	21.48%	94.36%
6M	18.73%	95.63%
9M	21.46%	93.81%
1Y	21.41%	93.39%
1.5Y	18.06%	96.60%
2Y	18.15%	98.38%

Table 4.4 Errors between implied B-S volatilities for the EURO STOXX 50 Index and those calculated within the calibrated model (in bp) expressed as function of maturities in fraction of years and relatives strikes. The risk free rate is equal to 4.08%.

T/K	95%	97.5%	100%	102.5%	105%
3M	9	25	14	-16	54
6M	2	-5	1	7	-13
9M	8	-6	-3	5	-8
1Y	22	1	-2	2	-13
1.5Y	22	-4	-4	4	-15
2Y	30	2	-2	2	-20

The errors show that our model is a good model for the Index EURO STOXX 50. Within our relevant algorithm, we are able to fit a 6×5 grid of quoted prices of Index options in less than **400 milliseconds**.

5 Infinitesimal Perturbation

In the following, we make some notations that will be used throughout the theorems and the proofs. Then, we perform a general expansion of the price $g(1)$ at any order, making explicit the order of magnitude of each term. After detailing some formal calculus for smooth payoff, we show how to extend this result to Lipschitz payoffs (case of call, put) under suitable conditions related to the ellipticity of σ. Finally, we extend the result to irregular payoffs (case of binary options) under similar suitable conditions. We discuss the form of the accuracy of the three analysis.

Notations.

Notation 5.1 *Let $n \in \mathbb{N}$, $i \in \mathbb{N}$, , $l_1, \cdots, l_n \in \mathscr{C}^i(\mathbb{R})$, $a \in \mathbb{R}^+$ and $\psi_1, \cdots, \psi_n \in \mathscr{C}^i(\mathbb{R}^2)$, we set:*

$$M^i_{l_1,\cdots,l_n} = \max(|l_1^{(j)}|_\infty, \cdots, |l_n^{(j)}|_\infty, j \leq i),$$
$$M^i_{\psi_1,\cdots,\psi_n,a} = \max(|\psi_1^{(0,j)}|_\infty, \cdots, |\psi_n^{(0,j)}|_\infty, a, j \leq i).$$

Notation 5.2
- *if $(Z)_{t\in[0,T]}$ is a càdlàg process, we define Z^* by $Z^*_t = \sup_{s\leq t} |Z_s|, \forall t \in [0,T]$.*
- *The L_p norm of a process $(Z)_{t\in[0,T]}$ at time t is denoted as usually by $\|Z_t\|_p = \mathbb{E}[(|Z_t|)^p]^{1/p}$.*

Notation 5.3 ***Generic Constants and Polynomials.***
We keep the same notation C for all non negative non decreasing constant depending on a number $p \geq 1$ arising in L_p estimates, on $|\sigma|_\infty$, $|\mu|_\infty$, $|\sigma^{(0,1)}|_\infty$, $|\mu^{(0,1)}|_\infty$, λ, $|\eta_J|$, γ_J, T and on universal constants.
We denote by P_{gen} any polynomial whose coefficients are generic constants and whose variables are $(M^j_{\sigma,\mu,\lambda})_j$.

Notation 5.4 ***General Differentiation.***
If these derivatives have a meaning, we write:

- $X^{\varepsilon}_{i,t} = \frac{\partial^i X^{\varepsilon}_t}{\partial \varepsilon^i}$, $Y^{\varepsilon}_T = X^{\varepsilon}_T - (x_0 + \varepsilon X_{1,T})$, $Y^{\varepsilon}_{k,i,T} = \frac{\partial^i ((Y^{\varepsilon}_T)^k)}{\partial \varepsilon^i}$, $Y_{k,i,T} = Y^0_{k,i,T}$;
- $R_{k,i,T} = \frac{\int_0^1 Y^{(1-\lambda)}_{k,i+1,T} \lambda^i d\lambda}{(i)!}$.

5.1 Smooth Payoff

Theorem 5.1 ***Asymptotic expansion for the price of smooth payoff (*** $h \in \mathscr{C}^{\infty}_0(R)$***).*** *For $m \geq 2$ assume that (R_{m+2}) holds. Then one has*

$$\mathbb{E}[h(X^1_T)] = \mathbb{E}[h(x_0 + X_{1,T})] + \sum_{i=2}^{m} Ord_i + Resid_m,$$

where different terms are as follows.

- *The contribution for order $i \in [\![2,m]\!]$ is : $Ord_i = \sum_{k=1}^{[\frac{i}{2}]} \mathbb{E}[h^{(k)}(x_0 + X_{1,T}) \frac{Y_{k,i,T}}{k!i!}]$ and it is estimated by*

$$|Ord_i| \leq P_{\text{gen}} M^{[\frac{i}{2}]-1}_{h^{(1)}} \prod_{j=1}^{i} (M^j_{\sigma,\mu,\lambda} \sqrt{T}).$$

- *The residual for order m is : $Resid_m = \mathbb{E}[\sum_{k=1}^{[\frac{m}{2}]} h^{(k)}(x_0 + X_{1,T}) \frac{R_{k,m,T}}{k!}$ $+ \frac{(Y^1_T)^{[\frac{m}{2}]+1}}{([\frac{m}{2}])!} \int_0^1 h^{([\frac{m}{2}]+1)}(vX^1_T + (1-v)(x_0 + X_{1,T}))(1-v)^{[\frac{m}{2}]} dv]$, such that*

$$|Resid_m| \leq P_{\text{gen}} M^{[\frac{m}{2}]}_{h^{(1)}} \prod_{j=1}^{m+1} (M^j_{\sigma,\mu,\lambda} \sqrt{T}).$$

Remark 5.2 **Multiplicative case** ($\sigma(t,x) = \Delta s(t,x)$, $\mu(t,x) = \Delta m(t,x)$ and $\lambda \leq \Delta$) In that case, $M^j_{\sigma,\mu,\lambda} = O(\Delta)$. Thus, one has for all $i \in [\![2,m]\!]$:

$$Ord_i = O((\Delta\sqrt{T})^i),$$
$$Resid_m = O((\Delta\sqrt{T})^m).$$

For example, at the second order the price can be approximated in the multiplicative case by:

$$\mathbb{E}[h(X^1_T)] = \mathbb{E}[h(X_{1,T})] + \mathbb{E}[h^{(1)}(X_{1,T}) \frac{X_{2,T}}{2}] + O((\Delta\sqrt{T})^3).$$

We recall that the second term is explicitly computed in Theorem 2.1. For higher terms, see [Miri].

IDEA OF THE PROOF :

Let Δ be a small positive number parameterizing the function σ, μ and the size of the jump frequency λ: $\sigma(t,x) = \Delta s(t,x)$, $\mu(t,x) = \Delta m(t,x)$ and $\lambda \leq \Delta$.

Under assumption (R_{m+2}), X^{ε}_t is C^{m+1} w.r.t ε. From Equation (2.1) the first derivative satisfies:

$$dX^{\varepsilon}_{1,t} = \Delta(s(t,X^{\varepsilon}_{t^-})dW_t + m(t,X^{\varepsilon}_{t^-})dt) + dJ_t + \Delta(\varepsilon X^{\varepsilon}_{1,t^-}(s^{(0,1)}(t,X^{\varepsilon}_{t^-})dW_t + m^{(0,1)}(t,X^{\varepsilon}_{t^-})dt)), X^{\varepsilon}_{1,0} = 0.$$

Then, it is not difficult to show that the first derivative is of the order of $\Delta\sqrt{T}$: for any $p \geq 1$,

$$\sup_{\varepsilon\in[0,1]} \|X^{\varepsilon}_{1,T}\|_p = O(\Delta\sqrt{T}).$$

We can also prove similar results for the second and third derivatives :

$$\sup_{\varepsilon\in[0,1]} \|X^{\varepsilon}_{2,T}\|_p = O((\Delta\sqrt{T})^2),\ \sup_{\varepsilon\in[0,1]} \|X^{\varepsilon}_{3,T}\|_p = O((\Delta\sqrt{T})^3).$$

