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Ali Hassani

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Strichartz estimates for the wave equation on Riemannian symmetric manifolds*

A. Hassani[†]

Abstract

We prove Strichartz type estimates for solutions of the homogeneous wave equation on Riemannian symmetric spaces. Our results generalize those of Ginibre and Velo in [7].

1 Introduction

The wave equation on a manifold with symmetries is a fundamental partial differential equation problem. In particular, it is important to have explicit estimates and dispersive properties for solutions of the following Cauchy problem:

$$\begin{cases} \Delta u = \frac{\partial^2}{\partial t^2} u, \\ u(t=0) = u_0 \\ \frac{\partial}{\partial t} u(t=0) = u_1 \end{cases} \quad (1.1)$$

where Δ denotes the Laplace operator on the manifold.

In the case of the euclidian space \mathbb{R}^n , $n \geq 1$, Segal proved some estimates in [16] which were improved by Strichartz in [21]. It turns out that these estimates play a crucial role in the modern theory of local and global well posed problems. Actually, one has the following *Strichartz type estimates* for (1.1) with non trivial Cauchy data u_0 and u_1 (see [7]):

$$\|u\|_{L^r(I, \dot{B}_p^{s,2}(\mathbb{R}^n))} \leq c_r (\|u_0\|_{L^2(\mathbb{R}^n)} + \|u_1\|_{\dot{H}^{-1}(\mathbb{R}^n)}) \quad (1.2)$$

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[†]Modal'X, UFR SEGMI, Université Paris X, 200 Avenue de la République, 92001 Nanterre cedex, FRANCE

where $I = [0; +\infty[$, $2 \leq p \leq \infty$, $s = \frac{n+1}{4}(\frac{2}{p} - 1)$ and $2/r = \frac{n-1}{2}(1 - \frac{2}{p})$. Here $\dot{B}_p^{s,q}(\mathbb{R}^n)$ is the *homogeneous Besov space* on \mathbb{R}^n defined by the norm:

$$\|f\|_{\dot{B}_p^{s,q}(\mathbb{R}^n)}^q = \sum_{j \in \mathbb{Z}} 2^{jsq} \|\varphi_j * f\|_{L^p(\mathbb{R}^n)}^q \text{ for } s \in \mathbb{R}, 1 \leq p, q \leq \infty$$

where $(\varphi_j)_{j \in \mathbb{Z}}$ is a sequence of smooth functions such that its Fourier transform is a *Paley-Littlewood partition of unity*. The space \dot{H}^s , $s \in \mathbb{R}$, is the *homogeneous Sobolev space* defined by the norm:

$$\|f\|_{\dot{H}^s(\mathbb{R}^n)}^2 = \sum_{j \in \mathbb{Z}} 2^{2js} \|\varphi_j * f\|_{L^2(\mathbb{R}^n)}^2.$$

It should be noted that estimates analogous to (1.2) have been proved on Heisenberg groups [3], hyperbolic spaces [22] and on Damek-Ricci spaces [15]. Moreover some *dispersive properties* for solutions of (1.1) may be deduced (see [4][7][20]):

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} \leq c(1 + |t|)^{-(n-1)/2} (\|u_0\|_{\dot{B}_1^{s,1}(\mathbb{R}^n)} + \|u_1\|_{\dot{B}_1^{s-1,1}(\mathbb{R}^n)}) \quad (1.3)$$

where $s = \frac{n+1}{2}$. The traditional interpretation for these estimates is the *decay of solutions* as $t \rightarrow \infty$. In the general context, where X is a manifold equipped with a pseudo-Riemannian metric, a measure σ and $\Delta = L$ is a second order differential operator on X which satisfies some integrability conditions, Lohoué proved some L^p -estimates of solutions of wave equation in terms of norms of the initial data (see [14]).

In this paper, we prove estimates analogous to (1.2) and (1.3) in the case of Riemannian symmetric space of the non-compact type. Our strategy is based on the Harish-Chandra Plancherel formula in the context of non-commutative harmonic analysis.

More precisely, let X be the G -homogeneous manifold G/K , where G is a non-compact connected semisimple Lie group with finite center and K a maximal compact subgroup of G . The Riemannian metric on X is induced by the Killing form of G and the (negative) Laplacian Δ on X is related to the Casimir element of G . To establish our estimates in this context, we first need to define Besov spaces on X . For general Riemannian manifolds, Besov spaces were defined by Triebel with purely geometric methods in [23]. However, in the particular case of Riemannian symmetric spaces, we shall proceed as follows. Let \mathfrak{g} (resp. \mathfrak{k}) be the Lie algebra of G (resp. K). Consider the corresponding Cartan decomposition of \mathfrak{g} :

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

where \mathfrak{p} is some \mathfrak{k} -invariant vector subspace of \mathfrak{g} . Let MAN be a minimal parabolic subgroup of G , where M is a closed subgroup of K and A (resp. N) an abelian (resp. nilpotent). The Lie algebra \mathfrak{a} of A is a maximal abelian subspace of \mathfrak{p} equipped with a norm $\| \cdot \|$ induced by the Killing form. Then the *Helgason-Fourier transform* \tilde{f} of some function f on X is given by:

$$\tilde{f}(\lambda, b) = \int_X f(x) e^{(-i\lambda + \rho)A(x,b)} dx, \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*, b \in K/M$$

where $\mathfrak{a}_{\mathbb{C}}^*$ is the complexification of the vector dual of \mathfrak{a} , ρ is the half-sum of positive restricted roots counted with their multiplicities and $A(x, b) \in \mathfrak{a}$ (see Section 2 for more detail). On the other hand, we define the following resolution of unity on \mathfrak{a}^* . Consider the set $\{\varphi_{0,N}, \varphi_t^N\}_{0 < t \leq 1, N \in \mathbb{N}^*}$ of compactly supported smooth K -invariant functions on X satisfying:

$$\mathcal{F}\varphi_{0,N}(\lambda) + \int_0^1 (\mathcal{F}\varphi_t^N)^2(\lambda) \frac{dt}{t} = 1, \quad \forall \lambda \in \mathfrak{a}_{\mathbb{C}}^*$$

and

$$f(x) = (f \times \varphi_{0,N})(x) + \int_0^1 (f \times \varphi_t^N \times \varphi_t^N)(x) \frac{dt}{t}, \quad \forall f \in L^2(X)$$

where \times denote the convolution product on X and \mathcal{F} is the usual *spherical transform* for K -invariant functions on X . Then the Besov space $B_p^{s,q}(X)$ on X is the space of tempered distributions on X satisfying:

$$\|f\|_{B_p^{s,q}(X)} = \|f \times \varphi_{0,N}\|_{L^p(X)} + \left(\int_0^1 t^{-sq} \|f \times \varphi_t^N\|_{L^p(X)}^q \frac{dt}{t} \right)^{1/q} < \infty$$

for $2N > |s|$. It should be noted that our definition differs slightly from that in [19].

We can now state our first result.

Theorem 1 : *Let N be a positive integer such that $n = \dim(X) \geq 4N$. Let $\alpha = \dim(\mathfrak{a}) \geq 1$ and $I = [0, +\infty[$. Let u be a solution of the wave equation (1.1) on X and, let p and r be two real numbers such that $2/r = \alpha(1 - \frac{2}{p})$ and $2 < p < \min(\frac{2n}{n-4N}, \frac{2\alpha}{\alpha-1})$. Then there exists a positive number c_r , depending on r , such that if u_0 and u_1 belong to $L^2(X)$, we have:*

$$\|u\|_{L^r(I, B_p^{s,2}(X))} \leq c_r \left(\|u_0\|_{L^2(X)} + \frac{1}{\|\rho\|} \|u_1\|_{L^2(X)} \right)$$

where $s = \frac{n}{2}(\frac{2}{p} - 1)$.

The proof rests on an argument of duality ([7][24]) and on a Hardy-Littlewood-Sobolev inequality ([13]). We find it useful to formulate solutions of (1.1) using the theory of propagators.

Our second result deals with dispersive properties. More precisely, we prove the following theorem:

Theorem 2 : *Let u be a solution of the wave equation (1.1) on X and write $\alpha = \dim(\mathfrak{a})$.*

1. *We assume that both u_0 and $(-\Delta)^{-1/2}u_1$ belong to $B_1^{n,1}(X)$. Then there exist a constant $c > 0$ and a sufficiently large number T such that for all $t \geq T$, we have:*

$$\|u(t, \cdot)\|_{L^\infty(X)} \leq ct^{-\alpha} (\|u_0\|_{B_1^{n,1}(X)} + \|(-\Delta)^{-1/2}u_1\|_{B_1^{n,1}(X)})$$

With $n = \dim X < 2N$, for some positive integer N .

