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► **To cite this version:**

Ali Hassani. Strichartz estimates for the wave equation on Riemannian symmetric manifolds.. 2007.  
hal-00200151

**HAL Id: hal-00200151**

**<https://hal.science/hal-00200151>**

Preprint submitted on 20 Dec 2007

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# Strichartz estimates for the wave equation on Riemannian symmetric manifolds\*

A. Hassani<sup>†</sup>

## Abstract

We prove Strichartz type estimates for solutions of the homogeneous wave equation on Riemannian symmetric spaces. Our results generalize those of Ginibre and Velo in [7].

## 1 Introduction

The wave equation on a manifold with symmetries is a fundamental partial differential equation problem. In particular, it is important to have explicit estimates and dispersive properties for solutions of the following Cauchy problem:

$$\begin{cases} \Delta u = \frac{\partial^2}{\partial t^2} u, \\ u(t=0) = u_0 \\ \frac{\partial}{\partial t} u(t=0) = u_1 \end{cases} \quad (1.1)$$

where  $\Delta$  denotes the Laplace operator on the manifold.

In the case of the euclidian space  $\mathbb{R}^n$ ,  $n \geq 1$ , Segal proved some estimates in [16] which were improved by Strichartz in [21]. It turns out that these estimates play a crucial role in the modern theory of local and global well posed problems. Actually, one has the following *Strichartz type estimates* for (1.1) with non trivial Cauchy data  $u_0$  and  $u_1$  (see [7]):

$$\|u\|_{L^r(I, \dot{B}_p^{s,2}(\mathbb{R}^n))} \leq c_r (\|u_0\|_{L^2(\mathbb{R}^n)} + \|u_1\|_{\dot{H}^{-1}(\mathbb{R}^n)}) \quad (1.2)$$

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\*This work is part of the author's PhD thesis.

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where  $I = [0; +\infty[$ ,  $2 \leq p \leq \infty$ ,  $s = \frac{n+1}{4}(\frac{2}{p} - 1)$  and  $2/r = \frac{n-1}{2}(1 - \frac{2}{p})$ . Here  $\dot{B}_p^{s,q}(\mathbb{R}^n)$  is the *homogeneous Besov space* on  $\mathbb{R}^n$  defined by the norm:

$$\|f\|_{\dot{B}_p^{s,q}(\mathbb{R}^n)}^q = \sum_{j \in \mathbb{Z}} 2^{jsq} \|\varphi_j * f\|_{L^p(\mathbb{R}^n)}^q \text{ for } s \in \mathbb{R}, 1 \leq p, q \leq \infty$$

where  $(\varphi_j)_{j \in \mathbb{Z}}$  is a sequence of smooth functions such that its Fourier transform is a *Paley-Littlewood partition of unity*. The space  $\dot{H}^s$ ,  $s \in \mathbb{R}$ , is the *homogeneous Sobolev space* defined by the norm:

$$\|f\|_{\dot{H}^s(\mathbb{R}^n)}^2 = \sum_{j \in \mathbb{Z}} 2^{2js} \|\varphi_j * f\|_{L^2(\mathbb{R}^n)}^2.$$

It should be noted that estimates analogous to (1.2) have been proved on Heisenberg groups [3], hyperbolic spaces [22] and on Damek-Ricci spaces [15]. Moreover some *dispersive properties* for solutions of (1.1) may be deduced (see [4][7][20]):

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} \leq c(1 + |t|)^{-(n-1)/2} (\|u_0\|_{\dot{B}_1^{s,1}(\mathbb{R}^n)} + \|u_1\|_{\dot{B}_1^{s-1,1}(\mathbb{R}^n)}) \quad (1.3)$$

where  $s = \frac{n+1}{2}$ . The traditional interpretation for these estimates is the *decay of solutions* as  $t \rightarrow \infty$ . In the general context, where  $X$  is a manifold equipped with a pseudo-Riemannian metric, a measure  $\sigma$  and  $\Delta = L$  is a second order differential operator on  $X$  which satisfies some integrability conditions, Lohoué proved some  $L^p$ -estimates of solutions of wave equation in terms of norms of the initial data (see [14]).

In this paper, we prove estimates analogous to (1.2) and (1.3) in the case of Riemannian symmetric space of the non-compact type. Our strategy is based on the Harish-Chandra Plancherel formula in the context of non-commutative harmonic analysis.

More precisely, let  $X$  be the  $G$ -homogeneous manifold  $G/K$ , where  $G$  is a non-compact connected semisimple Lie group with finite center and  $K$  a maximal compact subgroup of  $G$ . The Riemannian metric on  $X$  is induced by the Killing form of  $G$  and the (negative) Laplacian  $\Delta$  on  $X$  is related to the Casimir element of  $G$ . To establish our estimates in this context, we first need to define Besov spaces on  $X$ . For general Riemannian manifolds, Besov spaces were defined by Triebel with purely geometric methods in [23]. However, in the particular case of Riemannian symmetric spaces, we shall proceed as follows. Let  $\mathfrak{g}$  (resp.  $\mathfrak{k}$ ) be the Lie algebra of  $G$  (resp.  $K$ ). Consider the corresponding Cartan decomposition of  $\mathfrak{g}$ :

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

where  $\mathfrak{p}$  is some  $\mathfrak{k}$ -invariant vector subspace of  $\mathfrak{g}$ . Let  $MAN$  be a minimal parabolic subgroup of  $G$ , where  $M$  is a closed subgroup of  $K$  and  $A$  (resp.  $N$ ) an abelian (resp. nilpotent). The Lie algebra  $\mathfrak{a}$  of  $A$  is a maximal abelian subspace of  $\mathfrak{p}$  equipped with a norm  $\| \cdot \|$  induced by the Killing form. Then the *Helgason-Fourier transform*  $\tilde{f}$  of some function  $f$  on  $X$  is given by:

$$\tilde{f}(\lambda, b) = \int_X f(x) e^{(-i\lambda + \rho)A(x,b)} dx, \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*, b \in K/M$$

where  $\mathfrak{a}_{\mathbb{C}}^*$  is the complexification of the vector dual of  $\mathfrak{a}$ ,  $\rho$  is the half-sum of positive restricted roots counted with their multiplicities and  $A(x, b) \in \mathfrak{a}$  (see Section 2 for more detail). On the other hand, we define the following resolution of unity on  $\mathfrak{a}^*$ . Consider the set  $\{\varphi_{0,N}, \varphi_t^N\}_{0 < t \leq 1, N \in \mathbb{N}^*}$  of compactly supported smooth  $K$ -invariant functions on  $X$  satisfying:

$$\mathcal{F}\varphi_{0,N}(\lambda) + \int_0^1 (\mathcal{F}\varphi_t^N)^2(\lambda) \frac{dt}{t} = 1, \quad \forall \lambda \in \mathfrak{a}_{\mathbb{C}}^*$$

and

$$f(x) = (f \times \varphi_{0,N})(x) + \int_0^1 (f \times \varphi_t^N \times \varphi_t^N)(x) \frac{dt}{t}, \quad \forall f \in L^2(X)$$

where  $\times$  denote the convolution product on  $X$  and  $\mathcal{F}$  is the usual *spherical transform* for  $K$ -invariant functions on  $X$ . Then the Besov space  $B_p^{s,q}(X)$  on  $X$  is the space of tempered distributions on  $X$  satisfying:

$$\|f\|_{B_p^{s,q}(X)} = \|f \times \varphi_{0,N}\|_{L^p(X)} + \left( \int_0^1 t^{-sq} \|f \times \varphi_t^N\|_{L^p(X)}^q \frac{dt}{t} \right)^{1/q} < \infty$$

for  $2N > |s|$ . It should be noted that our definition differs slightly from that in [19].

We can now state our first result.

**Theorem 1 :** *Let  $N$  be a positive integer such that  $n = \dim(X) \geq 4N$ . Let  $\alpha = \dim(\mathfrak{a}) \geq 1$  and  $I = [0, +\infty[$ . Let  $u$  be a solution of the wave equation (1.1) on  $X$  and, let  $p$  and  $r$  be two real numbers such that  $2/r = \alpha(1 - \frac{2}{p})$  and  $2 < p < \min(\frac{2n}{n-4N}, \frac{2\alpha}{\alpha-1})$ . Then there exists a positive number  $c_r$ , depending on  $r$ , such that if  $u_0$  and  $u_1$  belong to  $L^2(X)$ , we have:*

$$\|u\|_{L^r(I, B_p^{s,2}(X))} \leq c_r \left( \|u_0\|_{L^2(X)} + \frac{1}{\|\rho\|} \|u_1\|_{L^2(X)} \right)$$

where  $s = \frac{n}{2}(\frac{2}{p} - 1)$ .

The proof rests on an argument of duality ([7][24]) and on a Hardy-Littlewood-Sobolev inequality ([13]). We find it useful to formulate solutions of (1.1) using the theory of propagators.

