Degenerate complex Monge-Ampère equations over compact Kahler manifolds

Jean-Pierre Demailly, Nefton Pali

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Abstract

We prove the existence and uniqueness of the solutions of some very general type of degenerate complex Monge-Ampère equations, and investigate their regularity. This type of equations is precisely what is needed in order to construct Kähler-Einstein metrics over irreducible singular Kähler spaces with ample or trivial canonical sheaf.

Contents

1 Introduction

In a celebrated paper [Yau] published in 1978, Yau settled all cases of the Calabi conjecture. As is well known, the problem of prescribing the Ricci curvature can be formulated in terms of non degenerate complex Monge-Ampère equations.

Theorem 1.1 (Yau). *Let X be a compact Kähler manifold of complex dimension n and let χ be a Kähler class. Then for any smooth density $v > 0$ on X such that $\int_X v = \int_X \chi^n$, there exists a unique (smooth) Kähler metric $\omega \in \chi$ (i.e. $\omega = \omega_0 + i\partial\bar\partial\varphi$ with $\omega_0 \in \chi$) such that $\omega^n = (\omega_0 + i\partial\bar\partial\varphi)^n = v$.*

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Another breakthrough concerning the study of complex Monge-Ampère equations was achieved by Bedford-Taylor [Be-Te]. Their work opened the way to the study of very degenerate complex Monge-Ampère equations. In fact, by combining these results, Kołodziej [Kol1] proved the existence of solutions for equations of type $(\omega + i\partial\bar\partial\varphi)^n = v$, where ω a Kähler metric and $v \geq 0$ a density in L^p or in some complicated Orlicz spaces. However, in various geometric applications, it is necessary to consider the case where ω is merely semipositive. This more difficult situation has been examined first by Tsuji [Ts], and his technique has been reconsidered in the recent works [Ca-La], [Ti-Zha], [E-G-Z] and [Pau].

In this paper we push further the techniques developed so far and we obtain some very general and sharp results on the existence, uniqueness and regularity of the solutions of degenerate complex Monge-Ampère equations. In order to define the relevant concept of uniqueness of the solutions, we introduce a suitable subset of the space of closed $(1,1)$-currents, namely the domain of definition BT of the complex Monge-Ampère operator "in the sense of Bedford-Taylor": a current Θ is in BT if the the successive exterior powers can be computed as $\Theta^{k+1} = i\partial\bar\partial(\varphi\Theta^k)$ where φ is a potential of Θ and $\varphi\Theta^k$ is locally of finite mass. Then for every pseudoeffective $(1,1)$-cohomology class χ, we prove a monotone convergence result for exterior powers of currents in the subset $\mathrm{BT}_\chi := \mathrm{BT} \cap \chi$.

The uniqueness of the solutions of the degenerate complex Monge-Ampère equations in a reasonable class of unbounded potentials has been a big issue and the object of intensive studies, see e.g. [Ts], [Ti-Zha], [Blo1], [E-G-Z]. In this direction, we introduce the subset $\mathrm{BT}^{\log}_\chi$ of (closed positive) currents $T \in \mathrm{BT}_\chi$ which have a Monge-Ampère product T^n possessing a L^1-density such that $\int_X -\log(T^n/\Omega)\,\Omega < +\infty$ for some smooth volume form $\Omega > 0$. For example this is the case when the current T^n possesses a L^1-density with complex analytic singularities (see theorem 6.1). We observe that the Ricci operator is well defined in the class $\mathrm{BT}^{\log}_\chi$.

In the last section we prove existence and fine regularity properties of the solutions of complex Monge-Ampère equations with respect to a given degenerate metric $\omega \geq 0$, when the right hand side possesses a $L\log^{n+\varepsilon} L$-density or a density carrying complex analytic singularities (see theorems 6.2 and 6.1). As a consequence of this results, we derive the following generalization of Yau's theorem.

Theorem 1.2 *Let X be a compact Kähler manifold of complex dimension n and let χ be $(1,1)$-cohomology class admitting a smooth closed semipositive $(1,1)$-form ω such that $\{\omega^n = 0\}$ is a set of measure zero.*
A). *For any $L\log^{n+\varepsilon} L$-density $v \geq 0$, $\varepsilon > 0$ such that $\int_X v = \int_X \chi^n$, there exists a unique closed positive current $T \in \mathrm{BT}_\chi$ such that $T^n = v$. Moreover, this current possesses bounded local potentials over X and continuous local potentials outside a complex analytic set $\Sigma_\chi \subset X$. This set depend only on*

the class χ and is empty if and only if the class χ is Kähler.
B). *In the particular case of a density $v \geq 0$ possessing complex analytic singularities the current T is also smooth outside the complex analytic subset $\Sigma_\chi \cup Z(v) \subset X$, where $Z(v)$ is the set of zeros and poles of v.*

We wish to point out that the main examples of Orlicz spaces considered by Kołodziej are contained in some space $L\log^{n+\varepsilon} L$. The type of complex Monge-Ampère equation solved in theorem 6.1 is precisely what is needed in order to construct Kähler-Einstein metrics over irreducible singular Kähler spaces with ample or trivial canonical sheaf. This allows us also to solve generalized equations of the form $\mathrm{Ric}(\omega) = -\lambda\omega + \rho$, $\lambda \geq 0$. The proof of our Laplacian estimate in theorem 6.1, which is obtained as a combination of the ideas of in [Yau], [Ts], [Blo2], provides in particular a drastic simplification of Yau's most general argument for complex Monge-Ampère equations with degenerate right hand side. Moreover, it can be applied immediately to certain singular situations considered in [Pau] and it reduces the Laplacian estimate in [Pau] to a simple consequence (however, one should point out that the argument in [Pau] contains a gap due to the fact that the L^p-norm of the exponential $\exp(\psi_{1,\varepsilon} - \psi_{2,\varepsilon})$ of ε-regularized quasi-plurisubharmonic functions need not be uniformly bounded in ε under the assumption that $\exp(\psi_1 - \psi_2)$ is L^p, as our lemma 5.4 clearly shows if we do not choose carefully the constant A there). Theorem 6.1 gives also some metric results for the geometry of varieties of general type. In this direction, we obtain the following result.

Theorem 1.3 *Let X be a smooth complex projective variety of general type. If the canonical bundle is nef, then there exists a unique closed positive current $\omega_{_E} \in \mathrm{BT}^{\log}_{2\pi c_1(K_X)}$ solution of the Einstein equation $\mathrm{Ric}(\omega_{_E}) = -\omega_{_E}$. This current possesses bounded local potentials and defines a smooth Kähler metric outside a complex analytic subset, which is empty if and only if the canonical bundle is ample.*

The existence part has been studied in [Ts], [Ca-La] and [Ti-Zha] by a Kähler-Ricci flow method. Quite recently Tian and Kołodziej [Ti-Ko] proved a very particular case of our C^0-estimate. Their method, which is completely different, is based on an idea developed in [De-Pa]. Our C^0-estimate allows us to completely solve a conjecture of Tian stated in [Ti-Ko] (see Appendix D).

2 General C^0-estimates for the solutions.

Let X be a compact connected complex manifold of complex dimension n and let γ be a closed real $(1,1)$-current with continuous local potentials or a closed positive $(1,1)$-current with bounded local potentials. Then to any

distribution Ψ on X such that $\gamma + i\partial\bar\partial\Psi \geq 0$ we can associate a unique locally integrable and bounded from above function $\psi : X \to [-\infty, +\infty)$ such that the corresponding distribution coincides with Ψ and such that for any continuous or plurisubharmonic local potential h of γ the function $h + \psi$ is plurisubharmonic. The set of functions ψ obtained in this way will be denoted by $\mathcal{P}_\gamma$. We set $\mathcal{P}^0_\gamma := \{\psi \in \mathcal{P}_\gamma \mid \sup_X \psi = 0\}$. A closed positive $(1,1)$-current with bounded local potentials such that $\{\gamma\}^n := \int_X \gamma^n > 0$, will be called big. If X is compact Kähler, one knows by [De-Pa] that the class $\{\gamma\}$ is big if and only if it contains a Kähler current $T = \gamma + i\partial\bar\partial\psi \geq \varepsilon\omega$, for some Kähler metric ω on X and $\varepsilon > 0$. We refer to the Appendix A and to [Ra-Re], [Iw-Ma] for the basic definitions of Orlicz norms and Orlicz spaces.

Theorem 2.1 *Let X be a compact Kähler manifold of complex dimension n, let $\Omega > 0$ be a smooth volume form, let γ be a big closed positive $(1,1)$-current with continuous local potentials. Let also $\psi \in \mathcal{P}_\gamma \cap L^\infty(X)$ be a solution of the degenerate complex Monge-Ampère equation*

$$(\gamma + i\partial\bar\partial\psi)^n = f\,\Omega\,,$$

with $f \in L\log^{n+\varepsilon_0} L(X)$ for some $\varepsilon_0 > 0$. Then the following conclusions hold.
(A) *There exist a uniform constant $C_1 = C_1(\varepsilon_0, \gamma, \Omega) > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ we have an estimate*

$$\mathrm{Osc}(\psi) \leq (C_1/\varepsilon)^{n^2/\varepsilon}\, I_{\gamma,\varepsilon}(f)^{\frac{n}{\varepsilon}} + \,1\,,$$

where

$$I_{\gamma,\varepsilon}(f) := \{\gamma\}^{-n} \int\limits_X f \log^{n+\varepsilon}\left(e + \{\gamma\}^{-n} f\right)\Omega\,.$$

(B) *Assume that the solution ψ is normalized by the condition $\sup_X \psi = 0$ and consider also a solution $\varphi \in \mathcal{P}_\gamma \cap L^\infty(X)$, $\sup_X \varphi = 0$ of the degenerate complex Monge-Ampère equation*

$$(\gamma + i\partial\bar\partial\varphi)^n = g\,\Omega\,,$$

with $g \in L\log^{n+\varepsilon_0} L(X)$. Assume also $I_{\gamma,\varepsilon_0}(f), I_{\gamma,\varepsilon_0}(g) \leq K$ for some constant $K > 0$. Then there exists a constant $C_2 = C_2(\varepsilon_0, \gamma, \Omega, K) > 0$ such that

$$\begin{aligned} \|\varphi - \psi\|_{C^0(X)} &\leq 2C_2^{\alpha_0}\, \left(\log \|\varphi - \psi\|^{-1}_{L^1(X,\,\Omega)}\right)^{-\alpha_0}\,, \\ \alpha_0 &:= \frac{1}{(n + 1 + n^2/\varepsilon_0)}\,, \end{aligned}$$

provided that the inequality $\|\varphi-\psi\|_{L^1(X,\,\Omega)} \leq \min\{1/2, e^{-C_2}\}$ *holds.*
(C) *Let* $(\gamma_t)_{t>0}$ *be a family of currents satisfying the same properties as* γ*, fix a finite covering* $(U_\alpha)_\alpha$ *of coordinate starshaped open sets, and let us write* $\gamma_t = i\partial\bar\partial h_{t,\alpha}$ *with* $\sup_{U_\alpha} h_{t,\alpha} = 0$ *over* U_α *and* $C_{1,t} := C_1(\varepsilon_0, \gamma_t, \Omega)$*,* $C_{2,t} = C_2(\varepsilon_0, \gamma_t, \Omega, K)$*. Assume*
(C1) $\sup_{t>0} \max_\alpha \|h_{t,\alpha}\|_{L^\infty(U_\alpha)} < +\infty$ *and*
(C2a) *there exist a decomposition of the type* $\gamma_t = \theta_t + i\partial\bar\partial u_t$*, with* θ_t *smooth,* $\min_X u_t = 0$*,* $\sup_{t>0} \max_X u_t < +\infty$ *and* $\theta_t \leq (\{\gamma_t\}^n)^{1/n}\omega$ *for some Kähler metric* $\omega > 0$ *on* X*,*
or
(C2b) *the distributions* γ_t^n/Ω *are represented by* L^1*-functions and*

$$\sup_{t>0}\ \{\gamma_t\}^{-n} \int_X \log\left(e + \{\gamma_t\}^{-n}\gamma_t^n/\Omega\right)\gamma_t^n < +\infty\,.$$

Then $\sup_{t>0} C_{j,t} < +\infty$ *for* $j = 1, 2$*.*

Statement (C) will follow from the arguments of the proof of theorem 2.1.

Remark 1. As an application of his estimates, Kołodziej considers in Example 1, page 91 of [Kol1] Monge-Ampère equations with non degenerate left hand side and with right hand side taking values in the Orlicz space $L^\Psi(X)$, with $\Psi(t) := t\log^n(e+t)\log^{n+\delta}(e+\log(1+t))$, $\delta > 0$. If we take $\varepsilon_0 = 1/k$ with an integer $k > 1$ we obtain

$$\lim_{t\to+\infty} \frac{\log^{n+\delta}(e+\log(1+t))}{\log^{\varepsilon_0}(e+t)} = +\infty\,.$$

This implies $L^\Psi(X) \subset L\log^{n+\varepsilon_0} L(X)$.

Let X be a compact complex manifold of complex dimension n, let γ be a big closed positive $(1,1)$-current with bounded local potentials. Set $\mathcal{P}_\gamma[0,1] := \{\varphi \in \mathcal{P}_\gamma \mid 0 \leq \varphi \leq 1\}$, $\gamma_\varphi := \gamma + i\partial\bar\partial\varphi$ and

$$\mathrm{Cap}_\gamma(E) := \sup_{\varphi\in\mathcal{P}_\gamma[0,1]} \{\gamma\}^{-n} \int_E \gamma_\varphi^n\,,$$

for all Borel sets $E \subset X$. We remark that if $(E_j)_j$, $E_j \subset E_{j+1} \subset X$ is a family of Borel sets and $E = \bigcup_j E_j$ then clearly, we have

$$\mathrm{Cap}_\gamma(E) = \lim_{j\to+\infty} \mathrm{Cap}_\gamma(E_j)\,. \tag{2.1}$$

Lemma 2.2 *Let* X *be a compact connected complex manifold of complex dimension* n*, let* γ *be a closed real* $(1,1)$*-current with continuous local potentials or a closed positive* $(1,1)$*-current with bounded local potentials and let* $\Omega > 0$ *be a smooth volume form. Then there exist constants* $\alpha = \alpha(\gamma,\Omega) > 0$*,* $C = C(\gamma,\Omega) > 0$ *such that* $\int_X -\psi\,\Omega \leq C$ *and* $\int_X e^{-\alpha\psi}\,\Omega \leq C$ *for all* $\psi \in \mathcal{P}_\gamma^0$*.*

The first two integral estimates of lemma 2.2 are quite standard in the elementary theory of plurisubharmonic functions and the dependence of the constants α and C on γ is only on the L^∞ bound of its local potentials. To be more precise concerning the uniform estimate $\int_X e^{-\alpha\psi}\,\Omega \leq C$ one can make the constant α depending only on the cohomology class of γ as in [Ti1], but in this case the constant C will depend on the L^∞ bound of the local potentials of γ and on the volume form Ω. One can also make C depending only on the volume form Ω, but in this case α will depend on the L^∞ bound of the local potentials of γ and on the volume form Ω.

Lemma 2.3 *Let X be a compact connected Kähler manifold of complex dimension n, let γ be a big closed positive $(1,1)$-current with continuous local potentials.*
(A) *There exists a constant $C = C(\gamma) > 0$ such that $\mathrm{Cap}_\gamma(\{\psi < -t\}) \leq C/t$ for all $\psi \in \mathcal{P}^0_\gamma$ and $t > 0$. Moreover the constant C stay bounded for perturbations of γ satisfying the hypothesis* (C1) *and* (C2a) *of statement* (C) *in theorem 2.1.*
(B) *If $\gamma^n/\Omega \in L\log L(X)$, for a smooth volume form $\Omega > 0$ then the conclusion of statement* (A) *hold with a constant $C = C(\gamma, \Omega) > 0$ which stays bounded for perturbations of γ satisfying the hypothesis* (C1) *and* (C2b) *of statement* (C) *in theorem 2.1.*

Proof. We first notice the obvious inequality

$$\int\limits_{\psi<-t} \gamma_\varphi^n \;\leq \frac{1}{t}\int\limits_X -\psi\,\gamma_\varphi^n$$

which implies

$$\mathrm{Cap}_\gamma(\{\psi < -t\}) \leq \frac{1}{t}\sup_{\varphi\in\mathcal{P}_\gamma[0,1]}\{\gamma\}^{-n}\int\limits_X -\psi\,\gamma_\varphi^n\,, \tag{2.2}$$

and we prove the following elementary claim.

Claim 2.4 *Let γ be a closed positive $(1,1)$-current with bounded local potentials over a compact complex manifold X of complex dimension n and let φ, $\psi \in \mathcal{P}_\gamma$ such that $0 \leq \varphi \leq 1$ and $\psi \leq 0$. Then*

$$\int\limits_X -\psi\,\gamma_\varphi^n \;\leq \int\limits_X -\psi\,\gamma^n + n\int\limits_X \gamma^n\,. \tag{2.3}$$

Proof. The fact that the current γ is positive implies $\psi_c := \max\{\psi, c\} \in \mathcal{P}_\gamma$, $c \in \mathbb{R}_{<0}$, so by the monotone convergence theorem it is sufficient to prove inequality (2.3) for $\psi \in \mathcal{P}_\gamma \cap L^\infty(X)$. So assume this and let $\omega > 0$ be a hermitian metric over X. By a result of Greene-Wu [Gr-Wu] there exist a

family of functions $(\psi_\varepsilon)_{\varepsilon>0}$, $\psi_\varepsilon \in \mathcal{P}_{\gamma+\varepsilon\omega}\cap C^\infty(X)$ such that $\psi_\varepsilon \downarrow \psi$ as $\varepsilon \to 0^+$. Consider now the integrals $I_j := \int_X -\psi\,\gamma^j \wedge \gamma_\varphi^{n-j}$ for all $j = 0, ..., n$. Then $I_j \leq I_{j+1} + \int_X \gamma^n$. In fact by Stokes formula

$$\begin{aligned} I_j &= I_{j+1} - \lim_{\varepsilon\to 0^+} \int_X \psi_\varepsilon\,\gamma^j \wedge i\partial\bar\partial\varphi \wedge \gamma_\varphi^{n-j-1} \\ &= I_{j+1} - \lim_{\varepsilon\to 0^+} \int_X i\partial\bar\partial\psi_\varepsilon \wedge \varphi\,\gamma^j \wedge \gamma_\varphi^{n-j-1} \\ &\leq I_{j+1} + \int_X \varphi\,\gamma^{j+1} \wedge \gamma_\varphi^{n-j-1} \leq I_{j+1} + \int_X \gamma^n\,. \end{aligned}$$

In this way we deduce the required inequality $I_0 \leq I_n + n\int_X \gamma^n$. □

The following claim will be very useful for the rest of the paper.

Claim 2.5 *Let (X,ω) be a polarized compact connected Kähler manifold of complex dimension n and let γ, T be closed positive $(1,1)$-currents with respectively continuous, bounded local potentials. Then for all $l = 0, ..., n$*

$$C_l := \sup_{\psi\in\mathcal{P}_\gamma^0} \int_X -\psi\,T^l \wedge \omega^{n-l} < +\infty$$

and $\gamma_\psi \wedge T^l = T^l \wedge \gamma_\psi$ for all $\psi \in \mathcal{P}_\gamma$.