2. *Let p and p' be two integers such that $1/p + 1/p' = 1$ and $2 \leq p < \frac{2n}{n-2N}$. We suppose that u_0 and $(-\Delta)^{-1/2}u_1$ are in $B_{p'}^{s,1}(X)$ with $s = n(1 - \frac{2}{p})$. Then there exist a constant $c > 0$ and a sufficiently large number T such that for all $t \geq T$, we have:*

$$\|u(t, \cdot)\|_{L^p(X)} \leq ct^{\alpha\gamma} (\|u_0\|_{B_{p'}^{s,1}(X)} + \|(-\Delta)^{-1/2}u_1\|_{B_{p'}^{s,1}(X)})$$

With $n = \dim X > 2N$ and $\gamma = 2/p - 1$.

The proof combines inverse Helgason-Fourier transform and interpolation theorems for L^p -spaces on X .

Our paper is organized as follows: in Section 2, we recall some definitions and basic results of harmonic analysis. Section 3 is devoted to the definition of Besov spaces on symmetric spaces and propagators. Section 4 contains the proof of our first theorem is contained in Section 4, while the proof of the second theorem is contained in Section 5.

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2 Preliminaries

2.1 Roots, decompositions and norms.

Let G be a noncompact connected semisimple Lie group with finite center and K a maximal compact subgroup of G . Let \mathfrak{g} (resp. \mathfrak{k}) be the Lie algebra of G (resp. K) and consider the Cartan decomposition of \mathfrak{g} :

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

where \mathfrak{p} is the (K -invariant) orthogonal complement of \mathfrak{k} in \mathfrak{g} with respect to the Killing form of \mathfrak{g} :

$$\mathcal{K} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}, (X, Y) \mapsto \text{Tr}(\text{ad}(X) \circ \text{ad}(Y)).$$

Here ad denotes the differential of the adjoint action Ad_G of G . Moreover, if θ denotes the corresponding Cartan involution, the Killing form defines the following G -invariant inner product on \mathfrak{g} :

$$\langle X, Y \rangle = -\mathcal{K}(X, \theta(Y))$$

which is positive definite on \mathfrak{p} and negative definite on \mathfrak{k} . This in turn induces a Riemannian structure on the homogeneous (symmetric) manifold:

$$X = G/K$$

whose tangent space at the origin eK is identified with \mathfrak{p} . On the other hand, since G is semisimple, the Killing form enables us to identify \mathfrak{g} with its vector dual \mathfrak{g}^* , as well as subspaces of \mathfrak{g} with subspaces in \mathfrak{g}^* . Let \mathfrak{a} be a maximal abelian subspace of \mathfrak{p} . The real dimension of \mathfrak{a} is known as the real rank $\text{rank}_{\mathbb{R}}(G)$ of G . Let Σ be the set of restricted roots with respect to \mathfrak{a} :

$$\Sigma = \{\lambda \in \mathfrak{a}^* \mid \lambda \neq 0 \text{ and } \mathfrak{g}_{\lambda} \neq \{0\}\}$$

where

$$\mathfrak{g}_{\lambda} = \{X \in \mathfrak{g} \mid [H, X] = \lambda(H)X \quad \forall H \in \mathfrak{a}\}.$$

Fix a positive Weyl chamber \mathfrak{a}_+ in \mathfrak{a}^* , and write Σ_+ for the corresponding set of positive restricted roots. Define the nilpotent subalgebra \mathfrak{n} of \mathfrak{g} :

$$\mathfrak{n} = \bigoplus_{\lambda \in \Sigma_+} \mathfrak{g}_{\lambda}$$

so that \mathfrak{g} decomposes as

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

known as the Iwasawa decomposition. The corresponding decomposition at the group level is:

$$G = KAN$$

where A and N are the analytic subgroups of G with Lie algebras \mathfrak{a} and \mathfrak{n} respectively. Finally, there is another decomposition of G that will be of interest to us. The Cartan decomposition of G is given by:

$$G = K\bar{A}_+K$$

where \overline{A}_+ is the closure of $A_+ = \exp(\mathfrak{a}_+)$. Any element g of G will be written as

$$g = k_1 A(g) k_2$$

in the Cartan decomposition, and as

$$g = k(g) \exp(H(g)) n(g)$$

in the Iwasawa decomposition. It should be noted that $k(g)$, $n(g)$, $H(g)$ and $A(g)$ are uniquely determined. Now, the Killing form induces norms on \mathfrak{g} and \mathfrak{g}^* , which we shall denote by the same symbol $\| \cdot \|$, as well as a norm on G defined by:

$$\| g \| = \| A(g) \| .$$

In particular one has:

$$\| g^{-1} \| = \| g \| \text{ and } \| k g k' \| = \| g \| \text{ for all } g \in G \text{ and } k, k' \in K.$$

Next, let M (resp. M') denote the centralizer (resp. the normalizer) of A in K . The quotient group $W = M'/M$ is the so-called Weyl group associated with Σ . It is a finite group that acts on \mathfrak{a} as a group of linear transformations by the operators $\text{Ad}_G(k)$, $k \in M'$. The group W acts also freely and transitively on the set of Weyl chambers:

$$s \cdot \lambda(H) = \lambda(s^{-1}H), \quad \forall s \in W, H \in \mathfrak{a}.$$

The following compact homogeneous manifold, known as the boundary of X ,

$$B = K/M = G/MAN$$

plays a crucial role in the harmonic analysis on X .

Finally, the Killing form of G induces euclidean measures on A , \mathfrak{a} and \mathfrak{a}^* . These measures remain invariant when multiplied by $(2\pi)^{-\text{rank}_{\mathbb{R}}(G)}$. The Haar measures dm on M and dk on K are normalized such that the total mass is 1. As is customary, Haar measures on G and N will be normalized so that:

$$\int_G f(g) dg = \int_{K \times A \times N} f(kan) e^{2\rho(\log a)} dk dadn$$

and

$$\int_G f(g) dg = \int_{G/K} \int_K f(gk) dk d(gK)$$

where

$$\rho = \frac{1}{2} \sum_{\lambda \in \Sigma_+} m_\lambda \lambda,$$

is the half-sum of positive restricted roots counted with their multiplicities m_λ and \log is the inverse of $\exp : \mathfrak{a} \mapsto A$. Moreover the invariant measure $db = d(kM)$ on $B = K/M$ is normalized by [11]:

$$\int_B db = \int_{K/M} d(kM) = 1.$$

Extend the Killing form of G linearly to the complexification $\mathfrak{a}_\mathbb{C}$ and keep the same symbol to denote this extension. For $\lambda \in \mathfrak{a}_\mathbb{C}^*$, let H_λ be the unique element in $\mathfrak{a}_\mathbb{C}$ defined by:

$$\lambda(H) = \langle H_\lambda, H \rangle, \quad \forall H \in \mathfrak{a}_\mathbb{C}.$$

Put

$$\langle \lambda, \mu \rangle = \langle H_\lambda, H_\mu \rangle, \quad \forall \lambda, \mu \in \mathfrak{a}_\mathbb{C}^*.$$