Then, the residual $R_{1,2,T}$ is of order of $(\Delta\sqrt{T})^3$.
Using a Taylor expansion twice (see the beginning of Section 2), one has:

$$\begin{aligned}
\mathbb{E}[h(X^1_T)] =& \mathbb{E}[h(x_0 + X_{1,T} + \frac{X_{2,T}}{2} + R_{1,2,T})] \\
=& \mathbb{E}[h(X_{1,T})] + \mathbb{E}[h^{(1)}(x_0 + X_{1,T})\frac{X_{2,T}}{2}] \\
&+ \mathbb{E}[h^{(1)}(x_0 + X_{1,T})R_{1,2,T}] + \int_0^1 \mathbb{E}[h^{(2)}((1-v)(x_0 + X_{1,T}) + vX^1_T)\frac{(\frac{X_{2,T}}{2} + R_{1,2,T})^2}{2}](1-v)dv \\
=& \mathbb{E}[h(x_0 + X_{1,T})] + \mathbb{E}[h^{(1)}(x_0 + X_{1,T})\frac{X_{2,T}}{2}] + O((\Delta\sqrt{T})^3).
\end{aligned} \tag{5.1}$$

Therefore, the main term of the price is $\mathbb{E}[h(x_0 + X_{1,T})]$ and the first correction term is $\mathbb{E}[h^{(1)}(x_0 + X_{1,T})\frac{X_{2,T}}{2}]$ which is already of order $(\Delta\sqrt{T})^2$.
The advantage of the multiplicative case is to give a clear view on how computations should be carried. Nevertheless, this is a simplified setting because in general the derivatives of functions σ, μ may be not of the same magnitude than the functions themselves (for example $|\sigma^{(0,1)}|_\infty$ is not necessary of the same magnitude than $|\sigma|_\infty$). We show in the postponed proof in the paragraph 6.1 that the order $(\Delta\sqrt{T})^i$ has to be replaced by $\prod_{j=1}^i (M^j_{\sigma,\mu,\lambda}\sqrt{T})$ in the general case. □

5.2 Vanilla Options

The payoff h for this kind of option is not necessarily smooth, it is almost everywhere differentiable and belongs to the space $\mathscr{H}$. Therefore we introduce some new variables in order to represent higher contributions only using $h^{(1)}$ (and not higher order derivatives).

Lemma 5.3 *Assume (E) and (R_N) for $N \geq 4$. Let $i \in [\![1, [\frac{N+2}{3}]]\!], v \in [0,1]$. There exist three variables $G_i, S_i, I_{i,v} \in \cap_{p\geq 1} L_p$ such that for any $l \in \mathscr{C}^\infty_0(\mathbb{R})$, one has*

$$\begin{aligned}
\mathbb{E}[l^{(1)}(x_0 + X_{1,T})G_i] &= \sum_{k=1}^{i-1} \frac{1}{k!}\mathbb{E}[l^{(k)}(x_0 + X_{1,T})\frac{Y_{k,k+i-1,T}}{(k+i-1)!}], \\
\mathbb{E}[l^{(1)}(x_0 + X_{1,T})S_i] &= \sum_{k=1}^{i-1} \frac{1}{k!}\mathbb{E}[l^{(k)}(x_0 + X_{1,T})R_{k,k+i-1,T}], \\
\mathbb{E}[l^{(1)}(vX^1_T + (1-v)(x_0 + X_{1,T}))I_{i,v}] =& \mathbb{E}[\frac{(Y^1_T)^i}{(i-1)!}l^{(i)}(vX^1_T + (1-v)(x_0 + X_{1,T}))],
\end{aligned}$$

and they are estimated in the L_p norm by:

$$\|G_i\|_p \leq P_{\text{gen}} \frac{(M^0_{\sigma,\mu,\lambda})^{(i-2)(2i-3)} (\prod_{l=1}^{2(i-1)} M^l_{\sigma,\mu,\lambda})^{\frac{i-2}{2(i-1)}}}{I_\sigma^{2(i-1)(i-2)}} (\prod_{l=1}^{2(i-1)} (M^l_{\sigma,\mu,\lambda}\sqrt{T}))^{\frac{i}{2(i-1)}},$$

$$\|S_i\|_p + \sup_{v\in[0,1]} \|I_{i,v}\|_p \leq P_{\text{gen}} \frac{(M^0_{\sigma,\mu,\lambda})^{(i-1)(2i-1)} (\prod_{l=1}^{2(i-1)+1} M^l_{\sigma,\mu,\lambda})^{\frac{i-1}{2(i-1)+1}}}{I_\sigma^{2(i-1)i}} (\prod_{l=1}^{2(i-1)+1} (M^l_{\sigma,\mu,\lambda}\sqrt{T}))^{\frac{i+1}{2(i-1)+1}}.$$

Theorem 5.4 ***Asymptotic expansion for the price of vanilla payoff ($h \in \mathscr{H}$ and $h' \in \mathscr{H}$).***
For $m \geq 2$ assume that (R_{3m-2}) holds. Then one has

$$\mathbb{E}[h(X_T^1)] = \mathbb{E}[h(x_0 + X_{1,T})] + \sum_{i=2}^{m} Ord_i + Resid_m, \tag{5.2}$$

where different terms are as follows.

– *The contribution for order $i \in [\![2,m]\!]$ is $Ord_i = \mathbb{E}[h^{(1)}(x_0 + X_{1,T})G_i]$ and it is estimated by:*

$$|Ord_i| \leq P_{\text{gen}} \|h^{(1)}(x_0 + X_{1,T})\|_2 \frac{(M^0_{\sigma,\mu,\lambda})^{(i-2)(2i-3)} (\prod_{l=1}^{2(i-1)} M^l_{\sigma,\mu,\lambda})^{\frac{i-2}{2(i-1)}}}{I_\sigma^{2(i-1)(i-2)}} (\prod_{l=1}^{2(i-1)} (M^l_{\sigma,\mu,\lambda}\sqrt{T}))^{\frac{i}{2(i-1)}}. \tag{5.3}$$

– *The residual for order m is $Resid_m = \mathbb{E}[h^{(1)}(x_0 + X_{1,T})S_m] + \int_0^1 \mathbb{E}[h^{(1)}(vX_T^1 + (1-v)(x_0 + X_{1,T}))I_{m,v}](1-v)^{m-1}dv$, such that*

$$|Resid_m| \leq P_{\text{gen}} (\|h^{(1)}(x_0 + X_{1,T})\|_2 + \sup_{v\in[0,1]} \|h^{(1)}(vX_T^1 + (1-v)(x_0 + X_{1,T}))\|_2) \frac{(M^0_{\sigma,\mu,\lambda})^{(m-1)(2m-1)} (\prod_{l=1}^{2(m-1)+1} M^l_{\sigma,\mu,\lambda})^{\frac{m-1}{2(m-1)+1}}}{I_\sigma^{2(m-1)m}} (\prod_{l=1}^{2(m-1)+1} (M^l_{\sigma,\mu,\lambda}\sqrt{T}))^{\frac{m+1}{2(m-1)+1}}. \tag{5.4}$$

Remark 5.5 – We have brought together correction terms in a different way than in the case of smooth payoff. Indeed, the hierarchy (in terms of magnitude) is modified by the smoothness of the payoff. To get an intuitive and clear view of this hierarchy, one should have in mind the multiplicative case ($\sigma(t,x) = \Delta s(t,x)$, $\mu(t,x) = \Delta m(t,x)$, $\lambda \leq \Delta$ and $M^j_{\sigma,\mu,\lambda} = O(\Delta)$) :

$$Ord_i = O((\Delta\sqrt{T})^i),$$
$$Resid_m = O((\Delta\sqrt{T})^m).$$

- We stress the fact that the formula $Ord_i = \mathbb{E}[h^{(1)}(x_0 + X_{1,T})G_i]$ is just a convenient representation of the correction term of order i (in order to control it) but to compute it explicitly, we proceed as in Theorem 2.1 by decomposing Ord_i as a sum of Greeks in the Merton model.
- Note that Ord_1 does not appear because we use the Merton model as a proxy (this gives $\mathbb{E}[(x_0 + X_{1,T})]$ as the main term).
- The contribution $Ord_2 = \mathbb{E}[h^{(1)}(x_0 + X_{1,T})\frac{X_{2,T}}{2}]$ coincides with that already computed in the case of smooth payoff (Theorems 2.1 and 5.1).
- For higher contributions, there is no more coincidence with the smooth case.
- Finally to accommodate irregular payoff, we require extra smoothness properties on μ and σ.

