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Duality properties of Gorrington-Leach equations

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In the category of motions preserving the angular momentum's direction, Gorrington and Leach exhibited two classes of differential equations having elliptical orbits. After enlarging slightly these classes, we show that they are related by a duality correspondence of the Arnold-Vassiliev type. The specific associated conserved quantities (Laplace-Runge-Lenz vector and Fradkin-Jauch-Hill tensor) are then dual reflections one of the other.

PACS numbers:

I. INTRODUCTION

In 1993, Gorrington and Leach [1], exhibited two classes of differential equations incorporating drag terms which have closed elliptical orbits, generalizing then previous results of Jezewski and Mittleman [2, 3] and Leach [4]. Both possess conserved quantities which extend the Laplace-Runge-Lenz vector and Fradkin-Jauch-Hill tensor respectively. These two classes belong to a broader category of planar motions incorporating velocity dependent terms submitted to certain constraint on their coefficients [5, 6, 7, 8].

In this paper we show that the above classes can be slightly enlarged. To these generalized Gorrington-Leach equations are associated two conserved quantities : a pseudo-energy and a pseudo-angular momentum. We obtain two types (H and K) of generalized Gorrington-Leach equations presenting closed orbits for every values of these quantities. In the spherically symmetrical case, we obtain compact analytical formulas for the periods which, if restricted to the standard case, recover the results of ref. [1]. The H and K types belongs to a larger category of generalized Gorrington-Leach equations possessing duality properties (in the Arnold-Vassiliev sense [9, 10]). In this category, the equations can be gathered in classes indexed by a characteristic real parameter ν . Each ν -class possesses an associated dual μ -class, with $\mu = -\frac{\nu}{1+\frac{\nu}{2}}$. H type and K type generalized Gorrington-Leach equations are then shown to be dual of each other. As in the conservative case [10], the pseudo Laplace-Runge-Lenz vector associated to the K type equations is also proportional to the dual transform of the pseudo Fradkin-Jauch-Hill tensor of the H type equations.

II. COMPLEX DESCRIPTION OF MOTIONS WITH CONSERVATION OF THE DIRECTION OF ANGULAR MOMENTUM

As was shown in [8], the most general form for the equation of a 3-dimensional motion $\vec{r}(t)$ for which the angular momentum $\vec{L} = L\vec{e}_L$ conserves its direction is given by

$$\ddot{\vec{r}} + h\dot{\vec{r}} + g\vec{r} = 0, \quad (1)$$

where h and g are two arbitrary scalars depending on time t .

As the motion is confined to the plane orthogonal to \vec{e}_L , we can adopt a complex representation for the position $\vec{r}(t) \rightarrow z(t)$.

The functions h and g are then represented by two **real** arbitrary functions of t through $z(t)$ and $\dot{z}(t)$, not necessarily analytical (that is potentially dependent on $\bar{z}(t)$ and $\dot{\bar{z}}(t)$).

We apply our interest more specifically to the autonomous case (g and h do not depend explicitly on t), to which we add the constraint that the last term is a function of z and \bar{z} only. We then arrive to the following equation for our planar motion $z(t)$:

$$\ddot{z} + h(z, \bar{z}, \dot{z}, \dot{\bar{z}})\dot{z} + g(z, \bar{z})z = 0 \quad (2)$$

g and h being two arbitrary real-valued functions.

III. EULER-SUNDMAN REPARAMETRIZATION

We now perform an Euler-Sundman reparametrization $t \in \mathbb{R}^+ \rightarrow s \in \mathbb{R}^+$ (where the correspondence is one to one and increasing) of our motion, $z(t)$. We put

$$s = s(t, z(t), \bar{z}(t)) \quad (3)$$

with

$$\frac{ds}{dt} = \dot{s} > 0. \quad (4)$$

If we write $\frac{df}{ds} = f'$, Eq. 2 becomes

$$z'' + \frac{\ddot{s} + \dot{s}h(z, \bar{z}, \dot{s}z', \dot{s}\bar{z}')}{(\dot{s})^2} z' + \frac{g(z, \bar{z})}{(\dot{s})^2} z = 0 \quad (5)$$

in which s is called the **pseudotime** and $z(s)$ the **pseudomotion**.