For $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and $i = \sqrt{-1}$, we write

$$\lambda = \mathcal{R}e\lambda + i\mathcal{I}m\lambda,$$

where $\mathcal{R}e\lambda$ and $\mathcal{I}m\lambda \in \mathfrak{a}^*$ are respectively the real part and the imaginary part of λ , with

$$\|\lambda\|^2 = \|\mathcal{R}e\lambda\|^2 + \|\mathcal{I}m\lambda\|^2.$$

2.2 Helgason-Fourier transform, inversion formula, Plancherel formula

For this section, we refer to [11][12]. We view functions on X as functions on G which are K -invariant on the right. Then the Helgason-Fourier transform of a function f on X is the map \tilde{f} on $\mathfrak{a}_\mathbb{C}^* \times B$ given by:

$$\tilde{f}(\lambda, b) = \int_X f(x) e^{(-i\lambda + \rho)A(x,b)} dx$$

where

$$A(x, b) = A(gK, kM) = -H(g^{-1}k) \in \mathfrak{a}$$

for $x = gK \in X$ and $b = kM \in B$. Moreover, if f belongs to the space $\mathcal{D}(X) = C_0^\infty(X)$ of compactly supported smooth functions on X , then its inverse Helgason-Fourier transform is:

$$f(x) = \frac{1}{|W|} \int_{\mathfrak{a}^* \times B} e^{(i\lambda + \rho)A(x,b)} \tilde{f}(\lambda, b) \frac{d\lambda db}{|c(\lambda)|^2} \quad (2.4)$$

where $|W|$ denotes the order of the Weyl group. The function c is called the *Harish-Chandra function* and satisfies the following inequality:

$$|c(\lambda)|^{-2} \leq c'(1 + \|\lambda\|^2)^d \quad (2.5)$$

with $d = \dim(X) - \text{rank}_{\mathbb{R}}(G)$ and c' is a positive constant. The Helgason-Fourier transform

$$L^2(X, dx) \rightarrow L^2(\mathfrak{a}_+^* \times B, \frac{d\lambda db}{|c(\lambda)|^2}), f \mapsto \tilde{f}$$

is an isometry and we have the Plancherel formula:

$$\int_X |f(x)|^2 dx = \int_{\mathfrak{a}_+^* \times B} |\tilde{f}(\lambda, b)|^2 \frac{d\lambda db}{|c(\lambda)|^2} \quad (2.6)$$

where $\mathfrak{a}_+^* = \{\lambda \in \mathfrak{a}^* \mid H_\lambda \in \mathfrak{a}_+\}$.

2.3 Spherical transform, euclidean Fourier transform, Abel transform

Let f be a K -invariant function on X , i.e. left and right K -invariant on G . The Helgason-Fourier transform of f is constant in the B -variable, and is W -invariant on \mathfrak{a}^* . The spherical transform $\mathcal{F}f$ of f is given by:

$$\mathcal{F}f(\lambda) = \int_X f(x) \varphi_{-\lambda}(x) dx$$

where φ_λ is the *spherical function* defined by:

$$\varphi_\lambda(x) = \varphi_\lambda(gK) = \int_K e^{(i\lambda + \rho)A(kg)} dk.$$

It will be useful to write the spherical transform as follows:

$$\mathcal{F}f = \widehat{\mathcal{A}f},$$

where $\widehat{}$ is the *euclidean Fourier transform*:

$$\widehat{f}(\lambda) = \int_{\mathfrak{a}} f(H) e^{-i\lambda(H)} dH, \quad \lambda \in \mathfrak{a}^*$$

and \mathcal{A} is the *Abel transform*:

$$\mathcal{A}f(H) = e^{\rho(H)} \int_N f((\exp H)n) dn, \quad H \in \mathfrak{a}.$$

2.4 Schwartz spaces

The Schwartz space on X is the space $\mathcal{S}(X)$ of smooth functions on X such that for any $m \in \mathbb{N}$ and any (left or right) invariant differential operator D on G , we have:

$$\sup | (Df)(g) | (1 + \sigma(g))^m \Xi(g)^{-1} < \infty$$

where

$$\sigma(g) = \| Y \|, \quad g = k \exp(Y) \in G, \quad Y \in \mathfrak{p}$$

and

$$\Xi(g) = \int_K e^{-\rho H(gk)} dk.$$

The following inclusions are standard:

$$\mathcal{D}(X) \subset \mathcal{S}(X) \subset L^p(X), \quad p \geq 2.$$

Moreover $\mathcal{D}(X)$ is dense in $\mathcal{S}(X)$. Similarly $\mathcal{S}(K \backslash X)$ is the Schwartz space on $K \backslash X$. Write $\mathcal{S}(\mathfrak{a})$ (resp. $\mathcal{S}(\mathfrak{a}^*)$) for the (classical) Schwartz space on \mathfrak{a} (resp. \mathfrak{a}^*). Recall the Weyl group action on \mathfrak{a} and \mathfrak{a}^* , and let $\mathcal{S}_W(\mathfrak{a})$ (resp. $\mathcal{S}_W(\mathfrak{a}^*)$) be the subspace of W -invariant elements of $\mathcal{S}(\mathfrak{a})$ (resp. $\mathcal{S}(\mathfrak{a}^*)$). In particular, $\mathcal{S}_W(\mathfrak{a}^*)$ is the space of W -invariant complex functions h such that h and all its derivatives extend continuously to \mathfrak{a}^* and, for any polynomial function P on \mathfrak{a} and any integer $m \geq 0$, we have:

$$\sup_{\lambda \in \mathfrak{a}^*} (\|\lambda\| + 1)^m | P(\frac{\partial}{\partial \lambda}) h(\lambda) | < +\infty.$$

The following diagram is commutative (up to a normalizing constant) and each arrow is an isomorphism:

$$\begin{array}{ccc} \mathcal{S}(K \backslash X) & \xrightarrow{\mathcal{A}} & \mathcal{S}_W(\mathfrak{a}) \\ & \searrow \mathcal{F} & \swarrow \hat{\phantom{\mathcal{A}}} \\ & \mathcal{S}_W(\mathfrak{a}^*) & \end{array}$$

The fact that \mathcal{F} is an isomorphism is a fundamental result of Harish-Chandra [8, 9, 10]. A simpler proof was obtained by Anker [2]. For the isomorphisms \mathcal{A} and $\hat{\phantom{\mathcal{A}}}$, we refer to [1]. Actually, a Paley-Wiener theorem of Helgason implies that \mathcal{A} is an isomorphism between the space $\mathcal{D}(K \backslash X)$ of compactly supported smooth functions on $K \backslash X$ and the space $\mathcal{D}_W(\mathfrak{a}) = C_0^\infty(\mathfrak{a})_W$ of W -invariant compactly supported smooth functions on \mathfrak{a} [12].

2.5 Distributions on symmetric spaces

For $\psi \in \mathcal{D}(X)$ and $k \in K$, define ψ_k to be the left translation of ψ :

$$\psi_k(x) = \psi(k^{-1} \cdot x).$$

A distribution f on X is said to be K -invariant if

$$f(\psi_k) = f(\psi), \quad \forall \psi \in \mathcal{D}(X)$$

A distribution on X is said to be tempered if it can be extended to a continuous functional on $\mathcal{S}(X)$. Since $\mathcal{D}(X)$ is continuously embedded and dense in $\mathcal{S}(X)$, the space $\mathcal{S}'(X)$ of tempered distribution can be regarded as the dual space to $\mathcal{S}(X)$. It is clear that every compactly supported distribution on X is tempered:

$$\mathcal{E}'(X) \subset \mathcal{S}'(X)$$

and that $L^2(X)$ is continuously embedded in $\mathcal{S}'(X)$. The Helgason-Fourier transform is a topological isomorphism of $\mathcal{S}'(X)$ onto the space $\mathcal{Z}'(\mathfrak{a}^* \times B)$ dual to $\mathcal{Z}(\mathfrak{a}^* \times B)$ (both equipped with their weak topologies, see [6]).

2.6 Convolution product on symmetric spaces

The convolution product on X is denoted \times and is defined by:

$$(f_1 \times f_2) \circ \pi = (f_1 \circ \pi) \star (f_2 \circ \pi)$$

where

$$\pi : G \mapsto X = G/K$$

is the natural projection and \star is the convolution product on G . It should be noted that the product of convolution on X is not commutative, and therefore the Helgason-Fourier transform does not turn it into a multiplication. However this will be the case whenever the second factor is K -invariant (see [12]).

3 Besov spaces. Propagators

In this section we define Besov spaces and we state some results we will need later. First, we define a resolution of unity on \mathfrak{a}^* slightly different from the one given in [19].

3.1 Resolution of unity

Let $\varphi \in \mathcal{D}(K \setminus X)$ a real function such that $\text{supp} \varphi \subset \overline{B(eK, 1)}$ and $\mathcal{F}\varphi(0) \neq 0$, where $B(eK, 1)$ is the unit geodesic ball.

We define a function ψ on X such that:

$$\psi = \mathcal{A}\varphi \text{ and } \psi^N = \mathcal{A}\varphi^N$$

where $\varphi^N = \Gamma^N \varphi$ for $N \in \mathbb{N}$ and $\Gamma = -\Delta - \|\rho\|^2$.