IDEA OF THE PROOF :

Take $m = 2$. We use here the same context of multiplicative coefficients ($\sigma(t,x) = \Delta s(t,x), \mu(t,x) = \Delta m(t,x), \lambda \leq \Delta$ and Δ a small real parameter).
Rewriting equation (5.1):

$$\begin{aligned}\mathbb{E}[h(X_T^1)] =& \mathbb{E}[h(x_0 + X_{1,T})] + \mathbb{E}[h^{(1)}(x_0 + X_{1,T})\frac{X_{2,T}}{2}] \\ &+ \mathbb{E}[h^{(1)}(x_0 + X_{1,T})R_{1,2,T}] + \int_0^1 \mathbb{E}[h^{(2)}((1-v)(x_0 + X_{1,T}) + vX_T^1)\frac{(Y_T^1)^2}{2}](1-v)dv.\end{aligned}$$

The difference with the case of smooth payoff yields in the term $\int_0^1 \mathbb{E}[h^{(2)}((1-v)(x_0 + X_{1,T}) + vX_T^1)\frac{(Y_T^1)^2}{2}](1-v)dv$ because $h^{(2)}$ does not necessarily exists anymore. However, using Lemma 5.3, one has:

$$\begin{aligned}&\int_0^1 \mathbb{E}[h^{(2)}((1-v)(x_0 + X_{1,T}) + vX_T^1)\frac{(Y_T^1)^2}{2}](1-v)dv \\ &= \int_0^1 \mathbb{E}[h^{(1)}((1-v)(x_0 + X_{1,T}) + vX_T^1)I_{2,v}](1-v)dv.\end{aligned}$$

Finally, since smooth payoffs are dense in the set of vanilla ones, we obtain the identity (5.2) for vanilla options. In the one hand, the Inequality (5.3) yields using the same reasoning like the previous proof. In the other hand, due to the ellipticity of σ, the order of $I_{2,v}$ is obtained by decreasing the order of $(Y_T^1)^2 = \frac{(\frac{X_{2,T}}{2} + R_{1,2,T})^2}{2}$ which is $(\Delta\sqrt{T})^4$ to $(\Delta\sqrt{T})^3$ and multiplying it by an ellipticity term of the shape $\frac{\Delta^4}{(I_\sigma)^4}$ (for technical details see Lemma 6.11). Therefore yields the Inequality (5.4). □

The rigorous proof of the above arguments and the general case are tricky, we need to combine the technical lemma used for Theorem 5.1, and an infinite analysis on Malliavin calculus. The proof is postponed to the paragraph 6.3.

5.3 Binary Options

The payoff for this kind of option, is not necessarily smooth, it is at least in $\mathscr{H}$. The results below are easy extensions of the case of vanilla options, we left the proof to the reader.

Lemma 5.6 *Assume (E) and (R_N) for $N \geq 5$. Let $i \in [\![1, [\frac{N-2}{3}]]\!], v \in [0,1]$. There exist three variables $P_i, Q_i, T_{i,v} \in \cap_{p\geq 1} L_p$ such that, for any $l \in \mathscr{C}_0^\infty(\mathbb{R})$, one has:*

$$\mathbb{E}[l(x_0 + X_{1,T})P_i] = \sum_{k=1}^{i} \frac{1}{k!}\mathbb{E}[l^{(k)}(x_0 + X_{1,T})\frac{Y_{k,k+i,T}}{(k+i)!}],$$

$$\mathbb{E}[l(x_0 + X_{1,T})Q_i] = \sum_{k=1}^{i} \frac{1}{k!}\mathbb{E}[l^{(k)}(x_0 + X_{1,T})R_{k,k+i,T}],$$

$$\mathbb{E}[l(vX_T^1 + (1-v)(x_0 + X_{1,T}))T_{i,v}] = \mathbb{E}[\frac{(Y_T^1)^{i+1}}{i!} l^{(i+1)}(vX_T^1 + (1-v)(x_0 + X_{1,T}))],$$

and they are estimated in the L_p norm by:

$$\|P_i\|_p \leq P_{\text{gen}} \frac{(M^0_{\sigma,\mu,\lambda})^{i(2i+1)}(\prod_{l=1}^{2i} M^l_{\sigma,\mu,\lambda})^{\frac{i}{2i}}}{I_\sigma^{2i(i+1)}} (\prod_{l=1}^{2i}(M^l_{\sigma,\mu,\lambda}\sqrt{T}))^{\frac{i}{2i}},$$

$$\|Q_i\|_p + \sup_{v\in[0,1]} \|T_{i,v}\|_p \leq P_{\text{gen}} \frac{(M^0_{\sigma,\mu,\lambda})^{(i+1)(2i+3)}(\prod_{l=1}^{2i+1} M^l_{\sigma,\mu,\lambda})^{\frac{i+1}{2i+1}}}{I_\sigma^{2(i+2)(i+1)}}$$
$$(\prod_{l=1}^{2i+1}(M^l_{\sigma,\mu,\lambda}\sqrt{T}))^{\frac{i+1}{2i+1}}.$$

Theorem 5.7 ***Asymptotic expansion for the price of binary payoff ($h \in \mathscr{H}$).*** *For $m \geq 1$ assume that (R_{3m+2}) holds. Then one has*

$$\mathbb{E}[h(X_T^1)] = \mathbb{E}[h(x_0 + X_{1,T})] + \sum_{i=1}^{m} Ord_i + Resid_m, \tag{5.5}$$

where different terms are as follows.

– *The contribution for order $i \in [\![1,m]\!]$ is $Ord_i = \mathbb{E}[h(x_0 + X_{1,T})P_i]$ and it is estimated by:*

$$|Ord_i| \leq P_{\text{gen}} \|h(x_0 + X_{1,T})\|_2 \frac{(M^0_{\sigma,\mu,\lambda})^{i(2i+1)}(\prod_{l=1}^{2i} M^l_{\sigma,\mu,\lambda})^{\frac{i}{2i}}}{I_\sigma^{2i(i+1)}} (\prod_{l=1}^{2i}(M^l_{\sigma,\mu,\lambda}\sqrt{T}))^{\frac{i}{2i}}. \tag{5.6}$$

– *The residual for order m is $Resid_m = \mathbb{E}[h(x_0 + X_{1,T})Q_m] + \int_0^1 \mathbb{E}[h(vX_T^1 + (1-v)(x_0 + X_{1,T}))T_{m,v}](1-v)^m dv$, such that*

$$|Resid_m| \leq P_{\text{gen}}(\|h(x_0 + X_{1,T})\|_{2p} + \sup_{v\in[0,1]} \|h(vX_T^1 + (1-v)(x_0 + X_{1,T}))\|_{2p})$$
$$\frac{(M^0_{\sigma,\mu,\lambda})^{(m+1)(2m+3)}(\prod_{l=1}^{2m+1} M^l_{\sigma,\mu,\lambda})^{\frac{m+1}{2m+1}}}{I_\sigma^{2(m+2)(m+1)}} (\prod_{l=1}^{2m+1}(M^l_{\sigma,\mu,\lambda}\sqrt{T}))^{\frac{m+1}{2m+1}}. \tag{5.7}$$

The second order for vanilla options and smooth payoff is a first order for binary options. The residual term in Theorem 2.1 is of order $(\Delta\sqrt{T})^2$ in the multiplicative case (instead of $(\Delta\sqrt{T})^3$ for previous payoffs).

6 Proofs

For the following, we use the same definitions and notations as in chapter 1 of [Nua05]. Before giving the proofs for the main theorems, we need to upper bound the L_p norm of the derivatives $X_{i,t}^{\varepsilon}$ to state Theorem 5.1, to upper bound also the L_p norm of the Malliavin derivatives $D^{j}_{t_1,\cdots,t_j}X_{i,t}^{\varepsilon}$, and use the key lemma 6.11 in order to state Theorems 5.4-5.7.

