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# Mixed exterior Laplace's problem

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## Abstract

In [3], authors study Dirichlet and Neumann problems for the Laplace operator in exterior domains of  $\mathbb{R}^n$ . This paper extends this study to the resolution of a mixed exterior Laplace's problem. Here, we give existence, unicity and regularity results in  $L^p$ 's theory with  $1 < p < \infty$ , in weighted Sobolev spaces.

*Keywords* : Weighted Sobolev spaces ; Laplacian ; Mixed boundary conditions ; Poincaré type inequality

## 1 Introduction and preliminaries

Let  $\omega_0$  and  $\omega_1$  be two compact, disconnected and not empty regions of  $\mathbb{R}^n$ ,  $n \geq 2$ , with Lipschitz-continuous boundaries, respectively,  $\Gamma_0$  and  $\Gamma_1$  and let  $\Omega$  be the complement of  $\omega_0 \cup \omega_1$ . We set  $\Gamma = \Gamma_0 \cup \Gamma_1 = \partial\Omega$ .

This paper is devoted to solve the following problem :

$$(\mathcal{P}) \begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g_0 & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial \mathbf{n}} = g_1 & \text{on } \Gamma_1. \end{cases}$$

Since these problems are setted in an exterior domain, we must complete their statements with adequate asymptotic conditions at infinity. We have chosen to impose such conditions by setting our problem in weighted Sobolev spaces where the growth or decay of functions at infinity are expressed by means of weights. These weighted Sobolev spaces provide a correct functional setting for the exterior Laplace equation, in particular because the functions in these spaces satisfy an optimal weighted Poincaré type inequality. This gives them a great advantage over the two families of spaces currently used for the Laplace operator, namely, the completion of  $\mathcal{D}(\overline{\Omega})$  for the norm of the gradient in  $L^p(\Omega)$  and the subspace in  $L^p_{loc}(\Omega)$  of functions whose gradients belong to  $L^p(\Omega)$ . On one hand, when  $p \geq n$ , some very treacherous Cauchy sequences exist in  $\mathcal{D}(\overline{\Omega})$  that do not converge to distributions, a behaviour carefully described in 1954 by Deny and Lions (*cf.* [5]) but unfortunately overlooked by many authors. These sequences are eliminated in our spaces because we equip

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them with the full Sobolev norm instead of the norm of the gradient alone. On the other hand, this full Sobolev norm avoids the imprecision at infinity inherent to the  $L_{loc}^p$  norm. In an unbounded region, it is important to describe sharply the behaviour of functions at infinity and not just their gradient. This is vital from the mathematical point of view, not only because it permits to characterize easily the data from which we can solve our problems, but also because the analysis done here for one exponent of the weight extends readily to a wide range of real exponents. This is even more crucial from the numerical point of view because in most formulations, the function itself is the primary unknown that engineers discretize, the gradient being only secondary and usually deduced from the function values.

This paper is organized as follows. Sections 2, 3 and 4 are devoted to the study of questions of existence and unicity of the solution respectively in cases  $p = 2$ ,  $p > 2$  and  $p < 2$  and the section 5 deals with different behaviours at the infinity of the solution according to the data.

We complete this introduction with a short review of the weighted Sobolev spaces and their isomorphisms that we shall use in the sequel. For any integer  $q$  we denote by  $\mathcal{P}_q$  the space of polynomials in  $n$  variables, smaller than or equal to  $q$ , with the convention that  $\mathcal{P}_q$  is reduced to  $\{0\}$  when  $q$  is negative. For any real number  $p \in ]1, +\infty[$ , we denote by  $p'$  the dual exponent of  $p$  :

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Let  $\mathbf{x} = (x_1, \dots, x_n)$  be a typical point of  $\mathbb{R}^n$  and let  $r = |\mathbf{x}| = (x_1^2 + \dots + x_n^2)^{1/2}$  denote its distance to the origin. We shall use two basic weights :

$$\rho(r) = (1 + r^2)^{1/2} \quad \text{and} \quad \lg r = \ln(2 + r^2)$$

Then, for any nonnegative integers  $n$  and  $m$  and real numbers  $p > 1$ ,  $\alpha$  and  $\beta$ , setting

$$k = k(m, n, p, \alpha) = \begin{cases} -1 & \text{if } \frac{n}{p} + \alpha \notin \{1, \dots, m\}, \\ m - \frac{n}{p} - \alpha & \text{if } \frac{n}{p} + \alpha \in \{1, \dots, m\}, \end{cases}$$

we define the following space :

$$\begin{aligned} W_{\alpha, \beta}^{m, p}(\Omega) &= \{u \in \mathcal{D}'(\Omega); \\ &\forall \lambda \in \mathbb{N}^n : 0 \leq |\lambda| \leq k, \rho^{\alpha - m + |\lambda|} (\lg r)^{\beta - 1} D^\lambda u \in L^p(\Omega); \\ &\forall \lambda \in \mathbb{N}^n : k + 1 \leq |\lambda| \leq m, \rho^{\alpha - m + |\lambda|} (\lg r)^\beta D^\lambda u \in L^p(\Omega)\}. \end{aligned}$$

It is a reflexive Banach space equipped with its natural norm :

$$\begin{aligned} \|u\|_{W_{\alpha, \beta}^{m, p}(\Omega)} &= \left( \sum_{0 \leq |\lambda| \leq k} \|\rho^{\alpha - m + |\lambda|} (\lg r)^{\beta - 1} D^\lambda u\|_{L^p(\Omega)}^p \right. \\ &\quad \left. + \sum_{k+1 \leq |\lambda| \leq m} \|\rho^{\alpha - m + |\lambda|} (\lg r)^\beta D^\lambda u\|_{L^p(\Omega)}^p \right)^{1/p}. \end{aligned}$$

We also define the semi-norm :

$$|u|_{W_{\alpha,\beta}^{m,p}(\Omega)} = \left( \sum_{|\lambda|=m} \|\rho^\alpha(lg r)^\beta D^\lambda u\|_{L^p(\Omega)}^p \right)^{1/p}.$$

When  $\beta = 0$ , we agree to drop the index  $\beta$  and denote simply the space by  $W_\alpha^{m,p}(\Omega)$ .

The weights defined previously are chosen so that the space  $\mathcal{D}(\bar{\Omega})$  is dense in  $W_{\alpha,\beta}^{m,p}(\Omega)$  and so that theorem 1.1 below is satisfied.

The constants 1 and 2 in  $\rho(r)$  and  $lg r$  are added so that they do not modify the behaviour of the functions near the origin, in case it belongs to  $\Omega$ . Thus, the functions of  $W_{\alpha,\beta}^{m,p}(\Omega)$  belong to  $W^{m,p}(\mathcal{O})$  on all bounded domains  $\mathcal{O}$  contained in  $\Omega$ . As a consequence, the traces of these functions on  $\Gamma$ ,  $\gamma_0, \gamma_1, \dots, \gamma_{m-1}$ , satisfy the usual trace theorems (*cf.* Adams [1] or Nečas [8]). This allows to define in particular the space

$$\overset{\circ}{W}_{\alpha,\beta}^{m,p}(\Omega) = \{v \in W_{\alpha,\beta}^{m,p}(\Omega); \gamma_0 v = \gamma_1 v = \dots = \gamma_{m-1} v = 0\}.$$

It can be proved that  $\mathcal{D}(\Omega)$  is dense in  $\overset{\circ}{W}_{\alpha,\beta}^{m,p}(\Omega)$  and therefore, its dual space,  $W_{-\alpha,-\beta}^{-m,p'}(\Omega)$  is a space of distributions.

If  $n \in \mathbb{N}^*$  is the dimension of spaces, we set  $\mathbf{E} = E^n$ . Now, we define the space  $Y^p(\Omega) = W_0^{-1,p}(\Omega) \cap L^p(\Omega)$  equipped with the following norm :

$$\|u\|_{Y^p(\Omega)} = (\|u\|_{W_0^{-1,p}(\Omega)}^p + \|u\|_{L^p(\Omega)}^p)^{1/p}.$$

We easily check that  $Y^p(\Omega)$  is complete.

We introduce the partition of unity (we use the lemma of Urysohn) :

$$\begin{aligned} \psi_1, \psi_2 &\in C^\infty(\mathbb{R}^n), \quad 0 \leq \psi_1, \psi_2 \leq 1, \quad \psi_1 + \psi_2 = 1 \text{ in } \mathbb{R}^n, \\ \psi_1 &= 1 \text{ in } B_R, \quad \text{supp } \psi_1 \subset B_{R+1}, \end{aligned}$$

where  $R > 0$  is such that  $\omega_0 \cup \omega_1 \subset B_R$ .

For any  $v \in W_0^{1,p'}(\Omega)$ , we set  $v_1 = \psi_1 v$  and  $v_2 = \psi_2 v$ . We have  $\text{supp } v_1 \subset \Omega_{R+1} = B_{R+1} \cap \Omega$  and so  $v_1 \in W^{1,p'}(\Omega_{R+1})$ . Furthermore  $v_2 = 0$  on  $\Gamma$  because  $\psi_2 = 0$  on  $\Omega_R = B_R \cap \Omega$ , so  $v_2 \in \overset{\circ}{W}_0^{1,p'}(\Omega)$ . For  $f \in Y^p(\Omega)$ , we set :

$$\forall v \in W_0^{1,p'}(\Omega), \quad T_f(v) = \int_{\Omega_{R+1}} f v_1 \, d\mathbf{x} + \langle f, v_2 \rangle,$$

where  $\langle \cdot, \cdot \rangle$  denotes the duality pairing between  $W_0^{-1,p}(\Omega)$  and  $\overset{\circ}{W}_0^{1,p'}(\Omega)$ .

We easily notice that  $T_f$  is well defined, linear and we check that :

$$\forall \varphi \in \mathcal{D}(\bar{\Omega}), \quad T_f(\varphi) = \int_{\Omega} f \varphi \, dx, \quad (1)$$

and for any  $f \in Y^p(\Omega)$  and  $v \in W_0^{1,p'}(\Omega)$ ,

$$|T_f(v)| \leq C \|f\|_{Y^p(\Omega)} \|v\|_{W_0^{1,p'}(\Omega)}, \quad (2)$$

where  $C > 0$  is a constant which does not depend of  $f$  and  $v$ .

The next results are demonstrated by Amrouche, Girault and Giroire [2] and [3] and will be used in the sequel.