Proof. The proof of the convergence of the constants C_l goes by induction on $l = 0, ..., n$. The statement is true for $l = 0$ by the first integral estimate of lemma 2.2. So we assume it is true for l and we prove it for $l+1$. Let $\psi_c := \max\{\psi, c\} \in \mathcal{P}_\gamma$, $c \in \mathbb{R}_{<0}$. By the result of Greene-Wu [Gr-Wu] let $(\psi_{c,\varepsilon})_{\varepsilon>0}$, $\psi_{c,\varepsilon} \in \mathcal{P}_{\gamma+\varepsilon\omega} \cap C^\infty(X)$ such that $\psi_{c,\varepsilon} \downarrow \psi_c$ as $\varepsilon \to 0^+$ and write $T = \theta + i\partial\bar\partial u$, with θ smooth, $\theta \leq K\omega$ and u bounded with $\inf_X u = 0$. By using the monotone convergence theorem and Stokes formula, we expand

the integral

$$\int_X -\psi\, T^{l+1} \wedge \omega^{n-l-1} = \lim_{c\to-\infty} \lim_{\varepsilon\to 0^+} \int_X -\psi_{c,\varepsilon}\, T^{l+1} \wedge \omega^{n-l-1}$$

$$= \lim_{c\to-\infty} \lim_{\varepsilon\to 0^+} \left[\int_X -\psi_{c,\varepsilon}\, \theta \wedge T^l \wedge \omega^{n-l-1} - \int_X \psi_{c,\varepsilon}\, i\partial\bar\partial u \wedge T^l \wedge \omega^{n-l-1} \right]$$

$$\leq \lim_{c\to-\infty} \lim_{\varepsilon\to 0^+} \left[\int_X -\psi_{c,\varepsilon}\, T^l \wedge K\omega^{n-l} - \int_X u\, i\partial\bar\partial \psi_{c,\varepsilon} \wedge T^l \wedge \omega^{n-l-1} \right]$$

$$= \int_X -\psi\, T^l \wedge K\omega^{n-l} + \lim_{c\to-\infty} \lim_{\varepsilon\to 0^+} \left[-\int_X u\, (\gamma_{\psi_{c,\varepsilon}} + \varepsilon\omega) \wedge T^l \wedge \omega^{n-l-1} \right.$$

$$\left. + \int_X u\, (\gamma + \varepsilon\omega) \wedge T^l \wedge \omega^{n-l-1} \right]$$

$$\leq KC_l + \sup_X u \int_X \gamma \wedge T^l \wedge \omega^{n-l-1} < +\infty\,,$$

by the inductive hypothesis. Concerning the symmetry of the exterior product we remark that the decreasing monotone convergence theorem implies

$$\lim_{c\to-\infty} \int_X (\psi_c - \psi) T^l \wedge \omega^{n-l} = 0\,,$$

which means the convergence of the mass $\|(\psi_c - \psi)T^l\|_\omega(X) \to 0$ as $c \to -\infty$, in particular $\psi_c T^l \to \psi T$ weakly as $c \to -\infty$. So by the weak continuity of the $i\partial\bar\partial$ operator we deduce

$$\gamma_{\psi_c} \wedge T^l \longrightarrow \gamma_\psi \wedge T^l\,, \tag{2.4}$$

weakly as $c \to -\infty$. Moreover the weak continuity of the $i\partial\bar\partial$ operator implies by induction on l

$$T^l \wedge \gamma_{\psi_c} \longrightarrow T^l \wedge \gamma_\psi\,,$$

weakly as $c \to -\infty$. This combined with (2.4) implies $\gamma_\psi \wedge T^l = T^l \wedge \gamma_\psi$.□

In the particular case $T = \gamma$ big, the constant

$$0 < C(\gamma) := n + \sup_{\psi\in\mathcal{P}^0_\gamma} \{\gamma\}^{-n} \int_X -\psi\, \gamma^n < +\infty$$

satisfies the capacity estimate of statement (A) in lemma 2.3, by inequality (2.2) and claim 2.4. Thus if $(\gamma_t)_{t>0}$ is a family satisfying the hypothesis $(C1)$ and $(C2a)$ of statement (C) in theorem 2.1 and $K_t = (\{\gamma_t\}^n)^{1/n}$, then the constant $C(\gamma)$ satisfies the stability properties of statement (A) of the lemma 2.3, and we can use the induction in the proof of claim 2.5 with $T = \gamma_t$, $\theta = \theta_t$, $u = u_t$ and $K = K_t$ to get

$$C_1 \leq K_t \int\limits_X -\psi\,\omega^n + \sup_X u_t \int\limits_X \gamma_t \wedge \omega^{n-1} \leq K_t \int\limits_X -\psi\,\omega^n + RK_t \int\limits_X \omega^n\,,$$

where $R \geq \sup_X u_t$ and in general

$$C_{l+1} \leq K_t C_l + R \int\limits_X \gamma_t^{l+1} \wedge \omega^{n-l-1} \leq K_t C_l + RK_t^{l+1} \int\limits_X \omega^n\,.$$

We deduce $C_n \leq K_t^n \int_X -\psi\,\omega^n + nRK_t^n \int_X \omega^n$. We now prove statement (B) of lemma 2.3. In fact let $f := \{\gamma\}^{-n}\gamma^n/\Omega \geq 0$. Then the uniform estimate for the integral

$$\{\gamma\}^{-n} \int\limits_X -\psi\,\gamma^n = \frac{1}{\alpha} \int\limits_X -\alpha\psi f\,\Omega$$

follows from the elementary inequality $-\alpha\psi f \leq e^{-\alpha\psi} - 1 + f\log(1+f)$ combined with the uniform estimate $\int_X e^{-\alpha\psi}\Omega \leq C$ of lemma 2.2. In this case the required stability properties of the constant $C(\gamma, \Omega) > 0$ in the capacity estimate are obvious. □

Lemma 2.6 (Degenerate Comparison Principle). *Let X be a compact Kähler manifold of complex dimension n, let γ be a closed real $(1,1)$-current with continuous local potentials or a closed positive $(1,1)$-current with bounded local potentials, and consider φ, $\psi \in \mathcal{P}_\gamma \cap L^\infty(X)$. Then*

$$\int\limits_{\varphi<\psi} \gamma_\psi^n \;\leq \int\limits_{\varphi<\psi} \gamma_\varphi^n\,.$$

Proof.
Step I. We assume first φ, $\psi \in \mathcal{P}_\gamma \cap C^0(X)$. We will denote by ∂S the boundary in X of a set $S \subset X$. By the continuity of the functions φ, ψ we deduce:
1) the set $\{\varphi < \psi\}$ is open and $\partial\{\varphi < \psi\} \subset \{\varphi = \psi\}$,
2) for all $\varepsilon > 0$ there exists an open neighborhood $\mathcal{V} \subset X$ of the set $\{\varphi \geq \psi\}$ such that $\max\{\varphi + \varepsilon, \psi\} = \varphi + \varepsilon$ over $\mathcal{V}$.
So $\partial\{\varphi < \psi\} \subset \mathcal{V}$ and the Stokes formula implies the equality

$$\int\limits_{\varphi<\psi} \gamma_\varphi^n \;= \int\limits_{\varphi<\psi} (\gamma + i\partial\bar\partial \max\{\varphi + \varepsilon, \psi\})^n\,,$$

for all $\varepsilon > 0$. Moreover by the monotone convergence theorem in pluripotential theory we deduce that the current $(\gamma + i\partial\bar\partial \max\{\varphi+\varepsilon, \psi\})^n$ converges weakly to the current γ_ψ^n over the open set $\{\varphi < \psi\}$ as $\varepsilon \to 0^+$. Thus

$$\int\limits_{\varphi<\psi} \gamma_\varphi^n \;=\; \liminf_{\varepsilon\to 0^+} \int\limits_{\varphi<\psi} (\gamma + i\partial\bar\partial \max\{\varphi+\varepsilon,\psi\})^n \geq \int\limits_{\varphi<\psi} \gamma_\psi^n\,.$$

Step II. Let now $(U_\alpha)_{\alpha=1}^N$ be a finite open covering of X such that $\gamma = i\partial\bar\partial h_\alpha$ over U_α with $\inf_{U_\alpha} h_\alpha = 0$. By the quasicontinuity of plurisubharmonic functions, for every $\delta > 0$ there exists an open set $G_{\delta,\alpha} \subset U_\alpha$ such that $u_\alpha := h_\alpha + \varphi$, $h_\alpha \in C^0(U_\alpha \setminus G_{\delta,\alpha})$ and $\mathrm{Cap}(G_{\delta,\alpha}, U_\alpha) < \delta$. In particular $\varphi \in C^0(U_\alpha \setminus G_{\delta,\alpha})$, and therefore $\varphi \in C^0(X \setminus G_\delta)$ where $G_\delta := \bigcup_\alpha G_{\delta,\alpha} \subset X$. We can also assume $\psi \in C^0(X \setminus G_\delta)$. Set $v_\alpha := h_\alpha + \psi$. Let ω be a Kähler metric over X and g_α smooth functions over U_α such that $\omega = i\partial\bar\partial g_\alpha$, $\inf_{U_\alpha} g_\alpha = 0$. By the result of Greene-Wu [Gr-Wu] there exists a sequence $(\varepsilon_j)_j \subset (0,\varepsilon)$, $\varepsilon_j \downarrow 0$ and ψ_j, $\varphi_j \in \mathcal{P}_{\gamma+\varepsilon_j\omega} \cap C^\infty(X)$, with $\psi_j \downarrow \psi$ and $\varphi_j \downarrow \varphi$. We can assume $0 \leq \inf_X \varphi$, $0 \leq \inf_X \psi$ and we set $u_{j,\alpha} := h_\alpha + g_\alpha + \varphi_j$, $v_{j,\alpha} := h_\alpha + g_\alpha + \psi_j$, $M := \max_\alpha\{\|u_{1,\alpha}\|_{L^\infty(U_\alpha)}, \|v_{1,\alpha}\|_{L^\infty(U_\alpha)}\}$. Then by step I

$$\int\limits_{\varphi_k<\psi_j} (\gamma + \varepsilon\omega + i\partial\bar\partial\psi_j)^n \leq \int\limits_{\varphi_k<\psi_j} (\gamma + \varepsilon\omega + i\partial\bar\partial\varphi_k)^n\,. \tag{2.5}$$

Let $f \in C^0(X)$ such that $f = \psi$ over $X \setminus G_\delta$. Then the set $\{\varphi_k < f\}$ is open and $\{\varphi_k < f\} \cup G_\delta = \{\varphi_k < \psi\} \cup G_\delta$. Thus

$$\begin{aligned}
\int\limits_{\varphi_k<\psi} \gamma_\psi^n \;&\leq\; \int\limits_{\varphi_k<f} \gamma_\psi^n + \int\limits_{G_\delta} \gamma_\psi^n \;\leq\; \int\limits_{\varphi_k<f} (\gamma + \varepsilon\omega + i\partial\bar\partial\psi)^n + \sum_\alpha \int\limits_{G_{\delta,\alpha}} (i\partial\bar\partial v_\alpha)^n \\
&\leq\; \liminf_{j\to+\infty} \int\limits_{\varphi_k<f} (\gamma + \varepsilon\omega + i\partial\bar\partial\psi_j)^n + M^n \sum_\alpha \int\limits_{G_{\delta,\alpha}} (i\partial\bar\partial M^{-1}\, v_\alpha)^n \\
&\leq\; \liminf_{j\to+\infty} \left(\int\limits_{\varphi_k<\psi_j} (\gamma + \varepsilon\omega + i\partial\bar\partial\psi_j)^n + \int\limits_{G_\delta} (\gamma + \varepsilon\omega + i\partial\bar\partial\psi_j)^n \right) + M^n N\delta
\end{aligned}$$

$$
\begin{aligned}
&\leq \liminf_{j\to+\infty}\left(\int\limits_{\varphi_k<\psi_j}(\gamma+\varepsilon\omega+i\partial\bar\partial\varphi_k)^n+M^n\sum_\alpha\int\limits_{G_{\delta,\alpha}}(i\partial\bar\partial M^{-1}\,v_{j,\alpha})^n\right)\\
&+ \quad M^nN\delta \qquad\qquad (\text{by } (2.5))\\
&\leq \lim_{j\to+\infty}\int\limits_{\varphi_k<\psi_j}(\gamma+\varepsilon\omega+i\partial\bar\partial\varphi_k)^n+2M^nN\delta\\
&= \int\limits_{\varphi_k\leq\psi}(\gamma+\varepsilon\omega+i\partial\bar\partial\varphi_k)^n+2M^nN\delta\,.
\end{aligned}
$$

Then by letting $k\to+\infty$ we get

$$
\int\limits_{\varphi<\psi}\gamma_\psi^n\;\leq\;\limsup_{k\to+\infty}\int\limits_{\varphi_k\leq\psi}(\gamma+\varepsilon\omega+i\partial\bar\partial\varphi_k)^n+2M^nN\delta\,. \tag{2.6}
$$

Now the set $\{\varphi\leq\psi\}\smallsetminus G_\delta$ is closed by the continuity of φ and ψ over $X\smallsetminus G_\delta$. Thus

$$
\begin{aligned}
&\int\limits_{\varphi\leq\psi}(\gamma+\varepsilon\omega+i\partial\bar\partial\varphi)^n\geq\int\limits_{\{\varphi\leq\psi\}\smallsetminus G_\delta}(\gamma+\varepsilon\omega+i\partial\bar\partial\varphi)^n\\
&\geq \limsup_{k\to+\infty}\int\limits_{\{\varphi\leq\psi\}\smallsetminus G_\delta}(\gamma+\varepsilon\omega+i\partial\bar\partial\varphi_k)^n\\
&\geq \limsup_{k\to+\infty}\left(\int\limits_{\varphi\leq\psi}(\gamma+\varepsilon\omega+i\partial\bar\partial\varphi_k)^n-\int\limits_{G_\delta}(\gamma+\varepsilon\omega+i\partial\bar\partial\varphi_k)^n\right)\\
&\geq \limsup_{k\to+\infty}\left(\int\limits_{\varphi\leq\psi}(\gamma+\varepsilon\omega+i\partial\bar\partial\varphi_k)^n-M^n\sum_\alpha\int\limits_{G_{\delta,\alpha}}(i\partial\bar\partial M^{-1}\,u_{k,\alpha})^n\right)\\
&\geq \limsup_{k\to+\infty}\int\limits_{\varphi_k\leq\psi}(\gamma+\varepsilon\omega+i\partial\bar\partial\varphi_k)^n-M^nN\delta\,.
\end{aligned}
$$

So by (2.6) we derive

$$
\begin{aligned}
\int\limits_{\varphi<\psi}\gamma_\psi^n\;&\leq\;\int\limits_{\varphi\leq\psi}(\gamma+\varepsilon\omega+i\partial\bar\partial\varphi)^n+3M^nN\delta\\
&\leq\int\limits_{\varphi\leq\psi}\gamma_\varphi^n\;+\;\sum_{l=1}^n\binom{n}{l}\int\limits_X\varepsilon^l\omega^l\wedge\gamma^{n-l}+3M^nN\delta\,.
\end{aligned}
$$

Then letting $\varepsilon\to0$ and $\delta\to0$ we get

$$
\int\limits_{\varphi<\psi}\gamma_\psi^n\;\leq\int\limits_{\varphi\leq\psi}\gamma_\varphi^n\,.
$$

Now the conclusion follows by replacing φ by $\varphi + t$, $t > 0$ in the previous formula and letting $t \to 0$. □

We recall now the following lemma due to Kołodziej [Kol1], (see also [Ti-Zhu1], [Ti-Zhu2]).

Lemma 2.7 *Let $a : (-\infty, 0] \to [0, 1]$, be a monotone non decreasing function such that for some $B > 0$, $\delta > 0$ the inequality*

$$t\, a(s) \leq B\, a(s + t)^{1+\delta}$$

holds for all $s \leq 0$, $t \in [0, 1]$, $s + t \leq 0$. Then for all $S < 0$ such that $a(S) > 0$ and all $D \in [0, 1]$, $S + D \leq 0$ we have the estimate

$$D \leq e(3 + 2/\delta)B\, a(S + D)^{\delta}\,.$$

The following lemma is a simple application of the main result in Bedford-Taylor [Be-Te] and of the monotone increasing convergence theorem in pluripotential theory.

Lemma 2.8 *Let X be a compact connected complex manifold of complex dimension n, let γ be a big closed positive $(1, 1)$-current with continuous local potentials and let $\Omega > 0$ be a smooth volume form. Then there exist constants $\alpha = \alpha(\gamma, \Omega) > 0$, $C = C(\gamma, \Omega) > 0$ such that for all Borel sets $E \subset X$ we have*

$$\int_E \Omega \leq e^{\alpha} C e^{-\alpha/\operatorname{Cap}_\gamma(E)^{1/n}}\,. \tag{2.7}$$

In particular $\operatorname{Cap}_\gamma(E) = 0$ implies $\int_E \Omega = 0$.

Proof. It is sufficient to prove this estimate for an arbitrary compact set. In fact assume (2.7) for compact sets and let $(K_j)_j$, $K_j \subset K_{j+1} \subset E$ be a family of compact sets such that $\int_{K_j} \Omega \to \int_E \Omega$ as $j \to +\infty$. Set $U := \cup_j K_j \subset E$ and take the limit in (2.7) with E replaced by K_j. By (2.1) we deduce

$$\int_E \Omega \leq e^{\alpha} C e^{-\alpha/\operatorname{Cap}_\gamma(U)^{1/n}} \leq e^{\alpha} C e^{-\alpha/\operatorname{Cap}_\gamma(E)^{1/n}}\,.$$

We prove now (2.7) for compact sets $K \subset X$. For this purpose, consider the function

$$\Psi_K(x) := \sup\{\varphi(x) \;\mid\; \varphi \in \mathcal{P}_\gamma\,,\, \varphi_{|K} \leq 0\} \geq 0\,.$$

Remark that $(\Psi_K)_{|K} = 0$ since $0 \in \mathcal{P}_\gamma$ by the positivity assumption on γ. Assume $\int_K \Omega \neq 0$, otherwise there is nothing to prove. In this case there exists a constant $C_K > 0$ such that $\sup_X \varphi \leq C_K$ for all $\varphi \in \mathcal{P}_\gamma\,,\, \varphi_{|K} \leq 0$.

In fact let $S_K := \{\varphi \in \mathcal{P}_\gamma \mid \varphi_{|K} \leq 0\}$ and set $\tilde{\varphi} := \varphi - \sup_X \varphi$. By contradiction we would get a sequence $\varphi_j \in S_K$ such that $\sup_X \varphi_j \to +\infty$. This implies $\sup_K \tilde{\varphi}_j \to -\infty$ and so $\int_K -\tilde{\varphi}_j \, \Omega \geq -(\int_K \Omega) \sup_K \tilde{\varphi}_j \to +\infty$, which contradicts the first integral estimate of lemma 2.2.
Then it follows from quite standard local arguments that the upper regularization $\Psi_K^* \in \mathcal{P}_\gamma$. (Here we use the assumption that the local potentials of γ are continuous.) Moreover $\Psi_K^* \in L^\infty(X)$, $\Psi_K^* \geq 0$ and $\Psi_K^* = 0$ over the interior K^0 of K. We recall now the following well known consequence of a result of Bedford and Taylor [Be-Te].

Theorem 2.9 *Let $\varphi \in \mathcal{P}_\gamma \cap L^\infty(X)$ and let B be an open coordinate ball. Then there exists $\hat{\varphi} \in \mathcal{P}_\gamma \cap L^\infty(X)$, $\hat{\varphi} \geq \varphi$ such that $\gamma_{\hat{\varphi}}^n = 0$ on B and $\hat{\varphi} = \varphi$ on $X \setminus B$. Moreover if $\varphi_1 \leq \varphi_2$, then $\hat{\varphi}_1 \leq \hat{\varphi}_2$.*

This implies the following quite standard fact in pluripotential theory.

Corollary 2.10 *The extremal function $\Psi_K^* \in \mathcal{P}_\gamma \cap L^\infty(X)$ satisfies $\Psi_K^* \geq 0$ over X, $\Psi_K^* = 0$ over the interior K^0 of K and $\gamma_{\Psi_K^*}^n = 0$ over $X \setminus K$.*

Proof. By the classical Choquet lemma there exists a sequence $(\varphi_j)_j \subset S_K$, $\varphi_j \geq 0$ such that $\Psi_K^* = (\sup_j \varphi_j)^*$. We can assume that this sequence is increasing. Otherwise, set $\tilde{\varphi}_1 := \varphi_1$ and $\tilde{\varphi}_j := \max\{\varphi_j, \tilde{\varphi}_{j-1}\} \in S_K$. Let B be an open coordinate ball in $X \setminus K$ and let $\hat{\varphi}_j \in S_K$ be a solution of the Dirichlet problem $\gamma_{\hat{\varphi}_j}^n = 0$ over B as in theorem 2.9. Thus the sequence $(\hat{\varphi}_j)_j \subset S_K$ is still increasing and $\Psi_K^* = (\sup_j \hat{\varphi}_j)^*$. Remember also that the plurisubharmonicity implies that $\Psi_K^* = \lim_j \hat{\varphi}_j$ almost everywhere. By the monotone increasing theorem from classical pluripotential theory, we infer $\gamma_{\Psi_K^*}^n = 0$ on B, and the conclusion follows from the fact that B is arbitrary. □

By using a basic fact about measure theory and the second integral estimate of lemma 2.2 we get

$$\int_K \Omega = \int_{K^0} \Omega = \int_{K^0} e^{-\alpha \, \Psi_K^*} \, \Omega \leq \int_X e^{-\alpha \, \Psi_K^*} \, \Omega \leq C e^{-\alpha \sup_X \Psi_K^*} .$$

Set $A_K := \sup_X \Psi_K^*$. If $A_K > 1$ set $\varphi := A_K^{-1} \Psi_K^*$. Then $0 \leq \gamma_{\Psi_K^*} \leq A_K \gamma_\varphi$ and so $\varphi \in \mathcal{P}_\gamma[0,1]$. By corollary 2.10 we deduce

$$\{\gamma\}^n A_K^{-n} = A_K^{-n} \int_K \gamma_{\Psi_K^*}^n \leq \int_K \gamma_\varphi^n \leq \{\gamma\}^n \, \mathrm{Cap}_\gamma(K) ,$$

thus $-\alpha A_K \leq -\alpha/\operatorname{Cap}_\gamma(K)^{1/n}$ by the bigness assumption on the current γ. If $A_K \leq 1$ then $\Psi_K^* \in \mathcal{P}_\gamma[0,1]$ and so

$$1 = \{\gamma\}^{-n} \int_K \gamma_{\Psi_K^*}^n \;\leq\; \operatorname{Cap}_\gamma(K) \leq \operatorname{Cap}_\gamma(X) = 1\,.$$