Our second result deals with dispersive properties. More precisely, we prove the following theorem:

**Theorem 2 :** *Let  $u$  be a solution of the wave equation (1.1) on  $X$  and write  $\alpha = \dim(\mathfrak{a})$ .*

1. *We assume that both  $u_0$  and  $(-\Delta)^{-1/2}u_1$  belong to  $B_1^{n,1}(X)$ . Then there exist a constant  $c > 0$  and a sufficiently large number  $T$  such that for all  $t \geq T$ , we have:*

$$\|u(t, \cdot)\|_{L^\infty(X)} \leq ct^{-\alpha} (\|u_0\|_{B_1^{n,1}(X)} + \|(-\Delta)^{-1/2}u_1\|_{B_1^{n,1}(X)})$$

*With  $n = \dim X < 2N$ , for some positive integer  $N$ .*

2. *Let  $p$  and  $p'$  be two integers such that  $1/p + 1/p' = 1$  and  $2 \leq p < \frac{2n}{n-2N}$ . We suppose that  $u_0$  and  $(-\Delta)^{-1/2}u_1$  are in  $B_{p'}^{s,1}(X)$  with  $s = n(1 - \frac{2}{p})$ . Then there exist a constant  $c > 0$  and a sufficiently large number  $T$  such that for all  $t \geq T$ , we have:*

$$\|u(t, \cdot)\|_{L^p(X)} \leq ct^{\alpha\gamma} (\|u_0\|_{B_{p'}^{s,1}(X)} + \|(-\Delta)^{-1/2}u_1\|_{B_{p'}^{s,1}(X)})$$

*With  $n = \dim X > 2N$  and  $\gamma = 2/p - 1$ .*

The proof combines inverse Helgason-Fourier transform and interpolation theorems for  $L^p$ -spaces on  $X$ .

Our paper is organized as follows: in Section 2, we recall some definitions and basic results of harmonic analysis. Section 3 is devoted to the definition of Besov spaces on symmetric spaces and propagators. Section 4 contains the proof of our first theorem is contained in Section 4, while the proof of the second theorem is contained in Section 5.

**Acknowledgements.** The author would like to thank S. Mehdi and N. Lohoué for suggesting the problem and for several helpful discussions.

## 2 Preliminaries

### 2.1 Roots, decompositions and norms.

Let  $G$  be a noncompact connected semisimple Lie group with finite center and  $K$  a maximal compact subgroup of  $G$ . Let  $\mathfrak{g}$  (resp.  $\mathfrak{k}$ ) be the Lie algebra of  $G$  (resp.  $K$ ) and consider the Cartan decomposition of  $\mathfrak{g}$ :

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

where  $\mathfrak{p}$  is the ( $K$ -invariant) orthogonal complement of  $\mathfrak{k}$  in  $\mathfrak{g}$  with respect to the Killing form of  $\mathfrak{g}$ :

$$\mathcal{K} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}, (X, Y) \mapsto \text{Tr}(\text{ad}(X) \circ \text{ad}(Y)).$$

Here  $\text{ad}$  denotes the differential of the adjoint action  $\text{Ad}_G$  of  $G$ . Moreover, if  $\theta$  denotes the corresponding Cartan involution, the Killing form defines the following  $G$ -invariant inner product on  $\mathfrak{g}$ :

$$\langle X, Y \rangle = -\mathcal{K}(X, \theta(Y))$$

which is positive definite on  $\mathfrak{p}$  and negative definite on  $\mathfrak{k}$ . This in turn induces a Riemannian structure on the homogeneous (symmetric) manifold:

$$X = G/K$$

whose tangent space at the origin  $eK$  is identified with  $\mathfrak{p}$ . On the other hand, since  $G$  is semisimple, the Killing form enables us to identify  $\mathfrak{g}$  with its vector dual  $\mathfrak{g}^*$ , as well as subspaces of  $\mathfrak{g}$  with subspaces in  $\mathfrak{g}^*$ . Let  $\mathfrak{a}$  be a maximal abelian subspace of  $\mathfrak{p}$ . The real dimension of  $\mathfrak{a}$  is known as the real rank  $\text{rank}_{\mathbb{R}}(G)$  of  $G$ . Let  $\Sigma$  be the set of restricted roots with respect to  $\mathfrak{a}$ :

$$\Sigma = \{\lambda \in \mathfrak{a}^* \mid \lambda \neq 0 \text{ and } \mathfrak{g}_{\lambda} \neq \{0\}\}$$

where

$$\mathfrak{g}_{\lambda} = \{X \in \mathfrak{g} \mid [H, X] = \lambda(H)X \quad \forall H \in \mathfrak{a}\}.$$

Fix a positive Weyl chamber  $\mathfrak{a}_+$  in  $\mathfrak{a}^*$ , and write  $\Sigma_+$  for the corresponding set of positive restricted roots. Define the nilpotent subalgebra  $\mathfrak{n}$  of  $\mathfrak{g}$ :

$$\mathfrak{n} = \bigoplus_{\lambda \in \Sigma_+} \mathfrak{g}_{\lambda}$$

so that  $\mathfrak{g}$  decomposes as

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

known as the Iwasawa decomposition. The corresponding decomposition at the group level is:

$$G = KAN$$

where  $A$  and  $N$  are the analytic subgroups of  $G$  with Lie algebras  $\mathfrak{a}$  and  $\mathfrak{n}$  respectively. Finally, there is another decomposition of  $G$  that will be of interest to us. The Cartan decomposition of  $G$  is given by:

$$G = K\bar{A}_+K$$

where  $\overline{A}_+$  is the closure of  $A_+ = \exp(\mathfrak{a}_+)$ . Any element  $g$  of  $G$  will be written as

$$g = k_1 A(g) k_2$$

in the Cartan decomposition, and as

$$g = k(g) \exp(H(g)) n(g)$$

in the Iwasawa decomposition. It should be noted that  $k(g)$ ,  $n(g)$ ,  $H(g)$  and  $A(g)$  are uniquely determined. Now, the Killing form induces norms on  $\mathfrak{g}$  and  $\mathfrak{g}^*$ , which we shall denote by the same symbol  $\| \cdot \|$ , as well as a norm on  $G$  defined by:

$$\| g \| = \| A(g) \| .$$

In particular one has:

$$\| g^{-1} \| = \| g \| \text{ and } \| k g k' \| = \| g \| \text{ for all } g \in G \text{ and } k, k' \in K.$$

Next, let  $M$  (resp.  $M'$ ) denote the centralizer (resp. the normalizer) of  $A$  in  $K$ . The quotient group  $W = M'/M$  is the so-called Weyl group associated with  $\Sigma$ . It is a finite group that acts on  $\mathfrak{a}$  as a group of linear transformations by the operators  $\text{Ad}_G(k)$ ,  $k \in M'$ . The group  $W$  acts also freely and transitively on the set of Weyl chambers:

$$s \cdot \lambda(H) = \lambda(s^{-1}H), \quad \forall s \in W, H \in \mathfrak{a}.$$

The following compact homogeneous manifold, known as the boundary of  $X$ ,

$$B = K/M = G/MAN$$

plays a crucial role in the harmonic analysis on  $X$ .

Finally, the Killing form of  $G$  induces euclidean measures on  $A$ ,  $\mathfrak{a}$  and  $\mathfrak{a}^*$ . These measures remain invariant when multiplied by  $(2\pi)^{-\text{rank}_{\mathbb{R}}(G)}$ . The Haar measures  $dm$  on  $M$  and  $dk$  on  $K$  are normalized such that the total mass is 1. As is customary, Haar measures on  $G$  and  $N$  will be normalized so that:

$$\int_G f(g) dg = \int_{K \times A \times N} f(kan) e^{2\rho(\log a)} dk dadn$$

and

$$\int_G f(g) dg = \int_{G/K} \int_K f(gk) dk d(gK)$$

where

$$\rho = \frac{1}{2} \sum_{\lambda \in \Sigma_+} m_\lambda \lambda,$$

is the half-sum of positive restricted roots counted with their multiplicities  $m_\lambda$  and  $\log$  is the inverse of  $\exp : \mathfrak{a} \mapsto A$ . Moreover the invariant measure  $db = d(kM)$  on  $B = K/M$  is normalized by [11]:

$$\int_B db = \int_{K/M} d(kM) = 1.$$

Extend the Killing form of  $G$  linearly to the complexification  $\mathfrak{a}_\mathbb{C}$  and keep the same symbol to denote this extension. For  $\lambda \in \mathfrak{a}_\mathbb{C}^*$ , let  $H_\lambda$  be the unique element in  $\mathfrak{a}_\mathbb{C}$  defined by:

$$\lambda(H) = \langle H_\lambda, H \rangle, \quad \forall H \in \mathfrak{a}_\mathbb{C}.$$

Put

$$\langle \lambda, \mu \rangle = \langle H_\lambda, H_\mu \rangle, \quad \forall \lambda, \mu \in \mathfrak{a}_\mathbb{C}^*.$$