**Theorem 1.1.** Let  $\alpha$  and  $\beta$  be two real numbers and  $m \geq 1$  an integer not satisfying simultaneously :

$$\frac{n}{p} + \alpha \in \{1, \dots, m\} \quad \text{and} \quad (\beta - 1)p = -1$$

Let  $q' = \min(q, m - 1)$ , where  $q$  is the highest degree of the polynomials contained in  $W_{\alpha, \beta}^{m, p}(\Omega)$ . Then :

i) the semi-norm  $|\cdot|_{W_{\alpha, \beta}^{m, p}(\Omega)}$  defined on  $W_{\alpha, \beta}^{m, p}(\Omega)/\mathcal{P}_{q'}$  is a norm equivalent to the quotient norm.

ii) the semi-norm  $|\cdot|_{W_{\alpha, \beta}^{m, p}(\Omega)}$  is a norm on  $\overset{\circ}{W}_{\alpha, \beta}^{m, p}(\Omega)$ , which is equivalent to the full norm  $\|\cdot\|_{W_{\alpha, \beta}^{m, p}(\Omega)}$ .

**Theorem 1.2.** The following Laplace operators are isomorphisms :

$$\begin{aligned} i) \quad & \Delta : W_0^{1, p}(\mathbb{R}^n)/\mathcal{P}_{[1-n/p]} \rightarrow W_0^{-1, p}(\mathbb{R}^n) \perp \mathcal{P}_{[1-n/p]}, \\ ii) \quad & \Delta : W_1^{2, p}(\mathbb{R}^n)/\mathcal{P}_{[1-n/p]} \rightarrow W_1^{0, p}(\mathbb{R}^n) \perp \mathcal{P}_{[1-n/p]}, \\ iii) \quad & \Delta : W_1^{2, \frac{n}{n-1}}(\mathbb{R}^n)/\mathcal{P}_{2-n} \rightarrow (W_1^{0, \frac{n}{n-1}}(\mathbb{R}^n) \cap W_0^{-1, \frac{n}{n-1}}(\mathbb{R}^n)) \perp \mathbb{R}, \end{aligned}$$

where the symbol  $\perp$  has the following meaning : for  $E$  and  $F$  two spaces such that  $E \subset F$ ,

$$F' \perp E = \{f \in F', \forall x \in E, \langle f, x \rangle_{F', F} = 0\}.$$

**Proposition 1.3.** Assume that  $p > 2$  and  $f \in W_0^{-1, p}(\mathbb{R}^n)$  with compact support and satisfying, if  $n = 2$ , the compatibility condition

$$\langle f, 1 \rangle_{W_0^{-1, 2}(\mathbb{R}^2), W_0^{1, 2}(\mathbb{R}^2)} = 0.$$

Then, the problem

$$-\Delta u = f \text{ in } \mathbb{R}^n,$$

has a solution  $u \in W_0^{1, 2}(\mathbb{R}^n) \cap W_0^{1, p}(\mathbb{R}^n)$ , unique up to an additive constant if  $n = 2$ .

## 2 Case $p=2$

We begin to introduce the space

$$V_2 = \{v \in W_0^{1, 2}(\Omega), v = 0 \text{ on } \Gamma_0\}.$$

and to establish a Poincaré type inequality :

**Proposition 2.1.** There exists a constant  $C > 0$  such that :

$$\forall u \in V_2, \|u\|_{W_0^{1, 2}(\Omega)} \leq C |u|_{W_0^{1, 2}(\Omega)}.$$

**Proof-** We use an absurd argument ; so, assume that

$$\forall n \in \mathbb{N}^*, \exists w_n \in V_2, \|w_n\|_{W_0^{1, 2}(\Omega)} > n |w_n|_{W_0^{1, 2}(\Omega)}.$$

Then the sequence defined by  $u_n = \frac{w_n}{\|w_n\|_{W_0^{1,2}(\Omega)}}$  satisfy

$$\|u_n\|_{W_0^{1,2}(\Omega)} = 1 \quad \text{and} \quad |u_n|_{W_0^{1,2}(\Omega)} < \frac{1}{n}. \quad (3)$$

Here, we define an other partition of unity :

$$\begin{aligned} \varphi_1, \varphi_2 &\in C^\infty(\mathbb{R}^n), \quad 0 \leq \varphi_1, \varphi_2 \leq 1, \quad \varphi_1 + \varphi_2 = 1 \text{ in } \mathbb{R}^n, \\ \varphi_1 &= 1 \text{ in } B_{R_1}, \quad \text{supp } \varphi_1 \subset B_{R_1+1}, \end{aligned}$$

where  $R_1 > 0$  is such that  $\omega_1 \subset B_{R_1}$  and  $\omega_0 \cap B_{R_1+1} = \emptyset$ . We set  $u_n^1 = \varphi_1 u_n$  and  $u_n^2 = \varphi_2 u_n$ , so that  $u_n = u_n^1 + u_n^2$ . We deduce by (3) the existence of  $u \in V_2$  such that :

$$u_n \rightharpoonup u \text{ in } W_0^{1,2}(\Omega) \quad \text{and} \quad \nabla u = 0 \text{ in } \Omega.$$

As  $\Omega$  is connected and  $u \in V_2$ , then  $u = 0$  in  $\Omega$  and

$$u_n \rightarrow 0 \text{ in } W_0^{1,2}(\Omega). \quad (4)$$

Thanks to the Rellich's compactness theorem,  $u_n \rightarrow 0$  in  $L^2(\Omega_{R_1+1})$  and thanks to (3), we easily deduce that  $u_n^1 \rightarrow 0$  in  $W_0^{1,2}(\Omega)$ . Now, we prove that  $u_n^2 \rightarrow 0$  in  $W_0^{1,2}(\Omega)$ . First, we notice that  $u_n^2 \in \overset{\circ}{W}_0^{1,2}(\Omega)$ . Setting  $\Omega_0 = \Omega \setminus \overline{B}_{R_1}$  and  $\Omega' = \Omega_0 \cup \omega_0$ , we call again  $u_n^2$  the restriction of  $u_n^2$  to  $\Omega_0$  and we define :

$$\tilde{u}_n^2 = u_n^2 \text{ on } \Omega_0, \quad \tilde{u}_n^2 = 0 \text{ on } \omega_0$$

We easily check that  $\tilde{u}_n^2 \in \overset{\circ}{W}_0^{1,2}(\Omega')$  with  $\|\tilde{u}_n^2\|_{W_0^{1,2}(\Omega')} = \|u_n^2\|_{W_0^{1,2}(\Omega_0)}$ . Noticing that  $\Omega' = \overline{B}_{R_1}^c$  and applying a result established by Giroire [6], we have :

$$\|\tilde{u}_n^2\|_{W_0^{1,2}(\Omega')} \leq C |\tilde{u}_n^2|_{W_0^{1,2}(\Omega')}.$$

We easily show that  $|\tilde{u}_n^2|_{W_0^{1,2}(\Omega')} \rightarrow 0$ , so in particular  $\|u_n^2\|_{W_0^{1,2}(\Omega_0)} \rightarrow 0$ . To finish, like  $\Omega_0 = \Omega \setminus \Omega_{R_1}$  and  $u_n^2 = 0$  on  $\Omega_{R_1}$ , we have :  $\|u_n^2\|_{W_0^{1,2}(\Omega)} = \|u_n^2\|_{W_0^{1,2}(\Omega_0)} \rightarrow 0$ . So,  $u_n^2 \rightarrow 0$  in  $W_0^{1,2}(\Omega)$  which implies that  $u_n = u_n^1 + u_n^2 \rightarrow 0$  in  $W_0^{1,2}(\Omega)$ , and which contradicts (3). In consequence, we have the result searched.  $\square$

**Theorem 2.2.** *For any  $f \in Y^2(\Omega)$ ,  $g_0 \in H^{\frac{1}{2}}(\Gamma_0)$  and  $g_1 \in H^{-\frac{1}{2}}(\Gamma_1)$ , there exists a unique  $u \in W_0^{1,2}(\Omega)$  solution of the problem  $(\mathcal{P})$  and a constant  $C > 0$  such that*

$$\|u\|_{W_0^{1,2}(\Omega)} \leq C (\|f\|_{Y^2(\Omega)} + \|g_0\|_{H^{\frac{1}{2}}(\Gamma_0)} + \|g_1\|_{H^{-\frac{1}{2}}(\Gamma_1)}).$$

**Proof-** First, according to [3], there exists a unique  $u_0 \in W_0^{1,2}(\Omega_1)$  where  $\Omega_1 = \Omega \cup \omega_1$ , solution of :

$$(\mathcal{P}_D) \begin{cases} \Delta u_0 = 0 & \text{in } \Omega_1, \\ u_0 = g_0 & \text{on } \Gamma_0, \end{cases}$$

and checking

$$\|u_{0|\Omega}\|_{W_0^{1,2}(\Omega)} \leq C \|g_0\|_{H^{\frac{1}{2}}(\Gamma_0)}.$$

We notice that like  $u_{0|\Omega} \in W_0^{1,2}(\Omega)$  and  $0 = \Delta u_0 \in L^2(\Omega)$ , then

$\frac{\partial u_0}{\partial \mathbf{n}} \in H^{-\frac{1}{2}}(\Gamma_1)$ . Moreover, we know that there exists a unique  $v \in V_2$  solution of the following problem  $(\mathcal{FV})$  :

$$(\mathcal{FV}) \quad \forall w \in V_2, \quad a(v, w) = L(w),$$

where for  $v, w \in V_2$ ,

$$a(v, w) = \int_{\Omega} \nabla v \cdot \nabla w \, d\mathbf{x} \quad \text{and} \quad L(w) = T_f(w) + \langle g_1 - \frac{\partial u_0}{\partial \mathbf{n}}, w \rangle_{\Gamma_1},$$

and where  $\langle \cdot, \cdot \rangle_{\Gamma_1}$  is the duality pairing  $H^{-\frac{1}{2}}(\Gamma_1), H^{\frac{1}{2}}(\Gamma_1)$ . Indeed, this result is a simply consequence of the Lax-Milgram theorem and of the propriety 2.1 which shows that the form  $a$  is coercive.

Then, we easily check that this solution  $v \in W_0^{1,2}(\Omega)$  satisfies

$$(\mathcal{P}') \quad \begin{cases} -\Delta v = f & \text{in } \Omega, \\ v = 0 & \text{on } \Gamma_0, \\ \frac{\partial v}{\partial \mathbf{n}} = g_1 - \frac{\partial u_0}{\partial \mathbf{n}} & \text{on } \Gamma_1, \end{cases}$$

and

$$\|v\|_{W_0^{1,2}(\Omega)} \leq C (\|f\|_{Y^2(\Omega)} + \|g_1 - \frac{\partial u_0}{\partial \mathbf{n}}\|_{H^{-\frac{1}{2}}(\Gamma_1)}).$$

Finally, the function  $u = u_{0|\Omega} + v$  is the solution of  $(\mathcal{P})$  and the estimate searched is a consequence of the two previous inequalities.  $\square$

### 3 Case $p > 2$

We propose the following approach : first we solve the harmonic problem, this will enable us to establish an "inf-sup" condition which in turn will solve the full problem thanks to the theorem of Babuška-Brezzi.