It is easy to see that $\mathcal{F}\varphi^N = \widehat{\mathcal{A}\varphi^N} = \widehat{\psi^N} \in \mathcal{S}_W(\mathfrak{a}^*)$.

We suppose that for all $\lambda \in \mathfrak{a}^* (\lambda \neq 0)$:

$$\int_0^\infty (\widehat{\psi^N})^2(t\lambda) \frac{dt}{t} = 1.$$

We define a smooth function $\psi_{0,N}$ on \mathfrak{a} by its Euclidean Fourier transform:

$$\widehat{\psi_{0,N}}(\lambda) = 1 - \int_0^1 (\widehat{\psi^N})^2(t\lambda) \frac{dt}{t}.$$

Then, since $\widehat{\psi^N} \in \mathcal{S}_W(\mathfrak{a}^*)$, we can easily deduce that $\widehat{\psi_{0,N}} \in \mathcal{S}_W(\mathfrak{a}^*)$ and $\psi_{0,N} \in \mathcal{D}_W(\mathfrak{a})$.

We consider the function:

$$\varphi_{0,N} = \mathcal{A}^{-1}\psi_{0,N}.$$

Then by using the fact that \mathcal{A} is an isomorphism between $\mathcal{D}(K \setminus X)$ and $\mathcal{D}_W(\mathfrak{a})$, we remark that $\varphi_{0,N}$ belongs to $\mathcal{D}(K \setminus X)$. A simple computation shows that:

$$\mathcal{F}\varphi_{0,N}(\lambda) + \int_0^1 (\mathcal{F}\varphi^N)^2(t\lambda) \frac{dt}{t} = 1.$$

Finally, let φ_t^N a function in $\mathcal{D}(K \setminus X)$ given by $\varphi_t^N = \mathcal{F}^{-1}(\mathcal{F}\varphi^N(t.))$.

Then we have the following formula of Calderon type [19]:

$$f(x) = (f \times \varphi_{0,N})(x) + \int_0^1 (f \times \varphi_t^N \times \varphi_t^N)(x) \frac{dt}{t}, \quad \forall f \in L^2(X). \quad (3.7)$$

We conclude that the system $\{\varphi_{0,N}, \varphi_t^N\}_{0 < t \leq 1}$ is a continuous resolution of unity on \mathfrak{a}^* .

Now, we are in the position to give a definition of inhomogeneous Besov spaces on X .

3.2 Besov spaces

Definition 3.1. Let $1 \leq p \leq \infty$, $1 \leq q < \infty$, and $s \in \mathbb{R}$. Let N be a positive integer such that $2N > |s|$. Let $\{\varphi_{0,N}, \varphi_t^N\}_{0 < t \leq 1}$ be the system of functions defined above. Then the Besov space is the set denoted by $B_p^{s,q}(X)$ and given by:

$$B_p^{s,q}(X) = \{f \in \mathcal{S}'(X); \|f\|_{B_p^{s,q}(X)} < \infty\},$$

where:

$$\|f\|_{B_p^{s,q}(X)} = \|f \times \varphi_{0,N}\|_{L^p(X)} + \left(\int_0^1 t^{-sq} \|f \times \varphi_t^N\|_{L^p(X)}^q \frac{dt}{t} \right)^{1/q}.$$

Remark 1.

1. We supposed that $\varphi \in \mathcal{D}(K \setminus X)$, while Skrzypczak have considered in [19] φ not to be necessarily a smooth function with compact support.
2. Observe that the spaces $B_2^{0,2}(X)$ and $L^2(X)$ are equivalent.
3. In [17] Skrzypczak proved that for $1 < p < \infty$:

$$B_p^{s,q}(X) = (H_p^{s_0}(X), H_p^{s_1}(X))_{\theta,q}$$

with $0 < \theta < 1$, $s = \theta.s_0 + (1 - \theta).s_1$, $H_p^s(X)$ is the Bessel-potential space and $(\cdot, \cdot)_{\theta,q}$ denotes the real interpolation method.

Then, we can deduce that the above defined Besov spaces coincide with the Besov spaces defined on X by uniform localization (see [23]) and $B_p^{s,q}(X)$ is a Banach space for $1 < p < \infty$.

4. It is also known that the dual $(H_p^s(X))'$ of $H_p^s(X)$ is equal to $H_{p'}^{-s}(X)$. Then by the duality theorem from real interpolation, we deduce that:

$$(B_p^{s,q}(X))' = B_{p'}^{-s,q'}(X)$$

with $1/p + 1/p' = 1/q + 1/q' = 1$.

3.3 Applications

As an application of (3.7) and the definition of Besov spaces, we prove the following result.

Proposition 3.2. *For all $p \geq 2$, we have $B_p^{0,1}(X) \subset L^p(X)$.*

Proof. Let u in $B_p^{0,1}(X)$ then, by formula (3.7), we have:

$$\begin{aligned} \|u\|_{L^p(X)} &\leq \|u \times \varphi_{0,N}\|_{L^p(X)} + \int_0^1 \|u \times \varphi_t^N \times \varphi_t^N\|_{L^p(X)} \frac{dt}{t} \\ &\leq \|u \times \varphi_{0,N}\|_{L^p(X)} + \int_0^1 \|u \times \varphi_t^N\|_{L^p(X)} \|\varphi_t^N\|_{L^1(X)} \frac{dt}{t} \\ &\leq \|u \times \varphi_{0,N}\|_{L^p(X)} + c \int_0^1 \|u \times \varphi_t^N\|_{L^p(X)} \frac{dt}{t} \\ &\leq c \|u\|_{B_p^{0,1}(X)} \end{aligned}$$

where we used the Young inequality and the fact $\|\varphi_t^N\|_{L^1(X)} \leq c$ for all $0 < t \leq 1$ (see [19]).

As an application of *the atomic decomposition* of Besov spaces, Skrzypczak in [19] showed the following result.

Proposition 3.3. *Let $1 \leq p, q, q_0, q_1 \leq \infty$ and $s, s_0, s_1 \in \mathbb{R}$.*

(i) *(Elementary embedding)*

$$B_p^{s,q_0}(X) \subset B_p^{s,q_1}(X) \quad q_0 \leq q_1 \quad (3.8)$$

$$B_p^{s_0,q}(X) \subset B_p^{s_1,q}(X) \quad s_1 \leq s_0. \quad (3.9)$$

(ii) *(Embedding with different metrics)*

$$B_p^{s_0,q}(X) \subset B_\infty^{s_1,q}(X) \quad s_1 = s_0 - n/p \quad (3.10)$$

$$B_1^{s_0,q}(X) \subset B_p^{s_1,q}(X) \quad s_0 - n = s_1 - n/p \quad (3.11)$$

$$B_1^{0,1}(X) \subset L^1(X) \subset B_1^{0,\infty}(X) \quad (3.12)$$

$$B_\infty^{0,1}(X) \subset C(X) \subset B_\infty^{0,\infty}(X) \quad (3.13)$$

where $C(X)$ denotes the space of bounded continuous functions on X .

3.4 Propagators on X

We start this paragraph by giving a definition concerning the powers of the Laplace-Beltrami operator on X .

Definition 3.4. For all integer μ , we define the operator $\Delta_0^{\mu/2}$ by

$$\widetilde{\Delta_0^{\mu/2} f}(\lambda, b) = Z_\rho(\lambda)^\mu \widetilde{f}(\lambda, b).$$

where $\Delta_0 = -\Delta$ and $Z_\rho(\lambda) = (\|\lambda\|^2 + \|\rho\|^2)^{1/2}$, for a smooth function f or $f \in \mathcal{S}'(X)$.

Lemma 3.5. (i) Let $\mathcal{K}_W(\mathfrak{a}_\mathbb{C}^*)$ be the space of functions ψ which can be extended to a W -invariant entire holomorphic functions on $\mathfrak{a}_\mathbb{C}^*$ such that: there exist a constant $R \geq 0$ and the integer $m \geq 0$ satisfying the following condition:

$$\sup_{\lambda \in \mathfrak{a}_\mathbb{C}^*} (1 + \|\lambda\|)^{-m} e^{-R\|\mathcal{I}m\lambda\|} |\psi(\lambda)| < +\infty. \quad (3.14)$$

Then the spherical transform \mathcal{F} is an isomorphism of $\mathcal{E}'(K \backslash X)$ to $\mathcal{K}_W(\mathfrak{a}_\mathbb{C}^*)$.