For  $\lambda \in \mathfrak{a}_\mathbb{C}^*$  and  $i = \sqrt{-1}$ , we write

$$\lambda = \mathcal{R}e\lambda + i\mathcal{I}m\lambda,$$

where  $\mathcal{R}e\lambda$  and  $\mathcal{I}m\lambda \in \mathfrak{a}^*$  are respectively the real part and the imaginary part of  $\lambda$ , with

$$\|\lambda\|^2 = \|\mathcal{R}e\lambda\|^2 + \|\mathcal{I}m\lambda\|^2.$$

## 2.2 Helgason-Fourier transform, inversion formula, Plancherel formula

For this section, we refer to [11][12]. We view functions on  $X$  as functions on  $G$  which are  $K$ -invariant on the right. Then the Helgason-Fourier transform of a function  $f$  on  $X$  is the map  $\tilde{f}$  on  $\mathfrak{a}_\mathbb{C}^* \times B$  given by:

$$\tilde{f}(\lambda, b) = \int_X f(x) e^{(-i\lambda + \rho)A(x,b)} dx$$

where

$$A(x, b) = A(gK, kM) = -H(g^{-1}k) \in \mathfrak{a}$$

for  $x = gK \in X$  and  $b = kM \in B$ . Moreover, if  $f$  belongs to the space  $\mathcal{D}(X) = C_0^\infty(X)$  of compactly supported smooth functions on  $X$ , then its inverse Helgason-Fourier transform is:

$$f(x) = \frac{1}{|W|} \int_{\mathfrak{a}^* \times B} e^{(i\lambda + \rho)A(x,b)} \tilde{f}(\lambda, b) \frac{d\lambda db}{|c(\lambda)|^2} \quad (2.4)$$



where  $|W|$  denotes the order of the Weyl group. The function  $c$  is called the *Harish-Chandra function* and satisfies the following inequality:

$$|c(\lambda)|^{-2} \leq c'(1 + \|\lambda\|^2)^d \quad (2.5)$$

with  $d = \dim(X) - \text{rank}_{\mathbb{R}}(G)$  and  $c'$  is a positive constant. The Helgason-Fourier transform

$$L^2(X, dx) \rightarrow L^2(\mathfrak{a}_+^* \times B, \frac{d\lambda db}{|c(\lambda)|^2}), f \mapsto \tilde{f}$$

is an isometry and we have the Plancherel formula:

$$\int_X |f(x)|^2 dx = \int_{\mathfrak{a}_+^* \times B} |\tilde{f}(\lambda, b)|^2 \frac{d\lambda db}{|c(\lambda)|^2} \quad (2.6)$$

where  $\mathfrak{a}_+^* = \{\lambda \in \mathfrak{a}^* \mid H_\lambda \in \mathfrak{a}_+\}$ .

## 2.3 Spherical transform, euclidean Fourier transform, Abel transform

Let  $f$  be a  $K$ -invariant function on  $X$ , i.e. left and right  $K$ -invariant on  $G$ . The Helgason-Fourier transform of  $f$  is constant in the  $B$ -variable, and is  $W$ -invariant on  $\mathfrak{a}^*$ . The spherical transform  $\mathcal{F}f$  of  $f$  is given by:

$$\mathcal{F}f(\lambda) = \int_X f(x) \varphi_{-\lambda}(x) dx$$

where  $\varphi_\lambda$  is the *spherical function* defined by:

$$\varphi_\lambda(x) = \varphi_\lambda(gK) = \int_K e^{(i\lambda + \rho)A(kg)} dk.$$

It will be useful to write the spherical transform as follows:

$$\mathcal{F}f = \widehat{\mathcal{A}f},$$

where  $\widehat{\phantom{x}}$  is the *euclidean Fourier transform*:

$$\widehat{f}(\lambda) = \int_{\mathfrak{a}} f(H) e^{-i\lambda(H)} dH, \quad \lambda \in \mathfrak{a}^*$$

and  $\mathcal{A}$  is the *Abel transform*:

$$\mathcal{A}f(H) = e^{\rho(H)} \int_N f((\exp H)n) dn, \quad H \in \mathfrak{a}.$$

## 2.4 Schwartz spaces

The Schwartz space on  $X$  is the space  $\mathcal{S}(X)$  of smooth functions on  $X$  such that for any  $m \in \mathbb{N}$  and any (left or right) invariant differential operator  $D$  on  $G$ , we have:

$$\sup | (Df)(g) | (1 + \sigma(g))^m \Xi(g)^{-1} < \infty$$

where

$$\sigma(g) = \| Y \|, \quad g = k \exp(Y) \in G, \quad Y \in \mathfrak{p}$$

and

$$\Xi(g) = \int_K e^{-\rho H(gk)} dk.$$

The following inclusions are standard:

$$\mathcal{D}(X) \subset \mathcal{S}(X) \subset L^p(X), \quad p \geq 2.$$

Moreover  $\mathcal{D}(X)$  is dense in  $\mathcal{S}(X)$ . Similarly  $\mathcal{S}(K \backslash X)$  is the Schwartz space on  $K \backslash X$ . Write  $\mathcal{S}(\mathfrak{a})$  (resp.  $\mathcal{S}(\mathfrak{a}^*)$ ) for the (classical) Schwartz space on  $\mathfrak{a}$  (resp.  $\mathfrak{a}^*$ ). Recall the Weyl group action on  $\mathfrak{a}$  and  $\mathfrak{a}^*$ , and let  $\mathcal{S}_W(\mathfrak{a})$  (resp.  $\mathcal{S}_W(\mathfrak{a}^*)$ ) be the subspace of  $W$ -invariant elements of  $\mathcal{S}(\mathfrak{a})$  (resp.  $\mathcal{S}(\mathfrak{a}^*)$ ). In particular,  $\mathcal{S}_W(\mathfrak{a}^*)$  is the space of  $W$ -invariant complex functions  $h$  such that  $h$  and all its derivatives extend continuously to  $\mathfrak{a}^*$  and, for any polynomial function  $P$  on  $\mathfrak{a}$  and any integer  $m \geq 0$ , we have:

$$\sup_{\lambda \in \mathfrak{a}^*} (\|\lambda\| + 1)^m | P(\frac{\partial}{\partial \lambda}) h(\lambda) | < +\infty.$$

The following diagram is commutative (up to a normalizing constant) and each arrow is an isomorphism:

$$\begin{array}{ccc} \mathcal{S}(K \backslash X) & \xrightarrow{\mathcal{A}} & \mathcal{S}_W(\mathfrak{a}) \\ & \searrow \mathcal{F} & \swarrow \hat{\phantom{\mathcal{A}}} \\ & \mathcal{S}_W(\mathfrak{a}^*) & \end{array}$$

The fact that  $\mathcal{F}$  is an isomorphism is a fundamental result of Harish-Chandra [8, 9, 10]. A simpler proof was obtained by Anker [2]. For the isomorphisms  $\mathcal{A}$  and  $\hat{\phantom{\mathcal{A}}}$ , we refer to [1]. Actually, a Paley-Wiener theorem of Helgason implies that  $\mathcal{A}$  is an isomorphism between the space  $\mathcal{D}(K \backslash X)$  of compactly supported smooth functions on  $K \backslash X$  and the space  $\mathcal{D}_W(\mathfrak{a}) = C_0^\infty(\mathfrak{a})_W$  of  $W$ -invariant compactly supported smooth functions on  $\mathfrak{a}$  [12].

## 2.5 Distributions on symmetric spaces

For  $\psi \in \mathcal{D}(X)$  and  $k \in K$ , define  $\psi_k$  to be the left translation of  $\psi$ :

$$\psi_k(x) = \psi(k^{-1} \cdot x).$$

A distribution  $f$  on  $X$  is said to be  $K$ -invariant if

$$f(\psi_k) = f(\psi), \quad \forall \psi \in \mathcal{D}(X)$$

A distribution on  $X$  is said to be tempered if it can be extended to a continuous functional on  $\mathcal{S}(X)$ . Since  $\mathcal{D}(X)$  is continuously embedded and dense in  $\mathcal{S}(X)$ , the space  $\mathcal{S}'(X)$  of tempered distribution can be regarded as the dual space to  $\mathcal{S}(X)$ . It is clear that every compactly supported distribution on  $X$  is tempered:

$$\mathcal{E}'(X) \subset \mathcal{S}'(X)$$

and that  $L^2(X)$  is continuously embedded in  $\mathcal{S}'(X)$ . The Helgason-Fourier transform is a topological isomorphism of  $\mathcal{S}'(X)$  onto the space  $\mathcal{Z}'(\mathfrak{a}^* \times B)$  dual to  $\mathcal{Z}(\mathfrak{a}^* \times B)$  (both equipped with their weak topologies, see [6]).

## 2.6 Convolution product on symmetric spaces

The convolution product on  $X$  is denoted  $\times$  and is defined by:

$$(f_1 \times f_2) \circ \pi = (f_1 \circ \pi) \star (f_2 \circ \pi)$$

where

$$\pi : G \mapsto X = G/K$$

is the natural projection and  $\star$  is the convolution product on  $G$ . It should be noted that the product of convolution on  $X$  is not commutative, and therefore the Helgason-Fourier transform does not turn it into a multiplication. However this will be the case whenever the second factor is  $K$ -invariant (see [12]).