In all this section we suppose  $p > 2$  (except for the subsection 3.5. where we suppose  $p \geq 2$ ) and  $\Gamma$  of class  $C^{1,1}$ .

#### 3.1 Resolution of the harmonic problem

Let  $g_0$  be in  $W^{1-\frac{1}{p},p}(\Gamma_0)$  and  $g_1$  be in  $W^{-\frac{1}{p},p}(\Gamma_1)$ . Here, we consider the problem : find  $u$  in  $W_0^{1,p}(\Omega) \cap W_0^{1,2}(\Omega)$  solution of

$$(\mathcal{P}_0) \quad \begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g_0 & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial \mathbf{n}} = g_1 & \text{on } \Gamma_1. \end{cases}$$

**Theorem 3.1.** *For any  $g_0 \in W^{1-\frac{1}{p},p}(\Gamma_0)$  and  $g_1 \in W^{-\frac{1}{p},p}(\Gamma_1)$ , there exists a unique  $u \in W_0^{1,p}(\Omega) \cap W_0^{1,2}(\Omega)$ , solution of  $(\mathcal{P}_0)$ , and we have :*

$$\|u\|_{W_0^{1,p}(\Omega)} + \|u\|_{W_0^{1,2}(\Omega)} \leq C (\|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W^{-\frac{1}{p},p}(\Gamma_1)}). \quad (5)$$

**Proof-** By [3], we know there exists  $u_0 \in W_0^{1,p}(\Omega_1) \cap W_0^{1,2}(\Omega_1)$  solution of  $(\mathcal{P}_D)$ , where we remind that  $\Omega_1 = \Omega \cup \omega_1$ , with the following estimate :

$$\|u_0\|_{W_0^{1,p}(\Omega_1)} + \|u_0\|_{W_0^{1,2}(\Omega_1)} \leq C \|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)}. \quad (6)$$

We notice that like  $u_0|_\Omega \in W_0^{1,p}(\Omega)$  and  $0 = \Delta u_0 \in L^p(\Omega)$ , we have  $\frac{\partial u_0}{\partial \mathbf{n}} \in W^{-\frac{1}{p},p}(\Gamma_1)$ . Moreover

$$\left\| \frac{\partial u_0}{\partial \mathbf{n}} \right\|_{W^{-\frac{1}{p},p}(\Gamma_1)} \leq C \|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)}$$

Then, we are going to show that there exists a unique  $v$  in  $W_0^{1,2}(\Omega) \cap W_0^{1,p}(\Omega)$  solution of the following problem :

$$(\mathcal{P}'_0) \begin{cases} \Delta v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \Gamma_0, \\ \frac{\partial v}{\partial \mathbf{n}} = g_1 - \frac{\partial u_0}{\partial \mathbf{n}} & \text{on } \Gamma_1. \end{cases}$$

with the estimate :

$$\|v\|_{W_0^{1,p}(\Omega)} + \|v\|_{W_0^{1,2}(\Omega)} \leq C (\|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W^{-\frac{1}{p},p}(\Gamma_1)}). \quad (7)$$

As  $p > 2$ ,  $g_1 - \frac{\partial u_0}{\partial \mathbf{n}} \in H^{-\frac{1}{2}}(\Gamma_1)$  and thanks to theorem 2.2, there exists a unique  $v \in W_0^{1,2}(\Omega)$  solution of  $(\mathcal{P}'_0)$  and checking (7) with  $p = 2$ . There stays to show that  $v \in W_0^{1,p}(\Omega)$ . For this, we use the partition of the unity previously defined :

$$\begin{aligned} \varphi_1, \varphi_2 &\in C^\infty(\mathbb{R}^n), \quad 0 \leq \varphi_1, \varphi_2 \leq 1, \quad \varphi_1 + \varphi_2 = 1 \text{ in } \mathbb{R}^n, \\ \varphi_1 &= 1 \text{ in } B_{R_1}, \quad \text{supp } \varphi_1 \subset B_{R_1+1}, \end{aligned}$$

where  $R_1 > 0$  is such that  $\omega_1 \subset B_{R_1}$  and  $\omega_0 \cap B_{R_1+1} = \emptyset$ . We set  $v_1 = \varphi_1 v$ ,  $v_2 = \varphi_2 v$  and

$$\tilde{v}_2 = v_2 \text{ in } \Omega, \quad \tilde{v}_2 = 0 \text{ in } \omega_1.$$

We remind that  $v_2 \in \mathring{W}_0^{1,2}(\Omega)$  and so that  $\tilde{v}_2 \in W_0^{1,2}(\Omega_1)$ . Moreover, we have :

$$-\Delta v_2 = \Delta v_1 = v \Delta \psi_1 + 2\nabla \psi_1 \cdot \nabla v := f_1 \text{ in } \Omega.$$

Setting

$$\tilde{f}_1 = f_1 \text{ in } \Omega, \quad \tilde{f}_1 = 0 \text{ in } \omega_1,$$

it is obvious that  $\tilde{f}_1 \in L^2(\Omega_1)$  with  $\text{supp } \tilde{f}_1 \subset \Omega_{R_1+1}$  and that  $-\Delta \tilde{v}_2 = \tilde{f}_1$  in  $\Omega_1$ .

Now, we set  $s = \tilde{v}_2$  and we are going to show that  $s \in W_0^{1,p}(\Omega_1)$ . For this, we define an other partition of the unity :

$$\begin{aligned} \xi_1, \xi_2 &\in C^\infty(\mathbb{R}^n), \quad 0 \leq \xi_1, \xi_2 \leq 1, \quad \xi_1 + \xi_2 = 1 \text{ in } \mathbb{R}^n, \\ \xi_1 &= 1 \text{ in } B_{R_0}, \quad \text{supp } \xi_1 \subset B_{R_0+1}, \end{aligned}$$



where  $R_0 > 0$  is such that  $\omega_0 \subset B_{R_0}$  and  $\omega_1 \cap B_{R_0+1} = \emptyset$ . We set  $s_1 = \xi_1 s$  and  $s_2 = \xi_2 s$ , and we notice that :

$$-\Delta s_2 = \tilde{f}_1 + \Delta s_1 = \tilde{f}_1 + s \Delta \xi_1 + 2\nabla \xi_1 \cdot \nabla s := F \text{ in } \Omega_1.$$

Finally, we set :

$$\tilde{s}_2 = \begin{cases} s_2 & \text{in } \Omega_1, \\ 0 & \text{in } \omega_0, \end{cases} \quad \text{and} \quad \tilde{F} = \begin{cases} F & \text{in } \Omega_1, \\ 0 & \text{in } \omega_0. \end{cases}$$

We have  $\tilde{s}_2 \in W_0^{1,2}(\mathbb{R}^n)$  because  $s_2 \in \mathring{W}_0^{1,2}(\Omega_1)$ ,  $\tilde{F} \in L^2(\mathbb{R}^n)$  with  $\text{supp } \tilde{F} \subset \Omega_{R_0+1} \cap \Omega_{R_1+1}$ , and also  $-\Delta \tilde{s}_2 = \tilde{F}$  in  $\mathbb{R}^n$ .

**i)** Case  $2 < p \leq \frac{2n}{n-2}$  and  $n \geq 3$  or  $p > 2$  and  $n = 2$ .

Thanks to the injections of Sobolev, , we have  $\tilde{F} \in W_0^{-1,p}(\mathbb{R}^n)$ . In consequence, thanks to the theorem 1.2 i), (there is no condition of compatibility because  $p > 2$ ), we show that  $\tilde{s}_2 \in W_0^{1,p}(\mathbb{R}^n)$  and so  $s_2 \in W_0^{1,p}(\Omega_1)$  and we easily check that :

$$\|\tilde{s}_2\|_{W_0^{1,p}(\mathbb{R}^n)/\mathcal{P}_{[1-\frac{2}{p}]}} \leq C \|g_1 - \frac{\partial u_0}{\partial \mathbf{n}}\|_{W^{-\frac{1}{p},p}(\Gamma_1)}. \quad (8)$$

Outside of  $B_{R_0+1}$ ,  $\xi_1 = 0$  so  $s = s_2$  and the trace of  $s_2$  on  $\partial B_{R_0+1}$  belongs to  $W^{1-\frac{1}{p},p}(\partial B_{R_0+1})$ . So  $s$  checks :

$$-\Delta s = \tilde{f}_1 \text{ in } \Omega_{R_0+1}, \quad s = s_2 \text{ on } \partial B_{R_0+1}, \quad s = 0 \text{ on } \Gamma_0.$$

Consequently, (see Lions and Magenes [7] ),  $s \in W^{1,p}(\Omega_{R_0+1})$  and :

$$\|s\|_{W^{1,p}(\Omega_{R_0+1})} \leq C (\|\tilde{f}_1\|_{W^{-1,p}(\Omega_{R_0+1})} + \|s_2\|_{W^{1-\frac{1}{p},p}(\partial B_{R_0+1})}) \quad (9)$$

We deduce of this that  $s \in W_0^{1,p}(\Omega_1)$ ; and with (8) and (9), we have :

$$\|s\|_{W_0^{1,p}(\Omega_1)} \leq C \|g_1 - \frac{\partial u_0}{\partial \mathbf{n}}\|_{W^{-\frac{1}{p},p}(\Gamma_1)}. \quad (10)$$

**ii)** Case  $n \geq 3$  and  $p > \frac{2n}{n-2}$ .

The argument used above with  $p = \frac{2n}{n-2}$  shows that  $s \in W_0^{1,\frac{2n}{n-2}}(\Omega_1)$  and we

use the same demonstration that **i)** with  $s \in W_0^{1,\frac{2n}{n-2}}(\Omega_1)$  instead of  $s \in W_0^{1,2}(\Omega_1)$ . So, we obtain the result for  $n = 3$ ,  $n = 4$  and  $n = 5$  if

$p < \frac{2n}{n-4}$ ; then we take  $\frac{2n}{n-4}$  instead of  $\frac{2n}{n-2}$ , and we start again; so we reach for all dimension, all values of  $p$ .