6.1 Expansion Analysis of X_T^{ε}

In this subsection, we give the general form of the derivatives via some polynomials (Proposition 6.3). After that we give some lemmas in order to state the recurrence Inequality from the i^{th} to the $(i+1)^{th}$ derivatives (Corollary 6.6). This preparatory work leads to Theorem 6.7 which states upper bounds for the derivatives $X_{i,t}^{\varepsilon}$.
We first recall that $\varepsilon \to X_t^{\varepsilon}$ is almost surely C^{N-1} w.r.t. ε under assumption (R_N).
We need to introduce two kinds of polynomials in order to make the derivatives of X_t^{ε} w.r.t. ε explicit.

Definition 6.1 Given $n \in \mathbb{N}$, take $\psi : [0,T]\times\mathbb{R}\to\mathbb{R}$ of class $C^{0,n+1}$. $P : [0,T]\times\mathbb{R}^{n+1}\to\mathbb{R}$ satisfies the property $\mathbb{P}_{\psi,n}$ (shortly written $P\in\mathbb{P}_{\psi,n}$) if P is a polynomial of n variables $x_1,\cdots,x_n$ of the form :

$$P(t,x_0,x_1,\cdots,x_n) = \sum_{\substack{\alpha=(\alpha_1,\cdots,\alpha_n)\in\mathbb{N}^n \\ \sum_{j=1}^n j\alpha_j = n+1}} c_\alpha(t,x_0)\prod_{j=1}^n x_j^{\alpha_j},$$

such that

$$\exists \lambda_{\alpha,0},\cdots,\lambda_{\alpha,n+1}\in\mathbb{R}^{n+2}, \qquad c_\alpha(t,x_0) = \sum_{j=0}^{n+1}\lambda_{\alpha,j}\psi^{(0,i)}(t,x_0).$$

Definition 6.2 Given $n\in\mathbb{N}$, take $\psi : [0,T]\times\mathbb{R}\to\mathbb{R}$ of class $C^{0,n}$. $Q : [0,T]\times\mathbb{R}^{n+1}\to\mathbb{R}$ satisfies the property $\mathbb{Q}_{\psi,n}$ (shortly written $Q\in\mathbb{Q}_{\psi,n}$) if Q is a polynomial of n variables $x_1,\cdots,x_n$ of the form:

$$Q(t,x_0,x_1,\cdots,x_n) = \sum_{\substack{\alpha=(\alpha_1,\cdots,\alpha_n)\in\mathbb{N}^n \\ \sum_{j=1}^n j\alpha_j = n}} c_\alpha(t,x_0)\prod_{j=1}^n x_j^{\alpha_j},$$

such that

$$\exists \lambda_{\alpha,0},\cdots,\lambda_{\alpha,n}\in\mathbb{R}^{n+1}, \qquad c_\alpha(t,x_0) = \sum_{j=0}^{n}\lambda_{\alpha,j}\psi^{(0,i)}(t,x_0).$$

Proposition 6.3 *If X_t^ε is solution of the SDE (1.2), then for each $i \in [\![1, N-1]\!]$:*

$$dX_{i,t}^\varepsilon = dH_{i,t}^\varepsilon + X_{i,t^-}^\varepsilon dL_t^\varepsilon, X_{i,0}^\varepsilon = 0, \tag{6.1}$$

$$dL_t^\varepsilon = \varepsilon(\sigma^{(0,1)}(t, X_{t^-}^\varepsilon)dW_t + \mu^{(0,1)}(t, X_{t^-}^\varepsilon)dt), \tag{6.2}$$

$$dH_{i,t}^\varepsilon = \delta_{i,1}J_t + dR_{i,t}^\varepsilon + \varepsilon dZ_{i+1,t}^\varepsilon,$$

$$dR_{i,t}^\varepsilon = Q_{\sigma,i}(t, X_{0,t^-}^\varepsilon, X_{1,t^-}^\varepsilon, \cdots, X_{i-1,t^-}^\varepsilon)dW_t + Q_{\mu,i}(t, X_{0,t^-}^\varepsilon, X_{1,t^-}^\varepsilon, \cdots, X_{i-1,t^-}^\varepsilon)dt,$$

$$dZ_{i,t}^\varepsilon = P_{\sigma,i}(t, X_{0,t^-}^\varepsilon, X_{1,t^-}^\varepsilon, \cdots, X_{i-1,t^-}^\varepsilon)dW_t + P_{\mu,i}(t, X_{0,t^-}^\varepsilon, X_{1,t^-}^\varepsilon, \cdots, X_{i-1,t^-}^\varepsilon)dt,$$

where δ is Kronecker's delta, $Q_{\sigma,i} \in \mathbb{Q}_{\sigma,i-1}$, $Q_{\mu,i} \in \mathbb{Q}_{\mu,i-1}$, $P_{\sigma,i} \in \mathbb{P}_{\sigma,i-1}$, and $P_{\mu,i} \in \mathbb{P}_{\mu,i-1}$. Notice that $(X_{i,t}^\varepsilon)_t$ are continuous process for $i \geq 2$.

Proof We show the proof by recurrence. For $i = 1$, we have

$$dX_{1,t}^\varepsilon = dJ_t + \sigma(t, X_{t^-}^\varepsilon)dW_t + \mu(t, X_{t^-}^\varepsilon)dt + \varepsilon X_{1,t^-}^\varepsilon(\sigma^{(0,1)}(t, X_{t^-}^\varepsilon)dW_t + \mu^{(0,1)}(t, X_{t^-}^\varepsilon)dt),$$

and putting

$$Q_{\sigma,1}(t,x_0) = \sigma(t,x_0), Q_{\mu,1}(t,x_0) = \mu(t,x_0), P_{\sigma,1}(t,x_0) = 0, P_{\mu,1}(t,x_0) = 0, \tag{6.3}$$

readily gives $Q_{\sigma,1} \in \mathbb{Q}_{\sigma,0}, Q_{\mu,1} \in \mathbb{Q}_{\mu,0}, P_{\sigma,1} \in \mathbb{P}_{\sigma,0}$ and $P_{\mu,1} \in \mathbb{P}_{\mu,0}$.
Now, suppose that Equation (6.1) is true for some $i \leq N-1$. By a direct differentiation of the SDE (6.1), we obtain

$$dX_{i+1,t}^\varepsilon = dH_{i+1,t}^\varepsilon + X_{i+1,t^-}^\varepsilon dL_t^\varepsilon,$$

where

$$dH_{i+1,t}^\varepsilon = dR_{i+1,t}^\varepsilon + \varepsilon dZ_{i+1,t}^\varepsilon.$$

The term in factor of dW_t (in $dR_{i+1,t}^\varepsilon$) can be written as $Q_{\sigma,i+1}(t, X_{0,t^-}^\varepsilon, X_{1,t^-}^\varepsilon, \cdots, X_{i,t^-}^\varepsilon)$ with $Q_{\sigma,i+1} \in \mathbb{Q}_{\sigma,i}$. The same observation holds for the other terms and this finishes the proof. □

The two following lemmas are standard (apply usual stochastic calculus inequalities, see [Pro90]).

Lemma 6.4 *Let $(U)_{t\in[0,T]}$ (resp. $(V)_{t\in[0,T]}$, $Y_{t\in[0,T]}$ and $A_{t\in[0,T]}$) be a continuous process such that U_T^* (resp. V_T^*,Y_T^* and N_T^*) has a finite moment of order $2p$ (resp. $4p$).*
We define Z by :

$$Z_t = U_t \int_0^t V_s(Y_s dW_s + A_s ds).$$

Then, for all $t \in [0,T]$:

$$\|Z_t^*\|_p \leq C\|U_t^*\|_{2p} \, \|V_t^*\|_{4p}\sqrt{t}(\|Y_t^*\|_{4p} + \|A_t^*\|_{4p}).$$

Lemma 6.5 *Let U^ε be the solution of the linear SDE:*

$$dU_t^\varepsilon = U_t^\varepsilon dL_t^\varepsilon, U_0^\varepsilon = 1,$$

where L^ε is given by the SDE (6.2). First $(U_t^\varepsilon)_t$ does not vanish. Second, $(U_T^\varepsilon)^*$ and $((U_T^\varepsilon)^{-1})^*$ have moments of all order bounded by a generic constant.