(ii) If $f \in \mathcal{S}'(X)$ and $h \in \mathcal{E}'(K \backslash X)$ then $f \times h \in \mathcal{S}'(X)$ and we have:

$$\widetilde{(f \times h)}(\lambda, b) = \widetilde{f}(\lambda, b) \cdot \mathcal{F}h(\lambda) \quad \forall (\lambda, b) \in \mathfrak{a}_\mathbb{C}^* \times B.$$

The first point is a Paley-Wiener theorem due to Helgason [12] concerning the compactly supported K -invariant distributions on X . Also we can find the proof of the second point in [18].

If we use Definition 3.4 and the above lemma one can find one solution of our Cauchy problem (1.1). We suppose that $u_0, u_1 \in \mathcal{S}'(X)$. Then reformulating (1.1) in Fourier variables, the solution of (1.1) may be written as:

$$\widetilde{u}(t, \lambda, b) = \widetilde{u}_0(\lambda, b) \cos Z_\rho(\lambda)t + \widetilde{u}_1(\lambda, b) \frac{\sin Z_\rho(\lambda)t}{Z_\rho(\lambda)}.$$

One can check that the function $\lambda \mapsto \psi_{\rho,t}(\lambda) = \frac{\sin Z_\rho(\lambda)t}{Z_\rho(\lambda)}$ is W -invariant, holomorphic on $\mathfrak{a}_\mathbb{C}^*$ and satisfies (3.14). In particular these function belongs to the space $\mathcal{K}_W(\mathfrak{a}_\mathbb{C}^*)$.

According to Lemma 3.5, there is one and only one $\mathcal{E}_t \in \mathcal{E}'(K \backslash X)$ such that:

$$\mathcal{F}\mathcal{E}_t(\lambda) = \psi_{\rho,t}(\lambda) \quad \forall \lambda \in \mathfrak{a}_\mathbb{C}^*.$$

Also, by the same lemma, we have $u_0 \times \mathcal{E}_t \in \mathcal{S}'(X)$ and that:

$$(\widetilde{u_0 \times \mathcal{E}_t})(\lambda, b) = \widetilde{u_0}(\lambda, b) \mathcal{F} \mathcal{E}_t(\lambda) \quad \forall (\lambda, b) \in \mathfrak{a}_{\mathbb{C}}^* \times B.$$

Then we deduce that:

$$\partial_t(\widetilde{u_0 \times \mathcal{E}_t})(\lambda, b) = [\partial_t \widetilde{u_0}(\lambda, b)] \mathcal{F} \mathcal{E}_t(\lambda) = \widetilde{u_0}(\lambda, b) [\partial_t \mathcal{F} \mathcal{E}_t(\lambda)].$$

Since we have:

$$\partial_t(\widetilde{u_0 \times \mathcal{E}_t})(\lambda, b) = (\widetilde{u_0 \times \partial_t \mathcal{E}_t})(\lambda, b) = \widetilde{u_0}(\lambda, b) [\mathcal{F} \partial_t \mathcal{E}_t(\lambda)],$$

we deduce that:

$$\partial_t \mathcal{F} \mathcal{E}_t(\lambda) = \mathcal{F} \partial_t \mathcal{E}_t(\lambda).$$

Then, if we denote

$$\partial_t \mathcal{E}_t = \mathcal{E}'_t,$$

the solution u is written:

$$u(t, x) = (u_0 \times \mathcal{E}'_t)(x) + (u_1 \times \mathcal{E}_t)(x). \quad (3.15)$$

Therefore we have the following result.

Proposition 3.6. *If $u_0, u_1 \in \mathcal{S}'(X)$, the solution of (1.1) is given by (3.15), where $\mathcal{E}_t \in \mathcal{E}'(K \setminus X)$ is a solution of the partial differential equation:*

$$\partial_t^2 \mathcal{E}_t - \Delta \mathcal{E}_t = 0, \quad \mathcal{E}_0 = 0, \quad \mathcal{E}'_0 = \delta_0.$$

Proof. The fact that $\partial_t^2 \mathcal{E}_t - \Delta \mathcal{E}_t = 0$, is a direct consequence of Definition 3.4 and the expression of $\mathcal{F} \mathcal{E}_t$.

Definition 3.7. For all $t \in \mathbb{R}$, we define the operator U_t by:

$$(\widetilde{U_t f})(\lambda, b) = e^{iZ_\rho(\lambda)t} \widetilde{f}(\lambda, b) \quad (\lambda, b) \in \mathfrak{a}_{\mathbb{C}}^* \times B,$$

for a smooth function f on X or $f \in \mathcal{S}'(X)$. In this case, we may view \mathcal{E}_t and \mathcal{E}'_t as the operators defined by:

$$\mathcal{E}_t = \frac{1}{2i}(-\Delta)^{-1/2}(U_t - U_{-t}) \text{ and } \mathcal{E}'_t = \frac{1}{2}(U_t + U_{-t})$$

The operator \mathcal{E}_t is called *the propagator* of the solution u given by (3.15).

4 Proof of Theorem 1

The proof of our first result (Theorem 1 mentioned in the introduction) will follow from the following proposition.

Proposition 4.1. *Let N be a positive integer such that $n = \dim(X) \geq 4N$. Let $\alpha = \dim(\mathfrak{a}) \geq 1$ and $I = [0, +\infty[$.*

1. *Let u be a solution of the wave equation (1.1) on X and, let p and r two real numbers such that $\frac{2}{r} = \alpha(1 - \frac{2}{p})$ and $2 < p < \frac{2n}{n-4N+2}$. Then there exists a positive number $c_r > 0$ depending of r such that if $u_0, (-\Delta)^{1/2}u_0$ and u_1 belong to $L^2(X)$, we have:*

$$\|u\|_{L^r(I, B_p^{s,2}(X))} + \|\partial_t u\|_{L^r(I, B_p^{s-1,2}(X))} \leq c_r \cdot [\|u_0\|_{L^2(X)} + \|(-\Delta)^{1/2}u_0\|_{L^2(X)} + (1 + \frac{1}{\|\rho\|})\|u_1\|_{L^2(X)}]$$

where $s = \frac{n}{2}(\frac{2}{p} - 1)$.

2. *Let f a distribution on X which also depend on time. Let p_i, p'_i, r_i and r'_i eight real numbers such that $\frac{1}{p_i} + \frac{1}{p'_i} = \frac{1}{r_i} + \frac{1}{r'_i} = 1$, $\frac{2}{r_i} = \alpha(1 - \frac{2}{p_i})$ and $2 < p_i < \frac{2n}{n-4N}$ for $i = 1, 2$. Then there exists a positive number $c > 0$ such that if f belongs to $L^{r'_2}(I, B_{p'_2}^{-s_2,2}(X))$, we have:*

$$\|\mathcal{E}'_{(\cdot)} *_{t} f\|_{L^{r_1}(I, B_{p_1}^{s_1,2}(X))} \leq c \cdot \|f\|_{L^{r'_2}(I, B_{p'_2}^{-s_2,2}(X))}$$

where $s_i = \frac{n}{2}(\frac{2}{p_i} - 1)$ for $i = 1, 2$.

4.1 Proof of proposition 4.1

We proceed in several steps.

Step 1. We recall a Paley-Wiener due to Helgason for compactly supported smooth functions on $K \backslash X$.

Lemma 4.2. *Let $\mathcal{H}_W(\mathfrak{a}_{\mathbb{C}}^{\star})$ the space of functions h which can be extended to a W -invariant entire holomorphic functions on $\mathfrak{a}_{\mathbb{C}}^{\star}$ such that: there exists a constant $R \geq 0$ such that for each integer $m \geq 0$:*

$$\sup_{\lambda \in \mathfrak{a}_{\mathbb{C}}^{\star}} (1 + \|\lambda\|)^m e^{-R\|\mathcal{I}m\lambda\|} |h(\lambda)| < +\infty.$$

Then the spherical transform \mathcal{F} is an isomorphism from $\mathcal{D}(K \setminus X)$ onto $\mathcal{H}_W(\mathfrak{a}_{\mathbb{C}}^*)$.

A proof of this lemma can be found in [11, page 450].