Consequently, we have  $s = \tilde{v}_2 \in W_0^{1,p}(\Omega_1)$ ,  $v_2 \in W_0^{1,p}(\Omega)$  and

$$\|v_2\|_{W_0^{1,p}(\Omega)} \leq C \|g_1 - \frac{\partial u_0}{\partial \mathbf{n}}\|_{W^{-\frac{1}{p},p}(\Gamma_1)}. \quad (11)$$

Outside of  $B_{R_1+1}$ ,  $\varphi_1 = 0$  and  $v = v_2$  and the trace of  $v_2$  on  $\partial B_{R_1+1}$  belongs to  $W^{1-\frac{1}{p},p}(\partial B_{R_1+1})$ . So  $v$  checks :

$$\Delta v = 0 \text{ in } \Omega_{R_1+1}, \quad v = v_2 \text{ on } \partial B_{R_1+1}, \quad \frac{\partial v}{\partial \mathbf{n}} = g_1 - \frac{\partial u_0}{\partial \mathbf{n}} \text{ on } \Gamma_1.$$

In consequence, (see Lions and Magenes [7] ),  $v \in W^{1,p}(\Omega_{R_1+1})$  and :

$$\|v\|_{W^{1,p}(\Omega_{R_1+1})} \leq C \left( \|g_1 - \frac{\partial u_0}{\partial \mathbf{n}}\|_{W^{-\frac{1}{p},p}(\Gamma_1)} + \|v_2\|_{W^{1-\frac{1}{p},p}(\partial B_{R_1+1})} \right) \quad (12)$$

We deduce of this that  $v \in W_0^{1,p}(\Omega)$  and with (11) and (12), we have :

$$\|v\|_{W_0^{1,p}(\Omega)} \leq C \|g_1 - \frac{\partial u_0}{\partial \mathbf{n}}\|_{W^{-\frac{1}{p},p}(\Gamma_1)}.$$

Then, we easily check (7). Finally the function

$u = u_{0|\Omega} + v \in W_0^{1,2}(\Omega) \cap W_0^{1,p}(\Omega)$  suits and with (6) and (7), we have (5).  
□

### 3.2 An “inf-sup” condition

Setting for any  $p > 1$ ,

$$V_p = \{v \in W_0^{1,p}(\Omega), v = 0 \text{ on } \Gamma_0\}.$$

we notice that, equipped with the norm  $\|\nabla \cdot\|_{\mathbf{L}^p(\Omega)}$ ,  $V_p$  is a reflexive Banach space. In this subsection, we are interested in the existence of  $\beta > 0$  a constant such that :

$$\inf_{\substack{w \in V_{p'} \\ w \neq 0}} \sup_{\substack{v \in V_p \\ v \neq 0}} \frac{\int_{\Omega} \nabla v \cdot \nabla w \, d\mathbf{x}}{\|\nabla v\|_{\mathbf{L}^p(\Omega)} \|\nabla w\|_{\mathbf{L}^{p'}(\Omega)}} \geq \beta$$

We define :

$$\mathring{H}_p(\Omega) = \{z \in \mathbf{L}^p(\Omega), \operatorname{div} z = 0 \text{ in } \Omega, z \cdot \mathbf{n} = 0 \text{ on } \Gamma_1\}$$

**Proposition 3.2.** *For any  $g \in \mathbf{L}^p(\Omega)$ , there exists  $z \in \mathring{H}_p(\Omega)$  and  $\varphi \in V_p$ , such that :*

$$\begin{aligned} g &= \nabla \varphi + z, \\ \|\nabla \varphi\|_{\mathbf{L}^p(\Omega)} &\leq C \|g\|_{\mathbf{L}^p(\Omega)} \end{aligned}$$

where  $C > 0$  is a constant which depends only on  $\Omega$  and  $p$ .

**Proof-** Let  $g$  be in  $\mathbf{L}^p(\Omega)$  and  $\tilde{g}$  the extension by 0 of  $g$  in  $\mathbb{R}^n$ ; so we have  $\tilde{g} \in \mathbf{L}^p(\mathbb{R}^n)$  and thanks to a result established in [2],

$\operatorname{div} \tilde{g} \in W_0^{-1,p}(\mathbb{R}^n) \perp \mathcal{P}_{[1-n/p']}$  with

$$\|\operatorname{div} \tilde{g}\|_{W_0^{-1,p}(\mathbb{R}^n)} \leq C \|\tilde{g}\|_{\mathbf{L}^p(\mathbb{R}^n)} = C \|g\|_{\mathbf{L}^p(\Omega)}.$$

According to the theorem 1.2 i), we know there exists  $v \in W_0^{1,p}(\mathbb{R}^n)$  such that  $\Delta v = \operatorname{div} \tilde{g}$  in  $\mathbb{R}^n$ , and we show that  $\|\nabla v\|_{\mathbf{L}^p(\mathbb{R}^n)} \leq C_1 \|g\|_{\mathbf{L}^p(\Omega)}$ . So, we have :

$$\tilde{g} - \nabla v \in \mathbf{L}^p(\mathbb{R}^n) \quad \text{and} \quad \operatorname{div}(\tilde{g} - \nabla v) = 0 \quad \text{in } \mathbb{R}^n.$$

Consequently,  $(g - \nabla v) \cdot \mathbf{n} \in W^{-\frac{1}{p},p}(\Gamma_1)$  and  $v|_{\Gamma_0} \in W^{1-\frac{1}{p},p}(\Gamma_0)$ . Here, we apply the results of the theorem 3.1. There exists a unique  $w \in W_0^{1,p}(\Omega) \cap W_0^{1,2}(\Omega)$  solution of :

$$\Delta w = 0 \text{ in } \Omega, \quad w = -v \text{ on } \Gamma_0, \quad \frac{\partial w}{\partial \mathbf{n}} = (g - \nabla v) \cdot \mathbf{n} \text{ on } \Gamma_1,$$

and we show that  $\|\nabla w\|_{\mathbf{L}^p(\Omega)} \leq C_2 \|\mathbf{g}\|_{\mathbf{L}^p(\Omega)}$ .

Finally the functions  $\varphi = v|_\Omega + w$  and  $\mathbf{z} = \mathbf{g} - \nabla\varphi$  comply with the question.

□

**Theorem 3.3.** *There exists a constant  $\beta > 0$  such that*

$$\inf_{\substack{w \in V_{p'} \\ w \neq 0}} \sup_{\substack{v \in V_p \\ v \neq 0}} \frac{\int_{\Omega} \nabla v \cdot \nabla w \, d\mathbf{x}}{\|\nabla v\|_{\mathbf{L}^p(\Omega)} \|\nabla w\|_{\mathbf{L}^{p'}(\Omega)}} \geq \beta \quad (13)$$

**Proof-** Let  $w$  be in  $V_{p'}$  with  $w \neq 0$ . We notice that  $\nabla w \neq 0$  because otherwise  $w$  is constant in the connected open region  $\Omega$ , ie  $w = 0$  in  $\Omega$  because  $w = 0$  on  $\Gamma_0$ . We have

$$\|\nabla w\|_{\mathbf{L}^{p'}(\Omega)} = \sup_{\substack{\mathbf{g} \in \mathbf{L}^p(\Omega) \\ \mathbf{g} \neq 0}} \frac{\int_{\Omega} \nabla w \cdot \mathbf{g} \, d\mathbf{x}}{\|\mathbf{g}\|_{\mathbf{L}^p(\Omega)}}.$$

We easily check that for any  $\mathbf{z} \in \mathring{H}_p(\Omega)$  and any  $v \in V_{p'}$ , we have

$\int_{\Omega} \mathbf{z} \cdot \nabla v \, d\mathbf{x} = 0$ , what means that superior boundary defined above can not

be reached for a  $\mathbf{g} \in \mathring{H}_p(\Omega)$ . Let  $\mathbf{g}$  be in  $\mathbf{L}^p(\Omega)$  with  $\mathbf{g} \neq 0$ . We can suppose

that  $\mathbf{g} \notin \mathring{H}_p(\Omega)$  and thanks to the proposition 3.2, there exists  $\mathbf{z} \in \mathring{H}_p(\Omega)$

and  $\varphi \in V_p$ , with  $\nabla\varphi \neq 0$  such that  $\mathbf{g} = \mathbf{z} + \nabla\varphi$  and  $\|\nabla\varphi\|_{\mathbf{L}^p(\Omega)} \leq C \|\mathbf{g}\|_{\mathbf{L}^p(\Omega)}$ .

Thus,

$$\frac{\int_{\Omega} \nabla w \cdot \mathbf{g} \, d\mathbf{x}}{\|\mathbf{g}\|_{\mathbf{L}^p(\Omega)}} \leq C \frac{\int_{\Omega} \nabla w \cdot \nabla\varphi \, d\mathbf{x}}{\|\nabla\varphi\|_{\mathbf{L}^p(\Omega)}} \leq C \sup_{\substack{\varphi \in V_p \\ \varphi \neq 0}} \frac{\int_{\Omega} \nabla w \cdot \nabla\varphi \, d\mathbf{x}}{\|\nabla\varphi\|_{\mathbf{L}^p(\Omega)}}.$$

This is checked for any  $\mathbf{g}$  in  $\mathbf{L}^p(\Omega)$ , so :

$$\|\nabla w\|_{\mathbf{L}^{p'}(\Omega)} = \sup_{\substack{\mathbf{g} \in \mathbf{L}^p(\Omega) \\ \mathbf{g} \neq 0}} \frac{\int_{\Omega} \nabla w \cdot \mathbf{g} \, d\mathbf{x}}{\|\mathbf{g}\|_{\mathbf{L}^p(\Omega)}} \leq C \sup_{\substack{\varphi \in V_p \\ \varphi \neq 0}} \frac{\int_{\Omega} \nabla w \cdot \nabla\varphi \, d\mathbf{x}}{\|\nabla\varphi\|_{\mathbf{L}^p(\Omega)}}.$$

We deduce the estimate (13) with  $\beta = \frac{1}{C} > 0$ . □

### 3.3 The full problem

We remind here the following result :

**Theorem 3.4.** *Let  $X$  and  $M$  be two reflexive Banach spaces and  $X'$  and  $M'$  their dual spaces. Let  $b$  be a bilinear form defined and continuous on  $X \times M$ , let  $B \in \mathcal{L}(X; M')$  and  $B' \in \mathcal{L}(M, X')$  be the operators defined by :*

$$\forall v \in X, \forall w \in M, b(v, w) = \langle Bv, w \rangle = \langle v, B'w \rangle$$

*The following statements are equivalent :*

i) There exists  $\beta > 0$ , such that  $\inf_{\substack{w \in M \\ w \neq 0}} \sup_{\substack{v \in X \\ v \neq 0}} \frac{b(v, w)}{\|v\|_X \|w\|_M} \geq \beta$ .

ii) The operator  $B$  is an isomorphism from  $X/Ker B$  to  $M'$  and  $\frac{1}{\beta}$  is the continuity constant of  $B^{-1}$ .

iii) The operator  $B'$  is an isomorphism from  $M$  to  $X' \perp Ker B$  and  $\frac{1}{\beta}$  is the continuity constant of  $B'^{-1}$ .