If we choose the reparametrization in such a way that

$$\ddot{s} + \dot{s}h(z, \bar{z}, \dot{s}z', \dot{s}\bar{z}') = 0, \quad (6)$$

then Eq. 5 for the pseudomotion becomes

$$z'' + g(z, \bar{z}) \left(e^{2 \int h(z, \bar{z}, \dot{z}, \dot{\bar{z}}) dt} \right)_{t=t(s)} z = 0. \quad (7)$$

The problem is considerably simplified if we restrict h to be a total derivative :

$$h(z, \bar{z}, \dot{z}, \dot{\bar{z}}) = \dot{H}(z, \bar{z}) \quad (8)$$

In this particular case the initial Eq.2 is written as

$$\ddot{z} + \dot{H}(z, \bar{z}) \dot{z} + g(z, \bar{z}) z = 0 \quad (9)$$

The reparametrization is given by :

$$ds = e^{-H(z, \bar{z})} dt \quad (10)$$

and the pseudomotion Eq. 7, takes the following simple form

$$z'' + g(z, \bar{z}) e^{2H(z, \bar{z})} z = 0 \quad (11)$$

which is the equation of an autonomous conservative motion. The term $g(z, \bar{z}) e^{2H(z, \bar{z})} z$ is called the **pseudoforce**.

From now we always place ourselves in this case.

IV. RADIAL PSEUDOFORCE

Consider the case in which the pseudoforce, $g(z, \bar{z}) e^{2H(z, \bar{z})} z$, is derived from a radial real-valued potential $U(r)$:

$$\vec{\nabla} U(\vec{r}) = 2 \frac{\partial U(r)}{\partial \bar{z}} = \varphi(r) z = g(z, \bar{z}) e^{2H(z, \bar{z})} z \quad (12)$$

where

$$\varphi(r) = \frac{1}{r} \frac{\partial U(r)}{\partial r}. \quad (13)$$

The initial motion Eq. 9, takes the form:

$$\ddot{z} + \dot{H}(z, \bar{z}) \dot{z} + \varphi(r) e^{-2H(z, \bar{z})} z = 0 \quad (14)$$

which we call the **generalized Gorrige-Leach equation** [1].

To the pseudomotion we can associate a **pseudo-angular momentum**,

$$\vec{\mathcal{L}} = \vec{r} \times \vec{r}' \equiv \frac{1}{2i} (z'\bar{z} - z\bar{z}') \vec{e}_L, \quad (15)$$

and a **pseudoenergy**,

$$\mathcal{E} = \frac{1}{2} |\vec{r}'|^2 + U(r) \equiv \frac{1}{2} |z'|^2 + U(r), \quad (16)$$

which are constants of the pseudomotion.

Consequently the original motion possesses also the two conserved quantities:

$$\begin{cases} \vec{\mathcal{L}} = \mathcal{L} \vec{e}_L, & \mathcal{L} = e^{H(z,\bar{z})} L \\ \mathcal{E} = e^{2H(z,\bar{z})} \frac{|z'|^2}{2} + U(r). \end{cases} \quad (17)$$

V. BERTRAND'S THEOREM

Applying Bertrand's theorem [11, 12] to the pseudomotion, we deduce immediately that the only pseudomotions of which orbits are closed for every value of the characteristic parameters (given by the pseudo-energy and the pseudo-angular momentum), are those associated to the Hooke and Kepler potentials,

$$U_H(r) = \frac{1}{2} k r^2 \quad (18)$$

and

$$U_K(r) = -\frac{k}{r} \quad (19)$$

In both cases the orbits are ellipses, centered in O in the H case and having a focus in O in the K case.

The orbits of the pseudomotion being also those of the initial motion (the two motions differing only by a reparametrization), we deduce that the only generalized Gorrige-Leach Eq.14, which always (that is for every value of the conserved pseudo-energy \mathcal{E} and pseudo-angular momentum \mathcal{L}) admit closed orbits, are those for which $g(z, \bar{z})$ is of the form:

$$g_H(z, \bar{z}) = k e^{-2H(z, \bar{z})} \quad (20)$$

or

$$g_K(z, \bar{z}) = \frac{k}{r^3} e^{-2H(z, \bar{z})}. \quad (21)$$

In other words, the initial equation must be of the form

$$\ddot{z} + \widehat{H}(z, \bar{z}) \dot{z} + k e^{-2H(z, \bar{z})} z = 0, \quad \text{type H equation,} \quad (22)$$

or:

$$\ddot{z} + \widehat{H}(z, \bar{z}) \dot{z} + k r^{-3} e^{-2H(z, \bar{z})} z = 0, \quad \text{type K equation.} \quad (23)$$