Note that $X^{\varepsilon}_{i,.}$ is solution of a linear SDE. Hence, it can be explicitly represented using the process U^{ε} (see Th. 52 in [Pro90]):

$$X^{\varepsilon}_{i,t} = U^{\varepsilon}_t \int_0^t (U^{\varepsilon}_s)^{-1}(dH^{\varepsilon}_{i,s} - d < H^{\varepsilon}_{i,.}, L^{\varepsilon}_{i,.} >_s). \tag{6.4}$$

Corollary 6.6 *Take $p \geq 1$. For all $i \in [\![1, N-1]\!]$, $\varepsilon \in [0,1]$ and $t \in [0,T]$, one has:*

$$\|(X^{\varepsilon}_{i,.})^*_t\|_p \leq C\sqrt{t}(\|(Q^{\varepsilon}_{\sigma,i,.})^*_t\|_{4p} + \|(P^{\varepsilon}_{\sigma,i,.})^*_t\|_{4p} + \delta_{i,1}\lambda + \|(Q^{\varepsilon}_{\mu,i,.})^*_t\|_{4p} + \|(P^{\varepsilon}_{\mu,i,.})^*_t\|_{4p}),$$

where $Q^{\varepsilon}_{\sigma,i,t} = Q_{\sigma,i}(t, X^{\varepsilon}_{0,t^-}, X^{\varepsilon}_{1,t^-}, \cdots, X^{\varepsilon}_{i-1,t^-})$ and analogously for $Q^{\varepsilon}_{\mu,i,t}, P^{\varepsilon}_{\sigma,i,t}, P^{\varepsilon}_{\mu,i,t}$.

Proof Consider first the case $i \geq 2$. We know from Proposition (6.3) and Equation (6.4) that $X^{\varepsilon}_{i,t} = U^{\varepsilon}_t \int_0^t ((U^{\varepsilon}_s)^{-1}(Y^{\varepsilon}_s dW_s + A^{\varepsilon}_s ds))$, where $Y^{\varepsilon}_t = Q^{\varepsilon}_{\sigma,i,t} + \varepsilon P^{\varepsilon}_{\sigma,i,t}$, and $A^{\varepsilon}_t = Q^{\varepsilon}_{\mu,i,t} + \varepsilon(P^{\varepsilon}_{\mu,i,t} - Y^{\varepsilon}_t \sigma^{(0,1)}(t, X^{\varepsilon}_{t^-}))$. An application of Lemma 6.4 and 6.5 gives the result for $i \geq 2$. The result for $i = 1$ is straightforward using the same inequalities. □

Theorem 6.7 *Take $p \geq 1$. For all $i \in [\![1, N-1]\!]$, $\varepsilon \in [0,1]$ and $t \in [0,T]$, one has:*

$$\|(X^{\varepsilon}_{i,.})^*_t\|_p \leq P_{\text{gen}} \prod_{j=1}^{i} (M^j_{\sigma,\mu,\lambda}\sqrt{t}), \tag{6.5}$$

Proof We need the lemma below.

Lemma 6.8 *Consider two sequences of real numbers $(a_n)_n$ and $(b_n)_n$ such that $0 < a_1 \leq a_2 \leq, \cdots, \leq a_n$ and $b_1 \geq b_2 \geq, \cdots, \geq b_n$. Then*

$$\prod_{j=1}^{n} a_j^{b_j} \leq \prod_{j=1}^{n} a_j^{\frac{\sum_{k=1}^{n} b_j}{n}}.$$

Proof Define the random variables A and B as follows: $\mathbb{P}(A = \ln a_i, B = b_j) = \frac{\delta_{i,j}}{n}$. Each one is uniformly distributed. Due to the monotonicity properties of the sequences $(a_n)_n$ and $(b_n)_n$, one has $\text{Cov}(A,B) \leq 0$. This writes $\sum_{i=1}^{n} \frac{b_i \ln a_i}{n} \leq \sum_{i=1}^{n} \frac{b_i}{n} \sum_{i=1}^{n} \frac{\ln a_i}{n}$ and passing to the exponential, we get the result. □

Theorem 6.7 is proved by strong recurrence.
Initialization of Recurrence. According to Corollary 6.6 and to Equations (6.3), we have

$$\|(X^{\varepsilon}_{1,.})^*_t\|_p \leq C\sqrt{t}(|\sigma|_\infty + 0 + \lambda + |\mu|_\infty + 0).$$

Then $\|(X^{\varepsilon}_{1,.})^*_t\|_p \leq P_{\text{gen}} M^1_{\sigma,\mu,\lambda}\sqrt{t}$.
Recurrence. Assume now that Inequality (6.5) holds up to index i. First, we can upper bound the coefficients c_α of the four polynomials $Q_{\sigma,i+1}, Q_{\mu,,i+1}, P_{\sigma,i+1}, P_{\mu,i+1}$ by a constant $CM^{i+1}_{\sigma,\mu,\lambda}$. Then, upper bound the absolute value of each monomial for the four polynomials. Each monomial of these polynomials is bounded by

$CM^{i+1}_{\sigma,\mu,\lambda}\prod_{l=1}^{i}((X^{\varepsilon}_{l,.})^*_t)^{\alpha_l}$, where $\sum_{l=1}^{i} l\alpha_l = i$ or $i+1$ (according to Proposition 6.3). Thus, we can deduce by Hölder Inequality ($\sum_{l=1}^{i}\frac{1}{ip}=\frac{1}{p}$)

$$\|(\prod_{l=1}^{i}((X^{\varepsilon}_{l,.})^*_.)^{\alpha_l})_t\|_p \le \prod_{l=1}^{i}(\|(X^{\varepsilon}_{l,.})^*_t\|_{\alpha_l ip})^{\alpha_l}.$$

Using the hypothesis of recurrence, one has:

$$\|(\prod_{l=1}^{i}((X^{\varepsilon}_{l,.})^*_.)^{\alpha_l})_t\|_p \le P_{\text{gen}}\prod_{l=1}^{i}(\prod_{j=1}^{l}M^j_{\sigma,\mu,\lambda}\sqrt{t})^{\alpha_l}.$$

Thus, we get that each monomial of the four polynomials $Q_{\sigma,i+1}, Q_{\mu,.,i+1}, P_{\sigma,i+1}, P_{\mu,i+1}$ is bounded in L_p-norms by

$$\begin{aligned}P_{\text{gen}}M^{i+1}_{\sigma,\mu,\lambda}\prod_{j=1}^{i}(M^j_{\sigma,\mu,\lambda}\sqrt{t})^{\frac{\sum_{l=1}^{i} l\alpha_l}{i}} \le& P_{\text{gen}}M^{i+1}_{\sigma,\mu,\lambda}(M^i_{\sigma,\mu,\lambda}\sqrt{T})^{\sum_{l=1}^{i} l\alpha_l - i}\prod_{j=1}^{i}(M^j_{\sigma,\mu,\lambda}\sqrt{t})\\ \le& P_{\text{gen}}M^{i+1}_{\sigma,\mu,\lambda}\prod_{j=1}^{i}(M^j_{\sigma,\mu,\lambda}\sqrt{t}),\end{aligned}$$

where we used for the first upper bound Lemma 6.8 ($M^1_{\sigma,\mu,\lambda} \le \cdots, \le M^i_{\sigma,\mu,\lambda}$ and $\sum_{l=1}^{i}\alpha_l \ge \cdots \ge \sum_{l=i}^{i}\alpha_l$) and for the second upper bound $M^j_{\sigma,\mu,\lambda} \le M^i_{\sigma,\mu,\lambda}$, $t \le T$ and $\sum_{l=1}^{i} l\alpha_l - i \ge 0$, and the third upper bound $\sum_{l=1}^{i} l\alpha_l - i \ge 0$. Using Corollary 6.6, we obtain immediately the result. □