In both cases we reach the required conclusion. □

Proof of theorem 2.1, part A.
We can assume $\sup_X \psi = 0$. Let $U_s := \{\psi < s\}$, $s \leq 0$, $t \in [0,1]$, $s+t \leq 0$, $\varphi \in \mathcal{P}_\gamma[-1,0]$ and set $V := \{\psi - s - t < t\varphi\}$. Then we have inclusions $U_s \subset V \subset U_{s+t}$. By using the Degenerate Comparison Principle we infer

$$t^n \int_{U_s} \gamma_\varphi^n \;\leq \int_{U_s} \gamma_{t\varphi}^n \;\leq \int_V \gamma_{t\varphi}^n \;\leq \int_V \gamma_\psi^n \;\leq \int_{U_{s+t}} \gamma_\psi^n\,,$$

thus combining this with Hölder inequality in Orlicz spaces (7.3), formula (7.2) in Appendix A and lemma 2.8 we obtain

$$\begin{aligned}
t^n \operatorname{Cap}_\gamma(U_s) &\leq \{\gamma\}^{-n} \int_{U_{s+t}} \gamma_\psi^n \;=\; \{\gamma\}^{-n} \int_{U_{s+t}} f\,\Omega \\
&\leq \{\gamma\}^{-n} C_{\varepsilon_0} \|f\|_{L\log^{n+\varepsilon} L(X)} \cdot \|1\|_{\operatorname{Exp}^{\frac{1}{n+\varepsilon}} L(U_{s+t})} \\
&= \frac{\{\gamma\}^{-n} C_{\varepsilon_0} \|f\|_{L\log^{n+\varepsilon} L(X)}}{\log^{n+\varepsilon}(1 + 1/\operatorname{Vol}_\Omega(U_{s+t}))} \\
&\leq \frac{\{\gamma\}^{-n} C_{\varepsilon_0} \|f\|_{L\log^{n+\varepsilon} L(X)}}{\log^{n+\varepsilon}\left(1 + e^{-\alpha} C^{-1} e^{\alpha/\operatorname{Cap}_\gamma(U_{s+t})^{1/n}}\right)} \\
&\leq C_{\varepsilon_0} (k/\alpha)^{n+\varepsilon} \{\gamma\}^{-n} \|f\|_{L\log^{n+\varepsilon} L(X)} \operatorname{Cap}_\gamma(U_{s+t})^{(n+\varepsilon)/n}\,.
\end{aligned}$$

(Here $k > 0$ is a constant such that $k^{-1}\alpha/x \leq \log(1 + e^{-\alpha} C^{-1} e^{\alpha/x})$ for all $x \in (0,1]$). So if we set $\delta := \varepsilon/n$ and

$$B := C_{\varepsilon_0}^{1/n} (k/\alpha)^{1+\varepsilon/n} I_{\gamma,\varepsilon}(f)^{1/n}\,,$$

we deduce that the function $a(s) := \operatorname{Cap}_\gamma(U_s)^{1/n}$, $s \leq 0$, satisfies the hypothesis of lemma 2.7. (We use here the inequality (7.1) in appendix A.) Consider now the function $\kappa(t) := K_\delta B\, t^\delta$, with constant $K_\delta := e(3 + 2/\delta)$. Remember also the uniform capacity estimate $a(s) \leq C\,(-s)^{-1/n}$ of lemma 2.3. Let now $\eta > 1$ be arbitrary. We claim that $a(S_\eta) = 0$ for

$$-S_\eta = C^n (K_\delta B\,\eta)^{n/\delta} + 1\,.$$

The fact that the function a is left continuous (by formula (2.1)) will imply that $a(S_1) = 0$ also. Remark that S_η is a solution of the equation

$$C(-S_\eta - 1)^{-1/n} = \kappa^{-1}(\eta^{-1})\,,$$

where κ^{-1} is the inverse of the function κ. So if we assume by contradiction that $a(S_\eta) > 0$ we deduce by lemmas 2.7 and 2.3

$$1 \leq \kappa(a(S_\eta + 1)) \leq \kappa(C(-S_\eta - 1)^{-1/n}) = \eta^{-1} < 1\,,$$

which is a contradiction. Thus if we set $-I := \max\{s \leq 0 \mid a(s) = 0\}$ we obtain $I \leq -S_1 \leq C^n(K_\delta B)^{n/\delta} + 1$, which by arranging the coefficients yields the right hand side of the estimate in statement A of theorem 2.1. Moreover by definition $\mathrm{Cap}_\gamma(U_{-I}) = 0$, thus $\mathrm{Vol}_\Omega(U_{-I}) = 0$ by lemma 2.8. The fact that the current γ has continuous local potentials implies that the function ψ is upper semicontinuous, so the set U_{-I} is open, thus empty. This implies the required conclusion. □

Proof of part B.
Set $a := \max\{\|\varphi\|_{L^\infty}(X),\, \|\psi\|_{L^\infty}(X)\}$, consider $\theta \in \mathcal{P}_\gamma[0,1]$, $s \geq 0$, $t \in [0,1]$ and set

$$V := \left\{\varphi < \frac{t}{1+a}\,\theta + \left(1 - \frac{t}{1+a}\right)\psi - s - t\right\}\,.$$

Then the obvious inequality $0 \leq -\frac{t}{1+a}\psi \leq \frac{at}{1+a}$ implies the inclusions $\{\varphi - \psi < -s - t\} \subset V \subset \{\varphi - \psi < -s\}$. Thus by applying the Degenerate Comparison Principle as in [Kol2] we obtain

$$\begin{aligned}\frac{t^n}{(1+a)^n} \int\limits_{\varphi-\psi<-s-t} \gamma_\theta^n \;&\leq\; \int\limits_V \left[\frac{t}{1+a}\,\gamma_\theta + \left(1 - \frac{t}{1+a}\right)\gamma_\psi\right]^n \\ &\leq\; \int\limits_V \gamma_\varphi^n \;\leq\; \int\limits_{\varphi-\psi<-s} \gamma_\varphi^n\,.\end{aligned}$$

By inverting the roles of φ and ψ in the previous inequality and by summing up we get

$$\frac{t^n}{(1+a)^n} \int\limits_{|\varphi-\psi|>s+t} \gamma_\theta^n \;\leq\; \int\limits_{|\varphi-\psi|>s} (f+g)\,\Omega\,.$$

By taking the supremum over θ we obtain the capacity estimate

$$t^n\,\mathrm{Cap}_\gamma(|\varphi - \psi| > s + t) \leq (1+a)^n\{\gamma\}^{-n} \int\limits_{|\varphi-\psi|>s} (f+g)\,\Omega\,, \qquad (2.8)$$

for all $s \geq 0$, $t \in [0,1]$. Set $U_s := \{|\varphi - \psi| > s\} \subset X$. By combining lemma 2.8 with a computation similar to the one in the proof of part A we obtain

$$\begin{aligned} t^n \operatorname{Cap}_\gamma(U_{s+t}) &\leq (1+a)^n \{\gamma\}^{-n} C'_{\varepsilon_0} \|f+g\|_{L\log^{n+\varepsilon_0} L(X)} \operatorname{Cap}_\gamma(U_s)^{(n+\varepsilon_0)/n} \\ &\leq B^n \operatorname{Cap}_\gamma(U_s)^{(n+\varepsilon_0)/n}, \end{aligned}$$

where the constant $B > 0$ depends on the same quantities as the constant C_2 in statement (B) of theorem 2.1. We deduce that the function $a(s) := \operatorname{Cap}_\gamma(U_{-s})^{1/n}$, $s \leq 0$, satisfies the hypothesis of lemma 2.7 with $\delta = \varepsilon_0/n$. On the other hand, the capacity estimate (2.8) combined with Hölder's inequality in Orlicz spaces implies for all $t \in [0,1]$ the inequalities

$$\begin{aligned} t^n \operatorname{Cap}_\gamma(|\varphi - \psi| > 2t) &\leq (1+a)^n \{\gamma\}^{-n} \int\limits_{|\varphi-\psi|>t} (f+g)\,\Omega \\ &\leq \frac{(1+a)^n\{\gamma\}^{-n}}{t} \int\limits_X |\varphi-\psi|(f+g)\,\Omega \\ &\leq \frac{2(1+a)^n\{\gamma\}^{-n}}{t} \|\varphi-\psi\|_{\operatorname{Exp} L(X)} \|f+g\|_{L\log L(X)} \\ &\leq \frac{4K(1+a)^n}{t} \|\varphi-\psi\|_{\operatorname{Exp} L(X)}\,. \end{aligned} \tag{2.9}$$

Claim 2.11 *If $\|\varphi - \psi\|_{L^1(X)} \leq 1/2$, then there exists a constant $C_a > 0$ such that*

$$\|\varphi-\psi\|_{\operatorname{Exp} L(X)} \leq C_a / \log \|\varphi-\psi\|^{-1}_{L^1(X)}\,.$$

Proof. We assume $\|\varphi - \psi\|_{L^1(X)} > 0$, otherwise there is nothing to prove. Set $C_{k,a} := k(e^{2a/k} - 1)/(2a)$, $k > 0$. Then for all $k > 0$ and all $x \in [0, 2a/k]$ the inequality $e^x - 1 \leq C_{k,a}\, x$ holds. Thus the inequality $|\varphi - \psi|/k \leq 2a/k$ implies

$$\int\limits_X \left(e^{|\varphi-\psi|/k} - 1\right)\Omega \leq C_{k,a} \int\limits_X \frac{|\varphi-\psi|}{k}\,\Omega\,.$$

We get from there the implication

$$\|\varphi-\psi\|_{L^1(X)} = k/C_{k,a} \quad \Longrightarrow \quad \|\varphi-\psi\|_{\operatorname{Exp} L(X)} \leq k\,, \tag{2.10}$$

since by definition

$$\|\varphi-\psi\|_{\operatorname{Exp} L(X)} := \inf\left\{ k > 0 \;\Big|\; \int\limits_X \left(e^{|\varphi-\psi|/k} - 1\right)\Omega \leq 1 \right\}.$$

So if we set $\mu(k) := k/C_{k,a} > 0$ we deduce by the implication (2.10)

$$\|\varphi - \psi\|_{\operatorname{Exp} L(X)} \leq \mu^{-1}\left(\|\varphi - \psi\|_{L^1(X)}\right) , \tag{2.11}$$

where $\mu^{-1} : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is the inverse function of μ. Explicitly $\mu^{-1}(y) = 2a/\log(1 + 2a/y)$, for all $y > 0$. Now there exists a constant $C_a > 0$ such that $\mu^{-1}(y) \leq C_a/\log(1/y)$ for all $y \in (0, 1/2]$. This combined with (2.11) implies the conclusion. □

Combining claim 2.11 with the estimate (2.9) we infer the capacity estimate

$$a(-t) \leq \frac{C}{t^{1+1/n}} \left(\log \|\varphi - \psi\|^{-1}_{L^1(X)}\right)^{-1/n} , \tag{2.12}$$

where the constant $C > 0$ depends on the same quantities as the constant C_2 in statement B. Set now $C_2 := C^n(2K_\delta B)^{n/\delta} > 0$ (with $K_\delta > 0$ as in the proof of part (A) and define

$$t := C_2^{\alpha_0} \left(\log \|\varphi - \psi\|^{-1}_{L^1(X)}\right)^{-\alpha_0} .$$

The hypothesis $t \in (0, 1]$ combined with the hypothesis of claim 2.11 forces the condition $\|\varphi - \psi\|_{L^1(X)} \leq \min\{1/2, e^{-C_2}\}$. Moreover t is solution of the equation

$$\frac{C}{t^{1+1/n}} \left(\log \|\varphi - \psi\|^{-1}_{L^1(X)}\right)^{-1/n} = \kappa^{-1}\left(\frac{t}{2}\right) ,$$

where κ^{-1} is the inverse of the function κ introduced in the proof of part A. We claim that $a(-2t) = 0$. Otherwise, by lemma 2.7 and inequality (2.12), we infer

$$0 < t \leq \kappa(a(-t)) \leq \kappa(\kappa^{-1}(t/2)) = t/2 ,$$

which is absurd. We deduce $\operatorname{Vol}_\Omega(|\varphi - \psi| > 2t) = 0$ by lemma 2.8. We prove now that the set $U_{2t} = \{|\varphi - \psi| > 2t\} \subset X$ is empty, which will imply the desired C^0-stability estimate. The fact that $|\varphi - \psi| \leq 2t$ a.e over X, implies $|⨍_{B(x,r)}(\varphi - \psi)\, d\lambda| \leq 2t$ for all coordinate open balls $B(x, r) \subset X$. (The symbol $⨍_B$ means mean value operator.) By elementary properties of plurisubharmonic functions follows

$$\varphi(x) - \psi(x) = \lim_{r \to 0^+} ⨍_{B(x,r)} (\varphi - \psi)\, d\lambda ,$$

for all $x \in X$. We infer $|\varphi - \psi| \leq 2t$ over X. □

Corollary 2.12 *Let (X, ω) be a polarized compact connected Kähler manifold of complex dimension n, let $\Omega > 0$ be a smooth volume form, let $\gamma \geq 0$ be a big closed smooth $(1,1)$-form. Let also $f \in L\log^{n+\delta} L(X)$, $\delta > 0$, such*

that $\int_X \gamma^n = \int_X f\,\Omega$ *and* $(f_\varepsilon)_{\varepsilon>0} \subset C^\infty(X)$ *be a family converging to* f *in the* $L\log^{n+\delta} L(X)$*-norm as* $\varepsilon \to 0^+$ *and satisfying the integral condition*

$$\int_X (\gamma + \varepsilon\omega)^n = \int_X f_\varepsilon\,\Omega\,, \tag{2.13}$$

Let $\lambda \geq 0$ *be a real number. Then the unique solution of the non degenerate complex Monge-Ampère equation*

$$(\gamma + \varepsilon\omega + i\partial\bar\partial\psi_\varepsilon)^n = f_\varepsilon\, e^{\lambda\,\psi_\varepsilon}\Omega\,, \tag{2.14}$$

given by the Aubin-Yau solution of the Calabi conjecture (which in the case $\lambda = 0$ *is normalized by* $\max_X \psi_\varepsilon = 0$*) satisfies the uniform* C^0*-estimate* $\|\psi_\varepsilon\|_{C^0(X)} \leq C(\delta,\gamma,\Omega)\, I_{\gamma,\delta}(f)^{\frac{n}{\delta}} + 1$*.*

Proof. The existence of a regularizing family f_ε of f in $L\log^{n+\delta} L(X)$ follows from [Ra-Re] page 364 or [Iw-Ma], theorem 4.12.2, page 79. We can always assume the integral condition (2.13) otherwise we multiply f_ε by a constant $c_\varepsilon > 0$ which converges to 1 by the normalizing condition $\int_X \gamma^n = \int_X f\,\Omega$. We distinguish two cases.

Case $\lambda = 0$. The hypothesis (C1) and (C2a) of statement (C) of theorem 2.1 are obviously satisfied for the family $(\gamma + \varepsilon\omega)_\varepsilon$. We deduce that the constant $C_1 = C_1(\delta\,,\,\gamma + \varepsilon\omega\,,\,\Omega) > 0$ in the statement of theorem 2.1, A does not blow up as $\varepsilon \to 0^+$. Moreover the uniform estimate

$$\|f_\varepsilon\|_{L\log^{n+\delta} L(X)} \leq C'\|f\|_{L\log^{n+\delta} L(X)} =: K \tag{2.15}$$

holds for all $\varepsilon \in (0,1)$. Thus by theorem 2.1, A we obtain the required uniform estimate $\|\psi_\varepsilon\|_{C^0(X)} \leq C := C(\delta,\gamma,\Omega)\, I_{\gamma,\delta}(f)^{\frac{n}{\delta}} + 1$.

Case $\lambda > 0$. We start by proving the following lemma, which is a particular case of a more general result due to Yau (see [Yau], sect. 6, page 376).

Lemma 2.13 *Let* (X,ω) *be a polarized compact Kähler manifold of complex dimension* n*, let* h *be a smooth function such that* $\int_X \omega^n = \int_X e^h\omega^n$ *and let* $\varphi \in \mathcal{P}_\omega$ *be the unique solution of the complex Monge-Ampère equation*

$$(\omega + i\partial\bar\partial\varphi)^n = e^{h+\lambda\varphi}\omega^n\,, \tag{2.16}$$

$\lambda > 0$*. Consider also two solutions* φ'*,* $\varphi'' \in \mathcal{P}_\omega$ *of the complex Monge-Ampère equation* $(\omega + i\partial\bar\partial\psi)^n = e^h\omega^n$ *such that* $\min_X \varphi' = 0 = \max_X \varphi''$*. Then* $\varphi'' \leq \varphi \leq \varphi'$*.*

Proof. The argument is a simplification, in our particular case, of Yau's original argument for the proof of Theorem 4, sect. 6 in [Yau]. Set $\varphi'_0 := \varphi'$, $\varphi''_0 :=$

φ'' and consider the solutions φ'_j, φ''_j of the complex Monge-Ampère equations given by the iteration

$$(\omega + i\partial\bar\partial\varphi'_j)^n = e^{h+(\lambda+1)\varphi'_j-\varphi'_{j-1}}\,\omega^n\,, \tag{2.17}$$

$$(\omega + i\partial\bar\partial\varphi''_j)^n = e^{h+(\lambda+1)\varphi''_j-\varphi''_{j-1}}\,\omega^n\,. \tag{2.18}$$

Notice that we can solve these equations even if the terms $e^{h-\varphi'_{j-1}}$, $e^{h-\varphi''_{j-1}}$ are not normalized, see Lemma 2 page 378 in [Yau]. Set $L := \lambda + 1$ and consider

$$(\omega+i\partial\bar\partial\varphi'_1)^n = e^{h+L(\varphi'_1-\varphi'_0)+\lambda\varphi'_0}\omega^n \geq e^{L(\varphi'_1-\varphi'_0)}e^h\omega^n = e^{L(\varphi'_1-\varphi'_0)}(\omega+i\partial\bar\partial\varphi'_0)^n\,.$$

At a maximum point of $\varphi'_1 - \varphi'_0$ we have the inequality

$$(\omega + i\partial\bar\partial\varphi'_0)^n \geq (\omega + i\partial\bar\partial\varphi'_1)^n\,.$$

By plugging this into the previous one, we deduce $\varphi'_1 \leq \varphi'_0$. We now prove by induction the inequality $\varphi'_j \leq \varphi'_{j-1}$. In fact by dividing $(2.17)_j$ with $(2.17)_{j-1}$ we get

$$\frac{(\omega + i\partial\bar\partial\varphi'_j)^n}{(\omega + i\partial\bar\partial\varphi'_{j-1})^n} = e^{L(\varphi'_j-\varphi'_{j-1})-(\varphi'_{j-1}-\varphi'_{j-2})} \geq e^{L(\varphi'_j-\varphi'_{j-1})}\,.$$

At a maximum point of $\varphi'_j - \varphi'_{j-1}$ we find again the inequality

$$(\omega + i\partial\bar\partial\varphi'_j)^n \leq (\omega + i\partial\bar\partial\varphi'_{j-1})^n\,.$$

Combining this with the previous one we deduce $\varphi'_j \leq \varphi'_{j-1}$. By applying a quite similar argument to (2.18) we obtain also $\varphi''_{j-1} \leq \varphi''_j$. We also prove by induction the inequality $\varphi''_j \leq \varphi'_j$, which is true by definition in the case $j = 0$. By dividing $(2.17)_j$ with $(2.18)_j$ we get

$$\frac{(\omega + i\partial\bar\partial\varphi'_j)^n}{(\omega + i\partial\bar\partial\varphi''_j)^n} = e^{L(\varphi'_j-\varphi''_j)-(\varphi'_{j-1}-\varphi''_{j-1})} \leq e^{L(\varphi'_j-\varphi''_j)}\,,$$

by the induction hypothesis $\varphi''_{j-1} \leq \varphi'_{j-1}$. At a minimum point of $\varphi'_j - \varphi''_j$ we get

$$(\omega + i\partial\bar\partial\varphi'_j)^n \geq (\omega + i\partial\bar\partial\varphi''_j)^n\,,$$

hence $\varphi''_j \leq \varphi'_j$. As a conclusion, we have proved the sequence of inequalities

$$\varphi''_0 \leq \varphi''_{j-1} \leq \varphi''_j \leq \varphi'_j \leq \varphi'_{j-1} \leq \varphi'_0\,. \tag{2.19}$$

We now prove a uniform estimate for the Laplacian of the potentials φ'_j. The inequalities 2.19 imply $0 < 2n + \Delta_\omega\varphi'_j \leq C\,B_j$, where $B_j > 0$ satisfies the uniform estimate

$$0 \geq C_1\,B_j^{\frac{1}{n-1}} - \left(2n + \max_X \Delta_\omega\varphi'_{j-1}\right)B_j^{-1} - C_0\,, \tag{2.20}$$

$C_0, C_1 > 0$, which is obtained by applying the maximum principle in a similar way as in Yau's proof of the second order estimate for the solution of the Calabi conjecture [Yau]. (It can also be obtained by setting $\delta = l = h = 0$ in step (I.B) in the proof of theorem 6.1, see Appendix C. In the case $n = 1$ the uniform estimate $0 < 2n + \Delta_\omega \varphi'_j \leq C'$ follows immediately from the inequalities (2.19).) Fix now a constant $C_2 > 0$ such that the inequality

$$C_1 \, x^{1+\frac{1}{n-1}} \geq (C_0 + 2C)x - C_2 \, ,$$

hold for all $x \geq 0$. This implies by (2.20) the estimate

$$2(2n + \Delta_\omega \varphi'_j) \leq 2C \, B_j \leq \left(2n + \max_X \Delta_\omega \varphi'_{j-1} \right) + C_2 \, ,$$

thus

$$2n + \max_X \Delta_\omega \varphi'_j \leq 2^{-j} \left(2n + \max_X \Delta_\omega \varphi'_0 \right) + C_2 \, ,$$

by iteration. By taking the derivative in the Green Formula (see [Aub], Th. 4.13 page 108) we get the identity

$$d_x \varphi'_j = - \int_X d_x G_\omega(x, \cdot) \, \Delta_\omega \varphi'_j \, \omega^n \, ,$$

which implies the estimate $|\nabla_\omega \varphi'_j|_\omega \leq C_\omega \max_X \Delta_\omega \varphi'_j \leq K$. By applying the complex version of the Evans-Krylov theory [Ti2] we deduce the uniform estimate $\|\varphi'_j\|_{C^{2,\alpha}(X)} \leq K'$. This combined with (2.17) implies that the monotone sequence $(\varphi'_j)_j$ converges in the $C^{2,\alpha}$-topology to the unique solution φ of the complex Monge-Ampère equation (2.16). Then the conclusion follows from the inequalities (2.19). □

Consider now the solutions ψ'_ε, ψ''_ε, $\min_X \psi'_\varepsilon = 0 = \max_X \psi''_\varepsilon$ of the complex Monge-Ampère equation (2.14) for $\lambda = 0$. By applying lemma 2.13 we deduce $\psi''_\varepsilon \leq \psi_\varepsilon \leq \psi'_\varepsilon$ for all $\varepsilon > 0$. By the argument in the case $\lambda = 0$, we infer $\|\psi'_\varepsilon\|_{C^0(X)}$, $\|\psi''_\varepsilon\|_{C^0(X)} \leq C$, thus $\|\psi_\varepsilon\|_{C^0(X)} \leq C$. □

3 The domain of definition of the complex Monge-Ampère operator.

In the situation we have to consider, the relevant class of currents which can be used as the input of Monge-Ampère operators is defined as follows.