VI. SPHERICALLY SYMMETRICAL CASE

In the spherically symmetrical case H and g depend only on the radial variable, $r = \sqrt{z\bar{z}}$, ie

$$\begin{cases} H(z, \bar{z}) = H(z\bar{z}) = \alpha(r) \\ g(z, \bar{z}) = g(z\bar{z}) = u(r), \end{cases} \quad (24)$$

and we are in the case in which the pseudoforce derives from a radial potential given by

$$U(r) = \int ru(r)e^{2\alpha(r)} dr. \quad (25)$$

In this case the initial Eq.14 is written as

$$\ddot{z} + \dot{\alpha}(r)\dot{z} + u(r)z = 0. \quad (26)$$

The two conserved quantities Eqs. ?? become:

$$\begin{cases} \vec{\mathcal{L}} = \mathcal{L}\vec{e}_L, & \mathcal{L} = e^{\alpha(r)}L \\ \mathcal{E} = e^{2\alpha(r)}\frac{|\dot{z}|^2}{2} + U(r) \end{cases} \quad (27)$$

and the type H, and K Eqs. 22,23 take the forms

$$\begin{cases} \ddot{z} + \dot{\alpha}(r)\dot{z} + ke^{-2\alpha(r)}z = 0, & \text{type H equation,} \\ \ddot{z} + \dot{\alpha}(r)\dot{z} + kr^{-3}e^{-2\alpha(r)}z = 0, & \text{type K equation.} \end{cases} \quad (28)$$

Note that, if we restrict ourselves to the case of a logarithmic function for $\alpha(r)$, we recover here in a very direct manner the original Gorryng-Leach results [1, 8].

VII. ORBITAL PERIOD IN THE SPHERICAL SYMMETRICAL CASE

The radial component $r(s)$ of the pseudomotion is given by integration of Barrow's differential formula applied to the radial pseudomotion [12], which gives

$$dt = \frac{ds}{\dot{s}} = \frac{1}{\sqrt{2}} \frac{e^{\alpha(r)} dr}{\sqrt{\mathcal{E} - V_{\mathcal{L}}(r)}}, \quad (29)$$

where the radial effective potential $V_{\mathcal{L}}(r)$ is

$$V_{\mathcal{L}}(r) = U(r) + \frac{\mathcal{L}^2}{2r^2}. \quad (30)$$

When the orbit is bounded, the radial variable r oscillates between the extremal values a et b (pericentral and apocentral radii) which are roots of the numerical equation:

$$\mathcal{E} - V_{\mathcal{L}}(r) = 0. \quad (31)$$

The orbital period is

$$T = \sqrt{2} \int_{r_<}^{r_>} \frac{e^{\alpha(r)} dr}{\sqrt{\mathcal{E} - V_{\mathcal{L}}(r)}}. \quad (32)$$

We are always in this case for the type H and K equations.

If, for the corresponding elliptical orbits, we note A the minor axis and B the major axis, we have :

* In the H case ($\mathcal{E} > 0$) $V_{\mathcal{L}}^H(r) = \frac{1}{2}kr^2 + \frac{\mathcal{L}^2}{2r^2}$,

$$\begin{cases} A = a \\ B = b \end{cases} \quad (33)$$

and

$$T_H = \frac{2}{\sqrt{k}} \int_a^b \frac{re^{\alpha(r)} dr}{\sqrt{(r^2 - a^2)(b^2 - r^2)}}. \quad (34)$$

* In the K case ($\mathcal{E} < 0$) $V_{\mathcal{L}}^K(r) = -\frac{k}{r} + \frac{\mathcal{L}^2}{2r^2}$,

$$\begin{cases} b = B + \sqrt{B^2 - A^2} \\ a = B - \sqrt{B^2 - A^2} \end{cases} \quad (35)$$

and

$$T_K = \frac{\sqrt{2}}{\sqrt{-\mathcal{E}}} \int_a^b \frac{r e^{\alpha(r)} dr}{\sqrt{(r-a)(b-r)}}. \quad (36)$$

By a straightforward changes of variables we finally obtain :

$$\begin{cases} T_H = \frac{1}{\sqrt{k}} \int_0^1 x^{-\frac{1}{2}} (1-x)^{-\frac{1}{2}} \exp\left(\alpha(\sqrt{(b^2-a^2)x+a^2})\right) dx \\ T_K = \frac{\sqrt{2}(b-a)}{\sqrt{-\mathcal{E}}} \int_0^1 x^{-\frac{1}{2}} (1-x)^{-\frac{1}{2}} \left(x + \frac{a}{b-a}\right) \exp\left(\alpha((b-a)x+a)\right) dx \end{cases} \quad (37)$$