Proof of Theorem 5.1 (Smooth payoff). One has

$$\begin{aligned}\mathbb{E}[h(X^1_T)] &= \mathbb{E}[h(x_0+X_{1,T})] + \sum_{k=1}^{[\frac{m}{2}]}\frac{1}{k!}\mathbb{E}[h^{(k)}(x_0+X_{1,T})(Y^1_T)^k]\\ &+\int_0^1 \mathbb{E}[\frac{(Y^1_T)^{[\frac{m}{2}]+1}(1-v)^{[\frac{m}{2}]}}{([\frac{m}{2}])!}h^{([\frac{m}{2}]+1)}(vX^1_T+(1-v)(x_0+X_{1,T}))]dv\\ &= \mathbb{E}[h(x_0+X_{1,T})] + \sum_{k=1}^{[\frac{m}{2}]}\frac{1}{k!}\mathbb{E}[h^{(k)}(x_0+X_{1,T})(\sum_{i=2k}^{m}\frac{Y_{k,i,T}}{i!}+R_{k,m,T})]\\ &+\int_0^1 \mathbb{E}[\frac{(Y^1_T)^{[\frac{m}{2}]+1}(1-v)^{[\frac{m}{2}]}}{([\frac{m}{2}])!}h^{([\frac{m}{2}]+1)}(vX^1_T+(1-v)(x_0+X_{1,T}))]dv\\ &= \mathbb{E}[h(x_0+X_{1,T})] + \sum_{i=2}^{m}\sum_{k=1}^{[\frac{i}{2}]}\frac{1}{k!}\mathbb{E}[h^{(k)}(x_0+X_{1,T})\frac{Y_{k,i,T}}{i!}]\\ &+\sum_{k=1}^{[\frac{m}{2}]}\frac{1}{k!}\mathbb{E}[h^{(k)}(x_0+X_{1,T})R_{k,m,T}]\\ &+\int_0^1 \mathbb{E}[\frac{(Y^1_T)^{[\frac{m}{2}]+1}(1-v)^{[\frac{m}{2}]}}{([\frac{m}{2}])!}h^{([\frac{m}{2}]+1)}(vX^1_T+(1-v)(x_0+X_{1,T}))]dv\\ &= \mathbb{E}[h(x_0+X_{1,T})] + \sum_{i=2}^{m}Ord_i + Resid_m,\end{aligned}$$

where we used a Taylor expansion twice for the two first identities (notice that $Y_{k,i,T} = 0$ for $i \leq 2k-1$), and we interchange the summations for the last one. Besides this, we have :

$$Y^{\varepsilon}_{k,i,T} = \frac{\partial^i((Y^{\varepsilon}_T)^k)}{\partial \varepsilon^i} = \sum_{\substack{\alpha=(\alpha_0,\cdots,\alpha_i) \\ \sum_{j=1}^i \alpha_j j = i}} c_{k,\alpha} \prod_{j=0}^{i} (Y^{\varepsilon}_{1,j,T})^{\alpha_j}.$$

Using Theorem 6.7 and since $M^j_{\sigma,\mu,\lambda}$ is an increasing function w.r.t. j, one can deduce for each $j \in [\![1,i]\!]$, for any $p \geq 1, \varepsilon \in [0,1]$ that

$$\|Y^{\varepsilon}_{1,j,T}\|_p \leq \|X^{\varepsilon}_{j,T}\|_p \leq P_{\text{gen}}(\prod_{l=1}^{i}(M^l_{\sigma,\mu,\lambda}\sqrt{T}))^{\frac{j}{i}}.$$

Then, using Holder Inequality, one has

$$\|Y^{\varepsilon}_{k,i,T}\|_p \leq P_{\text{gen}} \prod_{l=1}^{i}(M^l_{\sigma,\mu,\lambda}\sqrt{T}),$$

and necessarily

$$\|R_{k,m,T}\|_p \leq P_{\text{gen}} \prod_{l=1}^{m+1}(M^l_{\sigma,\mu,\lambda}\sqrt{T}).$$

Because $Y^1_T = X_{2,T} + R_{1,2,T}$ and since $M^j_{\sigma,\mu,\lambda}$ is an increasing function w.r.t. j, one has

$$\|Y^1_T\|_p \leq \|X_{2,T}\|_p + \|R_{1,2,T}\|_p \leq P_{\text{gen}}(\prod_{l=1}^{m+1}(M^l_{\sigma,\mu,\lambda}\sqrt{T}))^{\frac{2}{m+1}}.$$

Therefore

$$\|(Y^1_T)^{[\frac{m}{2}]+1}\|_p \leq P_{\text{gen}} \prod_{l=1}^{m+1}(M^l_{\sigma,\mu,\lambda}\sqrt{T}),$$

which finishes the proof.

6.2 Malliavin Expansion Analysis of X^{ε}_T

Corollary 6.9 *Assume (R_N) for $N \geq 2$. Then for any $t \in [0,T]$, X^{ε}_t belongs to $\mathbb{D}^{N,\infty}$. Regarding the j-first Malliavin derivatives of X^{ε}_t, one has the following estimates for any $p \geq 1$ and $j \in [\![1,N]\!]$:*

$$\sup_{(t_1,\cdots,t_j)\in[0,T]^j} \|(D^j_{t_1,\cdots,t_j} X^{\varepsilon}_{\cdot})^*_t\|_p \leq P_{\text{gen}} M^0_{\sigma,\mu,\lambda}. \tag{6.6}$$

For any $i \leq N-1$, $X^{\varepsilon}_{i,t}$ belongs to $\mathbb{D}^{N-1-i,\infty}$ for any $t \in [0,T]$. Regarding the j-first Malliavin derivatives of $X^{\varepsilon}_{i,t}$, for $j \in [\![1,N-1-i]\!]$, one has

$$\sup_{(t_1,\cdots,t_j)\in[0,T]^j} \|(D^j_{t_1,\cdots,t_j} X^{\varepsilon}_{i,\cdot})^*_t\|_p \leq P_{\text{gen}}(\prod_{k=1}^{i} M^k_{\sigma,\mu,\lambda}) t^{\frac{(i-j)^+}{2}}. \tag{6.7}$$

Remark 6.10 We can make better approximation than we do here for the Malliavin derivatives, but this is sufficient for our purpose. The reader could show more refinements for other purposes, using the same principle as for the proof in Theorem 6.7.

Proof **The upper bound for the Malliavin derivatives of X_t^ε.**
Begin by $j=1$, for $t_1 \in [0,t]$ using formula (2.59) in [Nua05] p.126, one has

$$D_{t_1}X_t^\varepsilon = \sigma(t_1,X_{t_1}^\varepsilon)e^{\int_{t_1}^t \varepsilon(\sigma^{(0,1)}(s,X_{s^-}^\varepsilon)dW_s+\mu^{(0,1)}-\varepsilon\frac{(\sigma^{(0,1)})^2}{2})(s,X_{s^-}^\varepsilon)ds)} = \sigma(t_1,X_{t_1}^\varepsilon)\frac{U_t^\varepsilon}{U_{t_1}^\varepsilon}.$$

So $\sup_{t_1\in[0,t]}\|(D_{t_1}X_\cdot^\varepsilon)_t^*\|_p \le P_{\text{gen}}M_{\sigma,\mu,\lambda}^0$. The result for $j\ge 2$ is easily obtained by recurrence.
The upper bound for the Malliavin derivatives of $X_{i,t}^\varepsilon$.
We prove the result by a strong recurrence on i (like in Theorem 6.7). The case $i=0$ has been established above. Consider now $i\ge 1$. We take the notation $\Gamma_t^\varepsilon = D_{t_1}X_{i,t}^\varepsilon$. From Proposition 6.3 and by Malliavin differentiation, one has :

$$\Gamma_{t_1}^\varepsilon = \varepsilon X_{i,t_1^-}^\varepsilon\sigma^{(0,1)}(t,X_{t_1^-}^\varepsilon)+Q_{\sigma,i,t_1}^\varepsilon+\varepsilon P_{\sigma,i,t_1}^\varepsilon,$$

and for $t\ge t_1$

$$\begin{aligned}d\Gamma_t^\varepsilon &= D_{t_1}(Q_{\sigma,i,t}^\varepsilon)dW_t+D_{t_1}(Q_{\mu,i,t}^\varepsilon)dt+\varepsilon(D_{t_1}(P_{\sigma,i,t}^\varepsilon)dW_t+D_{t_1}(P_{\mu,i,t}^\varepsilon)dt)+\\ &\quad+\Gamma_t^\varepsilon dL_t^\varepsilon+\varepsilon X_{i,t}^\varepsilon D_{t_1}(X_{t^-}^\varepsilon)(\sigma^{(0,2)}(t,X_{t^-}^\varepsilon)dW_t+\mu^{(0,2)}(t,X_{t^-}^\varepsilon)dt).\end{aligned}$$