Definition 3.1 *On a complex manifold, we consider the class* BT *of closed positive* $(1,1)$*-currents* Θ *whose exterior products* Θ^k, $0 \leq k \leq n$, *can be defined inductively in the sense of Bedford-Taylor, namely, if* $\Theta = i\partial\bar{\partial}\psi$ *on any open set, then* $\psi\Theta^k$ *is locally of finite mass and* $\Theta^{k+1} = i\partial\bar{\partial}(\psi\Theta^k)$ *for* $k < n$.

Notice that the local finiteness of the mass of $\psi\Theta^k$ is independent of the choice of the psh potential ψ, and that this assumption allows indeed to compute inductively $i\partial\bar\partial(\psi\Theta^k)$ in the sense of currents. Now, if χ is a $(1,1)$-cohomology class, we set

$$\mathrm{BT}_\chi = \mathrm{BT} \cap \chi. \tag{3.1}$$

Let $\gamma \geq 0$ be a closed positive $(1,1)$-current with continuous local potentials. We define corresponding classes of potentials

$$\begin{aligned} \mathcal{P}\,\mathrm{BT}_\gamma &:= \left\{\varphi \in \mathcal{P}_\gamma \,|\, \gamma + i\partial\bar\partial\varphi \in \mathrm{BT}_{\{\gamma\}}\right\}, \\ \mathcal{P}\,\mathrm{BT}^0_\gamma &:= \{\varphi \in \mathcal{P}\,\mathrm{BT}_\gamma \mid \sup_X \varphi = 0\}. \end{aligned}$$

Let $\varphi \in \mathcal{P}\,\mathrm{BT}_\gamma$ with zero Lelong numbers. It is well known from the work of the first author work [Dem2] (which becomes drastically simple in this particular case), that there exists a family $(\varphi_\varepsilon)_{\varepsilon>0}$, $\varphi_\varepsilon \in \mathcal{P}_{\gamma+\varepsilon\omega} \cap C^\infty(X)$, such that $\varphi_\varepsilon \downarrow \varphi$ as $\varepsilon \downarrow 0^+$. In the case the Lelong numbers of φ are not zero we can chose $R > 0$ sufficiently big such that $0 \leq \gamma + R\omega + i\partial\bar\partial\varphi_\varepsilon$ for all $\varepsilon \in (0,1)$ and $\varphi_\varepsilon \downarrow \varphi$ as $\varepsilon \to 0^+$. We have the following crucial result.

Theorem 3.2 (Degenerate monotone convergence result).
Let (X,ω) be a polarized compact Kähler manifold of complex dimension n and let γ, T be closed positive $(1,1)$-currents with continuous (resp. bounded) local potentials. Then the following statements hold true.
(A) *For all $\varphi \in \mathcal{P}\,\mathrm{BT}_\gamma$, $\varphi \leq 0$ and $k,l \geq 0$, $k+l \leq n$, $k \leq n-1$*

$$\int_X -\varphi\,\gamma_\varphi^k \wedge T^l \wedge \omega^{n-k-l} < +\infty, \quad \text{and} \quad \gamma_\varphi^{k+1} \wedge T^l = T^l \wedge \gamma_\varphi^{k+1}.$$

(B) *Let $\varphi \in \mathcal{P}\,\mathrm{BT}_\gamma$, $\varphi \leq 0$ with zero Lelong numbers and $\varphi_\varepsilon \in \mathcal{P}_{\gamma+\varepsilon\omega} \cap C^\infty(X)$, such that $\varphi_\varepsilon \downarrow \varphi$ as $\varepsilon \to 0^+$. Then for all $k,l \geq 0$, $k+l \leq n$, $k \leq n-1$*

$$\varphi_\varepsilon\,(\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^k \wedge T^l \longrightarrow \varphi\,\gamma_\varphi^k \wedge T^l, \tag{3.2}$$

$$(\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k+1} \wedge T^l \longrightarrow \gamma_\varphi^{k+1} \wedge T^l, \tag{3.3}$$

weakly as $\varepsilon \to 0^+$.
(C) *Let $\varphi \in \mathcal{P}\,\mathrm{BT}_\gamma$, $\varphi \leq 0$ and $\varphi_\varepsilon \in \mathcal{P}_{\gamma+R\omega} \cap C^\infty(X)$ such that $\varphi_\varepsilon \downarrow \varphi$ as $\varepsilon \to 0^+$. Then for all $k,l \geq 0$, $k+l \leq n$, $k \leq n-1$*

$$\varphi_\varepsilon\,(\gamma_{\varphi_\varepsilon} + R\omega)^k \wedge T^l \longrightarrow \varphi\,(\gamma_\varphi + R\omega)^k \wedge T^l, \tag{3.4}$$

$$(\gamma_{\varphi_\varepsilon} + R\omega)^{k+1} \wedge T^l \longrightarrow (\gamma_\varphi + R\omega)^{k+1} \wedge T^l, \tag{3.5}$$

weakly as $\varepsilon \to 0^+$.

As follows immediately from the proof, the statement of this theorem still holds if we replace T^l with a product $T_1 \wedge .... \wedge T_l$, where the currents T_j have the same properties as T. As a matter of fact, we wrote the statement in the previous special case only for the sake of notation simplicity. However during the proof it is useful to consider that statements concerning terms involving T^l are still valid if we replace T^l with $\gamma^r \wedge T^{l-r}$.

Proof. The convergence statement (3.3) follows from (3.2) by using the weak continuity of the $i\partial\bar{\partial}$ operator. The argument for statement (B) is the same as for (C).

Proof of (A). We denote by $A_{k,l}$ the assertion (A) in the statement of the theorem for the relative indices (k, l). We prove statements $A_{k,l}$, $l = 0, ..., n-k$ by using an induction on $k = 0, ..., n-1$. We remark that claim 2.5 asserts statement (A) in full generality for $k = 0$. So we assume statement $A_{k-1,\bullet}$ and we prove $A_{k,l}$, $l = 0, ..., n-k$ by using an induction on l. We remark that $A_{k,0}$ hold by hypothesis $\varphi \in \mathcal{P}\,\mathrm{BT}_\gamma$. So we assume $A_{k,l}$ and we prove $A_{k,l+1}$. In fact let $\varphi_c := \max\{\varphi, c\} \in \mathcal{P}_\gamma$, $c \in \mathbb{R}_{<0}$. By the result of Greene-Wu [Gr-Wu] let $(\varphi_{c,\varepsilon})_{\varepsilon>0}$, $\varphi_{c,\varepsilon} \in \mathcal{P}_{\gamma+\varepsilon\omega} \cap C^\infty(X)$ such that $\varphi_{c,\varepsilon} \downarrow \varphi_c$ as $\varepsilon \to 0^+$ and write $T = \theta + i\partial\bar{\partial}u$, with θ smooth, $\theta \leq K\omega$ and u bounded with $\inf_X u = 0$. By using the monotone convergence theorem, the symmetry of the wedge product provided by the inductive hypothesis in

k and Stokes formula, we expand the integral

$$
\begin{aligned}
&\int_X -\varphi\,\gamma_\varphi^k \wedge T^{l+1} \wedge \omega^{n-k-l-1} \\
= \;& \lim_{c\to-\infty} \lim_{\varepsilon\to 0^+} \int_X -\varphi_{c,\varepsilon}\,\gamma_\varphi^k \wedge T^{l+1} \wedge \omega^{n-k-l-1} \\
= \;& \lim_{c\to-\infty} \lim_{\varepsilon\to 0^+} \int_X -\varphi_{c,\varepsilon}\,T^{l+1} \wedge \gamma_\varphi^k \wedge \omega^{n-k-l-1} \\
= \;& \lim_{c\to-\infty} \lim_{\varepsilon\to 0^+} \int_X -\varphi_{c,\varepsilon}\,\theta \wedge T^l \wedge \gamma_\varphi^k \wedge \omega^{n-k-l-1} \\
- \;& \lim_{c\to-\infty} \lim_{\varepsilon\to 0^+} \int_X \varphi_{c,\varepsilon}\,i\partial\bar\partial u \wedge T^l \wedge \gamma_\varphi^k \wedge \omega^{n-k-l-1} \\
\le \;& \lim_{c\to-\infty} \lim_{\varepsilon\to 0^+} \int_X -\varphi_{c,\varepsilon}\,\gamma_\varphi^k \wedge T^l \wedge K\omega^{n-k-l} \\
- \;& \lim_{c\to-\infty} \lim_{\varepsilon\to 0^+} \int_X u\,i\partial\bar\partial\varphi_{c,\varepsilon} \wedge \gamma_\varphi^k \wedge T^l \wedge \omega^{n-k-l-1} \\
= \;& K \int_X -\varphi\,\gamma_\varphi^k \wedge T^l \wedge \omega^{n-k-l} \\
- \;& \lim_{c\to-\infty} \lim_{\varepsilon\to 0^+} \int_X u\,(\gamma_{\varphi_{c,\varepsilon}} + \varepsilon\omega) \wedge \gamma_\varphi^k \wedge T^l \wedge \omega^{n-k-l-1} \\
+ \;& \lim_{c\to-\infty} \lim_{\varepsilon\to 0^+} \int_X u\,(\gamma + \varepsilon\omega) \wedge \gamma_\varphi^k \wedge T^l \wedge \omega^{n-l-1} \\
\le \;& K \int_X -\varphi\,\gamma_\varphi^k \wedge T^l \wedge \omega^{n-k-l} + \sup_X u \int_X \gamma \wedge \gamma_\varphi^k \wedge T^l \wedge \omega^{n-l-1} < +\infty\,,
\end{aligned}
$$

by the inductive hypothesis in l. We now prove the symmetry relation

$$
\gamma_\varphi^{k+1} \wedge T^l = T^l \wedge \gamma_\varphi^{k+1}\,. \tag{3.6}
$$

The decreasing monotone convergence theorem implies

$$
\lim_{c\to-\infty} \int_X (\varphi_c - \varphi)\,\gamma_\varphi^k \wedge T^l \wedge \omega^{n-k-l} = 0\,,
$$

which means the convergence of the mass $\|(\varphi_c - \varphi)\,\gamma_\varphi^k \wedge T^l\|_\omega(X) \to 0$ as $c \to -\infty$. In particular

$$\varphi_c\,\gamma_\varphi^k \wedge T^l \longrightarrow \varphi\,\gamma_\varphi^k \wedge T\,,$$

weakly as $c \to -\infty$. So by the weak continuity of the $i\partial\bar\partial$ operator we deduce

$$\gamma_{\varphi_c} \wedge \gamma_\varphi^k \wedge T^l \longrightarrow \gamma_\varphi^{k+1} \wedge T^l\,, \tag{3.7}$$

weakly as $c \to -\infty$. The symmetry of the wedge product provided by the inductive hypothesis in k implies

$$\gamma_{\varphi_c} \wedge \gamma_\varphi^k \wedge T^l \;=\; \gamma_{\varphi_c} \wedge T^l \wedge \gamma_\varphi^k \;=\; T^l \wedge \gamma_{\varphi_c} \wedge \gamma_\varphi^k$$

By the other hand $(3.7)_{k,0}$ combined with the weak continuity of the $i\partial\bar\partial$ operator implies, by an induction on l

$$T^l \wedge \gamma_{\varphi_c} \wedge \gamma_\varphi^k \longrightarrow T^l \wedge \gamma_\varphi^{k+1}\,,$$

weakly as $c \to -\infty$. This combined with (3.7) implies the required symmetry (3.6).

Proof of (B). For all $k = 0, ..., n-1$ and $l = 0, ..., n-k$ we consider the following statement $B_{k,l}$: for all $p = 0, ..., k$

$$\varphi_\varepsilon\,\gamma_\varphi^p \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p} \wedge T^l \longrightarrow \varphi\,\gamma_\varphi^k \wedge T^l\,, \tag{3.8}$$

$$i\partial\bar\partial\varphi_\varepsilon \wedge \gamma_\varphi^p \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p} \wedge T^l \longrightarrow i\partial\bar\partial\varphi \wedge \gamma_\varphi^k \wedge T^l\,, \tag{3.9}$$

$$\gamma_\varphi^p \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p+1} \wedge T^l \longrightarrow \gamma_\varphi^{k+1} \wedge T^l\,, \tag{3.10}$$

$$\varphi\,\gamma_\varphi^p \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p} \wedge T^l \longrightarrow \varphi\,\gamma_\varphi^k \wedge T^l\,, \tag{3.11}$$

weakly as $\varepsilon \to 0^+$. We remark that (3.9) follows from (3.8) by the weak continuity of the $i\partial\bar\partial$ operator. By combining (3.9) with the weak continuity of the $i\partial\bar\partial$ operator we obtain

$$(\gamma_{\varphi_\varepsilon} + \varepsilon\omega) \wedge \gamma_\varphi^p \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p} \wedge T^l \longrightarrow \gamma_\varphi^{k+1} \wedge T^l\,,$$

weakly as $\varepsilon \to 0^+$. On the other hand the symmetry of the wedge product proved in part (A) of the theorem implies

$$\begin{aligned}(\gamma_{\varphi_\varepsilon} + \varepsilon\omega) \wedge \gamma_\varphi^p \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p} \wedge T^l \;&=\; (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p+1} \wedge T^l \wedge \gamma_\varphi^p \\ &=\; \gamma_\varphi^p \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p+1} \wedge T^l\,.\end{aligned}$$

In this way we deduce (3.10). The statements $B_{0,\bullet}$ are true by the proof of claim 2.5. We now prove by induction on $k = 0, ..., n-1$ that statements $B_{k,l}$, $l = 0, ..., n-k$ hold true. In fact we prove the following claim.

Claim 3.3 *If $B_{j,\bullet}$ hold true for all $j = 0, ..., k-1$, then $B_{k,l}$ hold also true for all $l = 0, ..., n-k$.*

As pointed out before in order to prove $B_{k,l}$ is sufficient to show (3.8) and (3.11). The proof of (3.11) is quite similar to the proof of (3.8) that we now explain. We first prove by induction on $s = 0, ..., k-p$ the inequality

$$\begin{aligned}
&\int_X -\varphi_\varepsilon\, \gamma_\varphi^p \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p} \wedge T^l \wedge \omega^{n-k-l} \\
\leq\ &\int_X -\varphi\, \gamma_\varphi^{p+s} \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p-s} \wedge T^l \wedge \omega^{n-k-l} \\
+\ &\sum_{r=0}^{s-1} \int_X (\varphi_\varepsilon - \varphi)\, \gamma_\varphi^{p+r} \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p-r-1} \wedge \gamma \wedge T^l \wedge \omega^{n-k-l} \\
-\ &\sum_{r=0}^{s-1} \int_X \varepsilon\varphi\, \gamma_\varphi^{p+r} \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p-r-1} \wedge T^l \wedge \omega^{n-k-l+1}. \qquad (3.12)
\end{aligned}$$

Inequality (3.12) is obviously true for $s = 0$. (Here we adopt the usual convention of neglecting a sum when it runs over an empty set of indices.) Before proceding to the proof of the inequality (3.12), we need to point out two useful remarks.

1) Let α be a smooth closed real (q,q)-form, R be a closed positive (r,r)-current, $v \geq 0$ be a measurable function such that $\int_X vR \wedge \omega^{n-r} < +\infty$. This implies that the currents $i\partial\bar\partial v \wedge R := i\partial\bar\partial(v\, R)$ and $i\partial\bar\partial v \wedge \alpha \wedge R := i\partial\bar\partial(v\alpha \wedge R)$ are well defined. Then the Leibnitz formula implies

$$\alpha \wedge i\partial\bar\partial v \wedge R = i\partial\bar\partial v \wedge \alpha \wedge R\,. \qquad (3.13)$$

2) Thanks to part (A) of the theorem we have

$$\int_X -\varphi\, \gamma_\varphi^{p+r} \wedge \gamma^h \wedge T^l \wedge \omega^{n-p-r-h-l} < +\infty$$

for all $h = 0, ..., k-p-r-1$. By (3.13) this implies

$$\int_X -\varphi\, \gamma_\varphi^{p+r} \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p-r-1} \wedge T^l \wedge \omega^{n-k-l+1} < +\infty\,,$$

so the current

$$S := \varphi\, \gamma_\varphi^{p+r} \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p-r-1} \wedge T^l$$

is well defined and we can define the current

$$i\partial\bar\partial\varphi \wedge \gamma_\varphi^{p+r} \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p-r-1} \wedge T^l := i\partial\bar\partial S\,.$$

Then the integration by parts formula

$$\int_X i\partial\bar\partial\varphi_\varepsilon \wedge S \wedge \omega^{n-k-l} \;=\; \int_X \varphi_\varepsilon\, i\partial\bar\partial S \wedge \omega^{n-k-l}$$

can be written explicitly as

$$\begin{aligned} &\int_X i\partial\bar\partial\varphi_\varepsilon \wedge \varphi\, \gamma_\varphi^{p+r} \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p-r-1} \wedge T^l \wedge \omega^{n-k-l} \\ = \;&\int_X \varphi_\varepsilon\, i\partial\bar\partial\varphi \wedge \gamma_\varphi^{p+r} \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p-r-1} \wedge T^l \wedge \omega^{n-k-l}\,. \quad (3.14) \end{aligned}$$

We suppose now the inequality (3.12) true for s and we prove it for $s+1$. We start by expanding, thanks to formula (3.13), the integral

$$\begin{aligned} I \;:=\; &\int_X -\varphi\, \gamma_\varphi^{p+s} \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p-s} \wedge T^l \wedge \omega^{n-k-l} \\ = \;&\int_X -\varphi\, \gamma_\varphi^{p+s} \wedge (\gamma + \varepsilon\omega) \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p-s-1} \wedge T^l \wedge \omega^{n-k-l} \\ + \;&\int_X -\varphi\, \gamma_\varphi^{p+s} \wedge i\partial\bar\partial\varphi_\varepsilon \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p-s-1} \wedge T^l \wedge \omega^{n-k-l} \\ = \;&\int_X -\varepsilon\varphi\, \gamma_\varphi^{p+s} \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p-s-1} \wedge T^l \wedge \omega^{n-k-l+1} \\ - \;&\int_X \varphi\, \gamma_\varphi^{p+s} \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p-s-1} \wedge \gamma \wedge T^l \wedge \omega^{n-k-l} \\ - \;&\int_X i\partial\bar\partial\varphi_\varepsilon \wedge \varphi\, \gamma_\varphi^{p+s} \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p-s-1} \wedge T^l \wedge \omega^{n-k-l}\,. \end{aligned}$$

By applying the integration by parts formula (3.14) to the last integral we deduce

$$\begin{aligned} I &= \int_X -\varphi_\varepsilon\, \gamma_\varphi^{p+s+1} \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p-s-1} \wedge T^l \wedge \omega^{n-k-l} \\ &+ \int_X \varphi_\varepsilon\, \gamma \wedge \gamma_\varphi^{p+s} \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p-s-1} \wedge T^l \wedge \omega^{n-k-l} \\ &- \int_X \varphi\, \gamma_\varphi^{p+s} \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p-s-1} \wedge \gamma \wedge T^l \wedge \omega^{n-k-l} \\ &- \int_X \varepsilon\varphi\, \gamma_\varphi^{p+s} \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p-s-1} \wedge T^l \wedge \omega^{n-k-l+1}\,. \end{aligned}$$

By combining the symmetry of the wedge product proved in part (A) with formula (3.13) we get

$$\begin{aligned} \gamma \wedge \gamma_\varphi^{p+s} \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p-s-1} \wedge T^l &= \gamma \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p-s-1} \wedge T^l \wedge \gamma_\varphi^{p+s} \\ &= (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p-s-1} \wedge \gamma \wedge T^l \wedge \gamma_\varphi^{p+s} \\ &= \gamma_\varphi^{p+s} \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p-s-1} \wedge \gamma \wedge T^l\,. \end{aligned}$$