## 3 Besov spaces. Propagators

In this section we define Besov spaces and we state some results we will need later. First, we define a resolution of unity on  $\mathfrak{a}^*$  slightly different from the one given in [19].

### 3.1 Resolution of unity

Let  $\varphi \in \mathcal{D}(K \setminus X)$  a real function such that  $\text{supp} \varphi \subset \overline{B(eK, 1)}$  and  $\mathcal{F}\varphi(0) \neq 0$ , where  $B(eK, 1)$  is the unit geodesic ball.

We define a function  $\psi$  on  $X$  such that:

$$\psi = \mathcal{A}\varphi \text{ and } \psi^N = \mathcal{A}\varphi^N$$

where  $\varphi^N = \Gamma^N \varphi$  for  $N \in \mathbb{N}$  and  $\Gamma = -\Delta - \|\rho\|^2$ .

It is easy to see that  $\mathcal{F}\varphi^N = \widehat{\mathcal{A}\varphi^N} = \widehat{\psi^N} \in \mathcal{S}_W(\mathfrak{a}^*)$ .

We suppose that for all  $\lambda \in \mathfrak{a}^* (\lambda \neq 0)$ :

$$\int_0^\infty (\widehat{\psi^N})^2(t\lambda) \frac{dt}{t} = 1.$$

We define a smooth function  $\psi_{0,N}$  on  $\mathfrak{a}$  by its Euclidean Fourier transform:

$$\widehat{\psi_{0,N}}(\lambda) = 1 - \int_0^1 (\widehat{\psi^N})^2(t\lambda) \frac{dt}{t}.$$

Then, since  $\widehat{\psi^N} \in \mathcal{S}_W(\mathfrak{a}^*)$ , we can easily deduce that  $\widehat{\psi_{0,N}} \in \mathcal{S}_W(\mathfrak{a}^*)$  and  $\psi_{0,N} \in \mathcal{D}_W(\mathfrak{a})$ .

We consider the function:

$$\varphi_{0,N} = \mathcal{A}^{-1}\psi_{0,N}.$$

Then by using the fact that  $\mathcal{A}$  is an isomorphism between  $\mathcal{D}(K \setminus X)$  and  $\mathcal{D}_W(\mathfrak{a})$ , we remark that  $\varphi_{0,N}$  belongs to  $\mathcal{D}(K \setminus X)$ . A simple computation shows that:

$$\mathcal{F}\varphi_{0,N}(\lambda) + \int_0^1 (\mathcal{F}\varphi^N)^2(t\lambda) \frac{dt}{t} = 1.$$

Finally, let  $\varphi_t^N$  a function in  $\mathcal{D}(K \setminus X)$  given by  $\varphi_t^N = \mathcal{F}^{-1}(\mathcal{F}\varphi^N(t.))$ .

Then we have the following formula of Calderon type [19]:

$$f(x) = (f \times \varphi_{0,N})(x) + \int_0^1 (f \times \varphi_t^N \times \varphi_t^N)(x) \frac{dt}{t}, \quad \forall f \in L^2(X). \quad (3.7)$$

We conclude that the system  $\{\varphi_{0,N}, \varphi_t^N\}_{0 < t \leq 1}$  is a continuous resolution of unity on  $\mathfrak{a}^*$ .

Now, we are in the position to give a definition of inhomogeneous Besov spaces on  $X$ .

### 3.2 Besov spaces

**Definition 3.1.** Let  $1 \leq p \leq \infty$ ,  $1 \leq q < \infty$ , and  $s \in \mathbb{R}$ . Let  $N$  be a positive integer such that  $2N > |s|$ . Let  $\{\varphi_{0,N}, \varphi_t^N\}_{0 < t \leq 1}$  be the system of functions defined above. Then the Besov space is the set denoted by  $B_p^{s,q}(X)$  and given by:

$$B_p^{s,q}(X) = \{f \in \mathcal{S}'(X); \|f\|_{B_p^{s,q}(X)} < \infty\},$$

where:

$$\|f\|_{B_p^{s,q}(X)} = \|f \times \varphi_{0,N}\|_{L^p(X)} + \left( \int_0^1 t^{-sq} \|f \times \varphi_t^N\|_{L^p(X)}^q \frac{dt}{t} \right)^{1/q}.$$

**Remark 1.**

1. We supposed that  $\varphi \in \mathcal{D}(K \setminus X)$ , while Skrzypczak have considered in [19]  $\varphi$  not to be necessarily a smooth function with compact support.
2. Observe that the spaces  $B_2^{0,2}(X)$  and  $L^2(X)$  are equivalent.
3. In [17] Skrzypczak proved that for  $1 < p < \infty$  :

$$B_p^{s,q}(X) = (H_p^{s_0}(X), H_p^{s_1}(X))_{\theta,q}$$

with  $0 < \theta < 1$ ,  $s = \theta.s_0 + (1 - \theta).s_1$ ,  $H_p^s(X)$  is the Bessel-potential space and  $(\cdot, \cdot)_{\theta,q}$  denotes the real interpolation method.

Then, we can deduce that the above defined Besov spaces coincide with the Besov spaces defined on  $X$  by uniform localization (see [23]) and  $B_p^{s,q}(X)$  is a Banach space for  $1 < p < \infty$ .

4. It is also known that the dual  $(H_p^s(X))'$  of  $H_p^s(X)$  is equal to  $H_{p'}^{-s}(X)$ . Then by the duality theorem from real interpolation, we deduce that:

$$(B_p^{s,q}(X))' = B_{p'}^{-s,q'}(X)$$

with  $1/p + 1/p' = 1/q + 1/q' = 1$ .

### 3.3 Applications

As an application of (3.7) and the definition of Besov spaces, we prove the following result.

**Proposition 3.2.** *For all  $p \geq 2$ , we have  $B_p^{0,1}(X) \subset L^p(X)$ .*

*Proof.* Let  $u$  in  $B_p^{0,1}(X)$  then, by formula (3.7), we have:

$$\begin{aligned} \|u\|_{L^p(X)} &\leq \|u \times \varphi_{0,N}\|_{L^p(X)} + \int_0^1 \|u \times \varphi_t^N \times \varphi_t^N\|_{L^p(X)} \frac{dt}{t} \\ &\leq \|u \times \varphi_{0,N}\|_{L^p(X)} + \int_0^1 \|u \times \varphi_t^N\|_{L^p(X)} \|\varphi_t^N\|_{L^1(X)} \frac{dt}{t} \\ &\leq \|u \times \varphi_{0,N}\|_{L^p(X)} + c \int_0^1 \|u \times \varphi_t^N\|_{L^p(X)} \frac{dt}{t} \\ &\leq c \|u\|_{B_p^{0,1}(X)} \end{aligned}$$

where we used the Young inequality and the fact  $\|\varphi_t^N\|_{L^1(X)} \leq c$  for all  $0 < t \leq 1$  (see [19]).

As an application of *the atomic decomposition* of Besov spaces, Skrzypczak in [19] showed the following result.

**Proposition 3.3.** *Let  $1 \leq p, q, q_0, q_1 \leq \infty$  and  $s, s_0, s_1 \in \mathbb{R}$ .*

(i) *(Elementary embedding)*

$$B_p^{s,q_0}(X) \subset B_p^{s,q_1}(X) \quad q_0 \leq q_1 \quad (3.8)$$

$$B_p^{s_0,q}(X) \subset B_p^{s_1,q}(X) \quad s_1 \leq s_0. \quad (3.9)$$

(ii) *(Embedding with different metrics)*

$$B_p^{s_0,q}(X) \subset B_\infty^{s_1,q}(X) \quad s_1 = s_0 - n/p \quad (3.10)$$

$$B_1^{s_0,q}(X) \subset B_p^{s_1,q}(X) \quad s_0 - n = s_1 - n/p \quad (3.11)$$

$$B_1^{0,1}(X) \subset L^1(X) \subset B_1^{0,\infty}(X) \quad (3.12)$$

$$B_\infty^{0,1}(X) \subset C(X) \subset B_\infty^{0,\infty}(X) \quad (3.13)$$

where  $C(X)$  denotes the space of bounded continuous functions on  $X$ .

### 3.4 Propagators on $X$

We start this paragraph by giving a definition concerning the powers of the Laplace-Beltrami operator on  $X$ .

**Definition 3.4.** For all integer  $\mu$ , we define the operator  $\Delta_0^{\mu/2}$  by

$$\widetilde{\Delta_0^{\mu/2} f}(\lambda, b) = Z_\rho(\lambda)^\mu \widetilde{f}(\lambda, b).$$

where  $\Delta_0 = -\Delta$  and  $Z_\rho(\lambda) = (\|\lambda\|^2 + \|\rho\|^2)^{1/2}$ , for a smooth function  $f$  or  $f \in \mathcal{S}'(X)$ .