Now, let $(\varphi_\varepsilon)_{0 < \varepsilon \leq 1}$ be a sequence of smooth functions on X with spherical transform given by:

$$\mathcal{F}\varphi_\varepsilon(\lambda) = \frac{\|\lambda\|^2}{\varepsilon + \|\lambda\|^2} e^{-\varepsilon(1 + \|\lambda\|^2)} \quad \text{for } \lambda \in \mathfrak{a}_{\mathbb{C}}^*.$$

Then by the previous lemma, one can show that $\mathcal{F}\varphi_\varepsilon$ and φ_ε belongs respectively to $\mathcal{H}_W(\mathfrak{a}_{\mathbb{C}}^*)$ and $\mathcal{D}(K \setminus X)$.

Step 2. We prove two properties for the sequence $(\varphi_\varepsilon)_{0 < \varepsilon \leq 1}$ defined above.

Lemma 4.3. *For the system of smooth functions $\{\varphi_{0,N}, \varphi_{t'}^N\}_{0 < t' \leq 1, N \in \mathbb{N}^*}$, defined in Section 3, we have:*

$$f \times \varphi_{0,N} = f \times \varphi_{0,N} \times \varphi_\varepsilon \quad \forall f \in \mathcal{S}'(X). \quad (4.16)$$

$$f \times \varphi_{t'}^N = f \times \varphi_{t'}^N \times \varphi_\varepsilon \quad \forall f \in \mathcal{S}'(X). \quad (4.17)$$

Proof. We follow the same idea used to show inequality (9) in [19].

Let $f \in C^\infty(X) \cap \mathcal{S}'(X)$. Then using the dominated convergence theorem, we can find a sequence $(f_n) \subset \mathcal{D}(X)$ such that $f_n \rightarrow f$ in $\mathcal{S}'(X)$.

Thus it is sufficient to prove (4.16) for $f \in \mathcal{D}(X)$ with the L^2 -convergence.

By Plancherel formula (2.6), we have:

$$\|f \times \varphi_{0,N} - f \times \varphi_{0,N} \times \varphi_\varepsilon\|_{L^2(X)}^2 = \int_{\mathfrak{a}_{\mathbb{C}}^* \times B} h_\varepsilon(\lambda, b) \frac{d\lambda db}{|c(\lambda)|^2}$$

where

$$h_\varepsilon(\lambda, b) = |\tilde{f}(\lambda, b) \mathcal{F}\varphi_{0,N}(\lambda)|^2 |1 - \mathcal{F}\varphi_\varepsilon(\lambda)|^2.$$

But, observe that:

$$0 \leq \mathcal{F}\varphi_\varepsilon(\lambda) \leq 1, \quad \forall \lambda \in \mathfrak{a}_{\mathbb{C}}^*,$$

and that

$$\mathcal{F}\varphi_\varepsilon(\lambda) \rightarrow 1 \text{ if } \varepsilon \rightarrow 0, \quad \forall \lambda \in \mathfrak{a}_{\mathbb{C}}^*.$$

We deduce that:

$$h_\varepsilon(\lambda, b) \leq 4|\tilde{f}(\lambda, b)\mathcal{F}\varphi_{0,N}(\lambda)|^2 \quad \forall \lambda \in \mathfrak{a}_\mathbb{C}^*.$$

$$h_\varepsilon(\lambda, b) \rightarrow 0 \text{ if } \varepsilon \rightarrow 0 \quad \forall \lambda \in \mathfrak{a}_\mathbb{C}^*.$$

Thus by the dominated convergence theorem, we conclude that:

$$\int_{\mathfrak{a}_+^* \times B} h_\varepsilon(\lambda, b) \frac{d\lambda db}{|c(\lambda)|^2} \rightarrow 0 \text{ if } \varepsilon \rightarrow 0.$$

Similarly, we prove equality (4.17).

Lemma 4.4. *There exists a positive number $c > 0$ such that for the operator U_t given in Definition 3.3, we have:*

$$\|U_t\varphi_\varepsilon\|_{L^\infty(X)} \leq ct^{-\alpha}I(t) \quad \forall t \geq 1. \quad (4.18)$$

Consequently we have:

$$\|U_t\varphi_\varepsilon\|_{L^\infty(X)} \leq ct^{-\alpha}.t'^{-n}I(t) \quad \forall t \geq 1, \forall 0 < t' \leq 1, \quad (4.19)$$

where $I(t) = \int_{\mathfrak{a}^*} |\mathcal{F}\varphi_\varepsilon(t^{-1}\lambda)|(1+t^{-2}\|\lambda\|^2)^d$ and $d = \dim(X) - \text{rank}_\mathbb{R}(G)$.

Proof. It is clear that the support of $U_t\varphi_\varepsilon$ is included in the support of φ_ε . Then it is sufficient to consider the L^∞ -norm on the support of φ_ε . According to inversion formula (2.4) and inequality (2.5), we write:

$$\begin{aligned} |U_t\varphi(x)| &= \left| \frac{1}{|W|} \int_{\mathfrak{a}^* \times B} e^{itZ_\rho(\lambda)} \mathcal{F}\varphi_\varepsilon(\lambda) e^{(i\lambda+\rho)A(x,b)} \frac{d\lambda db}{|c(\lambda)|^2} \right| \\ &\leq \frac{1}{|W|} \sup_{b \in B} e^{\rho A(x,b)} \int_{\mathfrak{a}^*} |\mathcal{F}\varphi_\varepsilon(\lambda)|(1+\|\lambda\|^2)^d d\lambda \quad \forall x \in X \\ &= \frac{1}{|W|} t^{-\alpha} \sup_{b \in B} e^{\rho A(x,b)} \int_{\mathfrak{a}^*} |\mathcal{F}\varphi_\varepsilon(t^{-1}\lambda)|(1+t^{-2}\|\lambda\|^2)^d d\lambda \quad \forall x \in X \end{aligned}$$

Then one has

$$\sup_{x \in \text{supp}\varphi_\varepsilon} |U_t\varphi_\varepsilon(x)| \leq ct^{-\alpha}I(t)$$

where

$$I(t) = \int_{\mathfrak{a}^*} |\mathcal{F}\varphi_\varepsilon(t^{-1}\lambda)|(1+t^{-2}\|\lambda\|^2)^d d\lambda$$

and

$$c = \frac{1}{|W|} \sup_{x \in \text{supp} \varphi_\varepsilon, b \in B} e^{\rho A(x,b)}.$$

proving (4.18). And Inequality (4.19) is a direct consequence of (4.18) if $0 < t' \leq 1$.

Corollary 4.5. *There exists a positive number $c > 0$ and a sufficiently large number T such that:*

$$\|U_t \varphi_\varepsilon\|_{L^\infty(X)} \leq c.t^{-\alpha} \quad \forall t \geq T \quad (4.20)$$

$$\|U_t \varphi_\varepsilon\|_{L^\infty(X)} \leq c.t^{-\alpha}.t'^{-n} \quad \forall t \geq T, \quad \forall 0 < t' \leq 1. \quad (4.21)$$

Proof. To show inequality (4.20), it suffices to prove that $I(t) \rightarrow 0$ when $t \rightarrow \infty$. Which is a direct consequence of the dominated convergence theorem, because $\mathcal{F}\varphi_\varepsilon(0) = 0$.

Next using(4.16), inequality (4.20) and Young inequality, we deduce that:

$$\begin{aligned} \|U_t(f \times \varphi_{0,N})\|_{L^\infty(X)} &= \|U_t(f \times \varphi_{0,N} \times \varphi_\varepsilon)\|_{L^\infty(X)} \\ &= \|f \times \varphi_{0,N} \times U_t \varphi_\varepsilon\|_{L^\infty(X)} \\ &\leq \|f \times \varphi_{0,N}\|_{L^1(X)} \cdot \|U_t \varphi_\varepsilon\|_{L^\infty(X)} \\ &\leq c.t^{-\alpha} \|f \times \varphi_{0,N}\|_{L^1(X)} \end{aligned} \quad (4.22)$$

In addition, we have:

$$\|U_t(f \times \varphi_{0,N})\|_{L^2(X)} = \|f \times \varphi_{0,N}\|_{L^2(X)} \quad (4.23)$$

Thus by interpolating inequalities (4.22) and (4.23), we obtain:

$$\|U_t(f \times \varphi_{0,N})\|_{L^p(X)} \leq c.t^{-\alpha.(1/p'-1/p)} \|f \times \varphi_{0,N}\|_{L^{p'}(X)} \quad (4.24)$$

for $1 \leq p' \leq 2 \leq p \leq \infty$ such that $1/p + 1/p' = 1$.