By plugging this into the previous expression of I we obtain

$$\begin{aligned} I &= \int_X -\varphi_\varepsilon\, \gamma_\varphi^{p+s+1} \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p-s-1} \wedge T^l \wedge \omega^{n-k-l} \\ &+ \int_X (\varphi_\varepsilon - \varphi)\, \gamma_\varphi^{p+s} \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p-s-1} \wedge \gamma \wedge T^l \wedge \omega^{n-k-l} \\ &- \int_X \varepsilon\varphi\, \gamma_\varphi^{p+s} \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p-s-1} \wedge T^l \wedge \omega^{n-k-l+1}\,, \end{aligned}$$

which implies inequality (3.12) for $s+1$. For $s = k-p$ the inequality (3.12) rewrites as

$$\begin{aligned} &\int_X -\varphi_\varepsilon\, \gamma_\varphi^{p} \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p} \wedge T^l \wedge \omega^{n-k-l} \;\leq\; \int_X -\varphi\, \gamma_\varphi^{k} \wedge T^l \wedge \omega^{n-k-l} \\ &+ \sum_{r=p}^{k-1} \int_X (\varphi_\varepsilon - \varphi)\, \gamma_\varphi^{r} \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-r-1} \wedge \gamma \wedge T^l \wedge \omega^{n-k-l} \\ &- \sum_{r=p}^{k-1} \int_X \varepsilon\varphi\, \gamma_\varphi^{r} \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-r-1} \wedge T^l \wedge \omega^{n-k-l+1}\,. \end{aligned}$$

By using the convergence inductive hypothesis $(3.8)_{j,\bullet}$, $(3.11)_{j,\bullet}$ in $B_{j,\bullet}$ for $j \leq k-1$ we deduce

$$\limsup_{\varepsilon\to 0^+} \int_X -\varphi_\varepsilon\, \gamma_\varphi^p \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p} \wedge T^l \wedge \omega^{n-k-l}$$

$$\leq \int_X -\varphi\, \gamma_\varphi^k \wedge T^l \wedge \omega^{n-k-l} < +\infty\,, \tag{3.15}$$

by statement A. (We can always arrange $\varphi_\varepsilon \leq 0$ for all $\varepsilon \in (0,1)$ by changing φ into $\varphi - C$.) Thus by weak compactness of the mass there exists a sequence $(\varepsilon_j)_j$, $\varepsilon_j \downarrow 0^+$ and a current of order zero $\Theta \in \mathcal{D}'_{n-k-l,n-k-l}(X)$ such that

$$\varphi_{\varepsilon_j}\, \gamma_\varphi^p \wedge (\gamma_{\varphi_{\varepsilon_j}} + \varepsilon_j\, \omega)^{k-p} \wedge T^l \longrightarrow \Theta\,,$$

weakly as $j \to +\infty$. So for any smooth and strongly positive form α of bidegree $(n-k-l, n-k-l)$, we have

$$\varphi_{\varepsilon_j}\, \gamma_\varphi^p \wedge (\gamma_{\varphi_{\varepsilon_j}} + \varepsilon_j\, \omega)^{k-p} \wedge T^l \wedge \alpha \longrightarrow \Theta \wedge \alpha\,,$$

weakly as $j \to +\infty$. The fact that $\varphi_{\varepsilon_j} \downarrow \varphi$ and

$$\gamma_\varphi^p \wedge (\gamma_{\varphi_{\varepsilon_j}} + \varepsilon_j\, \omega)^{k-p} \wedge T^l \wedge \alpha \longrightarrow \gamma_\varphi^k \wedge T^l \wedge \alpha\,,$$

weakly as $j \to +\infty$, by the convergence inductive hypothesis $(3.10)_{k-1,l}$, implies

$$\Theta \wedge \alpha \leq \varphi\, \gamma_\varphi^k \wedge T^l \wedge \alpha\,,$$

thanks to lemma (3.9), page 189 in [Dem1]. Thus $\Theta \leq \varphi\, \gamma_\varphi^k \wedge T^l$. Combining this with the inequality (3.15) we obtain

$$\begin{aligned}
\int_X \Theta \wedge \omega^{n-k-l} &\leq \int_X \varphi\, \gamma_\varphi^k \wedge T^l \wedge \omega^{n-k-l} \\
&\leq \liminf_{\varepsilon\to 0^+} \int_X \varphi_\varepsilon\, \gamma_\varphi^p \wedge (\gamma_{\varphi_\varepsilon} + \varepsilon\omega)^{k-p} \wedge T^l \wedge \omega^{n-k-l} \\
&\leq \lim_{j\to+\infty} \int_X \varphi_{\varepsilon_j}\, \gamma_\varphi^p \wedge (\gamma_{\varphi_{\varepsilon_j}} + \varepsilon_j\, \omega)^{k-p} \wedge T^l \wedge \omega^{n-k-l} \\
&= \int_X \Theta \wedge \omega^{n-k-l}\,.
\end{aligned}$$

We deduce $\mathrm{Tr}_\omega(\varphi\, \gamma_\varphi^k \wedge T^l - \Theta) = 0$, which implies $\varphi\, \gamma_\varphi^k \wedge T^l = \Theta$ since $0 \leq \varphi\, \gamma_\varphi^k \wedge T^l - \Theta$. This proves statement $B_{k,l}$. □

We introduce also the subsets

$$\hat{\mathcal{P}}\,\mathrm{BT}_\gamma := \left\{\varphi \in \mathcal{P}\,\mathrm{BT}^0_\gamma \mid \int_X -\varphi\,\gamma^n_\varphi < +\infty\right\} + \mathbb{R} \subset \mathcal{P}\,\mathrm{BT}_\gamma\,,$$

$$\hat{\mathcal{P}}\,\mathrm{BT}^0_\gamma := \{\varphi \in \hat{\mathcal{P}}\,\mathrm{BT}_\gamma \mid \sup_X \varphi = 0\}\,.$$

Without changes in the proof of theorem 3.2 we get the following corollary.

Corollary 3.4 *For all $\varphi \in \hat{\mathcal{P}}\,\mathrm{BT}_\gamma$, $\varphi \leq 0$, the assertions A), B) and C) of theorem 3.2 hold for all $k = 0, ..., n$.*

Let now Θ be a closed positive $(n-1, n-1)$-current and consider the L^2-space

$$L^2(X,\Theta) := \left\{\alpha \in \Gamma(X, \Lambda^{1,0}T^*_X) \mid \int_X i\alpha \wedge \bar{\alpha} \wedge \Theta < +\infty\right\} \Big/_{\Theta-a.e}\,,$$

equipped with the hermitian product $\langle \alpha, \beta\rangle_\Theta := \int_X i\alpha \wedge \bar{\beta} \wedge \Theta$, which is well defined by the polarization identity. The Θ-almost everywhere equality relation is defined by: $\alpha \sim \beta$ iff

$$\int_X i(\alpha - \beta) \wedge \overline{(\alpha - \beta)} \wedge \Theta = 0\,.$$

Let α_k, $\alpha \in L^2(X,\Theta)$. We say that the sequence α_k converges $L^2(X,\Theta)$-weakly to α if

$$\int_X i\alpha \wedge \bar{\beta} \wedge \Theta = \lim_{k\to+\infty} \int_X i\alpha_k \wedge \bar{\beta} \wedge \Theta\,,$$

for all $\beta \in L^2(X,\Theta)$. Let $\varphi \in \mathcal{P}^0_\gamma$ such that $\int_X -\varphi\,\Theta \wedge \omega < +\infty$. Then one can define $\partial\varphi \wedge \Theta := \partial(\varphi\Theta)$. We write $\partial\varphi \in L^2(X,\Theta)$ if there exists $\alpha \in L^2(X,\Theta)$ such that $\partial(\varphi\Theta) = \alpha \wedge \Theta$ in the sense of currents. In this case we write

$$\int_X i\partial\varphi \wedge \bar{\partial}\varphi \wedge \Theta := \int_X i\alpha \wedge \bar{\alpha} \wedge \Theta\,.$$

With these notations we have the following corollary.

Corollary 3.5 *Let (X,ω) be a polarized compact Kähler manifold of complex dimension n and let γ, T be closed positive $(1,1)$-currents with respectively continuous, bounded local potentials, let Θ be a closed positive*

$(n-1,n-1)$*-current and consider* $\varphi \in \hat{\mathcal{P}}\,\mathrm{BT}_\gamma$, $\varphi \leq 0$, $\psi \in \mathcal{P}_\gamma \cap L^\infty(X)$, $\psi \leq 0$. *Then for all* $k,l \geq 0$, $k+l \leq n-1$,

$$\int_X i\partial\varphi \wedge \bar{\partial}\varphi \wedge \gamma_\varphi^k \wedge T^l \wedge \omega^{n-k-l-1} < +\infty\,, \tag{3.16}$$

$$\int_X i\partial\psi \wedge \bar{\partial}\psi \wedge \Theta < +\infty\,. \tag{3.17}$$

Moreover let $(\varphi_\varepsilon)_{\varepsilon>0}$, $(\psi_\varepsilon)_{\varepsilon>0} \subset C^\infty(X)$, $\varphi_\varepsilon \in \mathcal{P}_{\gamma+R\omega}$, $\psi_\varepsilon \in \mathcal{P}_{\gamma+\varepsilon\omega}$ *such that* $\varphi_\varepsilon \downarrow \varphi$, $\psi_\varepsilon \downarrow \psi$ *as* $\varepsilon \to 0^+$. *Then*

$$\lim_{\varepsilon\to 0^+} \int_X i\partial(\varphi_\varepsilon - \varphi) \wedge \bar{\partial}(\varphi_\varepsilon - \varphi) \wedge \gamma_\varphi^k \wedge T^l \wedge \omega^{n-k-l-1} = 0\,, \tag{3.18}$$

$$\lim_{\varepsilon\to 0^+} \int_X i\partial(\psi_\varepsilon - \psi) \wedge \bar{\partial}(\psi_\varepsilon - \psi) \wedge \Theta = 0\,. \tag{3.19}$$

Proof. By integrating by parts we obtain

$$\begin{aligned}
&\int_X i\partial\varphi_\varepsilon \wedge \bar{\partial}\varphi_\varepsilon \wedge \gamma_\varphi^k \wedge T^l \wedge \omega^{n-k-l-1} \\
&= -\int_X \varphi_\varepsilon\, i\partial\bar{\partial}\varphi_\varepsilon \wedge \gamma_\varphi^k \wedge T^l \wedge \omega^{n-k-l-1} \\
&= \int_X \varphi_\varepsilon\,(\gamma + R\omega) \wedge \gamma_\varphi^k \wedge T^l \wedge \omega^{n-k-l-1} \\
&- \int_X \varphi_\varepsilon\,(\gamma_{\varphi_\varepsilon} + R\omega) \wedge \gamma_\varphi^k \wedge T^l \wedge \omega^{n-k-l-1}\,.
\end{aligned}$$

By the proof of theorem 3.2, B we can take the limit, so

$$\begin{aligned}
0 &\leq \lim_{\varepsilon\to 0^+} \int_X i\partial\varphi_\varepsilon \wedge \bar{\partial}\varphi_\varepsilon \wedge \gamma_\varphi^k \wedge T^l \wedge \omega^{n-k-l-1} \\
&= \int_X \varphi\,(\gamma - \gamma_\varphi) \wedge \gamma_\varphi^k \wedge T^l \wedge \omega^{n-k-l-1} < +\infty\,.
\end{aligned} \tag{3.20}$$

On the other hand the weak convergence of the sequence

$$\varphi_\varepsilon\,\gamma_\varphi^k \wedge T^l \wedge \omega^{n-k-l-1} \longrightarrow \varphi\,\gamma_\varphi^k \wedge T^l \wedge \omega^{n-k-l-1}\,,$$

combined with the weak continuity of the ∂ operator implies

$$\partial\varphi_\varepsilon \wedge \gamma_\varphi^k \wedge T^l \wedge \omega^{n-k-l-1} \longrightarrow \partial\varphi \wedge \gamma_\varphi^k \wedge T^l \wedge \omega^{n-k-l-1}\,,$$

weakly as $\varepsilon \to 0^+$. Then the $L^2(X, \gamma_\varphi^k \wedge T^l \wedge \omega^{n-k-l-1})$-weak compactness provided by (3.20) implies (3.16) and the $L^2(X, \gamma_\varphi^k \wedge T^l \wedge \omega^{n-k-l-1})$-weak convergence $\partial\varphi_\varepsilon \to \partial\varphi$ as $\varepsilon \to 0^+$. This implies

$$\begin{aligned}
&\int_X i\partial\varphi \wedge \bar\partial\varphi \wedge \gamma_\varphi^k \wedge T^l \wedge \omega^{n-k-l-1} \\
&= \lim_{\varepsilon\to 0^+} \int_X i\partial\varphi_\varepsilon \wedge \bar\partial\varphi \wedge \gamma_\varphi^k \wedge T^l \wedge \omega^{n-k-l-1} \\
&= \lim_{\varepsilon\to 0^+} \int_X -\varphi_\varepsilon\, i\partial\bar\partial\varphi \wedge \gamma_\varphi^k \wedge T^l \wedge \omega^{n-k-l-1} \\
&= \lim_{\varepsilon\to 0^+} \int_X -\varphi_\varepsilon\, (\gamma - \gamma_\varphi) \wedge \gamma_\varphi^k \wedge T^l \wedge \omega^{n-k-l-1} \\
&= \int_X -\varphi\, (\gamma - \gamma_\varphi) \wedge \gamma_\varphi^k \wedge T^l \wedge \omega^{n-k-l-1} \\
&= \lim_{\varepsilon\to 0^+} \int_X i\partial\varphi_\varepsilon \wedge \bar\partial\varphi_\varepsilon \wedge \gamma_\varphi^k \wedge T^l \wedge \omega^{n-k-l-1}\,,
\end{aligned}$$

by identity (3.20). This implies (3.18) by elementary facts about Hilbert spaces. The proof of (3.17) and (3.19) is quite similar. □

The conclusion of the corollary 3.5 still holds true if we replace the current $\gamma_\varphi^k \wedge T^l \wedge \omega^{n-k-l-1}$ with a sum of currents

$$\Xi := \sum_{k+l\le n-1} C_{k,l}\, \gamma_\varphi^k \wedge T^l \wedge \omega^{n-k-l-1}\,,$$

where $C_{k,l} \in \mathbb{R}$ such that $\Xi \ge 0$. We infer the linearity formula

$$\int_X i\partial\varphi \wedge \bar\partial\varphi \wedge \Xi \;=\; \sum_{k+l\le n-1} C_{k,l} \int_X i\partial\varphi \wedge \bar\partial\varphi \wedge \gamma_\varphi^k \wedge T^l \wedge \omega^{n-k-l-1}\,. \quad (3.21)$$

4 Uniqueness of the solutions.

Theorem 4.1 *Let X be a compact connected Kähler manifold of complex dimension n, let γ be a closed positive $(1,1)$-current with continuous local potentials, let θ be a continuous closed positive $(1,1)$-form such that $\{\theta^n = 0\}$ is a set of measure zero and $\gamma \ge \theta$.*
(A) *Let $\psi \in \mathcal{P}_\gamma^0 \cap L^\infty(X)$ and $\varphi \in \mathcal{P}\,\mathrm{BT}_\gamma^0$ such that*

$$(\gamma + i\partial\bar\partial\psi)^n = (\gamma + i\partial\bar\partial\varphi)^n\,.$$

Then $\psi = \varphi$.
(B) *Let* $\psi, \varphi \in \mathcal{P}_\gamma \cap L^\infty(X)$ *such that*

$$e^{-\lambda\psi}(\gamma + i\partial\bar\partial\psi)^n = e^{-\lambda\varphi}(\gamma + i\partial\bar\partial\varphi)^n\,.$$

with $\lambda > 0$. *Then* $\psi = \varphi$.

Proof of A. The identity $\gamma_\varphi^n = \gamma_\psi^n$ implies $\varphi \in \hat{\mathcal{P}}\,\mathrm{BT}_\gamma^0$ by claim 2.5. Let φ_ε, ψ_ε be as in the statement of corollary 3.5 and set $u := \psi - \varphi$, $u_\varepsilon := \psi_\varepsilon - \varphi_\varepsilon$. Let us also recall the formula

$$\alpha^k - \beta^k = (\alpha - \beta) \wedge \sum_{l=0}^{k-1} \alpha^l \wedge \beta^{k-l-1}\,.$$

From this we deduce

$$\begin{aligned}
0 &= \int_X -u(\gamma_\psi^n - \gamma_\varphi^n) = \lim_{\varepsilon\to 0^+} \int_X -u_\varepsilon(\gamma_\psi^n - \gamma_\varphi^n) \\
&= \lim_{\varepsilon\to 0^+} \sum_{l=0}^{n-1} \int_X -u_\varepsilon\, i\partial\bar\partial u \wedge \gamma_\psi^l \wedge \gamma_\varphi^{n-l-1} \\
&= \lim_{\varepsilon\to 0^+} \sum_{l=0}^{n-1} \int_X i\partial u_\varepsilon \wedge \bar\partial u \wedge \gamma_\psi^l \wedge \gamma_\varphi^{n-l-1} \\
&= \sum_{l=0}^{n-1} \int_X i\partial u \wedge \bar\partial u \wedge \gamma_\psi^l \wedge \gamma_\varphi^{n-l-1} =: I\,, \qquad (4.1)
\end{aligned}$$

since $\partial u_\varepsilon \to \partial u$ in $L^2(X, \gamma_\psi^l \wedge \gamma_\varphi^{n-l-1})$ by corollary 3.5. Inspired by an idea of S. Błocki [Blo1], we will prove by induction on $k = 0, ..., n-1$ that

$$\int_X i\partial u \wedge \bar\partial u \wedge \gamma_\psi^r \wedge \gamma_\varphi^s \wedge \gamma^k = 0 \qquad (4.2)$$

for all $r, s \geq 0$, $r + s = n - k - 1$. For $k = 0$ this follows from (4.1). So we assume (4.2) for $k-1$ and we prove it for k. In fact consider the identity

$$\gamma^k = \gamma_\psi^k - i\partial\bar\partial\psi \wedge \sum_{l=0}^{k-1} \gamma_\psi^l \wedge \gamma^{k-l-1} \quad \text{and set} \quad \Xi := \gamma_\psi^r \wedge \gamma_\varphi^s \wedge \sum_{l=0}^{k-1} \gamma_\psi^l \wedge \gamma^{k-l-1}\,.$$

By applying several times corollary 3.5 and by integrating by parts we derive

$$\begin{aligned}
\int_X i\partial u\wedge\bar\partial u\wedge\gamma_\psi^r\wedge\gamma_\varphi^s\wedge\gamma^k &= \lim_{\varepsilon\to 0^+}\int_X i\partial u_\varepsilon\wedge\bar\partial u\wedge\gamma_\psi^r\wedge\gamma_\varphi^s\wedge\gamma^k\\
&= \lim_{\varepsilon\to 0^+}\left[\int_X i\partial u_\varepsilon\wedge\bar\partial(u\gamma_\psi^{r+k}\wedge\gamma_\varphi^s) - \int_X i\partial u_\varepsilon\wedge\bar\partial(u\, i\partial\bar\partial\psi\wedge\Xi)\right]\\
&= \lim_{\varepsilon\to 0^+}\left[\int_X i\partial u_\varepsilon\wedge\bar\partial u\wedge\gamma_\psi^{r+k}\wedge\gamma_\varphi^s + \int_X u_\varepsilon\, i\partial\bar\partial u\wedge i\partial\bar\partial\psi\wedge\Xi\right]\\
&= \int_X i\partial u\wedge\bar\partial u\wedge\gamma_\psi^{r+k}\wedge\gamma_\varphi^s - \lim_{\varepsilon\to 0^+}\int_X u_\varepsilon\, i\partial\bar\partial\psi\wedge(\gamma_\varphi-\gamma_\psi)\wedge\Xi\\
&\le I + \lim_{\varepsilon\to 0^+}\int_X i\partial u_\varepsilon\wedge\bar\partial\,[\,\psi\,(\gamma_\varphi-\gamma_\psi)\wedge\Xi\,]
\end{aligned}$$

$$\begin{aligned}
&= \lim_{\varepsilon\to 0^+}\left[\int_X i\partial u_\varepsilon\wedge\bar\partial\psi\wedge\gamma_\varphi\wedge\Xi - \int_X i\partial u_\varepsilon\wedge\bar\partial\psi\wedge\gamma_\psi\wedge\Xi\right]\\
&= \int_X i\partial u\wedge\bar\partial\psi\wedge\gamma_\varphi\wedge\Xi - \int_X i\partial u\wedge\bar\partial\psi\wedge\gamma_\psi\wedge\Xi\,. \qquad (4.3)
\end{aligned}$$

Set $\chi=\varphi$ or $\chi=\psi$. Then the Cauchy-Schwartz inequality implies

$$\begin{aligned}
&\left|\int_X i\partial u\wedge\bar\partial\psi\wedge\gamma_\chi\wedge\Xi\right|\\
&\le \left(\int_X i\partial u\wedge\bar\partial u\wedge\gamma_\chi\wedge\Xi\right)^{1/2}\left(\int_X i\partial\psi\wedge\bar\partial\psi\wedge\gamma_\chi\wedge\Xi\right)^{1/2} = 0\,,
\end{aligned}$$

by the inductive hypothesis. This combined with (4.3) implies (4.2) for k. So at the end of the induction we get

$$0=\int_X i\partial u\wedge\bar\partial u\wedge\gamma^{n-1}\ge\int_X i\partial u\wedge\bar\partial u\wedge\theta^{n-1}\ge 0\,,$$

by the linearity formula (3.21). This implies $\varphi=\psi$ by elementary properties of plurisubharmonic functions.