**Lemma 3.5.** (i) Let  $\mathcal{K}_W(\mathfrak{a}_\mathbb{C}^*)$  be the space of functions  $\psi$  which can be extended to a  $W$ -invariant entire holomorphic functions on  $\mathfrak{a}_\mathbb{C}^*$  such that: there exist a constant  $R \geq 0$  and the integer  $m \geq 0$  satisfying the following condition:

$$\sup_{\lambda \in \mathfrak{a}_\mathbb{C}^*} (1 + \|\lambda\|)^{-m} e^{-R\|\mathcal{I}m\lambda\|} |\psi(\lambda)| < +\infty. \quad (3.14)$$

Then the spherical transform  $\mathcal{F}$  is an isomorphism of  $\mathcal{E}'(K \backslash X)$  to  $\mathcal{K}_W(\mathfrak{a}_\mathbb{C}^*)$ .

(ii) If  $f \in \mathcal{S}'(X)$  and  $h \in \mathcal{E}'(K \backslash X)$  then  $f \times h \in \mathcal{S}'(X)$  and we have:

$$\widetilde{(f \times h)}(\lambda, b) = \widetilde{f}(\lambda, b) \cdot \mathcal{F}h(\lambda) \quad \forall (\lambda, b) \in \mathfrak{a}_\mathbb{C}^* \times B.$$

The first point is a Paley-Wiener theorem due to Helgason [12] concerning the compactly supported  $K$ -invariant distributions on  $X$ . Also we can find the proof of the second point in [18].

If we use Definition 3.4 and the above lemma one can find one solution of our Cauchy problem (1.1). We suppose that  $u_0, u_1 \in \mathcal{S}'(X)$ . Then reformulating (1.1) in Fourier variables, the solution of (1.1) may be written as:

$$\widetilde{u}(t, \lambda, b) = \widetilde{u}_0(\lambda, b) \cos Z_\rho(\lambda)t + \widetilde{u}_1(\lambda, b) \frac{\sin Z_\rho(\lambda)t}{Z_\rho(\lambda)}.$$

One can check that the function  $\lambda \mapsto \psi_{\rho,t}(\lambda) = \frac{\sin Z_\rho(\lambda)t}{Z_\rho(\lambda)}$  is  $W$ -invariant, holomorphic on  $\mathfrak{a}_\mathbb{C}^*$  and satisfies (3.14). In particular these function belongs to the space  $\mathcal{K}_W(\mathfrak{a}_\mathbb{C}^*)$ .

According to Lemma 3.5, there is one and only one  $\mathcal{E}_t \in \mathcal{E}'(K \backslash X)$  such that:

$$\mathcal{F}\mathcal{E}_t(\lambda) = \psi_{\rho,t}(\lambda) \quad \forall \lambda \in \mathfrak{a}_\mathbb{C}^*.$$

Also, by the same lemma, we have  $u_0 \times \mathcal{E}_t \in \mathcal{S}'(X)$  and that:

$$(\widetilde{u_0 \times \mathcal{E}_t})(\lambda, b) = \widetilde{u_0}(\lambda, b) \mathcal{F}\mathcal{E}_t(\lambda) \quad \forall (\lambda, b) \in \mathfrak{a}_{\mathbb{C}}^* \times B.$$

Then we deduce that:

$$\partial_t(\widetilde{u_0 \times \mathcal{E}_t})(\lambda, b) = [\partial_t \widetilde{u_0}(\lambda, b)] \mathcal{F}\mathcal{E}_t(\lambda) = \widetilde{u_0}(\lambda, b) [\partial_t \mathcal{F}\mathcal{E}_t(\lambda)].$$

Since we have:

$$\partial_t(\widetilde{u_0 \times \mathcal{E}_t})(\lambda, b) = (\widetilde{u_0 \times \partial_t \mathcal{E}_t})(\lambda, b) = \widetilde{u_0}(\lambda, b) [\mathcal{F}\partial_t \mathcal{E}_t(\lambda)],$$

we deduce that:

$$\partial_t \mathcal{F}\mathcal{E}_t(\lambda) = \mathcal{F}\partial_t \mathcal{E}_t(\lambda).$$

Then, if we denote

$$\partial_t \mathcal{E}_t = \mathcal{E}'_t,$$

the solution  $u$  is written:

$$u(t, x) = (u_0 \times \mathcal{E}'_t)(x) + (u_1 \times \mathcal{E}_t)(x). \quad (3.15)$$

Therefore we have the following result.

**Proposition 3.6.** *If  $u_0, u_1 \in \mathcal{S}'(X)$ , the solution of (1.1) is given by (3.15), where  $\mathcal{E}_t \in \mathcal{E}'(K \setminus X)$  is a solution of the partial differential equation:*

$$\partial_t^2 \mathcal{E}_t - \Delta \mathcal{E}_t = 0, \quad \mathcal{E}_0 = 0, \quad \mathcal{E}'_0 = \delta_0.$$

*Proof.* The fact that  $\partial_t^2 \mathcal{E}_t - \Delta \mathcal{E}_t = 0$ , is a direct consequence of Definition 3.4 and the expression of  $\mathcal{F}\mathcal{E}_t$ .

**Definition 3.7.** For all  $t \in \mathbb{R}$ , we define the operator  $U_t$  by:

$$(\widetilde{U_t f})(\lambda, b) = e^{iZ_\rho(\lambda)t} \widetilde{f}(\lambda, b) \quad (\lambda, b) \in \mathfrak{a}_{\mathbb{C}}^* \times B,$$

for a smooth function  $f$  on  $X$  or  $f \in \mathcal{S}'(X)$ . In this case, we may view  $\mathcal{E}_t$  and  $\mathcal{E}'_t$  as the operators defined by:

$$\mathcal{E}_t = \frac{1}{2i}(-\Delta)^{-1/2}(U_t - U_{-t}) \text{ and } \mathcal{E}'_t = \frac{1}{2}(U_t + U_{-t})$$

The operator  $\mathcal{E}_t$  is called *the propagator* of the solution  $u$  given by (3.15).



## 4 Proof of Theorem 1

The proof of our first result (Theorem 1 mentioned in the introduction) will follow from the following proposition.

**Proposition 4.1.** *Let  $N$  be a positive integer such that  $n = \dim(X) \geq 4N$ . Let  $\alpha = \dim(\mathfrak{a}) \geq 1$  and  $I = [0, +\infty[$ .*

1. *Let  $u$  be a solution of the wave equation (1.1) on  $X$  and, let  $p$  and  $r$  two real numbers such that  $\frac{2}{r} = \alpha(1 - \frac{2}{p})$  and  $2 < p < \frac{2n}{n-4N+2}$ . Then there exists a positive number  $c_r > 0$  depending of  $r$  such that if  $u_0, (-\Delta)^{1/2}u_0$  and  $u_1$  belong to  $L^2(X)$ , we have:*

$$\|u\|_{L^r(I, B_p^{s,2}(X))} + \|\partial_t u\|_{L^r(I, B_p^{s-1,2}(X))} \leq c_r \cdot [\|u_0\|_{L^2(X)} + \|(-\Delta)^{1/2}u_0\|_{L^2(X)} + (1 + \frac{1}{\|\rho\|})\|u_1\|_{L^2(X)}]$$

where  $s = \frac{n}{2}(\frac{2}{p} - 1)$ .

2. *Let  $f$  a distribution on  $X$  which also depend on time. Let  $p_i, p'_i, r_i$  and  $r'_i$  eight real numbers such that  $\frac{1}{p_i} + \frac{1}{p'_i} = \frac{1}{r_i} + \frac{1}{r'_i} = 1$ ,  $\frac{2}{r_i} = \alpha(1 - \frac{2}{p_i})$  and  $2 < p_i < \frac{2n}{n-4N}$  for  $i = 1, 2$ . Then there exists a positive number  $c > 0$  such that if  $f$  belongs to  $L^{r'_2}(I, B_{p'_2}^{-s_2,2}(X))$ , we have:*

$$\|\mathcal{E}'_{(\cdot)} *_{t} f\|_{L^{r_1}(I, B_{p_1}^{s_1,2}(X))} \leq c \cdot \|f\|_{L^{r'_2}(I, B_{p'_2}^{-s_2,2}(X))}$$

where  $s_i = \frac{n}{2}(\frac{2}{p_i} - 1)$  for  $i = 1, 2$ .

### 4.1 Proof of proposition 4.1

We proceed in several steps.

**Step 1.** We recall a Paley-Wiener due to Helgason for compactly supported smooth functions on  $K \backslash X$ .

**Lemma 4.2.** *Let  $\mathcal{H}_W(\mathfrak{a}_{\mathbb{C}}^{\star})$  the space of functions  $h$  which can be extended to a  $W$ -invariant entire holomorphic functions on  $\mathfrak{a}_{\mathbb{C}}^{\star}$  such that: there exists a constant  $R \geq 0$  such that for each integer  $m \geq 0$ :*

$$\sup_{\lambda \in \mathfrak{a}_{\mathbb{C}}^{\star}} (1 + \|\lambda\|)^m e^{-R\|\mathcal{I}m\lambda\|} |h(\lambda)| < +\infty.$$

Then the spherical transform  $\mathcal{F}$  is an isomorphism from  $\mathcal{D}(K \setminus X)$  onto  $\mathcal{H}_W(\mathfrak{a}_{\mathbb{C}}^*)$ .