Here, we apply this theorem with  $X = V_p$ ,  $M = V_{p'}$  and :

$$b(v, w) = \int_{\Omega} \nabla v \cdot \nabla w \, dx.$$

According to (13),

$$B \text{ is an isomorphism from } V_p/Ker B \text{ to } (V_{p'})'. \quad (14)$$

Then, we define for  $f \in Y^p(\Omega)$  and  $g \in W^{-\frac{1}{p}, p}(\Gamma_1)$  the linear form  $T$  by

$$\forall w \in V_{p'}, \quad T(w) = T_f(w) + \langle g, w \rangle_{\Gamma_1},$$

where  $\langle \cdot, \cdot \rangle_{\Gamma_1}$  denotes the duality pairing between  $W^{-\frac{1}{p}, p}(\Gamma_1)$ ,  $W^{1-\frac{1}{p}, p'}(\Gamma_1)$ . We check that  $T \in (V_{p'})'$  and thanks to (14), we deduce the existence of  $v \in V_p$ , unique up to an element of  $Ker B$ , such that  $Bv = T$ , ie :

$$(\mathcal{FV}) \quad \forall w \in V_{p'}, \quad \int_{\Omega} \nabla v \cdot \nabla w \, dx = T_f(w) + \langle g, w \rangle_{\Gamma_1}.$$

**Corollary 3.5.** For any  $f \in Y^p(\Omega)$  and  $g \in W^{-\frac{1}{p}, p}(\Gamma_1)$ , there exists a unique  $v \in W_0^{1,p}(\Omega)/Ker B$ , solution of

$$-\Delta v = f \text{ in } \Omega, \quad v = 0 \text{ on } \Gamma_0, \quad \frac{\partial v}{\partial \mathbf{n}} = g \text{ on } \Gamma_1.$$

and checking the following inequality :

$$\|v\|_{W_0^{1,p}(\Omega)/Ker B} \leq C (\|f\|_{Y^p(\Omega)} + \|g\|_{W^{-\frac{1}{p}, p}(\Gamma_1)}). \quad (15)$$

**Proof-** As we have done in the theorem 2.2, we show that the solution of the problem  $(\mathcal{FV})$  is also solution of this problem.  $\square$

**Theorem 3.6.** For any  $f \in Y^p(\Omega)$ ,  $g_0 \in W^{1-\frac{1}{p}, p}(\Gamma_0)$  and  $g_1 \in W^{-\frac{1}{p}, p}(\Gamma_1)$ , there exists a unique  $u \in W_0^{1,p}(\Omega)/Ker B$ , solution of  $(\mathcal{P})$  and there exists a constant  $C > 0$  such that :

$$\|u\|_{W_0^{1,p}(\Omega)/Ker B} \leq C (\|f\|_{Y^p(\Omega)} + \|g_0\|_{W^{1-\frac{1}{p}, p}(\Gamma_0)} + \|g_1\|_{W^{-\frac{1}{p}, p}(\Gamma_1)}).$$

**Proof-** First of all, thanks to [3], we know there exists a unique  $u_0 \in W_0^{1,p}(\Omega_1)$  where  $\Omega_1 = \Omega \cup \omega_1$ , solution of  $(\mathcal{P}_D)$  and checking

$$\|u_0\|_{W_0^{1,p}(\Omega_1)} \leq C \|g_0\|_{W^{1-\frac{1}{p}, p}(\Gamma_0)}. \quad (16)$$

Thanks to corollary 3.5, we deduce that there exists a unique  $v \in W_0^{1,p}(\Omega)/Ker B$ , solution of  $(\mathcal{P}')$ . Finally, the function  $u = u_0|_{\Omega} + v$  is solution of  $(\mathcal{P})$  and the estimate comes from (15) and (16).  $\square$

### 3.4 Characterization of the kernel of the operator B

We set :

$$\mathcal{M}_0^p(\Omega) = \{v \in W_0^{1,p}(\Omega); \Delta v = 0 \text{ in } \Omega, v = 0 \text{ on } \Gamma_0, \frac{\partial v}{\partial \mathbf{n}} = 0 \text{ on } \Gamma_1\}.$$

Thanks to the density of  $\mathcal{D}(\bar{\Omega})$  in  $W_0^{1,p'}(\Omega)$ , we easily check that

$$\text{Ker } B = \mathcal{M}_0^p(\Omega).$$

Now, we characterize  $\mathcal{M}_0^p(\Omega)$ . For this, first of all, we define :  $\mu_0 = U * (\frac{1}{|\Gamma|} \delta_\Gamma)$ ,

where  $U = \frac{1}{2\pi} \ln(r)$  is the fundamental solution of Laplace's equation in  $\mathbb{R}^2$  and  $\delta_\Gamma$  is the distribution defined by :

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^2), \langle \delta_\Gamma, \varphi \rangle = \int_\Gamma \varphi \, d\sigma$$

**Proposition 3.7.** *We have the following statements :*

i) *If  $p < n$ , then  $\mathcal{M}_0^p(\Omega) = \{0\}$ .*

ii) *If  $p \geq n \geq 3$ , then  $\mathcal{M}_0^p(\Omega) = \{c(\lambda - 1), c \in \mathbb{R}\}$  where  $\lambda$  is the only solution in  $W_0^{1,2}(\Omega) \cap W_0^{1,p}(\Omega)$  of the following problem ( $\mathcal{P}_1$ ) :*

$$\Delta \lambda = 0 \text{ in } \Omega, \lambda = 1 \text{ on } \Gamma_0, \frac{\partial \lambda}{\partial \mathbf{n}} = 0 \text{ on } \Gamma_1.$$

iii) *If  $p > n = 2$ , then  $\mathcal{M}_0^p(\Omega) = \{c(\mu - \mu_0), c \in \mathbb{R}\}$  where  $\mu$  is the only solution in  $W_0^{1,2}(\Omega) \cap W_0^{1,p}(\Omega)$  of the following problem ( $\mathcal{P}_2$ ) :*

$$\Delta \mu = 0 \text{ in } \Omega, \mu = \mu_0 \text{ on } \Gamma_0, \frac{\partial \mu}{\partial \mathbf{n}} = \frac{\partial \mu_0}{\partial \mathbf{n}} \text{ on } \Gamma_1.$$

**Proof-** Let  $z \in \mathcal{M}_0^p(\Omega)$  and let  $\eta$  be the trace of  $z$  on  $\Gamma_1$ . We have

$\eta \in W^{-\frac{1}{p},p}(\Gamma_1)$ . We know there exists a unique  $\xi \in W^{1,p}(\overset{\circ}{\omega}_1)$ , where  $\overset{\circ}{\omega}_1$  is the interior of the compact  $\omega_1$ , checking :

$$\Delta \xi = 0 \text{ in } \overset{\circ}{\omega}_1, \xi = \eta \text{ on } \Gamma_1.$$

Let  $\tilde{z}$  be defined by

$$\tilde{z} = z \text{ in } \Omega, \tilde{z} = \xi \text{ in } \omega_1, \tilde{z} = 0 \text{ in } \omega_0.$$

It is obvious that  $\tilde{z} \in W_0^{1,p}(\mathbb{R}^n)$  and  $\Delta \tilde{z} \in W_0^{-1,p}(\mathbb{R}^n)$ . Moreover, for any  $\varphi \in \mathcal{D}(\mathbb{R}^n)$ , we have :

$$\langle \Delta \tilde{z}, \varphi \rangle_{\mathcal{D}'(\mathbb{R}^n), \mathcal{D}(\mathbb{R}^n)} = - \langle \frac{\partial z}{\partial \mathbf{n}}, \varphi \rangle_{\Gamma_0} - \langle \frac{\partial \xi}{\partial \mathbf{n}}, \varphi \rangle_{\Gamma_1}.$$

We set  $h = \Delta \tilde{z}$ . Then,  $h \in W_0^{-1,p}(\mathbb{R}^n)$  and  $h$  has a compact support. At this stage, the discussion splits into two parts according to the dimension  $n$  :

i) Case  $n \geq 3$ . Thanks to the proposition 1.3, we know there exists a unique  $w$  such that :

$$w \in W_0^{1,p}(\mathbb{R}^n) \cap W_0^{1,2}(\mathbb{R}^n) \text{ and } \Delta w = h \text{ in } \mathbb{R}^n.$$

The difference  $\tilde{z} - w$  is in  $W_0^{1,p}(\mathbb{R}^n)$  and is harmonic in  $\mathbb{R}^n$ . If  $p < n$ , then  $w = \tilde{z}$  in  $\mathbb{R}^n$ , the restriction of  $w$  to  $\Omega$  is in  $W_0^{1,p}(\Omega) \cap W_0^{1,2}(\Omega)$ , and like  $z = w$  in  $\Omega$ , the section 2 implies that  $w = 0$  in  $\Omega$  i.e  $\mathcal{M}_0^p(\Omega) = \{0\}$ . If  $p \geq n \geq 3$ , we have  $w = \tilde{z} + c$  in  $\mathbb{R}^n$  so  $w \in W_0^{1,p}(\Omega) \cap W_0^{1,2}(\Omega)$  is the only solution of the problem :

$$\Delta w = 0 \text{ in } \Omega, \quad w = c \text{ on } \Gamma_0, \quad \frac{\partial w}{\partial \mathbf{n}} = 0 \text{ on } \Gamma_1.$$

Consequently  $\mathcal{M}_0^p(\Omega) = \{c(\lambda - 1), c \in \mathbb{R}\}$  where  $\lambda$  is the solution of  $(\mathcal{P}_1)$ .

ii) Case  $n = 2$ . The problem

$$\Delta w = h \text{ in } \mathbb{R}^2,$$

does not have a solution in  $W_0^{1,2}(\mathbb{R}^2)$  unless  $h$  satisfies the necessary condition  $\langle h, 1 \rangle = 0$ . In this case, with the arguments above, we obtain  $z = c(\lambda - 1)$ . However, when  $n = 2$ , the constant functions are in  $W_0^{1,p}(\Omega) \cap W_0^{1,2}(\Omega)$  so  $\lambda = 1 \in W_0^{1,p}(\Omega) \cap W_0^{1,2}(\Omega)$  is solution of  $(\mathcal{P}_1)$ . So  $z = 0$ , which is the trivial case. Thus, we suppose that  $\langle h, 1 \rangle \neq 0$  and we consider the problem :

$$\Delta w = h - \langle h, 1 \rangle \Delta \mu_0 \text{ in } \mathbb{R}^2. \quad (17)$$

We know that  $\mu_0 \in W_0^{1,q}(\mathbb{R}^2)$  for any  $q > 2$  and moreover

$$\Delta \mu_0 = 0 \text{ in } \Omega \cup \overset{\circ}{\omega}_0 \cup \overset{\circ}{\omega}_1 \quad \text{and} \quad \langle \Delta \mu_0, 1 \rangle = 1.$$