By using inequality (4.21) and the same arguments as those used to prove (4.24), we obtain that:

$$\|U_t(f \times \varphi_{t'}^N)\|_{L^p(X)} \leq c.t^{-\alpha.(1/p'-1/p)} t'^{-n.(1/p'-1/p)} \|f \times \varphi_{t'}^N\|_{L^{p'}(X)}. \quad (4.25)$$

Therefore (4.24) and (4.25) give us the following inequality:

$$\|U_t f\|_{B_p^{s,q}(X)} \leq c.t^{-\alpha.(1-2/p)} \|f\|_{B_{p'}^{n(1-2/p)+s,q}(X)} \quad \forall t \geq T \quad (4.26)$$

with $2N > \max(|s|, |n(1 - 2/p) + s|)$, $1 \leq p' \leq 2 \leq p \leq \infty$ and $1 \leq q \leq \infty$.

Step 3. We recall two results we will use in the last step. The first is an argument of duality due to Ginibre-Velo [7] and Yajima [24], and the second is the Hardy-Littlewood-Sobolev inequality [13, page 117].

Lemma 4.6. *Let \mathcal{H} be a Hilbert space, Y a Banach space, Y^* the dual of Y , and \mathcal{D} a vector space densely contained in Y . Let $A \in \mathcal{L}_a(\mathcal{D}, \mathcal{H})$ and let $A^* \in \mathcal{L}_a(\mathcal{H}, \mathcal{D}_a^*)$ be its adjoint, defined by*

$$\langle A^*v, f \rangle_{\mathcal{D}} = \langle v, Af \rangle_{\mathcal{H}}, \quad \forall f \in \mathcal{D}, \forall v \in \mathcal{H}.$$

where $\mathcal{L}_a(Z_1, Z_2)$ is the space of linear maps from a vector space Z_1 to a vector space Z_2 , \mathcal{D}_a^* is the algebraic dual of \mathcal{D} and $\langle \varphi, f \rangle_{\mathcal{D}}$ is the pairing between \mathcal{D}_a^* and \mathcal{D} (with $f \in \mathcal{D}$ and $\varphi \in \mathcal{D}_a^*$). Then the following three conditions are equivalent.

1. There exist a real number $c_1 \geq 0$ such that for all $f \in \mathcal{D}$,

$$\|Af\|_{\mathcal{H}} \leq c_1 \|f\|_Y .$$

2. $\mathcal{R}(A^*) \subset Y^*$, and there exist a real number $c_2 \geq 0$ such that for all $v \in \mathcal{H}$,

$$\|A^*v\|_{Y^*} \leq c_2 \|v\|_{\mathcal{H}} .$$

3. $\mathcal{R}(A^*A) \subset Y^*$, and there exist a real number $c_3 \geq 0$ such that for all $f \in \mathcal{D}$,

$$\|A^*Af\|_{Y^*} \leq c_3^2 \|f\|_Y .$$

The constants c_1, c_2 and c_3 are equal. If one of these conditions is satisfied, the operators A and A^*A extend by continuously to bounded operators from Y to \mathcal{H} and from Y to Y^* respectively.

We use the following corollary to show the second part of Proposition 4.1.

Corollary 4.7. *Let \mathcal{H} , \mathcal{D} and $(Y_i, A_i), i = 1, 2$, satisfy any of the conditions of Lemma 4.6. Then for all choices of $i, j = 1, 2$, $\mathcal{R}(A_i^*A_j) \subset Y_i^*$, and for all $f \in \mathcal{D}$, one has:*

$$\|A_i^* A_j f\|_{Y^*} \leq c_i c_j \|f\|_{Y_j}.$$

where c_i, c_j for $i, j = 1, 2$ are the constants given in Lemma 4.6.

Lemma 4.8. (*Hardy-Littlewood-Sobolev inequality*).

Let $k_a(y) = |y|^{-m/a}$. If $1 < a < \infty$ and $1 < p, q < \infty$ such that $1/p + 1/a = 1 + 1/q$, then $\|k_a * u\|_{L^q(\mathbb{R}^m)} \leq c_{p,a} \|u\|_{L^p(\mathbb{R}^m)}$, u continuous with compact support.

We shall apply Lemma 4.6 and Corollary 4.7 in the following basic situation. Let \mathcal{H} be a Hilbert space, and U a unitary strongly continuous one parameter group in \mathcal{H} . We define the bounded operator A from $L^1(I, \mathcal{H})$ to \mathcal{H} by

$$Af = \int_I U(-t)f(t)dt. \quad (4.27)$$

Then its adjoint A^* is the operator

$$A^*v(t) = U(t)v \quad (4.28)$$

from \mathcal{H} to $L^\infty(I, \mathcal{H})$, where the duality is defined by the scalar products in \mathcal{H} and in $L^2(I, \mathcal{H})$. In particular A^*A is the bounded operator from $L^1(I, \mathcal{H})$ to $L^\infty(I, \mathcal{H})$ given by

$$A^*Af = \int_I U(t-t')f(t')dt'. \quad (4.29)$$

Step 4. Let f be a function defined on X which also depend on time. Let t and t' two real such that $t - t' \geq T$. Let τ be a real number such that $t \geq \tau + T$. Then inequality (4.26) gives:

$$\|U_{t-t'}f(t', \cdot)\|_{B_p^{s,q}(X)} \leq c(t-t')^{\alpha\gamma} \|f(t', \cdot)\|_{B_{p'}^{-n\gamma+s,q}(X)}$$

with $\gamma = \frac{2}{p} - 1$, $2N > \max(|s|, | -n\gamma + s|)$, $1 \leq p' \leq 2 \leq p \leq \infty$ and $1 \leq q \leq \infty$. Thus integrating in the variable t' on the interval $I_\tau =]-\infty, \tau]$, we obtain that:

$$\int_{-\infty}^{\tau} \|U_{t-t'}f(t', \cdot)\|_{B_p^{s,q}(X)} dt' \leq c((\cdot)^{\alpha\gamma} *_t \chi_{I_\tau} \|f\|_{B_{p'}^{-n\gamma+s,q}(X)})(t) \quad (4.30)$$

since,

$$\begin{aligned} \|(U_{(\cdot)} *_t \chi_{I_\tau} f)(t)\|_{B_p^{s,q}(X)} &= \left\| \int_{-\infty}^{\tau} U_{t-t'}f(t', \cdot) dt' \right\|_{B_p^{s,q}(X)} \\ &\leq \int_{-\infty}^{\tau} \|U_{t-t'}f(t', \cdot)\|_{B_p^{s,q}(X)} dt' \end{aligned}$$

then, by (4.30) and taken the L^r -norm on $I = [\tau + T, +\infty[$, we find that:

$$\left\| \|U_{(\cdot)} *_t \chi_{I_\tau} f\|_{B_p^{s,q}(X)} \right\|_{L^r(I)} \leq \left\| c(\cdot)^{\alpha\gamma} *_t \chi_{I_\tau} \|f(\cdot)\|_{B_{p'}^{-n\gamma+s,q}(X)} \right\|_{L^r(I)}$$

for $1 < r < \infty$. Therefore,

$$\|U_{(\cdot)} *_t \chi_{I_\tau} f\|_{L^r(I, B_p^{s,q}(X))} \leq c_{\beta, \alpha, p} \|\chi_{I_\tau} f\|_{L^\beta(I, B_{p'}^{-n\gamma+s,q}(X))} \quad (4.31)$$

where we used Hardy-Littlewood-Sobolev inequality, with $1/\beta - \alpha\gamma = 1 + 1/r$. To use Lemma 4.6, we choose β, s and q so that $L^\beta(I, B_{p'}^{-n\gamma+s,q}(X))$ is the dual of $L^r(I, B_p^{s,q}(X))$. We take $\beta = r', q = 2$ and $-n\gamma + s = -s$ i.e $s = \frac{n}{2}\gamma$. In this case we will have $2/r = -\alpha\gamma$.