A proof of this lemma can be found in [11, page 450].

Now, let  $(\varphi_\varepsilon)_{0 < \varepsilon \leq 1}$  be a sequence of smooth functions on  $X$  with spherical transform given by:

$$\mathcal{F}\varphi_\varepsilon(\lambda) = \frac{\|\lambda\|^2}{\varepsilon + \|\lambda\|^2} e^{-\varepsilon(1 + \|\lambda\|^2)} \quad \text{for } \lambda \in \mathfrak{a}_{\mathbb{C}}^*.$$

Then by the previous lemma, one can show that  $\mathcal{F}\varphi_\varepsilon$  and  $\varphi_\varepsilon$  belongs respectively to  $\mathcal{H}_W(\mathfrak{a}_{\mathbb{C}}^*)$  and  $\mathcal{D}(K \setminus X)$ .

**Step 2.** We prove two properties for the sequence  $(\varphi_\varepsilon)_{0 < \varepsilon \leq 1}$  defined above.

**Lemma 4.3.** *For the system of smooth functions  $\{\varphi_{0,N}, \varphi_{t'}^N\}_{0 < t' \leq 1, N \in \mathbb{N}^*}$ , defined in Section 3, we have:*

$$f \times \varphi_{0,N} = f \times \varphi_{0,N} \times \varphi_\varepsilon \quad \forall f \in \mathcal{S}'(X). \quad (4.16)$$

$$f \times \varphi_{t'}^N = f \times \varphi_{t'}^N \times \varphi_\varepsilon \quad \forall f \in \mathcal{S}'(X). \quad (4.17)$$

*Proof.* We follow the same idea used to show inequality (9) in [19].

Let  $f \in C^\infty(X) \cap \mathcal{S}'(X)$ . Then using the dominated convergence theorem, we can find a sequence  $(f_n) \subset \mathcal{D}(X)$  such that  $f_n \rightarrow f$  in  $\mathcal{S}'(X)$ .

Thus it is sufficient to prove (4.16) for  $f \in \mathcal{D}(X)$  with the  $L^2$ -convergence.

By Plancherel formula (2.6), we have:

$$\|f \times \varphi_{0,N} - f \times \varphi_{0,N} \times \varphi_\varepsilon\|_{L^2(X)}^2 = \int_{\mathfrak{a}_{\mathbb{C}}^* \times B} h_\varepsilon(\lambda, b) \frac{d\lambda db}{|c(\lambda)|^2}$$

where

$$h_\varepsilon(\lambda, b) = |\tilde{f}(\lambda, b) \mathcal{F}\varphi_{0,N}(\lambda)|^2 |1 - \mathcal{F}\varphi_\varepsilon(\lambda)|^2.$$

But, observe that:

$$0 \leq \mathcal{F}\varphi_\varepsilon(\lambda) \leq 1, \quad \forall \lambda \in \mathfrak{a}_{\mathbb{C}}^*,$$

and that

$$\mathcal{F}\varphi_\varepsilon(\lambda) \rightarrow 1 \text{ if } \varepsilon \rightarrow 0, \quad \forall \lambda \in \mathfrak{a}_{\mathbb{C}}^*.$$

We deduce that:

$$h_\varepsilon(\lambda, b) \leq 4|\tilde{f}(\lambda, b)\mathcal{F}\varphi_{0,N}(\lambda)|^2 \quad \forall \lambda \in \mathfrak{a}_\mathbb{C}^*.$$

$$h_\varepsilon(\lambda, b) \rightarrow 0 \text{ if } \varepsilon \rightarrow 0 \quad \forall \lambda \in \mathfrak{a}_\mathbb{C}^*.$$

Thus by the dominated convergence theorem, we conclude that:

$$\int_{\mathfrak{a}_+^* \times B} h_\varepsilon(\lambda, b) \frac{d\lambda db}{|c(\lambda)|^2} \rightarrow 0 \text{ if } \varepsilon \rightarrow 0.$$

Similarly, we prove equality (4.17).

**Lemma 4.4.** *There exists a positive number  $c > 0$  such that for the operator  $U_t$  given in Definition 3.3, we have:*

$$\|U_t\varphi_\varepsilon\|_{L^\infty(X)} \leq ct^{-\alpha}I(t) \quad \forall t \geq 1. \quad (4.18)$$

Consequently we have:

$$\|U_t\varphi_\varepsilon\|_{L^\infty(X)} \leq ct^{-\alpha}.t'^{-n}I(t) \quad \forall t \geq 1, \forall 0 < t' \leq 1, \quad (4.19)$$

where  $I(t) = \int_{\mathfrak{a}^*} |\mathcal{F}\varphi_\varepsilon(t^{-1}\lambda)|(1+t^{-2}\|\lambda\|^2)^d$  and  $d = \dim(X) - \text{rank}_\mathbb{R}(G)$ .

*Proof.* It is clear that the support of  $U_t\varphi_\varepsilon$  is included in the support of  $\varphi_\varepsilon$ . Then it is sufficient to consider the  $L^\infty$ -norm on the support of  $\varphi_\varepsilon$ . According to inversion formula (2.4) and inequality (2.5), we write:

$$\begin{aligned} |U_t\varphi(x)| &= \left| \frac{1}{|W|} \int_{\mathfrak{a}^* \times B} e^{itZ_\rho(\lambda)} \mathcal{F}\varphi_\varepsilon(\lambda) e^{(i\lambda+\rho)A(x,b)} \frac{d\lambda db}{|c(\lambda)|^2} \right| \\ &\leq \frac{1}{|W|} \sup_{b \in B} e^{\rho A(x,b)} \int_{\mathfrak{a}^*} |\mathcal{F}\varphi_\varepsilon(\lambda)|(1+\|\lambda\|^2)^d d\lambda \quad \forall x \in X \\ &= \frac{1}{|W|} t^{-\alpha} \sup_{b \in B} e^{\rho A(x,b)} \int_{\mathfrak{a}^*} |\mathcal{F}\varphi_\varepsilon(t^{-1}\lambda)|(1+t^{-2}\|\lambda\|^2)^d d\lambda \quad \forall x \in X \end{aligned}$$

Then one has

$$\sup_{x \in \text{supp}\varphi_\varepsilon} |U_t\varphi_\varepsilon(x)| \leq ct^{-\alpha}I(t)$$

where

$$I(t) = \int_{\mathfrak{a}^*} |\mathcal{F}\varphi_\varepsilon(t^{-1}\lambda)|(1+t^{-2}\|\lambda\|^2)^d d\lambda$$

and

$$c = \frac{1}{|W|} \sup_{x \in \text{supp} \varphi_\varepsilon, b \in B} e^{\rho A(x,b)}.$$

proving (4.18). And Inequality (4.19) is a direct consequence of (4.18) if  $0 < t' \leq 1$ .

**Corollary 4.5.** *There exists a positive number  $c > 0$  and a sufficiently large number  $T$  such that:*

$$\|U_t \varphi_\varepsilon\|_{L^\infty(X)} \leq c.t^{-\alpha} \quad \forall t \geq T \quad (4.20)$$

$$\|U_t \varphi_\varepsilon\|_{L^\infty(X)} \leq c.t^{-\alpha}.t'^{-n} \quad \forall t \geq T, \quad \forall 0 < t' \leq 1. \quad (4.21)$$

*Proof.* To show inequality (4.20), it suffices to prove that  $I(t) \rightarrow 0$  when  $t \rightarrow \infty$ . Which is a direct consequence of the dominated convergence theorem, because  $\mathcal{F}\varphi_\varepsilon(0) = 0$ .

Next using(4.16), inequality (4.20) and Young inequality, we deduce that:

$$\begin{aligned} \|U_t(f \times \varphi_{0,N})\|_{L^\infty(X)} &= \|U_t(f \times \varphi_{0,N} \times \varphi_\varepsilon)\|_{L^\infty(X)} \\ &= \|f \times \varphi_{0,N} \times U_t \varphi_\varepsilon\|_{L^\infty(X)} \\ &\leq \|f \times \varphi_{0,N}\|_{L^1(X)} \cdot \|U_t \varphi_\varepsilon\|_{L^\infty(X)} \\ &\leq c.t^{-\alpha} \|f \times \varphi_{0,N}\|_{L^1(X)} \end{aligned} \quad (4.22)$$

In addition, we have:

$$\|U_t(f \times \varphi_{0,N})\|_{L^2(X)} = \|f \times \varphi_{0,N}\|_{L^2(X)} \quad (4.23)$$

Thus by interpolating inequalities (4.22) and (4.23), we obtain:

$$\|U_t(f \times \varphi_{0,N})\|_{L^p(X)} \leq c.t^{-\alpha.(1/p'-1/p)} \|f \times \varphi_{0,N}\|_{L^{p'}(X)} \quad (4.24)$$

for  $1 \leq p' \leq 2 \leq p \leq \infty$  such that  $1/p + 1/p' = 1$ .