Besides this, by the recurrence assumptions, one gets:

$$\|D_{t_1}(Q_{\sigma,i,\cdot}^\varepsilon)_t^*\|_p \le P_{\text{gen}}(\prod_{k=1}^i M_{\sigma,\mu,\lambda}^k)t^{\frac{(i-2)^+}{2}}.$$

The same Inequality can be done analogously for $D_{t_1}P_{\sigma,i,t}^\varepsilon, D_{t_1}Q_{\mu,i,t}^\varepsilon, D_{t_1}P_{\mu,i,t}^\varepsilon$. We also have

$$\|\varepsilon X_{i,t_1^-}^\varepsilon\sigma^{(0,1)}(t,X_{t_1^-}^\varepsilon)+Q_{\sigma,i,t_1}^\varepsilon+\varepsilon P_{\sigma,i,t_1}^\varepsilon\|_p \le P_{\text{gen}}(M_{\sigma,\mu,\lambda}^1(\prod_{k=1}^i M_{\sigma,\mu,\lambda}^k)t^{\frac{i}{2}}+(\prod_{k=1}^i M_{\sigma,\mu,\lambda}^k)t^{\frac{i-1}{2}}),$$

and

$$\|(\varepsilon X_{i,\cdot}^\varepsilon D_{t_1}(X_\cdot^\varepsilon)(|\sigma^{(0,2)}(t,X_\cdot^\varepsilon)|+|\mu^{(0,2)}(t,X_\cdot^\varepsilon)|))_t^*\|_p \le P_{\text{gen}}(\prod_{k=1}^i M_{\sigma,\mu,\lambda}^k)t^{\frac{i}{2}}.$$

These estimates are sufficient to prove the announced result by using results analogous to Corollary 6.6.
For higher derivatives of $X_{i,t}^\varepsilon$, we proceed in the same way. We left the details to the reader. □

To make control for vanilla options, we need this technical lemma related to integration by parts formula.

Lemma 6.11 *Let p and k be two positive numbers and Z be in $\mathbb{D}^{k,\infty}$. Assume (E) and (R_{k+1}). For any $v \in [0,1]$, there exists Z_k^v such that for any $l \in \mathscr{C}_0^\infty(\mathbb{R})$, one has*

$$\mathbb{E}[l^{(k)}(vX_T^1 + (1-v)(x_0 + X_{1,T}))Z] = \mathbb{E}[l(vX_T^1 + (1-v)(x_0 + X_{1,T}))Z_k^v],$$

and it is estimated in the L_p norm by

$$\|Z_k^v\|_p \leq P_{\text{gen}} \frac{(M^0_{\sigma,\mu,\lambda})^{k(2k+1)}}{I_\sigma^{2k(k+1)}\sqrt{T^k}} \|Z\|_{k,2p}.$$

Proof **Step 1:** Show that $F_v = vX_T^1 + (1-v)(x_0 + X_{1,T})$ is a non degenerate random variable.
Since the coefficients σ and μ are bounded with $k+1$ bounded derivatives w.r.t. the space variable, we deduce that F_v is in $\mathbb{D}^{k+1,\infty}$ (Theorem 2.2.2 of [Nua05]). Besides this, using formula 2.59 in [Nua05] p.126, one has:

$$\gamma_{F_v} = \int_0^T (D_s F_v)^2 ds = \int_0^T (v\sigma(s, X_s^1) U_T^1 (U_s^1)^{-1} + (1-v)\sigma(s, x_0))^2 ds.$$

In view of Assumption (E), γ_{F_v} is positive and one has $\gamma_{F_v}^{-1} \leq (\sqrt{T} I_\sigma)^{-2} ((U_T^1)^{-1} (U_\cdot^1)_T^*)^2$, which gives that $\gamma_{F_v}^{-1}$ belongs to any L_p. In addition, for any $p \geq 1$, one obtain

$$\|\gamma_{F_v}^{-1}\|_p \leq C(\sqrt{T} I_\sigma)^{-2}.$$

Step 2: Using Proposition 2.1.4 and Proposition 1.5.6 in [Nua05], one gets the existence of Z_k^v in L_p with

$$\|Z_k^v\|_p \leq C \|\gamma_{F_v}^{-1}\|_{k,2^{k+1}p}^k \|DF_v\|_{k,2^{k+1}p}^k \|Z\|_{k,2p}.$$

Step 3: Upper bound of $\|DF_v\|_{k,q}, \|\gamma_{F_v}^{-1}\|_{k,q}$ for $q \geq 1$.
Due to the first part of Corollary 6.9, we immediately obtain $\|DF_v\|_{k,q} \leq P_{\text{gen}} M^0_{\sigma,\mu,\lambda} \sqrt{T}$.
Analogously and after long computations, we obtain

$$\|\gamma_{F_v}^{-1}\|_{k,q} \leq P_{\text{gen}} \frac{(M^0_{\sigma,\mu,\lambda})^{2k}}{(I_\sigma)^{2(k+1)} T}.$$

These two estimates complete our proof. □

6.3 Proof of Lemma 5.3

One has :

$$Y^\varepsilon_{k,i+k-1,T} = \frac{\partial^{i+k-1}((Y_T^\varepsilon)^k)}{\partial \varepsilon^{i+k-1}} = \sum_{\substack{\alpha=(\alpha_0,\cdots,\alpha_{i+k-1}) \\ \sum_{j=1}^{i+k-1} \alpha_j j = i+k-1}} c_{k,\alpha} \prod_{j=0}^{i+k-1} (Y^\varepsilon_{1,j,T})^{\alpha_j}.$$

Using Corollary 6.9, one deduces for each $j \in [\![0, i+k-1]\!]$ and any $\varepsilon \in [0,1]$ that $Y^{\varepsilon}_{1,j,T} \in \mathbb{D}^{k-1,\infty}$ (because $i+k-1+k-1 \leq N$). In addition,

$$\|Y^{\varepsilon}_{1,j,T}\|_{k-1,p} \leq P_{\text{gen}} \prod_{l=1}^{j} (M^{l}_{\sigma,\mu,\lambda}\sqrt{T}) \leq P_{\text{gen}} (\prod_{l=1}^{2(i-1)} (M^{l}_{\sigma,\mu,\lambda}\sqrt{T}))^{\frac{j}{2(i-1)}},$$

because $M^{j}_{\sigma,\mu,\lambda}$ is an increasing function w.r.t. j and $j \leq i+k-1 \leq 2(i-1)$. Then, using the Holder Inequality for the spaces $\mathbb{D}^{k-1,\infty}$ (see Proposition 1.5.6 in [Nua05]), one has $Y^{\varepsilon}_{k,i+k-1,T} \in \mathbb{D}^{k-1,\infty}$, and

$$\|Y^{\varepsilon}_{k,i+k-1,T}\|_{k-1,p} \leq P_{\text{gen}} (\prod_{l=1}^{2(i-1)} (M^{l}_{\sigma,\mu,\lambda}\sqrt{T}))^{\frac{i+k-1}{2(i-1)}}. \tag{6.8}$$

Then, Lemma 6.11 insures the existence of G_i in L_p. Its L_p norm is estimated using Lemma 6.11 and Inequality (6.8):

$$\begin{aligned}
\|G_i\|_p &\leq P_{\text{gen}} \sum_{k=1}^{i-1} \frac{(M^{0}_{\sigma,\mu,\lambda})^{(k-1)(2k-1)}}{I_{\sigma}^{2k(k-1)}\sqrt{T^{k-1}}} (\prod_{l=1}^{2(i-1)} (M^{l}_{\sigma,\mu,\lambda}\sqrt{T}))^{\frac{i+k-1}{2(i-1)}} \\
&\leq P_{\text{gen}} \sum_{k=1}^{i-1} \frac{(M^{0}_{\sigma,\mu,\lambda})^{(k-1)(2k-1)} (\prod_{l=1}^{2(i-1)} M^{l}_{\sigma,\mu,\lambda})^{\frac{k-1}{2(i-1)}}}{I_{\sigma}^{2k(k-1)}} (\prod_{l=1}^{2(i-1)} (M^{l}_{\sigma,\mu,\lambda}\sqrt{T}))^{\frac{i}{2(i-1)}} \\
&\leq P_{\text{gen}} \frac{(M^{0}_{\sigma,\mu,\lambda})^{(i-2)(2i-3)} (\prod_{l=1}^{2(i-1)} M^{l}_{\sigma,\mu,\lambda})^{\frac{i-2}{2(i-1)}}}{I_{\sigma}^{2(i-1)(i-2)}} (\prod_{l=1}^{2(i-1)} (M^{l}_{\sigma,\mu,\lambda}\sqrt{T}))^{\frac{i}{2(i-1)}}.
\end{aligned}$$

For S_i and $I_{i,v}$, we proceed analogously.