Proof of B. By applying the comparison principle as in [E-G-Z] we get

$$\int_{\varphi<\psi} \gamma_\psi^n \leq \int_{\varphi<\psi} \gamma_\varphi^n = \int_{\varphi<\psi} e^{\lambda(\varphi-\psi)}\gamma_\psi^n\,,$$

which implies $\int_{\varphi<\psi}\gamma_\psi^n = 0$ since $e^{\lambda(\varphi-\psi)} < 1$. This implies that the inequality $\varphi \geq \psi$ holds γ_ψ^n-almost everywhere, thus the inequality

$$\gamma_\varphi^n = e^{\lambda(\varphi-\psi)}\gamma_\psi^n \geq \gamma_\psi^n\,,$$

holds γ_ψ^n-almost everywhere. By symmetry we also deduce that $\gamma_\psi^n \geq \gamma_\varphi^n$ holds γ_φ^n-almost everywhere. The fact that the potentials φ and ψ satisfies

$$\gamma_\varphi^n = e^{\lambda(\varphi-\psi)}\gamma_\psi^n\,, \tag{4.4}$$

implies that a property holds γ_ψ^n-almost everywhere if and only if it holds γ_φ^n-almost everywhere. We infer $\gamma_\psi^n = \gamma_\varphi^n$, hence $\psi - \varphi = \text{Const}$ by part (A), and equality (4.4) now implies $\psi = \varphi$. □

5 Generalized Kodaira lemma.

We first recall a few standard definitions of algebraic and analytic geometry which will be useful in our situation.

Definition 5.1 *Let (X,ω) be a compact Kähler manifold.*
(A) *A modification of X is a bimeromorphic morphism $\mu : \tilde{X} \to X$ of compact complex manifolds with connected fibers. Then there is a smallest analytic set $Z \subset X$ such that the restriction $\mu : \tilde{X} \setminus \mu^{-1}(Z) \to X \setminus Z$ is a biholomorphism; we say that $\mathrm{Exc}(\mu) = \mu^{-1}(Z)$ is the exceptional locus of μ.*
(B) *A class $\chi \in H^{1,1}(X,\mathbb{R})$ is called big if there exist a current $T \in \chi$ such that $T \geq \varepsilon\omega$, for some $\varepsilon > 0$.*

By a result of [De-Pa], a nef class χ on a compact Kähler manifold is big if and only if $\int_X \chi^n > 0$. By the proof of theorem 3.4 in [De-Pa] we obtain the following generalization of Kodaira's lemma.

Lemma 5.2 *Let X be a compact Kähler manifold and $\chi \in H^{1,1}(X,\mathbb{R})$ be a big class. Then there exist a modification $\mu : \tilde{X} \to X$ of X, an effective integral divisor D on $\tilde{X}$ with support $|D| \supset \mathrm{Exc}(\mu)$ and such that the class $\mu^*\chi - \delta\{D\}$ is Kähler for some $\delta \in \mathbb{Q}_{>0}$.*

We associate to χ the set I_χ of couples (μ, D) satisfying the generalized Kodaira lemma 5.2 and the complex analytic set

$$\Sigma_\chi := \bigcap_{(\mu,D)\in I_\chi} \mu(|D|)\,. \tag{5.1}$$

A trivial approximation argument shows that the set Σ_χ depends only on the half line $\mathbb{R}_{>0}\chi$. In the case the class χ is Kähler, $(I,0) \in I_\chi$, thus $\Sigma_\chi = \emptyset$. In the case the class χ is integral (or rational), the set Σ_χ can be characterized as follows.

Lemma 5.3 *Let L be a big line bundle over a compact Kähler manifold. Then the class $\chi := c_1(L)$ satisfies*

$$SB(L) \subset \Sigma_\chi = \bigcap_{\substack{E \in \mathrm{Div}^+(X),\, \delta \in \mathbb{Q}_{>0}, \\ \chi - \delta\{E\} \text{ ample}}} |E|\,, \tag{5.2}$$

where $SB(L)$ is the stable base locus of L, i.e. the intersection of the base loci of all line bundles kL, and E runs over all effective integral divisors of X.

Proof. First notice that the existence of a big line bundle implies that X is Moishezon. This combined with the assumption that X is Kähler shows that X must in fact be projective (see [Moi], and also [Pet1], [Pet2] for a simple proof). The inclusion $SB(L) \subset \Sigma_\chi$ in (5.2) is quite easy: Let $(\mu, D) \in I_\chi$. Then Kodaira's theorem implies that $\{\alpha\} := \mu^*\chi - \rho\{D\}$, $\rho \in \mathbb{Q}_{>0}$ is a $\mathbb{Q}$-ample class on $\tilde{X}$ and so the integer multiples $k\alpha$ are base point free for k large enough. Therefore the base locus of $k\mu^*L$ is contained in $|D|$. This shows that $SB(L)$ is contained in the intersection of the sets $\mu(|D|)$, which is precisely equal to Σ_χ by definition. Now, if H is an ample divisor on X, we have

$$\mu^*(\chi - \varepsilon\{H\}) = \rho\{D\} + \{\alpha\} - \varepsilon\{\mu^*H\}$$

and, again, $\alpha - \varepsilon\mu^*H$ is ample for $\varepsilon \in \mathbb{Q}_{>0}$ small. We infer that the base locus of $k(L - \varepsilon H)$ is contained in Σ_χ for k large and sufficiently divisible. If we pick any divisor E in the linear system of $k(L - \varepsilon H)$, then $L - \frac{1}{k}E \equiv \varepsilon H$ is an ample class, and the intersection of all these divisors E is contained in Σ_χ. Therefore

$$\bigcap_{\substack{E \in \mathrm{Div}^+(X),\, \delta \in \mathbb{Q}_{>0}, \\ \chi - \delta\{E\} \text{ ample}}} |E| \subset \Sigma_\chi.$$

The opposite inclusion is obvious. □

The following lemma gives us an important class of densities which will be allowable as the right hand side of degenerate complex Monge-Ampère equations.

Lemma 5.4 *Let X be a compact complex manifold, let $\Omega > 0$ be a smooth volume form and let $\sigma_j \in H^0(X, E_j)$, $\tau_r \in H^0(X, F_r)$, $j = 1, ..., N$, $r =$*

$1, ..., M$ *be, non identically zero, holomorphic sections of some holomorphic vector bundles over* X *such that the integral condition*

$$\int_X \prod_{j=1}^{N} |\sigma_j|^{2l_j} \cdot \prod_{r=1}^{M} |\tau_r|^{-2h_r}\, \Omega < +\infty$$

holds for some real numbers $l_j \geq 0$, $h_r \geq 0$. *Then the integrand function belongs to some* L^p *space,* $p > 1$, *and for* $A \geq A_0 \geq 0$ *large enough, the family of functions*

$$G_\varepsilon := \prod_{j=1}^{N} (|\sigma_j|^2 + \varepsilon^A)^{l_j} \cdot \prod_{r=1}^{M} (|\tau_r|^2 + \varepsilon)^{-h_r}\,, \qquad \varepsilon \in [0, 1)$$

converges in L^p*-norm to the function* G_0 *when* $\varepsilon \to 0$. *In fact, for* $N \neq 0$ *and* $l_j > 0$, *one can take* $A_0 := (\sum_r h_r)/(\min_j l_j)$.

Proof. By blowing-up the coherent ideals generated by the components of any of the sections σ_j, τ_r, we obtain a modification $\mu : \tilde{X} \to X$ such that the pull-back of these ideals by μ is a divisorial ideal. Using Hironaka's desingularization theorem, we can even assume that all divisors obtained in this way form a family of normal crossing divisors in $\tilde{X}$. Then each square $|\sigma_j \circ \mu|^2$ (resp. $|\tau_r \circ \mu|^2$) can be written as the square $|z^\alpha|^2$ (resp. $|z^\beta|^2$) of a monomial in suitable local coordinates U on a neighborhood of any point of $\tilde{X}$, up to invertible factors. The Jacobian of μ can also be assumed to be equal to a monomial z^γ, up to an invertible factor. In restriction to such a neighborhood U, the convergence of the integral is equivalent to that of

$$\int_U |z^\gamma|^2 \prod_{j=1}^{N} |z^{\alpha_j}|^{2l_j} \prod_{r=1}^{M} |z^{\beta_r}|^{-2h_r}\, dz.$$

Notice also that $\tilde{X}$ can be covered by finitely many such neighborhoods, by compactness. Now it is clear that if the integral is convergent, then the integrand must be in some L^p, $p > 1$, because the integrability condition is precisely that each coordinate z_j appears with an exponent > -1 in the n-tuple $\gamma + \sum l_j\alpha_j - \sum h_r\beta_j$ (so that we can still replace l_j, h_r with pl_j, ph_r with p close to 1). In order to prove the convergence of the functions G_ε in the L^p norm we distinguish two cases. In the case where $l_j = 0$ for all j, the claim follows immediately from the monotone convergence theorem. The other possible case is $l_j > 0$ for all j. In this case the convergence statement will follow if we can prove that for A large enough the functions

$$|z^\gamma|^2 \prod_{j=1}^{N} (|z^{\alpha_j}|^2 + \varepsilon^A)^{l_j} \prod_{r=1}^{M} (|z^{\beta_r}|^2 + \varepsilon)^{-h_r}$$

converge in L^p-norm as $\varepsilon \to 0$. This is trivial my monotonicity when $N = 0$. When $N > 0$ and $l_j > 0$, we have

$$\prod_{j=1}^{N}(|z^{\alpha_j}|^2+\varepsilon^A)^{l_j} \leq C\Big(\prod_{j=1}^{N}(|z^{\alpha_j}|^{2l_j}+\varepsilon^{A\min l_j}\Big), \quad \prod_{r=1}^{M}(|z^{\beta_r}|^2+\varepsilon)^{-h_r} \leq \varepsilon^{-\sum h_r},$$

so it is sufficient to take $A \geq (\sum h_r)/(\min l_j)$ to obtain the desired uniform L^p-integrability in ε. □

6 Existence and higher order regularity of the solutions.

We are ready to prove the following fundamental existence theorem.

Theorem 6.1 *Let X be a compact connected Kähler manifold of complex dimension $n \geq 2$, let $\omega \geq 0$ be a big closed smooth $(1,1)$-form such that $\{\omega^n = 0\}$ is a set of measure zero and let $\Omega > 0$ be a smooth volume form. Consider also $\sigma_j \in H^0(X, E_j)$, $\tau_r \in H^0(X, F_r)$, $j = 1, ..., N$, $r = 1, ..., M$ be non identically zero holomorphic sections of some holomorphic vector bundles over X, such that the integral condition*

$$\int_X \prod_{j=1}^{N} |\sigma_j|^{2l_j} \cdot \prod_{r=1}^{M} |\tau_r|^{-2h_r}\, \Omega = \int_X \omega^n \tag{6.1}$$

holds for some real numbers $l_j \geq 0$, $h_r \geq 0$. Then there exists a unique solution $\varphi \in \mathcal{P}\,\mathrm{BT}_\omega$ of the degenerate complex Monge-Ampère equation

$$(\omega + i\partial\bar\partial\varphi)^n = \prod_{j=1}^{N} |\sigma_j|^{2l_j} \cdot \prod_{r=1}^{M} |\tau_r|^{-2h_r}\, e^{\lambda\varphi}\, \Omega\,, \quad \lambda \geq 0\,, \tag{6.2}$$

which in the case $\lambda = 0$ is normalized by $0 = \sup_X \varphi$. Moreover there exists a complex analytic set $\Sigma_\omega \subset X$ depending only on the $(1,1)$-cohomology class of ω possessing the following properties.

(A) *The set Σ_ω is empty if and only if the $(1,1)$-cohomology class of ω is Kähler.*

(B) *If L is a holomorphic line bundle over X such that $\{\omega\} = 2\pi c_1(L)$ then the set Σ_ω contains the stable base locus of L.*

(C) *If we define the complex analytic sets*

$$S' := \Sigma_\omega \cup \left(\bigcup_r \{\tau_r = 0\}\right), \quad S := S' \cup \left(\bigcup_j \{\sigma_j = 0\}\right),$$

then $\varphi \in \mathcal{P}_\omega \cap L^\infty(X) \cap C^0(X \smallsetminus \Sigma_\omega) \cap C^\alpha(X \smallsetminus S') \cap C^\infty(X \smallsetminus S)$, *for all* $\alpha \in (0,1)$.

Proof.
Step I. We first assume the existence of an effective divisor D and $\delta > 0$ small such that $\{\omega\} - \delta\{D\}$ is a Kähler class. We infer from the classic Kodaira's lemma that this is the case if X is projective and $\{\omega\} \in H^{1,1}(X, \mathbb{Q})$ is big and nef (as follows from the assumptions of theorem 6.1). So by using the Lelong-Poincaré formula we deduce the existence of a smooth hermitian metric on $\mathcal{O}(D)$ such that $0 < \omega_\delta = \omega - 2\pi\delta[D] + \delta\, i\partial\bar\partial \log|s|^2$ with $\mathrm{div}(s) = D$. By convention we will put $\delta = 0$ if $\omega > 0$ and we will denote by $|D|$ the support of the divisor D.

(I.A) Setup of Step I.
As will be clear from the proof, we can assume without any lost of generality $N = M = 1$. Let $\alpha > 0$ be a Kähler metric, let $\varepsilon \in (0,1)$ and let c_ε be a normalizing constant for the integral condition

$$e^{c_\varepsilon} \int\limits_X \frac{(|\sigma|^2 + \varepsilon^A)^l}{(|\tau|^2 + \varepsilon)^h}\, \Omega = \int\limits_X (\omega + \varepsilon\alpha)^n\,, \tag{6.3}$$

with $A := (h+1)/l$. The condition (6.1) combined with lemma 5.4 implies $c_\varepsilon \to 0$, when $\varepsilon \to 0^+$. Consider the standard solutions $\varphi_\varepsilon \in C^\infty(X)$ of the complex Monge-Ampère equations

$$(\omega + \varepsilon\alpha + i\partial\bar\partial\varphi_\varepsilon)^n = e^{c_\varepsilon}\, \frac{(|\sigma|^2 + \varepsilon^A)^l}{(|\tau|^2 + \varepsilon)^h}\, e^{\lambda\varphi_\varepsilon}\, \Omega\,, \tag{6.4}$$

given by the Aubin-Yau solution of the Calabi conjecture. As usual, in the case $\lambda = 0$, we normalize the solution φ_ε with the condition $0 = \max_X \varphi_\varepsilon$. Notice that the integral condition (6.3) implies that a non identically zero solution φ_ε changes signs in the case $\lambda > 0$. By combining lemma 5.4 with the estimate of corollary 2.12 we deduce a uniform bound for the oscillations, $\mathrm{Osc}(\varphi_\varepsilon) \leq C$. Set now $\omega_{\delta,\varepsilon} := \omega_\delta + \varepsilon\alpha$ and $\psi_\varepsilon := \varphi_\varepsilon - \delta \log|s|^2$. Then

$$0 < \omega + \varepsilon\alpha + i\partial\bar\partial\varphi_\varepsilon = \omega_{\delta,\varepsilon} + i\partial\bar\partial\psi_\varepsilon \tag{6.5}$$

over $X \smallsetminus |D|$, and equation (6.4) can be rewritten as

$$(\omega_{\delta,\varepsilon} + i\partial\bar\partial\psi_\varepsilon)^n = e^{F^\varepsilon + \lambda\delta\log|s|^2 + \lambda\psi_\varepsilon}\, \omega_{\delta,\varepsilon}^n \tag{6.6}$$

on $X \smallsetminus |D|$, with $F^\varepsilon := f^\varepsilon + l \cdot a^\varepsilon - h \cdot b^\varepsilon$, and with

$$f^\varepsilon := c_\varepsilon + \log(\Omega/\omega_{\delta,\varepsilon}^n)\,, \quad a^\varepsilon := \log(|\sigma|^2 + \varepsilon^A)\,, \quad b^\varepsilon := \log(|\tau|^2 + \varepsilon)\,.$$

(Here the superscripts in ε are indices and not powers.) Let $\mathcal{C}_{\omega_{\delta,\varepsilon}}$ be the Chern curvature form of the Kähler metric $\omega_{\delta,\varepsilon} > 0$ and let

$$\gamma_{\delta,\varepsilon} := \min_{x\in X} \min_{\xi,\eta\in T_{X,x}\smallsetminus 0_x} \mathcal{C}_{\omega_{\delta,\varepsilon}}(\xi\otimes\eta,\xi\otimes\eta)|\xi|^{-2}_{\omega_{\delta,\varepsilon}}|\eta|^{-2}_{\omega_{\delta,\varepsilon}} .$$

(We remark that the minimum is always achieved by an easy compactness argument, see e.g. [Kat], chap II, sect. 5.1, theorem 5.1, page 107.) We observe that the family of metrics $(\omega_{\delta,\varepsilon})_\varepsilon$ has bounded geometry for δ fixed and $0\le\varepsilon\le 1$. In particular for all $\varepsilon\in(0,1]$

$$\gamma_{\delta,\varepsilon}\ge C_\delta\,,\quad |f^\varepsilon|\le K_{0,\delta}\,,\quad \lambda(\omega_\delta-\omega)+i\partial\bar\partial f^\varepsilon\ge -K_{0,\delta}\,\omega_{\delta,\varepsilon}\,.$$

Moreover we can assume $i\partial\bar\partial a^\varepsilon$, $i\partial\bar\partial b^\varepsilon\ge -K_{0,\delta}\,\omega_{\delta,\varepsilon}$, (see the Appendix B.)

(I.B) The Laplacian estimate.
This estimate is obtained as a combination of ideas of Yau, Błocki and Tsuji, [Yau], [Blo2], [Ts]. Consider the continuous function $\Lambda_\varepsilon : X\to(0,+\infty)$ given by the maximal eigenvalue of $\omega_{\delta,\varepsilon}+i\partial\bar\partial\psi_\varepsilon$ with respect to the Kähler metric $\omega_{\delta,\varepsilon}$,

$$\Lambda_\varepsilon(x) := \max_{\xi\in T_{X,x}\smallsetminus 0_x}(\omega_{\delta,\varepsilon}+i\partial\bar\partial\psi_\varepsilon)(\xi,J\xi)|\xi|^{-2}_{\omega_{\delta,\varepsilon}}\,,$$

i.e. we extend Λ_ε over $|D|$ by continuity, as is permitted by (6.5). Consider also the continuous function over $X\smallsetminus|D|$,

$$A_\varepsilon := \log\Lambda_\varepsilon - k\psi_\varepsilon + hb^\varepsilon\,,$$

with $0<k:=2(1+h\,K_{0,\delta}/2-K_1)$ and

$$K_1 := \min\{-[\lambda+(1+l)K_{0,\delta}/(2n)]\,,\,C_\delta\}<-\lambda\,.$$

The reason for this crucial choice will be clear in a moment. The singularity of the function ψ_ε implies the existence of a maximum of the function A_ε at a certain point $x_\varepsilon\in X\smallsetminus|D|$. Let g be a smooth real valued function in a neighborhood of x_ε in $X\smallsetminus|D|$ such that $\omega_{\delta,\varepsilon}=\frac{i}{2}\partial\bar\partial g$, and let $u:=g+2\psi_\varepsilon$. Then

$$\omega_{\delta,\varepsilon}+i\partial\bar\partial\psi_\varepsilon=\frac{i}{2}\partial\bar\partial u\,.$$

In the following calculations we use the notation $u_{l,\bar r}:=\frac{\partial^2 u}{\partial z_l\partial\bar z_r}$. Let $(z_1,\dots,z_n)$ be $\omega_{\delta,\varepsilon}$-geodesic holomorphic coordinates with center the point x_ε such that the metric $\omega_{\delta,\varepsilon}+i\partial\bar\partial\psi_\varepsilon$ can be written in diagonal form in x_ε. Explicitly $\omega_{\delta,\varepsilon}=\frac{i}{2}\sum_{l,r}g_{l,\bar r}\,dz_l\wedge d\bar z_r$, with

$$g_{l,\bar r}=\delta_{l,r}-\sum_{j,k}C^{j,\bar k}_{l,\bar r}z_j\bar z_k+O(|z|^3)\,,\quad g_{j,\bar k,l,\bar r}(x_\varepsilon)=-C^{j,\bar k}_{l,\bar r}\,,$$

$$\mathcal{C}_{\omega_{\delta,\varepsilon}}(x_\varepsilon)=\sum_{j,k,l,r}C^{j,\bar k}_{l,\bar r}\,dz_j\otimes dz_l\otimes d\bar z_k\otimes d\bar z_r\,.$$

and $\frac{i}{2}\partial\bar\partial u = \frac{i}{2}\sum_l u_{l,\bar l}\, dz_l\wedge d\bar z_l$, with $0<u_{1,\bar 1}\le ...\le u_{n,\bar n}$ at the point x_ε. For every $\zeta\in\mathbb{C}^n$ we set $g_{\zeta,\bar\zeta}:=\sum_{l,r} g_{l,\bar r}\,\zeta_l\,\bar\zeta_r$. Then

$$\Lambda_\varepsilon(x)=\max_{\xi\in T_{X,x}\smallsetminus 0_x}\frac{\partial\bar\partial u(\xi^{1,0},\xi^{0,1})}{\partial\bar\partial g(\xi^{1,0},\xi^{0,1})}=\max_{|\zeta|=1}\frac{u_{\zeta,\bar\zeta}}{g_{\zeta,\bar\zeta}}\,,$$

and so $\Lambda_\varepsilon(x_\varepsilon)=u_{n,\bar n}(x_\varepsilon)$, $\frac{u_{n,\bar n}}{g_{n,\bar n}}\le\Lambda_\varepsilon$. We also set

$$\tilde A_\varepsilon:=\log\frac{u_{n,\bar n}}{g_{n,\bar n}}-k\psi_\varepsilon+hb^\varepsilon\,.$$