By using inequality (4.21) and the same arguments as those used to prove (4.24), we obtain that:

$$\|U_t(f \times \varphi_{t'}^N)\|_{L^p(X)} \leq c.t^{-\alpha.(1/p'-1/p)} t'^{-n.(1/p'-1/p)} \|f \times \varphi_{t'}^N\|_{L^{p'}(X)}. \quad (4.25)$$

Therefore (4.24) and (4.25) give us the following inequality:

$$\|U_t f\|_{B_p^{s,q}(X)} \leq c.t^{-\alpha.(1-2/p)} \|f\|_{B_{p'}^{n(1-2/p)+s,q}(X)} \quad \forall t \geq T \quad (4.26)$$

with  $2N > \max(|s|, |n(1 - 2/p) + s|)$ ,  $1 \leq p' \leq 2 \leq p \leq \infty$  and  $1 \leq q \leq \infty$ .

**Step 3.** We recall two results we will use in the last step. The first is an argument of duality due to Ginibre-Velo [7] and Yajima [24], and the second is the Hardy-Littlewood-Sobolev inequality [13, page 117].

**Lemma 4.6.** *Let  $\mathcal{H}$  be a Hilbert space,  $Y$  a Banach space,  $Y^*$  the dual of  $Y$ , and  $\mathcal{D}$  a vector space densely contained in  $Y$ . Let  $A \in \mathcal{L}_a(\mathcal{D}, \mathcal{H})$  and let  $A^* \in \mathcal{L}_a(\mathcal{H}, \mathcal{D}_a^*)$  be its adjoint, defined by*

$$\langle A^*v, f \rangle_{\mathcal{D}} = \langle v, Af \rangle_{\mathcal{H}}, \quad \forall f \in \mathcal{D}, \forall v \in \mathcal{H}.$$

where  $\mathcal{L}_a(Z_1, Z_2)$  is the space of linear maps from a vector space  $Z_1$  to a vector space  $Z_2$ ,  $\mathcal{D}_a^*$  is the algebraic dual of  $\mathcal{D}$  and  $\langle \varphi, f \rangle_{\mathcal{D}}$  is the pairing between  $\mathcal{D}_a^*$  and  $\mathcal{D}$  (with  $f \in \mathcal{D}$  and  $\varphi \in \mathcal{D}_a^*$ ). Then the following three conditions are equivalent.

1. There exist a real number  $c_1 \geq 0$  such that for all  $f \in \mathcal{D}$ ,

$$\|Af\|_{\mathcal{H}} \leq c_1 \|f\|_Y .$$

2.  $\mathcal{R}(A^*) \subset Y^*$ , and there exist a real number  $c_2 \geq 0$  such that for all  $v \in \mathcal{H}$ ,

$$\|A^*v\|_{Y^*} \leq c_2 \|v\|_{\mathcal{H}} .$$

3.  $\mathcal{R}(A^*A) \subset Y^*$ , and there exist a real number  $c_3 \geq 0$  such that for all  $f \in \mathcal{D}$ ,

$$\|A^*Af\|_{Y^*} \leq c_3^2 \|f\|_Y .$$

The constants  $c_1, c_2$  and  $c_3$  are equal. If one of these conditions is satisfied, the operators  $A$  and  $A^*A$  extend by continuously to bounded operators from  $Y$  to  $\mathcal{H}$  and from  $Y$  to  $Y^*$  respectively.

We use the following corollary to show the second part of Proposition 4.1.

**Corollary 4.7.** *Let  $\mathcal{H}$ ,  $\mathcal{D}$  and  $(Y_i, A_i), i = 1, 2$ , satisfy any of the conditions of Lemma 4.6. Then for all choices of  $i, j = 1, 2$ ,  $\mathcal{R}(A_i^*A_j) \subset Y_i^*$ , and for all  $f \in \mathcal{D}$ , one has:*

$$\|A_i^* A_j f\|_{Y^*} \leq c_i c_j \|f\|_{Y_j}.$$

where  $c_i, c_j$  for  $i, j = 1, 2$  are the constants given in Lemma 4.6.

**Lemma 4.8.** (Hardy-Littlewood-Sobolev inequality).

Let  $k_a(y) = |y|^{-m/a}$ . If  $1 < a < \infty$  and  $1 < p, q < \infty$  such that  $1/p + 1/a = 1 + 1/q$ , then  $\|k_a * u\|_{L^q(\mathbb{R}^m)} \leq c_{p,a} \|u\|_{L^p(\mathbb{R}^m)}$ ,  $u$  continuous with compact support.

We shall apply Lemma 4.6 and Corollary 4.7 in the following basic situation. Let  $\mathcal{H}$  be a Hilbert space, and  $U$  a unitary strongly continuous one parameter group in  $\mathcal{H}$ . We define the bounded operator  $A$  from  $L^1(I, \mathcal{H})$  to  $\mathcal{H}$  by

$$Af = \int_I U(-t)f(t)dt. \quad (4.27)$$

Then its adjoint  $A^*$  is the operator

$$A^*v(t) = U(t)v \quad (4.28)$$

from  $\mathcal{H}$  to  $L^\infty(I, \mathcal{H})$ , where the duality is defined by the scalar products in  $\mathcal{H}$  and in  $L^2(I, \mathcal{H})$ . In particular  $A^*A$  is the bounded operator from  $L^1(I, \mathcal{H})$  to  $L^\infty(I, \mathcal{H})$  given by

$$A^*Af = \int_I U(t-t')f(t')dt'. \quad (4.29)$$

**Step 4.** Let  $f$  be a function defined on  $X$  which also depend on time. Let  $t$  and  $t'$  two real such that  $t - t' \geq T$ . Let  $\tau$  be a real number such that  $t \geq \tau + T$ . Then inequality (4.26) gives:

$$\|U_{t-t'}f(t', \cdot)\|_{B_p^{s,q}(X)} \leq c(t-t')^{\alpha\gamma} \|f(t', \cdot)\|_{B_{p'}^{-n\gamma+s,q}(X)}$$

with  $\gamma = \frac{2}{p} - 1$ ,  $2N > \max(|s|, | -n\gamma + s|)$ ,  $1 \leq p' \leq 2 \leq p \leq \infty$  and  $1 \leq q \leq \infty$ . Thus integrating in the variable  $t'$  on the interval  $I_\tau = ]-\infty, \tau]$ , we obtain that:

$$\int_{-\infty}^{\tau} \|U_{t-t'}f(t', \cdot)\|_{B_p^{s,q}(X)} dt' \leq c((\cdot)^{\alpha\gamma} *_t \chi_{I_\tau} \|f\|_{B_{p'}^{-n\gamma+s,q}(X)})(t) \quad (4.30)$$

since,

$$\begin{aligned} \|(U_{(\cdot)} *_t \chi_{I_\tau} f)(t)\|_{B_p^{s,q}(X)} &= \left\| \int_{-\infty}^{\tau} U_{t-t'}f(t', \cdot) dt' \right\|_{B_p^{s,q}(X)} \\ &\leq \int_{-\infty}^{\tau} \|U_{t-t'}f(t', \cdot)\|_{B_p^{s,q}(X)} dt' \end{aligned}$$

then, by (4.30) and taken the  $L^r$ -norm on  $I = [\tau + T, +\infty[$ , we find that:

$$\left\| \|U_{(\cdot)} *_t \chi_{I_\tau} f\|_{B_p^{s,q}(X)} \right\|_{L^r(I)} \leq \left\| c(\cdot)^{\alpha\gamma} *_t \chi_{I_\tau} \|f(\cdot)\|_{B_{p'}^{-n\gamma+s,q}(X)} \right\|_{L^r(I)}$$

for  $1 < r < \infty$ . Therefore,

$$\|U_{(\cdot)} *_t \chi_{I_\tau} f\|_{L^r(I, B_p^{s,q}(X))} \leq c_{\beta, \alpha, p} \|\chi_{I_\tau} f\|_{L^\beta(I, B_{p'}^{-n\gamma+s,q}(X))} \quad (4.31)$$

where we used Hardy-Littlewood-Sobolev inequality, with  $1/\beta - \alpha\gamma = 1 + 1/r$ . To use Lemma 4.6, we choose  $\beta, s$  and  $q$  so that  $L^\beta(I, B_{p'}^{-n\gamma+s,q}(X))$  is the dual of  $L^r(I, B_p^{s,q}(X))$ . We take  $\beta = r', q = 2$  and  $-n\gamma + s = -s$  i.e  $s = \frac{n}{2}\gamma$ . In this case we will have  $2/r = -\alpha\gamma$ .