The right-hand side of (17) is orthogonal to constants, has compact support and belongs to  $W_0^{-1,p}(\mathbb{R}^2)$ . So, thanks to proposition 1.3, the problem (17) has a solution (unique up to an additive constant)  $w \in W_0^{1,2}(\mathbb{R}^2) \cap W_0^{1,p}(\mathbb{R}^2)$ . Moreover, the function  $w + \langle h, 1 \rangle \mu_0 - \tilde{z}$  is harmonic in  $\mathbb{R}^2$ . So, there exists  $c > 0$  such that  $w + \langle h, 1 \rangle \mu_0 - \tilde{z} = c$ . The restriction of  $w$  to  $\Omega$  is in  $W_0^{1,p}(\Omega) \cap W_0^{1,2}(\Omega)$  and  $w = c + w_1$  where  $w_1$  is the only solution in  $W_0^{1,2}(\Omega) \cap W_0^{1,p}(\Omega)$  of the problem :

$$\Delta w_1 = 0 \text{ in } \Omega, \quad w_1 = -\langle h, 1 \rangle \mu_0 \text{ on } \Gamma_0, \quad \frac{\partial w_1}{\partial \mathbf{n}} = -\langle h, 1 \rangle \frac{\partial \mu_0}{\partial \mathbf{n}} \text{ on } \Gamma_1.$$

The function  $\mu$  being the only solution in  $W_0^{1,2}(\Omega) \cap W_0^{1,p}(\Omega)$  of the problem  $(\mathcal{P}_2)$ . We have  $\mathcal{M}_0^p(\Omega) = \{c(\mu - \mu_0), c \in \mathbb{R}\}$ .  $\square$

### 3.5 A regularity result

We suppose, in this subsection, that  $p \geq 2$ . Here, we propose to study the question of the regularity of the solutions when the data are more regular. More precisely, we suppose that :

$$g_0 \in W^{2-\frac{1}{p},p}(\Gamma_0), \quad g_1 \in W^{1-\frac{1}{p},p}(\Gamma_1) \quad \text{and} \quad f \in X_1^{0,p}(\Omega)$$

where

$$X_1^{0,p}(\Omega) = \begin{cases} W_1^{0,p}(\Omega) & \text{if } p \neq \frac{n}{n-1}, \\ W_1^{0,p}(\Omega) \cap W_0^{-1,p}(\Omega) & \text{otherwise,} \end{cases}$$

equipped with its natural norm : (we remind that  $W_1^{0,p}(\Omega)$  is included in  $W_0^{-1,p}(\Omega)$  if and only if  $W_0^{1,p'}(\Omega) \subset W_0^{-1,p'}(\Omega)$ , this last inclusion taking place if and only if  $p \neq \frac{n}{n-1}$ ).

**Theorem 3.8.** *For any*

*$g_0 \in W^{2-\frac{1}{p},p}(\Gamma_0)$ ,  $g_1 \in W^{1-\frac{1}{p},p}(\Gamma_1)$ , and  $f \in X_1^{0,p}(\Omega)$ , there exists a unique  $u \in W_1^{2,p}(\Omega)/\mathcal{M}_0^p(\Omega)$  solution of  $(\mathcal{P})$ , and we have :*

$$\|u\|_{W_1^{2,p}(\Omega)/\mathcal{M}_0^p(\Omega)} \leq C (\|f\|_{X_1^{0,p}(\Omega)} + \|g_0\|_{W^{2-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W^{1-\frac{1}{p},p}(\Gamma_1)}) \quad (18)$$

**Proof-** For  $p \geq 2$  and  $g_0 \in W^{2-\frac{1}{p},p}(\Gamma_0)$ , we have, thanks to [3] that the solution  $u_0 \in W_0^{1,p}(\Omega_1)$  (where  $\Omega_1 = \Omega \cup \omega_1$ ) of the problem  $(\mathcal{P}_D)$  is in  $W_1^{2,p}(\Omega_1)$  and it checks :

$$\|u_0\|_{W_1^{2,p}(\Omega_1)} \leq C \|g_0\|_{W^{2-\frac{1}{p},p}(\Gamma_0)}. \quad (19)$$

now, we notice that  $\frac{\partial u_0}{\partial \mathbf{n}} \in W^{1-\frac{1}{p},p}(\Gamma_1)$  because  $u_0 \in W_1^{2,p}(\Omega_1)$ . Thus, as

$f \in X_1^{0,p}(\Omega) \subset Y^p(\Omega)$  and  $g_1 - \frac{\partial u_0}{\partial \mathbf{n}} \in W^{1-\frac{1}{p},p}(\Gamma_1) \subset W^{-\frac{1}{p},p}(\Gamma_1)$ , applying the corollary 3.5 when  $p > 2$  and the theorem 2.2 when  $p = 2$ , there exists, for  $p \geq 2$ , a unique  $v \in W_0^{1,p}(\Omega)$  solution of the problem  $(\mathcal{P}')$ . It stays to show that  $v \in W_1^{2,p}(\Omega)$  and that

$$\|v\|_{W_1^{2,p}(\Omega)/\mathcal{M}_0^p(\Omega)} \leq C (\|f\|_{X_1^{0,p}(\Omega)} + \|g_0\|_{W^{2-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W^{1-\frac{1}{p},p}(\Gamma_1)}). \quad (20)$$

For this, we follow the same reasoning as theorem 3.1 using theorem 1.2 ii) if  $p \neq \frac{n}{n-1}$  and theorem 1.2 iii) otherwise, and using regularity results in bounded open regions (see Lions and Magenes [7] for instance).

Finally, the function  $u = u_0|_{\Omega} + v \in W_1^{2,p}(\Omega)$  is solution of  $(\mathcal{P})$  and the estimate (18) is a consequence of (19) and (20).  $\square$

## 4 Case $p < 2$

We are going to proceed in two steps. First of all, thanks to an argument of duality, which allows us to use results of the previous section, we solve the problem in the case where  $f = 0$  and  $g_1 = 0$ . The sum of the solution of this problem and of a solution of a Neumann problem will permit us to solve the general problem  $(\mathcal{P})$ .

In all the section, we suppose that  $p < 2$  and that  $\Gamma$  is of class  $C^{1,1}$ .

### 4.1 Case where $f = 0$ and $g_1 = 0$ .

Let  $g_0 \in W^{1-\frac{1}{p},p}(\Gamma_0)$  checking the condition of compatibility

$$\forall z \in \mathcal{M}_0^{p'}(\Omega), \quad \langle \frac{\partial z}{\partial \mathbf{n}}, g_0 \rangle_{\Gamma_0} = 0, \quad (21)$$

where  $\langle \cdot, \cdot \rangle_{\Gamma_0}$  denotes the duality pairing  $W^{\frac{1}{p'},p'}(\Gamma_0), W^{1-\frac{1}{p},p}(\Gamma_0)$ .

With this hypothesis, we consider the problem : find  $v \in W_0^{1,p}(\Omega)$  solution of :

$$(\mathcal{Q}) \quad \begin{cases} \Delta v = 0 & \text{in } \Omega, \\ v = g_0 & \text{on } \Gamma_0, \\ \frac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } \Gamma_1. \end{cases}$$

and this other problem ( $\mathcal{Q}'$ ) : find  $v \in W_0^{1,p}(\Omega)$  such that for any  $u \in X_{p'}(\Omega)$  satisfying  $u = 0$  on  $\Gamma_0$  and  $\frac{\partial u}{\partial \mathbf{n}} = 0$  on  $\Gamma_1$ , we have :

$$T_{-\Delta u}(v) = - \langle \frac{\partial u}{\partial \mathbf{n}}, g_0 \rangle_{\Gamma_0} \quad (22)$$

where

$$X_{p'}(\Omega) = \{u \in W_0^{1,p'}(\Omega), \Delta u \in L^{p'}(\Omega)\}.$$

We easily check that the problem ( $\mathcal{Q}$ ) is equivalent to the problem ( $\mathcal{Q}'$ ).

**Theorem 4.1.** *For any  $g_0 \in W^{1-\frac{1}{p},p}(\Gamma_0)$  satisfying the compatibility condition (21), there exists a unique  $v \in W_0^{1,p}(\Omega)$  solution of the problem ( $\mathcal{Q}$ ) and we have the following estimate :*

$$\|v\|_{W_0^{1,p}(\Omega)} \leq \|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)}.$$

**Proof-** Let  $f$  be in  $Y^{p'}(\Omega)$ . It is obvious that  $T_f \in (W_0^{1,p}(\Omega))'$ . Like  $p' > 2$ , thanks to the theorem 3.6, we know there exists a unique  $u \in W_0^{1,p'}(\Omega)/\mathcal{M}_0^{p'}(\Omega)$  such that

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma_0, \quad \frac{\partial u}{\partial \mathbf{n}} = 0 \text{ on } \Gamma_1,$$

checking

$$\|u\|_{W_0^{1,p'}(\Omega)/\mathcal{M}_0^{p'}(\Omega)} \leq C \|T_f\|_{(W_0^{1,p}(\Omega))'} \quad (23)$$

Let  $L$  be the linear form defined on  $(W_0^{1,p}(\Omega))'$  by :

$$L(T_f) = - \langle \frac{\partial u}{\partial \mathbf{n}}, g_0 \rangle_{\Gamma_0}.$$

We are going to show that  $L$  is continuous. Let  $z$  be in  $\mathcal{M}_0^{p'}(\Omega)$ , then, thanks to (21), we have  $\langle \frac{\partial u}{\partial \mathbf{n}}, g_0 \rangle_{\Gamma_0} = \langle \frac{\partial(u+z)}{\partial \mathbf{n}}, g_0 \rangle_{\Gamma_0}$ . Now, let  $\theta$  be an open region of class  $C^{1,1}$  such that  $\omega_0 \subset \theta \subset B_R$  where  $R$  is the radius associated to the partition of the unity used in the definition of  $T_f$ . We set  $\Omega_\theta = \theta \setminus \omega_0$  and let  $\varphi \in W^{1,p}(\Omega_\theta)$  be such that  $\varphi = 0$  on  $\partial\theta$ . We have :

$$| \langle \frac{\partial(u+z)}{\partial \mathbf{n}}, \varphi \rangle_{\Gamma_0} | \leq \| \nabla(u+z) \|_{L^{p'}(\Omega_\theta)} \| \nabla \varphi \|_{L^p(\Omega_\theta)} + | \int_{\Omega_\theta} \varphi \Delta(u+z) \, d\mathbf{x} |.$$