VIII. GORRINGE-LEACH EQUATIONS

Gorringe-Leach equations [1] are equations of the type considered with

$$\alpha(r) = -\frac{\alpha}{2} \ln(r). \quad (38)$$

The initial equations of motion presenting closed orbits are then (see Eq. ??) :

$$\ddot{z} - \frac{\alpha}{2} \frac{\dot{r}}{r} \dot{z} + kr^\alpha z = 0, \quad \text{H type}, \quad (39)$$

and

$$\ddot{z} - \frac{\alpha}{2} \frac{\dot{r}}{r} \dot{z} + kr^{\alpha-3} z = 0, \quad \text{K type}. \quad (40)$$

As for the corresponding orbital periods Eq. 37, they are written in these cases as

$$T_H = \frac{\pi a^{-\frac{\alpha}{2}}}{\sqrt{k}} F\left(\frac{\alpha}{4}, \frac{1}{2}, 1, \frac{a^2 - b^2}{a^2}\right) \quad (41)$$

and

$$T_K = \frac{\sqrt{2}\pi a^{1-\frac{\alpha}{2}}}{\sqrt{-E'}} F\left(\frac{\alpha}{2} - 1, \frac{1}{2}, 1, \frac{a-b}{a}\right), \quad (42)$$

where F is an hypergeometric function.

Note that the hypergeometric functions of the form $F(\nu, \beta, 2\beta, z)$ can be rewritten in terms of Legendre functions via :

$$F(\nu, \beta, 2\beta, z) = 2^{2\beta-1} \Gamma(\beta + \frac{1}{2}) z^{\frac{1}{2}-\beta} (1-z)^{\frac{\beta-\nu-\frac{1}{2}}{2}} P_{\nu-\beta-\frac{1}{2}}^{\frac{1}{2}-\beta}\left(\frac{1-\frac{z}{2}}{\sqrt{1-z}}\right). \quad (43)$$

Under this form we recover the results of Gorringe and Leach [1], namely

$$T_H = \frac{\pi a^{-\frac{\alpha}{2}}}{\sqrt{k}} (1-z)^{-\frac{\alpha}{8}} P_{\frac{\alpha}{4}-1}^{\frac{\alpha}{4}-1}\left(\frac{1-\frac{z}{2}}{\sqrt{1-z}}\right) \quad (44)$$

and

$$T_K = \frac{\sqrt{2}\pi a^{1-\frac{\alpha}{2}}}{\sqrt{-E'}} (1-z)^{-\frac{\alpha-2}{2}} P_{\frac{\alpha}{2}-2}^{\frac{\alpha}{2}-2}\left(\frac{1-\frac{z}{2}}{\sqrt{1-z}}\right). \quad (45)$$

IX. ARNOLD-VASSILIEV DUALITY

In order for the pseudomotion potential be an Arnold-Vassiliev potential [9, 10] and then be dualizable in the Arnold-Vassiliev sense, it has to be of the type

$$U(z, \bar{z}) = k |u(z)|^2 \in \mathbb{R} \quad (46)$$

It is easy to show that, if we suppose $u(z)$ analytical, this necessitates

$$z \frac{u^{(1)}(z)}{u(z)} = cste = \frac{\nu}{2} \in \mathbb{R}, \quad (47)$$

that is, U must be a central power law potential namely

$$U(z, \bar{z}) = Ar^\nu \quad (48)$$

and the pseudomotion equation is ($k = \nu A$)

$$z'' + kr^{\nu-2}z = 0. \quad (49)$$

In other words the generalized Gorringe-Leach equations (Eq. 14) which are dualizable in the Arnold-Vassiliev sense have the form

$$\ddot{z} + \dot{H}(z, \bar{z})\dot{z} + kr^{\nu-2}e^{-2H(z, \bar{z})}z = 0. \quad (50)$$

Such an equation is said to be of class ν . In this nomenclature an H-type equation is then an equation of class 2 and a K-type equation an equation of class -1.

The pseudomotion corresponding to an equation of class ν admits a dual in the Arnold-Vassiliev sense, the associated potential of which is given by

$$V(w, \bar{w}) = B\rho^\mu, \quad (51)$$

where $\rho = |w|$, $B = -\mathcal{E} / \left(1 + \frac{\mu}{2}\right)^2$, $\left(1 + \frac{\mu}{2}\right) \left(1 + \frac{\nu}{2}\right) = 1$.