Then, taking  $Y = L^{r'}(I, B_{p'}^{-\frac{n}{2}\gamma, 2}(X))$  and  $Y^* = L^r(I, B_p^{\frac{n}{2}\gamma, 2}(X))$ , the inequality (4.31) can be written:

$$\|U_{(\cdot)} *_t g\|_{Y^*} \leq c_r \|g\|_Y; \quad (4.32)$$

which is exactly the condition 3 of Lemma 4.6 and who is equivalent to condition 2:

$$\|U_{(\cdot)} h\|_{L^r(I, B_p^{\frac{n}{2}\gamma, 2}(X))} \leq c_r \|h\|_{L^2(X)}. \quad (4.33)$$

By taking account the expression of  $u$ , (4.33) enables us to write:

$$\begin{aligned} \|u\|_{L^r(I, B_p^{\frac{n}{2}\gamma, 2}(X))} &\leq \|U_{(\cdot)} u_0\|_{L^r(I, B_p^{\frac{n}{2}\gamma, 2}(X))} + \|U_{(\cdot)} (-\Delta)^{-1/2} u_1\|_{L^r(I, B_p^{\frac{n}{2}\gamma, 2}(X))} \\ &\leq c_r (\|u_0\|_{L^2(X)} + \|(-\Delta)^{-1/2} u_1\|_{L^2(X)}) \\ &\leq c_r (\|u_0\|_{L^2(X)} + \frac{1}{\|\rho\|} \|u_1\|_{L^2(X)}) \end{aligned}$$

where we used the Plancherel formula in the last inequality.

Now, for the term concerning  $\partial_t u$ , we multiply (4.25) by  $t^{-s}$  with  $s = \frac{n}{2}\gamma - 1$ , to obtain:

$$\|U_{(\cdot)} f\|_{L^r(I, B_p^{\frac{n}{2}\gamma-1, 2}(X))} \leq c_r \|f\|_{L^2(X)}$$

showing that:

$$\|\partial_t u\|_{L^r(I, B_p^{\frac{n}{2}\gamma-1, 2}(X))} \leq c_r (\|u_1\|_{L^2(X)} + \|(-\Delta)^{1/2} u_0\|_{L^2(X)}).$$

This completes the proof of the first part of the proposition.

Now to prove the second part of the proposition, we consider the spaces:

$$Y_1 = L^{r_1}(I, B_{p_1}^{-\frac{n}{2}\gamma(p_1), 2}(X)) \quad Y_1^* = L^{r_1}(I, B_{p_1}^{\frac{n}{2}\gamma(p_1), 2}(X)).$$

$$Y_2 = L^{r'_2}(I, B_{p'_2}^{-\frac{n}{2}\gamma(p_2),2}(X)) \quad Y_2^* = L^{r_2}(I, B_{p_2}^{\frac{n}{2}\gamma(p_2),2}(X)).$$

with  $\gamma(p_i) = \frac{2}{p_i} - 1$  for  $i = 1, 2$ . And we remark that:

$$A_1^* A_2^* f = U_{(\cdot)} *_t f.$$

Thus according to Corollary 4.7,  $A_1^* A_2^*$  is bounded from  $Y_2$  onto  $Y_1^*$ , i.e.

$$\|U_{(\cdot)} *_t f\|_{L^{r_1}(I, B_{p_1}^{\frac{n}{2}\gamma(p_1),2}(X))} \leq c \|f\|_{L^{r'_2}(I, B_{p'_2}^{-\frac{n}{2}\gamma(p_2),2}(X))}.$$

This completes the proof of proposition 4.1.

## 4.2 Proof of Theorem 1

We can deduce easily Theorem 1 if we take  $\tau = -T$  used in the proof of the first part of proposition.

## 5 Proof of Theorem 2

We shall now prove our second result (Theorem 2 mentioned in the introduction). As mentioned in the previous section, we shall use Lemma 4.4. For the solution  $u$  of (1.1) given by expression (3.15), we write:

$$\begin{aligned} \|u(t, \cdot)\|_{L^\infty(X)} &\leq \|U_t u_0\|_{L^\infty(X)} + \|U_t (-\Delta)^{-1/2} u_1\|_{L^\infty(X)} \\ &\leq c (\|U_t u_0\|_{B_\infty^{0,1}(X)} + \|U_t (-\Delta)^{-1/2} u_1\|_{B_\infty^{0,1}(X)}) \\ &\leq ct^{-\alpha} (\|u_0\|_{B_1^{n,1}(X)} + \|(-\Delta)^{-1/2} u_1\|_{B_1^{n,1}(X)}) \end{aligned}$$

where we used inclusion (3.13) and inequality (4.26) for  $p = \infty, s = 0$  and  $q = 1$  in the last inequality. This proves the first result.

**Remark 2.** We can prove these results without using (4.26). In fact, by formula (3.7), we have:

$$\begin{aligned} \|U_t f\|_{L^\infty(X)} &\leq \|U_t(f \times \varphi_{0,N})\|_{L^\infty(X)} + \int_0^1 \|U_t(f \times \varphi_{t'}^N \times \varphi_{t'}^N)\|_{L^\infty(X)} \frac{dt'}{t'} \\ &\leq ct^{-\alpha} \|f \times \varphi_{0,N}\|_{L^1(X)} + c.t^{-\alpha} \int_0^1 t'^{-n} \|f \times \varphi_{t'}^N\|_{L^1(X)} \frac{dt'}{t'} \\ &= ct^{-\alpha} \|f\|_{B_1^{n,1}(X)}. \end{aligned}$$



Now if we use Proposition 3.2 and (4.26) for  $s = 0$ , we obtain:

$$\begin{aligned} \|u(t, \cdot)\|_{L^p(X)} &\leq \|U_t u_0\|_{L^p(X)} + \|U_t (-\Delta)^{-1/2} u_1\|_{L^p(X)} \\ &\leq c. (\|U_t u_0\|_{B_p^{0,1}(X)} + \|U_t (-\Delta)^{-1/2} u_1\|_{B_p^{0,1}(X)}) \\ &\leq c.t^{\alpha\gamma} (\|u_0\|_{B_{p'}^{-n\gamma,1}(X)} + \|(-\Delta)^{-1/2} u_1\|_{B_{p'}^{-n\gamma,1}(X)}) \end{aligned}$$

Which proves Theorem 2.

**Remark 3.** In Theorem 1, we have imposed that  $2 < p < \min(\frac{2n}{n-4N}, \frac{2\alpha}{\alpha-1})$ . This restriction comes from the assumptions  $1 < a = -\frac{1}{\alpha\gamma} < \infty$  in Hardy-Littlewood-Sobolev inequality and the definition of Besov spaces.

The restrictions on the dimension  $n$  of  $X$  and on  $L^p$ -spaces depend on integer  $N$  which comes from the definition of Besov spaces. We can prove other estimates not utilizing the dimension of  $X$ . Indeed, we have:

**Theorem 1':** *Let  $\alpha = \dim(\mathfrak{a}) \geq 1$  and  $I = [0, +\infty[$ . Let  $u$  be a solution of the wave equation (1.1) on  $X$  and, let  $p$  and  $r$  be two real numbers such that  $2/r = \alpha(1 - \frac{2}{p})$  and  $2 < p < \frac{2\alpha}{\alpha-1}$ . Then there exists a positive number  $c_r > 0$ , depending of  $r$ , such that if  $u_0$  and  $u_1$  belong to  $L^2(X)$ , we have:*

$$\|u\|_{L^r(I, B_p^{0,2}(X))} \leq c_r (\|u_0\|_{L^2(X)} + \frac{1}{\|\rho\|} \|u_1\|_{L^2(X)}).$$

For dispersive properties, we have:

**Theorem 2' :** *Let  $u$  be a solution of the wave equation (1.1) on  $X$  and write  $\alpha = \dim(\mathfrak{a})$ . Let  $p$  and  $p'$  two real numbers such that  $1/p + 1/p' = 1$  and  $1 \leq p' \leq 2 \leq p \leq \infty$ . We suppose that  $u_0$  and  $(-\Delta)^{-1/2} u_1$  are in  $B_{p'}^{0,1}(X)$ . Then there exist a constant  $c > 0$  and a sufficiently large number  $T$  such that for all  $t \geq T$ , we have:*

$$\|u(t, \cdot)\|_{L^p(X)} \leq ct^{\alpha\gamma} (\|u_0\|_{B_{p'}^{0,1}(X)} + \|(-\Delta)^{-1/2} u_1\|_{B_{p'}^{0,1}(X)}),$$

where  $\gamma = \frac{2}{p} - 1$ .

For the proof of these two theorems, it is enough to consider the inequality:

$$\|U_t \varphi_\varepsilon\|_{L^\infty(X)} \leq ct^{-\alpha} \text{ for all } t \geq T,$$

which implies that:

$$\|U_t f\|_{B_p^{s,q}(X)} \leq c.t^{-\alpha.(1-2/p)} \|f\|_{B_p^{s,q}(X)},$$

for  $2N > |s|$ ,  $2 \leq p \leq \infty$  and  $1 \leq q \leq \infty$ .

Then we use the same calculations as in the preceding proofs.

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