Then, taking $Y = L^{r'}(I, B_{p'}^{-\frac{n}{2}\gamma, 2}(X))$ and $Y^* = L^r(I, B_p^{\frac{n}{2}\gamma, 2}(X))$, the inequality (4.31) can be written:

$$\|U_{(\cdot)} *_t g\|_{Y^*} \leq c_r \|g\|_Y; \quad (4.32)$$

which is exactly the condition 3 of Lemma 4.6 and who is equivalent to condition 2:

$$\|U_{(\cdot)} h\|_{L^r(I, B_p^{\frac{n}{2}\gamma, 2}(X))} \leq c_r \|h\|_{L^2(X)}. \quad (4.33)$$

By taking account the expression of u , (4.33) enables us to write:

$$\begin{aligned} \|u\|_{L^r(I, B_p^{\frac{n}{2}\gamma, 2}(X))} &\leq \|U_{(\cdot)} u_0\|_{L^r(I, B_p^{\frac{n}{2}\gamma, 2}(X))} + \|U_{(\cdot)} (-\Delta)^{-1/2} u_1\|_{L^r(I, B_p^{\frac{n}{2}\gamma, 2}(X))} \\ &\leq c_r (\|u_0\|_{L^2(X)} + \|(-\Delta)^{-1/2} u_1\|_{L^2(X)}) \\ &\leq c_r (\|u_0\|_{L^2(X)} + \frac{1}{\|\rho\|} \|u_1\|_{L^2(X)}) \end{aligned}$$

where we used the Plancherel formula in the last inequality.

Now, for the term concerning $\partial_t u$, we multiply (4.25) by t^{-s} with $s = \frac{n}{2}\gamma - 1$, to obtain:

$$\|U_{(\cdot)} f\|_{L^r(I, B_p^{\frac{n}{2}\gamma-1, 2}(X))} \leq c_r \|f\|_{L^2(X)}$$

showing that:

$$\|\partial_t u\|_{L^r(I, B_p^{\frac{n}{2}\gamma-1, 2}(X))} \leq c_r (\|u_1\|_{L^2(X)} + \|(-\Delta)^{1/2} u_0\|_{L^2(X)}).$$

This completes the proof of the first part of the proposition.

Now to prove the second part of the proposition, we consider the spaces:

$$Y_1 = L^{r_1}(I, B_{p_1}^{-\frac{n}{2}\gamma(p_1), 2}(X)) \quad Y_1^* = L^{r_1}(I, B_{p_1}^{\frac{n}{2}\gamma(p_1), 2}(X)).$$

$$Y_2 = L^{r'_2}(I, B_{p'_2}^{-\frac{n}{2}\gamma(p_2),2}(X)) \quad Y_2^* = L^{r_2}(I, B_{p_2}^{\frac{n}{2}\gamma(p_2),2}(X)).$$

with $\gamma(p_i) = \frac{2}{p_i} - 1$ for $i = 1, 2$. And we remark that:

$$A_1^* A_2^* f = U_{(\cdot)} *_t f.$$

Thus according to Corollary 4.7, $A_1^* A_2^*$ is bounded from Y_2 onto Y_1^* , i.e.

$$\|U_{(\cdot)} *_t f\|_{L^{r_1}(I, B_{p_1}^{\frac{n}{2}\gamma(p_1),2}(X))} \leq c \|f\|_{L^{r'_2}(I, B_{p'_2}^{-\frac{n}{2}\gamma(p_2),2}(X))}.$$

This completes the proof of proposition 4.1.

4.2 Proof of Theorem 1

We can deduce easily Theorem 1 if we take $\tau = -T$ used in the proof of the first part of proposition.

5 Proof of Theorem 2

We shall now prove our second result (Theorem 2 mentioned in the introduction). As mentioned in the previous section, we shall use Lemma 4.4. For the solution u of (1.1) given by expression (3.15), we write:

$$\begin{aligned} \|u(t, \cdot)\|_{L^\infty(X)} &\leq \|U_t u_0\|_{L^\infty(X)} + \|U_t (-\Delta)^{-1/2} u_1\|_{L^\infty(X)} \\ &\leq c (\|U_t u_0\|_{B_\infty^{0,1}(X)} + \|U_t (-\Delta)^{-1/2} u_1\|_{B_\infty^{0,1}(X)}) \\ &\leq ct^{-\alpha} (\|u_0\|_{B_1^{n,1}(X)} + \|(-\Delta)^{-1/2} u_1\|_{B_1^{n,1}(X)}) \end{aligned}$$

where we used inclusion (3.13) and inequality (4.26) for $p = \infty, s = 0$ and $q = 1$ in the last inequality. This proves the first result.

Remark 2. We can prove these results without using (4.26). In fact, by formula (3.7), we have:

$$\begin{aligned} \|U_t f\|_{L^\infty(X)} &\leq \|U_t(f \times \varphi_{0,N})\|_{L^\infty(X)} + \int_0^1 \|U_t(f \times \varphi_{t'}^N \times \varphi_{t'}^N)\|_{L^\infty(X)} \frac{dt'}{t'} \\ &\leq ct^{-\alpha} \|f \times \varphi_{0,N}\|_{L^1(X)} + c.t^{-\alpha} \int_0^1 t'^{-n} \|f \times \varphi_{t'}^N\|_{L^1(X)} \frac{dt'}{t'} \\ &= ct^{-\alpha} \|f\|_{B_1^{n,1}(X)}. \end{aligned}$$

Now if we use Proposition 3.2 and (4.26) for $s = 0$, we obtain:

$$\begin{aligned} \|u(t, \cdot)\|_{L^p(X)} &\leq \|U_t u_0\|_{L^p(X)} + \|U_t (-\Delta)^{-1/2} u_1\|_{L^p(X)} \\ &\leq c. (\|U_t u_0\|_{B_p^{0,1}(X)} + \|U_t (-\Delta)^{-1/2} u_1\|_{B_p^{0,1}(X)}) \\ &\leq c.t^{\alpha\gamma} (\|u_0\|_{B_{p'}^{-n\gamma,1}(X)} + \|(-\Delta)^{-1/2} u_1\|_{B_{p'}^{-n\gamma,1}(X)}) \end{aligned}$$

Which proves Theorem 2.

Remark 3. In Theorem 1, we have imposed that $2 < p < \min(\frac{2n}{n-4N}, \frac{2\alpha}{\alpha-1})$. This restriction comes from the assumptions $1 < a = -\frac{1}{\alpha\gamma} < \infty$ in Hardy-Littlewood-Sobolev inequality and the definition of Besov spaces. The restrictions on the dimension n of X and on L^p -spaces depend on integer N which comes from the definition of Besov spaces. We can prove other estimates not utilizing the dimension of X . Indeed, we have:

Theorem 1': *Let $\alpha = \dim(\mathfrak{a}) \geq 1$ and $I = [0, +\infty[$. Let u be a solution of the wave equation (1.1) on X and, let p and r be two real numbers such that $2/r = \alpha(1 - \frac{2}{p})$ and $2 < p < \frac{2\alpha}{\alpha-1}$. Then there exists a positive number $c_r > 0$, depending of r , such that if u_0 and u_1 belong to $L^2(X)$, we have:*

$$\|u\|_{L^r(I, B_p^{0,2}(X))} \leq c_r (\|u_0\|_{L^2(X)} + \frac{1}{\|\rho\|} \|u_1\|_{L^2(X)}).$$

For dispersive properties, we have:

Theorem 2' : *Let u be a solution of the wave equation (1.1) on X and write $\alpha = \dim(\mathfrak{a})$. Let p and p' two real numbers such that $1/p + 1/p' = 1$ and $1 \leq p' \leq 2 \leq p \leq \infty$. We suppose that u_0 and $(-\Delta)^{-1/2} u_1$ are in $B_{p'}^{0,1}(X)$. Then there exist a constant $c > 0$ and a sufficiently large number T such that for all $t \geq T$, we have:*

$$\|u(t, \cdot)\|_{L^p(X)} \leq ct^{\alpha\gamma} (\|u_0\|_{B_{p'}^{0,1}(X)} + \|(-\Delta)^{-1/2} u_1\|_{B_{p'}^{0,1}(X)}),$$

where $\gamma = \frac{2}{p} - 1$.

For the proof of these two theorems, it is enough to consider the inequality:

$$\|U_t \varphi_\varepsilon\|_{L^\infty(X)} \leq ct^{-\alpha} \text{ for all } t \geq T,$$

which implies that:

$$\|U_t f\|_{B_p^{s,q}(X)} \leq c.t^{-\alpha.(1-2/p)} \|f\|_{B_p^{s,q}(X)},$$

for $2N > |s|$, $2 \leq p \leq \infty$ and $1 \leq q \leq \infty$.

Then we use the same calculations as in the preceding proofs.

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