The dual pseudomotion equation is then

$$w'' + \kappa r^{\mu-2}w = 0 \quad (52)$$

with $\kappa = \mu B = -\mu \mathcal{E}' / \left(1 + \frac{\mu}{2}\right)^2$.

The correspondence between the position variables is

$$w = \left(1 + \frac{\mu}{2}\right) z^{1+\frac{\mu}{2}} \quad (53)$$

and the dual pseudotimes are related by the Euler-Sundman reparametrization

$$ds = \left(1 + \frac{\nu}{2}\right)^{-\frac{\nu}{1+\frac{\nu}{2}}} \rho^{-\frac{\nu}{1+\frac{\nu}{2}}} d\sigma. \quad (54)$$

To the pseudo-motion, $w(\sigma)$, is also associated a "true" motion, $w(\tau)$, satisfying a generalized Gorringe-Leach equation of the form

$$\ddot{w} + \dot{\tilde{H}}(z, \bar{z})\dot{w} + \kappa r^{\mu-2}e^{-2\tilde{H}(z, \bar{z})}w = 0, \quad (55)$$

$\tilde{H}(z, \bar{z})$ being an arbitrary real valued function.

The dot indicates here the derivative with respect to τ , where the "true" time, τ , is related to the pseudotime, σ , by the Euler-Sundman reparametrization

$$d\sigma = e^{-\tilde{H}(z, \bar{z})} d\tau. \quad (56)$$

This equation is an equation of class μ . We can then see that the generalized Gorringer-Leach equations, if not strictly dualizable, can, however, be grouped into classes indexed by a real characteristic exponent, the ν class being in dual correspondence with the class μ , such that :

$$\left(1 + \frac{\mu}{2}\right) \left(1 + \frac{\nu}{2}\right) = 1. \quad (57)$$

The classes 2 and -1 are dual in the sense that they are linked by a Levi-Civita [10] change of coordinates, this preserving the elliptical structure of the orbits. The fact that the equations of H and K types possess the same orbital characteristics is not fortuitous, but is a direct consequence of this duality.

As was demonstrated in [10], in complex formulation the existence of an additional conserved quantity for a motion in the Hooke potential (the Fradkin-Jauch-Hill tensor) is evident. If we apply this result to a type H pseudomotion, we deduce immediately that a H type generalized Gorringer-Leach equation admits besides its pseudo-energy, \mathcal{E} , an additional conserved complex quantity, namely

$$\mathcal{T} = \frac{1}{2}(z')^2 + \frac{1}{2}kz^2 = e^{2H(z, \bar{z})} \frac{1}{2}(\dot{z})^2 + \frac{1}{2}kz^2. \quad (58)$$

(The components of the 2D associated pseudo-FJH tensor are given by $T_{xx} = \text{Re}(\mathcal{E} + \mathcal{T})$, $T_{yy} = \text{Re}(\mathcal{E} - \mathcal{T})$ and $T_{xy} = \text{Im}(\mathcal{T}) = T_{yx}$.)

Following [10] the dual K-type generalized Gorringer-Leach equation $\ddot{w} + \widehat{H}(w, \bar{w})\dot{w} + \tilde{k}\rho^{-3}e^{-2\tilde{H}(w, \bar{w})}w = 0$, admits in addition to its pseudo-energy $\tilde{\mathcal{E}}$, the following conserved complex quantity :

$$\mathcal{A} = \frac{\mathcal{T}}{2\tilde{k}} = \frac{1}{\tilde{k}}i\tilde{w}'\tilde{\mathcal{L}} - \frac{w}{\rho} = e^{2\tilde{H}(w, \bar{w})} \frac{m}{\tilde{k}}i\tilde{w}'\tilde{\mathcal{L}} - \frac{w}{\rho} \quad (59)$$

which is the image of \mathcal{T} via the dual transformation (Eqs.53 and 54), with $\tilde{\mathcal{E}} = -\frac{k}{2}$ and $\tilde{k} = -\frac{\xi}{2}$. We clearly recognize in \mathcal{A} the complex formulation of the pseudo-Laplace-Runge-Lenz vector [1] :

$$\vec{A} = \frac{e^{2\tilde{H}}}{\tilde{k}} \vec{L} \times \vec{\rho}' - \frac{\vec{\rho}}{\rho}. \quad (60)$$

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