Then $\tilde A_\varepsilon\le A_\varepsilon$, with $\tilde A_\varepsilon(x_\varepsilon)=A_\varepsilon(x_\varepsilon)$. This implies that $\tilde A_\varepsilon$ also reaches a maximum at x_ε, thus $\Delta_{\psi_\varepsilon}\tilde A_\varepsilon(x_\varepsilon)\le 0$, where $\Delta_{\psi_\varepsilon}$ is the Laplacian respect to the metric $\omega_{\delta,\varepsilon}+i\partial\bar\partial\psi_\varepsilon$. All the subsequent computations in this part of the proof will be made at the point x_ε. By the local expressions for the Ricci tensor we obtain

$$\begin{aligned}\partial^2_{n,\bar n}\log\det(u_{j,\bar k}) &= \sum_{l,p}\Big(u_{n,\bar n,l,\bar p}-\sum_{s,t}u_{n,l,\bar s}\,u^{s,\bar t}\,u_{\bar n,t,\bar p}\Big)u^{p,\bar l}\\ &= \sum_p\frac{u_{n,\bar n,p,\bar p}}{u_{p,\bar p}}-\sum_{p,q}\frac{|u_{n,p,\bar q}|^2}{u_{p,\bar p}\,u_{q,\bar q}}\,,\end{aligned}$$

and in a similar way $\partial^2_{n,\bar n}\log\det(g_{j,\bar k})=\sum_p g_{n,\bar n,p,\bar p}$. Then by differentiating with respect to the operator $\partial^2_{n,\bar n}$ the identity (6.6), which can be rewritten as

$$\log\det(u_{j,\bar k})=F^\varepsilon+\lambda\delta\log|s|^2+\lambda(u-g)/2+\log\det(g_{j,\bar k})\,,$$

we obtain

$$\begin{aligned}\sum_p\frac{u_{n,\bar n,p,\bar p}}{u_{p,\bar p}}-\sum_{p,q}\frac{|u_{n,p,\bar q}|^2}{u_{p,\bar p}\,u_{q,\bar q}} &= f^\varepsilon_{n,\bar n}+\lambda[(\omega_\delta)_{n,n}-\omega_{n,n}]/2\\ &+ la^\varepsilon_{n,\bar n}-hb^\varepsilon_{n,\bar n}+\lambda(u_{n,\bar n}-1)/2+\sum_p g_{n,\bar n,p,\bar p}\,.\end{aligned}$$

Combining this with the inequality $\Delta_{\psi_\varepsilon}\tilde A_\varepsilon(x_\varepsilon)\le 0$, we get

$$\begin{aligned}0 &\ge \sum_p\frac{\tilde A_{p,\bar p}}{u_{p,\bar p}}=\sum_p\left(\frac{u_{n,\bar n,p,\bar p}}{u_{p,\bar p}\,u_{n,\bar n}}-\frac{|u_{n,\bar n,p}|^2}{u_{p,\bar p}\,u^2_{n,\bar n}}+\frac{k/2+hb^\varepsilon_{p,\bar p}-g_{n,\bar n,p,\bar p}}{u_{p,\bar p}}\right)-nk/2\\ &= \sum_{p,q}\frac{|u_{n,p,\bar q}|^2}{u_{p,\bar p}\,u_{q,\bar q}\,u_{n,\bar n}}-\sum_p\frac{|u_{n,\bar n,p}|^2}{u_{p,\bar p}\,u^2_{n,\bar n}}\\ &+ \frac{f_{n,\bar n}+\lambda[(\omega_\delta)_{n,n}-\omega_{n,n}-1]/2+la^\varepsilon_{n,\bar n}-hb^\varepsilon_{n,\bar n}}{u_{n,\bar n}}\\ &+ \sum_p\left(\frac{g_{n,\bar n,p,\bar p}}{u_{n,\bar n}}+\frac{k/2+hb^\varepsilon_{p,\bar p}-g_{n,\bar n,p,\bar p}}{u_{p,\bar p}}\right)-(nk-\lambda)/2\,.\end{aligned}$$

We observe at this point that the sum of the two first terms following the second equality is nonnegative and the trivial inequality

$$-\frac{hb^{\varepsilon}_{n,\bar{n}}}{u_{n,\bar{n}}}+\sum_p \frac{hb^{\varepsilon}_{p,\bar{p}}}{u_{p,\bar{p}}} \;\geq\; \sum_p \frac{-hK_{0,\delta}/2}{u_{p,\bar{p}}}\,.$$

By plugging these inequalities in the previous computations and by using the definition of the constants k and K_1, we get

$$\begin{aligned} 0 \;&\geq\; \sum_p \left(\frac{K_1 - C^{n,\bar{n}}_{p,\bar{p}}}{u_{n,\bar{n}}} + \frac{-K_1 + C^{n,\bar{n}}_{p,\bar{p}}}{u_{p,\bar{p}}} + \frac{1}{u_{p,\bar{p}}} \right) - (nk-\lambda)/2 \\ &\geq\; \sum_p \frac{(C^{n,\bar{n}}_{p,\bar{p}} - K_1)(u_{n,\bar{n}} - u_{p,\bar{p}})}{u_{p,\bar{p}}\, u_{n,\bar{n}}} + \sum_p \frac{1}{u_{p,\bar{p}}} - C_0\,, \end{aligned}$$

where $C_0 > 0$ and all the following constants are independents of ε. Denote by $(x_1, ..., x_n)$ the real part of the complex coordinates $(z_1, ..., z_n)$. Then the inequality $C^{n,\bar{n}}_{p,\bar{p}} = \mathcal{C}_{\omega_{\delta,\varepsilon}}(\frac{\partial}{\partial x_n} \otimes \frac{\partial}{\partial x_p}, \frac{\partial}{\partial x_n} \otimes \frac{\partial}{\partial x_p})(x_\varepsilon) \geq \gamma_{\delta,\varepsilon} \geq C_\delta$ implies

$$0 \geq \sum_p \frac{1}{u_{p,\bar{p}}} - C_0 \geq \left(\frac{u_{n,\bar{n}}}{\prod_p u_{p,\bar{p}}} \right)^{\frac{1}{n-1}} - C_0 = e^{\frac{-\lambda\psi_\varepsilon - \lambda\delta \log|s|^2 - F^\varepsilon}{n-1}}\, u_{n,\bar{n}}^{\frac{1}{n-1}} - C_0\,.$$

Consider now the function $B_\varepsilon := e^{A_\varepsilon} = \Lambda_\varepsilon\, e^{-k\psi_\varepsilon + hb^\varepsilon}$. Then x_ε is also a maximum point for B_ε over $X \smallsetminus |D|$ and the previous inequality can be rewritten as

$$\begin{aligned} 0 \;&\geq\; e^{\frac{(k-\lambda)\psi_\varepsilon - \lambda\delta \log|s|^2 - hb^\varepsilon - F^\varepsilon}{n-1}(x_\varepsilon)}\; B_\varepsilon(x_\varepsilon)^{\frac{1}{n-1}} - C_0 \\ &=\; e^{\frac{(k-\lambda)\varphi_\varepsilon - \delta k \log|s|^2 - la^\varepsilon - f^\varepsilon}{n-1}(x_\varepsilon)}\; B_\varepsilon(x_\varepsilon)^{\frac{1}{n-1}} - C_0\,. \end{aligned}$$

Then by the inequalities $k - \lambda > 0$, $|s|^2 \leq C$, $a^\varepsilon \leq C$ and $|f^\varepsilon| \leq K_{0,\delta}$, it follows the estimate

$$0 \geq C_1\, e^{\frac{(k-\lambda)}{n-1} \min_X \varphi_\varepsilon}\; B_\varepsilon(x_\varepsilon)^{\frac{1}{n-1}} - C_0\,.$$

In conclusion we have found over $X \smallsetminus |D|$ the estimates

$$\begin{aligned} 0 \;&<\; 2n + \Delta_{\omega_{\delta,\varepsilon}}\varphi_\varepsilon - \delta\Delta_{\omega_{\delta,\varepsilon}} \log|s|^2 = \mathrm{Tr}_{\omega_{\delta,\varepsilon}}(\omega_{\delta,\varepsilon} + i\partial\bar{\partial}\psi_\varepsilon) \\ &\leq\; 2n\Lambda_\varepsilon \leq 2n\, e^{k\psi_\varepsilon - hb^\varepsilon} B_\varepsilon(x_\varepsilon) \leq \frac{C_2\, e^{k\varphi_\varepsilon - (k-\lambda)\min_X \varphi_\varepsilon}}{|s|^{2\delta k}(|\tau|^2+\varepsilon)^h} \leq \frac{C_2\, e^{k\,\mathrm{Osc}(\varphi_\varepsilon)}}{|s|^{2\delta k}\,|\tau|^{2h}}\,. \end{aligned}$$

The last inequality follows from the fact that $\lambda \min_X \varphi_\varepsilon \leq 0$, since a non identically zero solution φ_ε changes signs in the case $\lambda > 0$. Then using the inequality

$$\left|\delta\Delta_{\omega_{\delta,\varepsilon}} \log|s|^2\right| = |\,\mathrm{Tr}_{\omega_{\delta,\varepsilon}}(\omega - \omega_\delta)| \leq C$$

over $X \setminus |D|$ we deduce the singular Laplacian estimate

$$-C < 2n + \Delta_{\omega_{\delta,\varepsilon}} \varphi_\varepsilon \leq \frac{C_2\, e^{k\,\mathrm{Osc}(\varphi_\varepsilon)}}{|s|^{2\delta k}\,|\tau|^{2h}} + C\,.$$

(I.C) Higher order estimates
By the previous estimates we infer $0 < 2u_{l,\bar{l}} < \mathrm{Tr}_{\omega_{\delta,\varepsilon}}(\omega + \varepsilon\alpha + i\partial\bar{\partial}\varphi_\varepsilon) \leq 2\Upsilon_\varepsilon$, for all $l = 1, ..., n$, with

$$\Upsilon_\varepsilon := \frac{C_2\, e^{k\,\mathrm{Osc}(\varphi_\varepsilon)}}{|s|^{2\delta k}(|\tau|^2 + \varepsilon)^h}\,.$$

The equation (6.4) rewrites as $(\omega + \varepsilon\alpha + i\partial\bar{\partial}\varphi_\varepsilon)^n = e^{F^\varepsilon + \lambda\varphi_\varepsilon}\,\omega_{\delta,\varepsilon}^n$. We deduce $e^{F^\varepsilon+\lambda\varphi_\varepsilon} = \prod_l u_{l,\bar{l}} \leq \Upsilon_\varepsilon^{n-1} u_{k,\bar{k}}$, for all $k = 1, ..., n$. The fact that a non identically zero solution φ_ε changes signs in the case $\lambda > 0$ implies $\lambda \min_X \varphi_\varepsilon \geq -\lambda\,\mathrm{Osc}(\varphi_\varepsilon)$. Thus

$$e^{F^\varepsilon - \lambda\,\mathrm{Osc}(\varphi_\varepsilon)}\,\Upsilon_\varepsilon^{1-n}\,\omega_{\delta,\varepsilon} \leq \omega + \varepsilon\alpha + i\partial\bar{\partial}\varphi_\varepsilon\,.$$

Then an elementary computation yields the singular estimate

$$C_3^{-1}\,|s|^{2\delta k(n-1)}\,|\sigma|^{2l}\,|\tau|^{2h(n-2)}\,e^{-kn\,\mathrm{Osc}(\varphi_\varepsilon)}\,\omega_{\delta,\varepsilon}$$

$$\leq \quad \omega + \varepsilon\alpha + i\partial\bar{\partial}\varphi_\varepsilon \;\leq\; \frac{C_3\, e^{k\,\mathrm{Osc}(\varphi_\varepsilon)}}{|s|^{2\delta k}\,|\tau|^{2h}}\,\omega_{\delta,\varepsilon}\,. \tag{6.7}$$

Moreover the fact that $\varphi_\varepsilon \in \mathcal{P}_{\omega+\varepsilon\alpha}$ implies

$$2|\partial\bar{\partial}\varphi_\varepsilon|_{\omega_{\delta,\varepsilon}} \leq \Delta_{\omega_{\delta,\varepsilon}}\varphi_\varepsilon + 2\,\mathrm{Tr}_{\omega_{\delta,\varepsilon}}(\omega + \varepsilon\alpha)\,.$$

At this step of the proof we define

$$\Sigma_\omega := \bigcap_{\substack{\{\omega\}-\delta\{D\}>0 \\ D\geq 0,\ \delta>0}} |D|\,.$$

(The final definition of Σ_ω will be given in step II.) Then by the standard Schauder estimates [Gi-Tru] we find that for any coordinate compact set $K \subset X \setminus S'$ there are uniform constants $C_K > 0$ such that

$$\max_K |\nabla\varphi_\varepsilon| \leq C_K \left(\max_K \Delta\varphi_\varepsilon + \max_K |\varphi_\varepsilon|\right)\,.$$

Therefore, we can apply the complex version of Evans-Krylov theory [Ti2] on every compact set $K \subset X \setminus S$ to get uniform constants $C_{K,2} > 0$ such that $\|\varphi_\varepsilon\|_{C^{2,\alpha}(K)} \leq C_{K,2}$. Let now $U \subset X \setminus S$ be an open set and $\xi \in \mathcal{O}(T_X^{1,0})(U)$.

By deriving with respect to the complex vector field ξ the complex Monge-Ampère equation (6.4), which we rewrite under the form

$$(\omega + \varepsilon\alpha + i\partial\bar\partial\varphi_\varepsilon)^n = e^{H_\varepsilon + \lambda\varphi_\varepsilon}\alpha^n\,,$$

with

$$H_\varepsilon := c_\varepsilon + \log(\Omega/\alpha^n) + la^\varepsilon - hb^\varepsilon\,,$$

we obtain (see the proof of formula 11 in [Pal])

$$\Delta_{\varphi_\varepsilon}(\xi\,.\,\varphi_\varepsilon) - 2\lambda\,\xi\,.\,\varphi_\varepsilon = -\,\mathrm{Tr}_{\varphi_\varepsilon}\,L_\xi(\omega+\varepsilon\alpha) + \mathrm{Tr}_\alpha\,L_\xi\alpha + 2\xi\,.\,H_\varepsilon\,, \qquad (6.8)$$

where $\Delta_{\varphi_\varepsilon}$ and $\mathrm{Tr}_{\varphi_\varepsilon}$ are respectively the Laplacian and the trace operators with respect to the Kähler metric $\omega + \varepsilon\alpha + i\partial\bar\partial\varphi_\varepsilon > 0$. By the uniform estimates (6.7) and $\|\varphi_\varepsilon\|_{C^{2,\alpha}(K)} \leq C_{K,2}$ it follows that the operator $\Delta_{\varphi_\varepsilon}$ is uniformly elliptic with coefficients uniformly bounded in C^α-norm at least, over any compact set $K \subset U$. The right hand side of equation (6.8) is also uniformly bounded in C^α-norm at least, over K. By the standard regularity theory for linear elliptic equations [Gi-Tru] we deduce $\|\xi\,.\,\varphi_\varepsilon\|_{C^{2,\alpha}(K)} \leq C'_K$ for all $\varepsilon > 0$. By conjugation the same holds for $\bar\xi\,.\,\varphi_\varepsilon$. Thus we obtain the uniform estimate $\|\varphi_\varepsilon\|_{C^{3,\alpha}(K)} \leq C_{K,3}$.
In its turn this estimate implies that the coefficients of the Laplacian $\Delta_{\varphi_\varepsilon}$ and the right hand side of equation (6.8) are uniformly bounded in $C^{1,\alpha}$-norm at least. By iteration we get the uniform estimates $\|\varphi_\varepsilon\|_{C^{r,\alpha}(K)} \leq C_{K,r}$ for all $\varepsilon > 0$ and $r \in \mathbb{N}$. We deduce that the family $(\varphi_\varepsilon)_{\varepsilon>0} \subset C^\infty(X \smallsetminus S)$ is precompact in the smooth topology.
Chose now an exhaustion of $X \smallsetminus S$ by compact subsets K_j. The precompactness of the family $(\varphi_\varepsilon)_{\varepsilon>0} \subset C^\infty(X \smallsetminus S)$ in the smooth topology implies the existence of a family of sequences $(\varphi_{j,l})_l \subset (\varphi_\varepsilon)_{\varepsilon>0}$ such that $(\varphi_{j+1,l})_l \subset (\varphi_{j,l})_l$ and $\varphi_{j,l}$ converges in the $C^\infty(K_j)$-topology as $l \to +\infty$. We obtain in this way a function $\tilde\varphi \in (C^\infty \cap L^\infty)(X \smallsetminus S)$ such that $\omega + i\partial\bar\partial\tilde\varphi \geq 0$ over $X \smallsetminus S$ and solution of (6.2) over $X \smallsetminus S$. By theorem 5.24, page 54 in [Dem1] there exist a unique $\varphi \in \mathcal{P}_\omega \cap L^\infty(X)$ such that $\varphi = \tilde\varphi$ on $X \smallsetminus S$. This implies that the global complex Monge-Ampère measure $(\omega + i\partial\bar\partial\varphi)^n$ does not carry any mass on complex analytic sets. We infer that φ is a global bounded solution of the complex Monge-Ampère equation (6.2) which belongs to the class $\mathcal{P}_\omega \cap L^\infty(X) \cap C^\alpha(X \smallsetminus S') \cap C^\infty(X \smallsetminus S)$.

(I.D) Uniqueness of the solution
We now prove now the uniqueness of the solution φ in the class $\mathcal{P}\,\mathrm{BT}_\omega$. In the case $\lambda = 0$ this follows immediately from the theorem 4.1. In the case $\lambda > 0$ let $\psi \in \mathcal{P}\,\mathrm{BT}_\omega$ be an other solution and let $G_0 \geq 0$ be the integrand in the first integral in (6.1). The fact that $\psi \in \mathcal{P}_\omega$ implies that we can solve the degenerate complex Monge-Ampère equation

$$(\omega + i\partial\bar\partial u)^n = G_0\,e^{\lambda\psi}\,\Omega\,, \qquad (6.9)$$

with the methods so far explained, so as to obtain a solution $u \in \mathcal{P}^0_\omega \cap L^\infty(X)$. In fact, with the notations of lemma 5.4, we consider the solutions u_ε of the non degenerate complex Monge-Ampère equations

$$(\omega + \varepsilon\alpha + i\partial\bar\partial u_\varepsilon)^n = G_\varepsilon\, e^{\lambda\psi_\varepsilon + c_\varepsilon}\,\Omega\,,$$

with $\psi_\varepsilon \downarrow \psi$, $\psi_\varepsilon \in C^\infty(X)$, $\psi_\varepsilon \leq C$, $i\partial\bar\partial\psi_\varepsilon \geq -K_{0,\delta}\omega_{\delta,\varepsilon}$ and c_ε being a normalizing constant converging to 0 as $\varepsilon \to 0$. Moreover by combining lemma 5.4 with the dominated convergence theorem we infer that the family $G_\varepsilon\, e^{\lambda\psi_\varepsilon + c_\varepsilon}$ converges in L^p-norm to $G_0\, e^{\lambda\psi}$. These conditions are sufficient to provide the singular Laplacian estimate of step (I.B). Thus by replacing the C^∞-compactness argument of step (I.C) with a similar one using the C^α-compactness we infer the existence of the solution u of the degenerate complex Monge-Ampère equation (6.9).
By the uniqueness result in the case $\lambda = 0$ we infer $u = \psi - \sup_X \psi$, thus $\psi \in L^\infty(X)$. Then the required uniqueness follows immediately from theorem 4.1 B.