But,

$$| \int_{\Omega_\theta} \varphi \Delta(u+z) \, d\mathbf{x} | = | \int_{\Omega_\theta} \varphi \Delta u \, d\mathbf{x} | = | T_{-\Delta u}(\tilde{\varphi}) |,$$

where  $\tilde{\varphi} \in W_0^{1,p}(\Omega)$  is defined by  $\tilde{\varphi} = \varphi$  in  $\Omega_\theta$ ,  $\tilde{\varphi} = 0$  in  $\Omega \setminus \theta$ . In consequence,

$$| \langle \frac{\partial(u+z)}{\partial \mathbf{n}}, \varphi \rangle_{\Gamma_0} | \leq \| \nabla(u+z) \|_{L^{p'}(\Omega_\theta)} \| \nabla \varphi \|_{L^p(\Omega_\theta)} + \| T_f \|_{(W_0^{1,p}(\Omega))'} \| \varphi \|_{W^{1,p}(\Omega_\theta)}.$$

Now, for any  $\mu \in W^{1-\frac{1}{p},p}(\Gamma_0)$ , we know there exists  $\varphi \in W^{1,p}(\Omega_\theta)$  such that  $\varphi = \mu$  on  $\Gamma_0$  and  $\varphi = 0$  on  $\partial\theta$  checking

$$\| \varphi \|_{W^{1,p}(\Omega_\theta)} \leq C \| \mu \|_{W^{1-\frac{1}{p},p}(\Gamma_0)},$$



where  $C > 0$  is a constant which depends only on  $\Omega_\theta$  and on  $\mu$ . So

$$| \langle \frac{\partial(u+z)}{\partial \mathbf{n}}, \mu \rangle_{\Gamma_0} | \leq C ( \| \nabla(u+z) \|_{L^{p'}(\Omega_\theta)} + \| T_f \|_{(W_0^{1,p}(\Omega))'} ) \| \mu \|_{W^{1-\frac{1}{p},p}(\Gamma_0)}.$$

Thus, we deduce of (23) that

$$\inf_{z \in \mathcal{M}_0^{p'}(\Omega)} \| \frac{\partial(u+z)}{\partial \mathbf{n}} \|_{W^{-1/p',p'}(\Gamma_0)} \leq C \| T_f \|_{(W_0^{1,p}(\Omega))'}.$$

So

$$|L(T_f)| = | \langle \frac{\partial u}{\partial \mathbf{n}}, g_0 \rangle | \leq C \| T_f \|_{(W_0^{1,p}(\Omega))'} \| g_0 \|_{W^{1-\frac{1}{p},p}(\Gamma_0)},$$

and the linear form  $L$  is continuous on  $(W_0^{1,p}(\Omega))'$ . Like the space  $W_0^{1,p}(\Omega)$  is reflexive, we can identify  $L$  to an element of  $W_0^{1,p}(\Omega)$ , *i.e* there exists a unique  $v \in W_0^{1,p}(\Omega)$  such that :

$$T_f(v) = - \langle \frac{\partial u}{\partial \mathbf{n}}, g_0 \rangle_{\Gamma_0},$$

and checking the estimate searched. In consequence, the function  $v$  is solution of the problem  $(\mathcal{Q}')$  equivalent to the problem  $(\mathcal{Q})$ .  $\square$

## 4.2 The general problem when $p < 2$ .

Let  $f \in Y^p(\Omega)$ ,  $g_0 \in W^{1-\frac{1}{p},p}(\Gamma_0)$  and  $g_1 \in W^{-\frac{1}{p},p}(\Gamma_1)$ . We remind that we search  $u \in W_0^{1,p}(\Omega)$  solution of the problem  $(\mathcal{P})$ . Assuming that a such solution  $u \in W_0^{1,p}(\Omega)$  exists, for any  $\varphi \in \mathcal{M}_0^{p'}(\Omega)$ , we have thanks to the density of  $\mathcal{D}(\bar{\Omega})$  in  $W_0^{1,p'}(\Omega)$  :

$$T_f(\varphi) = \int_{\Omega} \nabla u \cdot \nabla \varphi \, d\mathbf{x} - \langle g_1, \varphi \rangle_{\Gamma_1}. \quad (24)$$

Like  $\mathcal{D}(\bar{\Omega})$  is also dense in  $W_0^{1,p}(\Omega)$ , we have, for any  $\varphi \in \mathcal{M}_0^{p'}(\Omega)$  :

$$\int_{\Omega} \nabla \varphi \cdot \nabla u \, d\mathbf{x} = \langle \frac{\partial \varphi}{\partial \mathbf{n}}, g_0 \rangle_{\Gamma_0}. \quad (25)$$

We deduce of (24) and (25) that if  $u \in W_0^{1,p}(\Omega)$  is solution of the problem  $(\mathcal{P})$ , the data must checking the following condition of compatibility :

$$\forall \varphi \in \mathcal{M}_0^{p'}(\Omega), \quad T_f(\varphi) = \langle \frac{\partial \varphi}{\partial \mathbf{n}}, g_0 \rangle_{\Gamma_0} - \langle g_1, \varphi \rangle_{\Gamma_1}. \quad (26)$$

**Theorem 4.2.** *Let  $p < 2$ ,  $f \in Y^p(\Omega)$ ,  $g_0 \in W^{1-\frac{1}{p},p}(\Gamma_0)$ ,  $g_1 \in W^{-\frac{1}{p},p}(\Gamma_1)$  be satisfying the condition of compatibility (26) if  $1 < p \leq \frac{n}{n-1}$ , then there exists a unique  $u \in W_0^{1,p}(\Omega)$  solution of the problem  $(\mathcal{P})$  and there exists  $C > 0$  such that*

$$\| u \|_{W_0^{1,p}(\Omega)} \leq C ( \| f \|_{Y^p(\Omega)} + \| g_0 \|_{W^{1-\frac{1}{p},p}(\Gamma_0)} + \| g_1 \|_{W^{-\frac{1}{p},p}(\Gamma_1)} ).$$

**Proof-** First of all, we notice that the condition (26) is always satisfied if  $\frac{n}{n-1} < p < 2$  because in this case  $p' < n$  and  $\mathcal{M}_0^{p'}(\Omega) = \{0\}$ . Let  $\theta$  be the constant defined by :

$$T_f(1) + \langle g_1, 1 \rangle_{\Gamma_1} + \langle \theta, 1 \rangle_{\Gamma_0} = 0. \quad (27)$$

According to [3], thanks to (27), there exists a unique  $w \in W_0^{1,p}(\Omega)$  such that :

$$-\Delta w = f \text{ in } \Omega, \quad \frac{\partial w}{\partial \mathbf{n}} = \theta \text{ on } \Gamma_0, \quad \frac{\partial w}{\partial \mathbf{n}} = g_1 \text{ on } \Gamma_1.$$

Moreover, for any  $\varphi \in \mathcal{M}_0^{p'}(\Omega)$ , we have, thanks to (26), that

$$\langle \frac{\partial \varphi}{\partial \mathbf{n}}, g_0 - w \rangle_{\Gamma_0} = 0. \quad (28)$$

So, we can apply theorem 4.1 which assures existence of a unique  $v \in W_0^{1,p}(\Omega)$  such that :

$$\Delta v = 0 \text{ in } \Omega, \quad v = g_0 - w \text{ on } \Gamma_0, \quad \frac{\partial v}{\partial \mathbf{n}} = 0 \text{ on } \Gamma_1,$$

satisfying

$$\|v\|_{W_0^{1,p}(\Omega)} \leq C (\|w\|_{W_0^{1,p}(\Omega)} + \|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)}). \quad (29)$$

Finally, the function  $u = v + w \in W_0^{1,p}(\Omega)$  is the solution searched and the inequality of continuous dependance comes from (29) and :

$$\|w\|_{W_0^{1,p}(\Omega)} \leq C (\|f\|_{Y^p(\Omega)} + \|g_1\|_{W^{-\frac{1}{p},p}(\Gamma_1)}). \quad \square$$

### 4.3 A regularity result

We suppose, in this subsection, that  $p < 2$ . Here, we study the regularity of solutions when the data are more regular.

**Theorem 4.3.** *For any  $g_0 \in W^{2-\frac{1}{p},p}(\Gamma_0)$ ,  $g_1 \in W^{1-\frac{1}{p},p}(\Gamma_1)$ ,  $f \in X_1^{0,p}(\Omega)$  satisfying the condition (26) if  $1 < p < \frac{n}{n-1}$ , there exists a unique  $u \in W_1^{2,p}(\Omega)$  solution of  $(\mathcal{P})$ , and we have :*

$$\|u\|_{W_1^{2,p}(\Omega)} \leq C (\|f\|_{X_1^{0,p}(\Omega)} + \|g_0\|_{W^{2-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W^{1-\frac{1}{p},p}(\Gamma_1)}) \quad (30)$$

**Proof-** We retake the proof of the theorem 4.2. We know, thanks to [3] that the solution  $w \in W_0^{1,p}(\Omega)$  of the problem  $(\mathcal{Q}_1)$  is, in this case, in  $W_1^{2,p}(\Omega)$  and checks :

$$\|w\|_{W_1^{2,p}(\Omega)} \leq C (\|f\|_{X_1^{0,p}(\Omega)} + \|g_1\|_{W^{1-\frac{1}{p},p}(\Gamma_1)}). \quad (31)$$

Now, we notice that, on  $\Gamma_0$ ,  $g_0 - w \in W^{2-\frac{1}{p},p}(\Gamma_0) \subset W^{1-\frac{1}{p},p}(\Gamma_0)$ . Thus, applying the theorem 4.2, the condition of compatibility (26) being checked if  $1 < p < \frac{n}{n-1}$ , there exists a unique  $v \in W_0^{1,p}(\Omega)$  solution of the problem  $(\mathcal{Q}_2)$ . It stays to show that  $v \in W_1^{2,p}(\Omega)$  and that

$$\|v\|_{W_1^{2,p}(\Omega)} \leq C (\|f\|_{X_1^{0,p}(\Omega)} + \|g_0\|_{W^{2-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W^{1-\frac{1}{p},p}(\Gamma_1)}). \quad (32)$$

For this, we follow the same reasoning that at the proposition 3.1 using the theorem 1.2 ii) if  $p \neq \frac{n}{n-1}$  and the theorem 1.2 iii) otherwise and also using regularity results in bounded open regions (see Lions and Magenes [7] for instance).

Finally,  $u = w + v \in W_1^{2,p}(\Omega)$  is the solution searched and by (31) and (32), we have (30).  $\square$

## 5 Solutions in homogeneous spaces

In all the section we suppose that  $p < n$ .