Proof of Theorem 5.4 (Vanilla options) In the one hand, one has for smooth function h:

$$\begin{aligned}
\mathbb{E}[h(X^1_T)] =& \mathbb{E}[h(x_0 + X_{1,T})] \\
&+ \sum_{k=1}^{m-1} \frac{1}{k!} \mathbb{E}[h^{(k)}(x_0 + X_{1,T}) (\sum_{i=k+1}^{m} \frac{Y_{k,k+i-1,T}}{(k+i-1)!} + R_{k,k+m-1,T})] \\
&+ \int_0^1 \mathbb{E}[\frac{(Y^1_T)^m (1-v)^{m-1}}{(m-1)!} h^{(m)}(vX^1_T + (1-v)(x_0 + X_{1,T}))] dv \\
=& \mathbb{E}[h(x_0 + X_{1,T})] \\
&+ \sum_{i=2}^{m} \frac{1}{k!} \sum_{k=1}^{i-1} \mathbb{E}[h^{(k)}(x_0 + X_{1,T}) \frac{Y_{k,k+i-1,T}}{(k+i-1)!}] \\
&+ \sum_{k=1}^{m-1} \frac{1}{k!} \mathbb{E}[h^{(k)}(x_0 + X_{1,T}) R_{k,k+m-1,T}] \\
&+ \int_0^1 \mathbb{E}[\frac{(Y^1_T)^m (1-v)^{m-1}}{(m-1)!} h^{(m)}(vX^1_T + (1-v)(x_0 + X_{1,T}))] dv
\end{aligned}$$

$$
\begin{aligned}
=&\mathbb{E}[h(x_0+X_{1,T})]+\sum_{i=2}^{m}\mathbb{E}[h^{(1)}(x_0+X_{1,T})G_i]\\
&+\mathbb{E}[h^{(1)}(x_0+X_{1,T})S_m]\\
&+\int_0^1\mathbb{E}[h^{(m)}(vX_T^1+(1-v)(x_0+X_{1,T}))I_{m,v}](1-v)^{m-1}dv,
\end{aligned}
$$

where we used a Taylor expansion in the first identity, we interchange the summations in the second equation, and use the Lemma 5.3 in the last equation. So yields the identity (5.2) for smooth payoff.
In the other hand, using Lemma 5.3, it is straightforward to show the inequalities (5.3) and (5.4).
It remains to extend the result (5.2) to vanilla options (instead of smooth function h). We use a standard density argument (see [GM05]). Denote by $\tilde{\mu}$ the measure defined by $\int_{\mathbb{R}} g(x)\tilde{\mu}(dx)=\mathbb{E}(g(X_T^1))+\mathbb{E}(g(x_0+X_{1,T}))+\int_0^1\mathbb{E}(g(vX_T^1+(1-v)(x_0+X_{1,T})))dv$. Since there exists a sequence $(h_n)_{n\in\mathbb{N}}$ of smooth functions converging to h in $L_2(\tilde{\mu})$ as well as its first derivative (see [Rud66] for instance), we easily get Equality (5.2) in the case of vanilla payoffs.

7 Technical results related to explicit correction terms

In this section, we bring together the results (and their proofs) which allow us to derive the explicit terms in the formula (2.2).
In the sequel, (u_t)(resp. $(v_t),(v_t)$) are square predictable (resp. deterministic) process and l is a smooth function with compact support.

Lemma 7.1 *For any continuous (or piecewise continuous) function f, any continuous semimartingale Z vanishing at t=0, one has:*

$$\int_0^T f(t)Z_t dt=\int_0^T \omega(T)_t^f dZ_t.$$

Proof this is an application of the Itô formula to the product $\omega(T)_t^f Z_t$. □

7.1 Proof of Lemma 2.2

We first give the proof in a particular case when u and v are equal to 1. By a usual integration by parts formula, one has:

$$\mathbb{E}[l(W_T)W_T]=\int_{-\infty}^{\infty} l(\sqrt{T}x)\sqrt{T}x\frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}dx=\int_{-\infty}^{\infty} Tl^{(1)}(\sqrt{T}x)\frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}dx=T\mathbb{E}[l^{(1)}(W_T)].$$

For the general proof: apply the duality relationship of Malliavin calculus (see Lemma 1.2.1 in [Nua05]), identifying Itô's integral and Skorohod operator for adapted integrands.

7.2 Proof of Lemma 2.3

Applying first Lemma 7.1 to $f(t) = v_t$ and $Z_t = X_{c,1,t}$, one has:

$$\begin{aligned}\mathbb{E}[(\int_0^T v_t X_{c,1,t} dt) l(\int_0^T v_t dW_t)] &= \mathbb{E}[(\int_0^T \omega(T)_t^v dX_{c,1,t}) l(\int_0^T v_t dW_t)] \\ &= \mathbb{E}[(\int_0^T \omega(T)_t^v (\sigma_t dW_t + \mu_t dt) l(\int_0^T v_t dW_t)] \\ &= (\int_0^T v_t \sigma_t \omega(T)_t^v dt) \mathbb{E}[l^{(1)}(\int_0^T v_t dW_t)] \\ &+ (\int_0^T \mu_t \omega(T)_t^v dt) \mathbb{E}[l(\int_0^T v_t dW_t)],\end{aligned}$$

and we have used Lemma 2.2 for the last equality.

7.3 Proof of Lemma 2.4

Using the independence of increments for J, one has:

$$\mathbb{E}[(\int_0^T v_t J_t dt) l(J_T)] = \int_0^T v_t \mathbb{E}[J_t l(J_T - J_t + J_t)] dt = \int_0^T v_t \mathbb{E}[\iota(J_T - J_t)] dt.$$

Using a conditioning argument and since $\sum_{j=1}^k Y_i$ is a Gaussian variable, one has:

$$\begin{aligned}\iota(x) &= \mathbb{E}[J_t l(x + J_t)] \\ &= \sum_{k \in \mathbb{N}^*} P(N_t = k) \mathbb{E}[\sum_{j=1}^k Y_i l(x + \sum_{j=1}^k Y_i)] \\ &= \sum_{k \in \mathbb{N}^*} P(N_t = k) k (\eta_J \mathbb{E}[l(x + \sum_{j=1}^k Y_i)] + \gamma_J \mathbb{E}[l^{(1)}(x + \sum_{j=1}^k Y_i)]) \\ &= \sum_{k \in \mathbb{N}} \lambda t P(N_t = k) (\eta_J \mathbb{E}[l(x + \sum_{j=1}^{k+1} Y_i)] + \gamma_J \mathbb{E}[l^{(1)}(x + \sum_{j=1}^{k+1} Y_i)]) \\ &= \lambda t (\eta_J \mathbb{E}[l(x + J_t + Y'] + \gamma_J \mathbb{E}[l^{(1)}(x + J_t + Y'],\end{aligned}$$

such that Y' is a random variable independent from the variables $(Y_i)_{i \in \mathbb{N}^*}$ with the same law as Y_1.

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