Step II.
In this step we apply to the class $\{\omega\}$ the considerations of section 5. In fact let $(\mu, D) \in I_{\{\omega\}}$. Then the integral condition (6.1) implies

$$0 < \int_{\tilde X} \mu^*\omega^n = \int_{\tilde X} (G_0\,\Omega\circ\mu)\cdot(\Lambda^n\partial\mu)\wedge\overline{(\Lambda^n\partial\mu)} = \int_{\tilde X} (G_0\circ\mu)\,|\Lambda^n\partial\mu|^2_{\alpha,\beta}\,\Omega'$$

where α and $\beta > 0$ are hermitian forms respectively over X and $\tilde X$,

$$|\Lambda^n\partial\mu|^2_{\alpha,\beta} := \frac{\alpha^n\cdot(\Lambda^n\partial\mu)\wedge\overline{(\Lambda^n\partial\mu)}}{\beta^n} \qquad \text{and} \qquad \Omega' = \left(\frac{\Omega}{\alpha^n}\circ\mu\right)\beta^n > 0\,.$$

Therefore by applying the fact that the class $\{\mu^*\omega\} - \delta\{D\}$ is Kähler, as in step I, we can solve the degenerate complex Monge-Ampère equation

$$(\mu^*\omega + i\partial\bar\partial\Phi)^n = (G_0\circ\mu)\,|\Lambda^n\partial\mu|^2_{\alpha,\beta}\,e^{\lambda\Phi}\,\Omega'\,,$$

so as to obtain a solution

$$\Phi \in \mathcal{P}_{\mu^*\omega} \cap L^\infty(\tilde X) \cap C^\alpha(\tilde X \smallsetminus (|D|\cup\mu^{-1}S'_0)) \cap C^\infty(\tilde X \smallsetminus (|D|\cup\mu^{-1}S_0))\,,$$

with

$$S'_0 := \bigcup_r \{\tau_r = 0\}\,, \qquad \text{and} \qquad S_0 := S'_0 \cup \left(\bigcup_j \{\sigma_j = 0\}\right)\,.$$

Remark that we have in fact $|\operatorname{div}(\Lambda^n\partial\mu)| \subset |D|$ by our definition of the set $I_{\{\omega\}}$. Moreover $j_q^*\mu^*\omega = 0$, where $j_q : \mu^{-1}(q) \hookrightarrow \tilde X$, $q \in \mu(|D|)$ is

the inclusion map. Thus $\Phi \circ j_q \in \mathrm{Psh}(\mu^{-1}(q))$ since $\Phi \in \mathcal{P}_{\mu^*\omega} \cap L^\infty(\tilde{X})$. By hypothesis $\mu^{-1}(q)$ is compact and connected, which implies that Φ is constant along the fibers $\mu^{-1}(q)$. Therefore we can define $\varphi := \pi_*\Phi \in \mathcal{P}_\omega \cap L^\infty(X)$. The fact that φ is bounded implies that the current $(\omega + i\partial\bar\partial\varphi)^n$ does not carry any mass on complex analytic sets. Thus φ is the unique solution in $\mathcal{P}_\omega \cap L^\infty(X)$ (see Theorem 4.1) of the complex Monge-Ampère equation (6.2) with the required C^α, C^∞-regularity over the adequate subsets of $X \setminus \mu(|D|)$. With the notations of section 5 we set finally $\Sigma_\omega := \Sigma_{\{\omega\}}$. Then the conclusion (C) about the $\mathcal{P}_\omega \cap L^\infty(X) \cap C^\alpha(X \setminus S') \cap C^\infty(X \setminus S)$ regularity of the solution φ follows by letting $(\mu, D) \in I_{\{\omega\}}$ vary. The proof of the uniqueness of the solution φ in the class $\mathcal{P}\,\mathrm{BT}_\omega$ is the same as in step I.
We remark now that if Σ_ω is empty then the class of ω is Kähler. In fact, let us choose the volume form $\Omega > 0$ so that $\int_X \omega^n = \int_X \Omega$. By the previous arguments we can find a unique solution φ of the equation $(\omega + i\partial\bar\partial\varphi)^n = \Omega > 0$, which is smooth, thus $\omega + i\partial\bar\partial\varphi > 0$ is a Kähler metric. This proves statement (A) of theorem 6.1. Statement (B) follows immediately from lemma 5.3.
The required $C^0(X \setminus \Sigma_\omega)$-regularity of the solution φ will follow immediately from the following theorem $\square$

Theorem 6.2 . *Let X be a compact connected Kähler manifold of complex dimension $n \geq 2$, let $\omega \geq 0$ be a big closed smooth $(1,1)$-form such that $\{\omega^n = 0\}$ is a set of measure zero and let $\Omega > 0$ be a smooth volume form. Let also $f \in L\log^{n+\delta} L(X)$, $\delta > 0$ such that $\int_X \omega^n = \int_X f\,\Omega$ and $\lambda \geq 0$ be a real number. Then there exists a unique solution $\varphi \in \mathcal{P}BT_\omega$ of the degenerate complex Monge-Ampère equation*

$$(\omega + i\partial\bar\partial\varphi)^n = f\, e^{\lambda\varphi}\Omega\,, \tag{6.10}$$

which in the case $\lambda = 0$ is normalized by $0 = \sup_X \varphi$. The solution φ is in the class $\mathcal{P}_\omega \cap L^\infty(X) \cap C^0(X \setminus \Sigma_\omega)$ and satisfies the C^0-estimate $\|\varphi\|_{C^0(X)} \leq C(\delta, \omega, \Omega)\, I_{\omega,\delta}(f)^{\frac{n}{\delta}} + 1$. Moreover the constant $C(\delta, \omega, \Omega) > 0$ stay bounded for perturbations of $\omega \geq 0$ as in the statement (C) of theorem 2.1.

Proof. We consider a regularizing family $(f_j)_j \subset C^\infty(X)$, $f_j > 0$ of f in $L\log^{n+\delta} L(X)$. (The existence of such family follows from [Ra-Re] page 364 or [Iw-Ma], theorem 4.12.2, page 79.) We can assume as usually $\int_X \omega^n = \int_X f_j\,\Omega$. By the proof of theorem 6.1 and the C^0-estimate in corollary 2.12 we deduce the existence of a unique solution of the degenerate complex Monge-Ampère equation

$$(\omega + i\partial\bar\partial\varphi_j)^n = f_j\, e^{\lambda\varphi_j}\Omega\,, \tag{6.11}$$

with the properties $\varphi_j \in \mathcal{P}_\omega \cap L^\infty(X) \cap C^\infty(X \setminus \Sigma_\omega)$ and

$$\|\varphi_j\|_{C^0(X)} \leq C := C(\delta, \omega, \Omega)\, I_{\omega,\delta}(f)^{\frac{n}{\delta}} + 1\,. \tag{6.12}$$

(With $\sup_X \varphi_j = 0$ in the case $\lambda = 0$.) We deduce in particular the uniform estimate

$$\|f_j\, e^{\lambda\,\varphi_j}\|_{L\log^{n+\delta}\,L(X)} \leq K e^{\lambda C} \|f\|_{L\log^{n+\delta}\,L(X)}\,, \tag{6.13}$$

for all j. (See [Ra-Re] page 364 or [Iw-Ma], theorem 4.12.2, page 79.) On the other hand the uniform estimate (6.12) implies by elementary properties of plurisubharmonic functions, (see [Dem1], chapter 1) the existence of a L^1-convergent subsequence $(\varphi_j)_j$ (which by abuse of notations we denote in the same way). We can apply the C^0-stability estimate of theorem 2.1, B to the complex Monge-Ampère equation (6.11) since we dispose of the estimates (6.12) and (6.13). Notice in fact that thanks to the estimate (6.12) the C^0-stability estimate of theorem 2.1, B apply even if in the case $\lambda > 0$ the solutions φ_j are not normalised by the supremum condition. We infer that the sequence $(\varphi_j)_j$ is a Cauchy sequence in the $C^0(X)$-norm, thus convergent to some function $\tilde{\varphi} \in L^\infty(X) \cap C^0(X \setminus \Sigma_\omega)$ such that $\omega + i\partial\bar{\partial}\tilde{\varphi} \geq 0$ over $X \setminus \Sigma_\omega$. This implies the convergence of the weak limits

$$(\omega + i\partial\bar{\partial}\tilde{\varphi})^n = \lim_{j\to+\infty} (\omega + i\partial\bar{\partial}\varphi_j)^n = \lim_{j\to+\infty} f_j\, e^{\lambda\,\varphi_j}\Omega = f\ e^{\lambda\,\tilde{\varphi}}\Omega\,,$$

over $X \setminus \Sigma_\omega$. The same argument in the proof of step (I.C) and step (I.D) in the proof of theorem 6.1 implies that the function $\tilde{\varphi}$ extends to a unique solution $\varphi \in \mathcal{P}BT_\omega$ of the degenerate complex Monge-Ampère equation (6.10) with the requited regularity and with $\|\varphi\|_{C^0(X)} \leq C$, as follows from elementary properties of plurisubharmonic functions. □

Proof of theorem 1.3.
A result of Kawamata [Kaw] claims that in our case the canonical bundle is base point free. By Noetherianity, for all $m \gg 0$ sufficiently big, mK_X has no base points. So we can fix m such that the pluricanonical map $f_m : X \to \mathbb{CP}^N$ is defined. Consider also the semipositive and big Kähler form $\omega_m := f_m^*\omega_{FS}/m \in 2\pi c_1(K_X)$, where ω_{FS} is the Fubini-Study metric of $\mathbb{CP}^N$. Let also $\Omega > 0$ be a smooth volume form over X, $\int_X \Omega = \int_X \omega_m^n$ such that $-\omega_m = \mathrm{Ric}(\Omega)$. According to theorem 6.1 we can find a unique solution $\varphi \in \mathcal{P}\,\mathrm{BT}_{\omega_m}$ of the degenerate complex Monge-Ampère equation

$$(\omega_m + i\partial\bar{\partial}\varphi)^n = e^{\varphi}\,\Omega\,.$$

Moreover $\varphi \in \mathcal{P}_{\omega_m} \cap L^\infty(X) \cap C^\infty(X \setminus \Sigma_{\omega_m})$, thus $\omega_{_E} := \omega_m + i\partial\bar{\partial}\varphi$ is the required unique Einstein current in the class $\mathrm{BT}^{\log}_{2\pi c_1(K_X)}$. □

7 Appendix

Appendix A. Basic facts about Orlicz spaces. Let $P : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $P(0) = 0$, be a convex increasing function and $\Omega > 0$ be a smooth volume

form over a manifold M and let $X \subset M$ be a Borel set of Ω-finite volume. According to [Ra-Re] we introduce the vector space

$$L^P(X) := \left\{ f : X \to \mathbb{R} \cup \{\pm\infty\} \,|\, \exists \lambda > 0 \,:\, \int_X P(|f|/\lambda)\, \Omega < +\infty \right\},$$

(with the usual identification of functions equal a.e.), equipped with the norm

$$\|f\|_{L^P(X)} := \inf \left\{ \lambda > 0 \,|\, \int_X P(|f|/\lambda)\, \Omega \leq 1 \right\}.$$

The space $L^P(X)$ equipped with this norm is called the Orlicz space associated with the convex function P. Moreover this norm is order preserving, i.e $\|f\|_{L^P(X)} \leq \|g\|_{L^P(X)}$ if $|f| \leq |g|$ a.e. If $P(t) = |t|^p$, $p \geq 1$, then $L^P(X)$ is the usual Lebesgue space. Other important examples of Orlicz spaces are given by the functions $P_\alpha := t \log^\alpha(e+t)$ and $Q_\alpha := e^{t^{1/\alpha}} - 1$, $\alpha \geq 1$. We set $L\log^\alpha L(X) := L^{P_\alpha}(X)$ and $\mathrm{Exp}^{1/\alpha} L(X) := L^{Q_\alpha}(X)$. An important class of Orlicz spaces is given by considering functions P satisfying the "doubling property": $P(2t) \leq 2^C P(t)$ for some constant $C \geq 1$. This is the case of the functions $|t|^p$ and $P_\alpha(t)$, but not the case of $Q_\alpha(t)$. For functions satisfying the doubling condition one has (see proposition 6 page 77 in [Ra-Re])

$$L^P(X) = \left\{ f : X \to \mathbb{R} \cup \{\pm\infty\} \,|\, \int_X P(|f|)\, \Omega < +\infty \right\},$$

and

$$\int_X P(\|f\|^{-1}_{L^P(X)} |f|)\, \Omega = 1$$

for all $f \in L^P(X) \setminus \{0\}$. So in the particular case of the function P_α, one obtains the inequality

$$\|f\|_{L\log^\alpha L(X)} \leq \int_X |f| \log^\alpha \left(e + \|f\|^{-1}_{L^1(X)} |f| \right) \Omega\,, \tag{7.1}$$

since $\|f\|_{L^1(X)} \leq \|f\|_{L\log^\alpha L(X)}$. It is quite hard to get estimates of the norm $\mathrm{Exp}^{1/\alpha} L(X)$, however it is easy to obtain

$$\|1\|_{\mathrm{Exp}^{1/\alpha} L(X)} = \frac{1}{\log^\alpha(1 + 1/\operatorname{Vol}_\Omega(X))}\,. \tag{7.2}$$

The relation between the Orlicz spaces $L\log^\alpha L(X)$ and $\mathrm{Exp}^{1/\alpha} L(X)$ is expressed by the Hölder inequality (see [Iw-Ma])

$$\left| \int_X f g\, \Omega \right| \leq 2C_\alpha \|f\|_{L\log^\alpha L(X)} \|g\|_{\mathrm{Exp}^{1/\alpha} L(X)}\,, \tag{7.3}$$

which follows from the inequality $xy \leq C_\alpha(P_\alpha(x) + Q_\alpha(y))$ for all $x, y \geq 0$. (Observe moreover that $C_1 = 1$.)

Appendix B. Computation of a complex Hessian. Let $\sigma \in H^0(X, E)$ be a holomorphic section of a holomorphic hermitian vector bundle (E, h) and set $S_\varepsilon := \log(|\sigma|^2 + \varepsilon)$, for some $\varepsilon > 0$. We denote by $\{\cdot, \cdot\}$ the exterior product of E-valued forms respect to the hermitian metric h. We have

$$i\partial S_\varepsilon = \frac{i\{\partial_h\sigma, \sigma\}}{|\sigma|^2 + \varepsilon}\,,$$

since σ is a holomorphic section. We compute now the complex hessian

$$\begin{aligned}
i\partial\bar\partial S_\varepsilon &= -\bar\partial \frac{i\{\partial_h\sigma, \sigma\}}{|\sigma|^2 + \varepsilon} \\
&= \frac{-i\{\bar\partial\partial_h\sigma, \sigma\} + i\{\partial_h\sigma, \partial_h\sigma\}}{|\sigma|^2 + \varepsilon} + i\{\partial_h\sigma, \sigma\} \wedge \bar\partial\left(\frac{1}{|\sigma|^2 + \varepsilon}\right) \\
&= \frac{i\{\partial_h\sigma, \partial_h\sigma\} - \{i\mathcal{C}_{E,h}\sigma, \sigma\}}{|\sigma|^2 + \varepsilon} - \frac{i\{\partial_h\sigma, \sigma\} \wedge \{\sigma, \partial_h\sigma\}}{(|\sigma|^2 + \varepsilon)^2} \\
&= \underbrace{\frac{(|\sigma|^2 + \varepsilon) i\{\partial_h\sigma, \partial_h\sigma\} - i\{\partial_h\sigma, \sigma\} \wedge \{\sigma, \partial_h\sigma\}}{(|\sigma|^2 + \varepsilon)^2}}_{iT(S_\varepsilon)} - \frac{\{i\mathcal{C}_{E,h}\sigma, \sigma\}}{|\sigma|^2 + \varepsilon}
\end{aligned}$$

where $\mathcal{C}_{E,h} \in C^\infty(X, \Lambda^{1,1}T^*X \otimes \mathrm{End}(E, E))$ is the curvature tensor of (E, h). We show that the $(1,1)$-form $iT(S_\varepsilon)$ is nonnegative. In fact by using twice the Lagrange inequality

$$i\{\partial_h\sigma, \sigma\} \wedge \{\sigma, \partial_h\sigma\} \leq |\sigma|^2\, i\{\partial_h\sigma, \partial_h\sigma\}$$

(which is an equality in the case of line bundles), we get

$$iT(S_\varepsilon) \geq \frac{\varepsilon i\{\partial_h\sigma, \partial_h\sigma\}}{(|\sigma|^2 + \varepsilon)^2} \geq \frac{\varepsilon i\{\partial_h\sigma, \sigma\} \wedge \{\sigma, \partial_h\sigma\}}{|\sigma|^2(|\sigma|^2 + \varepsilon)^2} = \frac{\varepsilon}{|\sigma|^2}\, i\partial S_\varepsilon \wedge \bar\partial S_\varepsilon \geq 0\,.$$

Observe that the last form is smooth. Consequently, we find the inequalities

$$\begin{aligned}
i\partial\bar\partial S_\varepsilon &\geq \frac{\varepsilon}{|\sigma|^2}\, i\partial S_\varepsilon \wedge \bar\partial S_\varepsilon - \frac{\{i\mathcal{C}_{E,h}\sigma, \sigma\}}{|\sigma|^2 + \varepsilon} \\
&\geq \frac{\varepsilon}{|\sigma|^2}\, i\partial S_\varepsilon \wedge \bar\partial S_\varepsilon - \|\mathcal{C}_{E,h}\|_{h,\omega} \frac{|\sigma|^2}{|\sigma|^2 + \varepsilon}\, \omega
\end{aligned}$$

where ω is a positive $(1,1)$-form.

Appendix C. Proof of estimate (2.20) **in lemma 2.13.** We will apply the computations of step (I.B) in the proof of Theorem 6.1 to the non degenerate complex Monge-Ampère equation

$$(\omega + i\partial\bar\partial\varphi'_j)^n = e^{h+L\varphi'_j-\varphi'_{j-1}}\,\omega^n\,.$$

In this setting, the notation of setup (I.A) in the proof of the Theorem 6.1 reduces to $\delta = l = h = 0$ and $i\partial\bar\partial h \geq -K_0\,\omega$. By replacing the term f with $h-\varphi'_{j-1}$ in the expansion of the term $\sum_p \tilde A_{p,\bar p}/u_{p,\bar p}$ in step (I.B) in the proof of Theorem 6.1, we infer

$$0 \geq e^{\frac{-L\varphi'_j-h+\varphi'_{j-1}}{n-1}}\,u_{n,\bar n}^{\frac{1}{n-1}} - \frac{(\varphi'_{j-1})_{n,\bar n}}{u_{n,\bar n}} - C'_0\,,$$

Thus

$$0 \geq C'_1\,u_{n,\bar n}^{\frac{1}{n-1}} - \frac{2n+\max_X \Delta_\omega\varphi'_{j-1}}{4u_{n,\bar n}} - C'_0\,, \tag{7.4}$$

by the estimates

$$\varphi''_0 \leq \varphi''_{j-1} \leq \varphi''_j \leq \varphi'_j \leq \varphi'_{j-1} \leq \varphi'_0\,. \tag{7.5}$$

This estimate implies also that at the maximum point x_j we have

$$u_{n,\bar n}(x_j) = \Lambda^\omega_{\varphi'_j} = e^{k\varphi'_j(x_j)}\mathcal{B}_j(x_j) \geq C'_2 B_j\,,$$

with $B_j := \max_X \mathcal{B}_j > 0$. Then estimate (2.20) in lemma 2.13 follows from (7.4) and the fact that

$$0 < 2n + \Delta_\omega\varphi'_j \leq 2ne^{k\max_X\varphi'_j}\,B_j \leq C\,B_j\,,$$

which is itself a consequence of (7.5). □

Appendix D. Proof of the conjecture of Tian. Let (X,ω_X) be a polarized compact Kähler manifold of complex dimension n, let (Y,ω_Y) be a compact irreducible Kähler space of complex dimension $m \leq n$, let $\pi : X \to Y$ be a surjective holomorphic map and let $0 \leq f \in L\log^{n+\varepsilon} L(X,\omega_X^n)$, for some $\varepsilon > 0$ such that $1 = \int_X f\omega_X^n$. Set $K_t := \{\pi^*\omega_Y + t\omega_X\}^n > 0$ for $t \in (0,1)$. Consider the complex Monge-Ampère equations

$$(\pi^*\omega_Y + t\omega_X + i\partial\bar\partial\psi_t)^n = K_t\,f\,\omega_X^n\,.$$

The hypothesis (C1) of statement (C) in theorem 2.1 is obviously satisfied. The hypothesis (C2b) is also satisfied since

$$\lim_{t\to 0}\frac{(\pi^*\omega_Y + t\omega_X)^n}{K_t\,\omega_X^n} = \left(\int_{y\in Y}\omega_Y^m(y)\cdot\int_{z\in\pi^{-1}(y)}\omega_X^{n-m}\right)^{-1}\frac{\pi^*\omega_Y\wedge\omega_X^{n-m}}{\omega_X^n} < +\infty.$$

We deduce $\mathrm{Osc}(\psi_t) \leq C < +\infty$ for all $t \in (0,1)$ by statements (C) and (A) of theorem 2.1. This solves in full generality a conjecture of Tian stated in [Ti-Ko].

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We wish to point out that in a quite recent preprint [Di-Zh] the authors claim (in Theorem 1.1) boundedness and continuity of the solutions of some particular type of degenerate complex Monge-Ampère equations. No proof of this claim seems to be provided. The authors also claim a stability result which is not sufficient to imply the continuity of solutions in the degenerate case. In fact a sequence of discontinuous functions converging in L^∞-norm does not have necessarily a continuous limit ! Moreover the same claim (Theorem 1.1) has been stated in [Zh1], [Zh2], but again no proof of continuity seem to be given (see page 12 in [Zh1] and page 146 in [Zh2]). The arguments for the boundedness of the solutions in [Zh1], [Zh2] are quite informal in the degenerate case and seem impossible to follow.

Concerning the stability of the solutions, the continuity assumption is quite natural and often available in the applications. In fact in the applications one works with smooth solutions provided by the Aubin-Yau solution of the Calabi conjecture with respect to variable Kähler forms of type $\omega + \varepsilon\alpha$ as in the proof of theorem 6.1. This perturbation process is one of the reasons of trouble for the continuity of the solutions. Moreover the stability with respect to the data f considered in [Di-Zh] is not essential in this context since one has L^1-compactness of quasi-plurisubharmonic functions normalized by the supremum condition. In fact a particular case of the stability result, namely Theorem 2.1 B, implies the continuity of the solution of the complex Monge-Ampère equation $(\omega + i\partial\bar\partial\varphi)^n = e^{\lambda\varphi} f\,\Omega$, whenever $\omega > 0$ is a Kähler metric and $f \in L\log^{n+\varepsilon} L$.

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