Let  $v_\infty$  be in  $\mathbb{R}$ . It is frequent to meet in the litterature the following problem : find  $v \in \mathcal{D}'(\Omega)$ , with  $\nabla v \in \mathbf{L}^p(\Omega)$  solution of

$$(\mathcal{R}) \begin{cases} -\Delta v = f & \text{in } \Omega, \\ v = g_0 & \text{on } \Gamma_0, \\ \frac{\partial v}{\partial \mathbf{n}} = g_1 & \text{on } \Gamma_1, \\ v \rightarrow v_\infty & \text{at infinity.} \end{cases}$$

where the sense of the convergence  $v \rightarrow v_\infty$  is specified in the following proposition (see [4]).

**Proposition 5.1.** *We suppose that  $1 < p < n$  and  $z \in \mathcal{D}'(\Omega)$  such that  $\nabla z \in \mathbf{L}^p(\Omega)$ . Then, there exists a unique constant  $z_\infty \in \mathbb{R}$  such that  $z - z_\infty \in W_0^{1,p}(\Omega)$ , where  $z_\infty$  is defined by :*

$$z_\infty = \lim_{|\mathbf{x}| \rightarrow +\infty} \frac{1}{|S_n|} \int_{S_n} z(\sigma|\mathbf{x}|) d\sigma$$

Moreover, we have the following proprieties :

$$\begin{aligned} z - z_\infty &\in L^{\frac{np}{n-p}}(\Omega), \\ \|z - z_\infty\|_{L^{\frac{np}{n-p}}(\Omega)} &\leq C \|\nabla z\|_{\mathbf{L}^p(\Omega)}, \\ \lim_{|\mathbf{x}| \rightarrow +\infty} \frac{1}{|S_n|} \int_{S_n} |z(\sigma|\mathbf{x}|) - z_\infty| d\sigma &= \lim_{|\mathbf{x}| \rightarrow +\infty} \frac{1}{|S_n|} \int_{S_n} |z(\sigma|\mathbf{x}|) - z_\infty|^p d\sigma = 0 \\ \int_{S_n} |z(r\sigma) - z_\infty|^p d\sigma &\leq Cr^{p-n} \int_{\{\mathbf{x} \in \Omega, |\mathbf{x}| > r\}} |\nabla z|^p d\mathbf{x}. \end{aligned}$$

Let  $z \in \mathcal{D}'(\Omega)$  be such that  $\nabla z \in \mathbf{L}^p(\Omega)$ . So, we say that  $z \rightarrow z_\infty$  if and only if :

$$\lim_{|\mathbf{x}| \rightarrow +\infty} \frac{1}{|S_n|} \int_{S_n} (z(\sigma|\mathbf{x}|) - z_\infty) d\sigma = 0.$$

It is obvious that if  $z \in \mathcal{D}'(\Omega)$ ,  $\nabla z \in \mathbf{L}^p(\Omega)$ , then  $z \rightarrow z_\infty$  is equivalent to  $z - z_\infty \in L^{\frac{np}{n-p}}(\Omega)$  or to  $z - z_\infty \in W_0^{1,p}(\Omega)$ .

**Proposition 5.2.** *For any  $f \in Y^p(\Omega)$ ,  $g_0 \in W^{1-\frac{1}{p},p}(\Gamma_0)$ ,  $g_1 \in W^{-\frac{1}{p},p}(\Gamma_1)$  and  $v_\infty \in \mathbb{R}$ , with  $1 < p < n$ , there exists a unique  $v \in \mathcal{D}'(\Omega)$  with  $\nabla v \in \mathbf{L}^p(\Omega)$  solution of  $(\mathcal{R})$  and we have the following estimate :*

$$\|v - v_\infty\|_{W_0^{1,p}(\Omega)} \leq C ( \|f\|_{Y^p(\Omega)} + \|g_0 - v_\infty\|_{W^{1-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W^{-\frac{1}{p},p}(\Gamma_1)} ).$$

**Proof-** The previous sections allow us to say that there exists a unique  $u \in W_0^{1,p}(\Omega)$  solution of

$$-\Delta u = f \text{ in } \Omega, \quad u = g_0 - v_\infty \text{ on } \Gamma_0, \quad \frac{\partial u}{\partial \mathbf{n}} = g_1 \text{ on } \Gamma_1,$$

and checking

$$\|u\|_{W_0^{1,p}(\Omega)} \leq C (\|f\|_{Y^p(\Omega)} + \|g_0 - v_\infty\|_{W^{1-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W^{-\frac{1}{p},p}(\Gamma_1)}). \quad (33)$$

It is obvious that  $u \rightarrow 0$ . Thus, the function  $v = u + v_\infty$  belongs to  $\mathcal{D}'(\Omega)$ , checks  $\nabla v \in \mathbf{L}^p(\Omega)$  and  $v \rightarrow v_\infty$ . So,  $v$  is solution of  $(\mathcal{R})$  and the estimate searched comes from the inequality (33).  $\square$

Now, let  $\mathbf{a}_\infty$  be in  $\mathbb{R}^n$ . We want to solve the following problem : find

$v \in \mathcal{D}'(\Omega)$ , such that for any  $i, j = 1, \dots, n$ , we have  $\frac{\partial^2 v}{\partial x_i \partial x_j} \in L^p(\Omega)$ , solution of

$$(\mathcal{R}') \quad \begin{cases} -\Delta v = f & \text{in } \Omega, \\ v = g_0 & \text{on } \Gamma_0, \\ \frac{\partial v}{\partial \mathbf{n}} = g_1 & \text{on } \Gamma_1, \\ \nabla v \rightarrow \mathbf{a}_\infty & \text{at infinity.} \end{cases}$$

**Proposition 5.3.** *For any  $f \in L^p(\Omega)$ ,  $g_0 \in W^{2-\frac{1}{p},p}(\Gamma_0)$ ,  $g_1 \in W^{1-\frac{1}{p},p}(\Gamma_1)$  and  $\mathbf{a}_\infty \in \mathbb{R}^n$ , with  $1 < p < n$ , there exists a unique  $v \in \mathcal{D}'(\Omega)$  with, for any  $i, j = 1, \dots, n$ ,  $\frac{\partial^2 v}{\partial x_i \partial x_j} \in L^p(\Omega)$ , solution of  $(\mathcal{R}')$  and we have the following estimate :*

$$\|v - \mathbf{a}_\infty \cdot \mathbf{x}\|_{W_0^{2,p}(\Omega)} \leq C (\|f\|_{L^p(\Omega)} + \|g_0 - \mathbf{a}_\infty \cdot \mathbf{x}\|_{W^{2-\frac{1}{p},p}(\Gamma_0)} + \|g_1 - \mathbf{a}_\infty \cdot \mathbf{n}\|_{W^{1-\frac{1}{p},p}(\Gamma_1)}).$$

**Proof-** First of all, we set  $\tilde{f}$  the extension of  $f$  by 0 on  $\mathbb{R}^n$ . So, we have  $\tilde{f} \in L^p(\mathbb{R}^n)$ . Moreover, we know that  $\Delta : W_0^{2,p}(\mathbb{R}^n)/\mathcal{P}_{[2-n/p]} \rightarrow L^p(\mathbb{R}^n)$  is an isomorphism, so there exists  $\tilde{u} \in W_0^{2,p}(\mathbb{R}^n)$  unique up to an additive constant, such that  $-\Delta \tilde{u} = \tilde{f}$  in  $\mathbb{R}^n$ . We have  $\tilde{u}|_\Omega \in W_0^{2,p}(\Omega)$  and  $-\Delta \tilde{u}|_\Omega = f$  in  $\Omega$ . Thanks to the regularity results of the previous sections, we know there exists a unique  $z \in W_1^{2,p}(\Omega) \subset W_0^{2,p}(\Omega)$  solution of the problem

$$\Delta z = 0 \text{ in } \Omega, \quad z = g_0 - \mathbf{a}_\infty \cdot \mathbf{x} - \tilde{u} \text{ on } \Gamma_0, \quad \frac{\partial z}{\partial \mathbf{n}} = g_1 - \mathbf{a}_\infty \cdot \mathbf{n} - \left(\frac{\partial \tilde{u}}{\partial \mathbf{n}}\right) \text{ on } \Gamma_1.$$

So the function  $u = \tilde{u}|_\Omega + z$  is in  $W_0^{2,p}(\Omega)$  and is solution of the problem

$$-\Delta u = f \text{ in } \Omega, \quad u = g_0 - \mathbf{a}_\infty \cdot \mathbf{x} \text{ on } \Gamma_0, \quad \frac{\partial u}{\partial \mathbf{n}} = g_1 - \mathbf{a}_\infty \cdot \mathbf{n} \text{ on } \Gamma_1,$$

checking the following estimate :

$$\|u\|_{W_0^{2,p}(\Omega)} \leq C (\|f\|_{L^p(\Omega)} + \|g_0 - \mathbf{a}_\infty \cdot \mathbf{x}\|_{W^{2-\frac{1}{p},p}(\Gamma_0)} + \|g_1 - \mathbf{a}_\infty \cdot \mathbf{n}\|_{W^{1-\frac{1}{p},p}(\Gamma_1)}). \quad (34)$$

We have also  $\nabla u \in W_0^{1,p}(\Omega)$  and  $\nabla u \rightarrow 0$ . Now, we set  $v = u + \mathbf{a}_\infty \cdot \mathbf{x}$ . We

have  $v \in \mathcal{D}'(\Omega)$  and  $\forall i, j = 1, \dots, n$ ,  $\frac{\partial^2 v}{\partial x_i \partial x_j} = \frac{\partial^2 u}{\partial x_i \partial x_j} \in L^p(\Omega)$  because

$u \in W_0^{2,p}(\Omega)$ . Moreover  $\nabla v = \nabla u + \mathbf{a}_\infty$  so  $\nabla v \rightharpoonup \mathbf{a}_\infty$ . So  $v$  is solution of  $(\mathcal{R}')$  and the estimate searched comes from (34).  $\square$

Remark : If we take  $f \in X_1^{0,p}(\Omega)$ , we can show that for any  $g_0 \in W^{2-\frac{1}{p},p}(\Gamma_0)$ ,  $g_1 \in W^{1-\frac{1}{p},p}(\Gamma_1)$ ,  $\mathbf{a}_\infty \in \mathbb{R}^n$  and  $b_\infty \in \mathbb{R}$ , there exists a unique  $v \in \mathcal{D}'(\Omega)$  with, for any  $i, j = 1, \dots, n$ ,  $\frac{\partial^2 v}{\partial x_i \partial x_j} \in L^p(\Omega)$ , solution of the following problem

$$(\mathcal{R}'') \left\{ \begin{array}{ll} -\Delta v = f & \text{in } \Omega, \\ v = g_0 & \text{on } \Gamma_0, \\ \frac{\partial v}{\partial \mathbf{n}} = g_1 & \text{on } \Gamma_1, \\ v - \mathbf{a}_\infty \cdot \mathbf{x} - b_\infty \rightharpoonup 0 & \text{at infinity.} \end{array} \right.$$

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