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L^p -Inequalities for Scalar Oseen Potential

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Abstract

The Oseen equations are obtained by linearizing the Navier-Stokes equations around a nonzero constant vector which is the velocity at infinity. We are interested with the study of the scalar problem corresponding to the anisotropic operator $-\Delta + \frac{\partial}{\partial x_1}$. The Marcinkiewicz interpolation's theorem and the Sobolev embeddings are used to give, in the L^p theory, the continuity's properties of the scalar Oseen potential. The contribution of the term $\frac{\partial}{\partial x_1}$ gives supplementary properties with regard to the Riesz potential.

Key words: Oseen potential, weighted Sobolev spaces, L^p -theory.

AMS Classification: 26D10, 35Q35, 42B20.

1 Introduction

The Oseen system is obtained by linearizing the Navier-Stokes equations around a nonzero constant vector $\mathbf{u} = \mathbf{u}_\infty$, where $\mathbf{u}_\infty = \lambda \mathbf{e}_1$ is the velocity at infinity, and can be written as follow (see [14], [15]):

$$\begin{aligned} -\nu \Delta \mathbf{u} + \lambda \frac{\partial \mathbf{u}}{\partial x_1} + \nabla \pi &= \mathbf{f} \quad \text{in } \mathbb{R}^n, \\ \operatorname{div} \mathbf{u} &= g \quad \text{in } \mathbb{R}^n, \end{aligned} \tag{1.1}$$

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where $n \geq 2$. The data are, the external forces \mathbf{f} , a function g , the positive real number λ and the viscosity of the fluid ν . The unknowns of this system are the velocity \mathbf{u} and the pressure π . One of the first works devoted to these equations is due to Finn [6], [7]. Specifically, Finn studied (1.1) Oseen's equations in three and two dimensional exterior domains when $(1 + |\mathbf{x}|)\mathbf{f} \in L^2(\mathbb{R}^3)$ and $g = 0$. He proved that this system has a unique solution \mathbf{u} such that $(1 + |\mathbf{x}|)^{-1}\mathbf{u} \in L^2(\mathbb{R}^3)$. When $(\mathbf{f}, g) \in \mathbf{W}^{m,p}(\mathbb{R}^n) \times W^{m+1,p}(\mathbb{R}^n)$ with $m \geq 0$, Galdi [8] proved that Problem (1.1) has a solution $(\mathbf{u}, \pi) \in \mathbf{W}_{loc}^{m,p}(\mathbb{R}^n) \times W_{loc}^{m+1,p}(\mathbb{R}^n)$ and some results on the derivatives of \mathbf{u} and π under conditions on the power p . Recently, Amrouche and Razafison [2] investigated this problem by working in weighted Sobolev spaces.

Taking the divergence of the first equation of (1.1), we observe that the pressure satisfies the Poisson's equation (see for instance [1]):

$$\Delta\pi = \operatorname{div}\mathbf{f} + \nu\Delta g - \lambda\frac{\partial g}{\partial x_1} \quad \text{in } \mathbb{R}^n, \quad (1.2)$$

and the vector field \mathbf{u} satisfies

$$-\nu\Delta\mathbf{u} + \lambda\frac{\partial\mathbf{u}}{\partial x_1} = \mathbf{f} - \nabla\pi \quad \text{in } \mathbb{R}^n. \quad (1.3)$$

Now observe that each component u_j of the velocity satisfies

$$-\nu\Delta u_j + \lambda\frac{\partial u_j}{\partial x_1} = f_j - \frac{\partial\pi}{\partial x_j} \quad \text{in } \mathbb{R}^n. \quad (1.4)$$

Hence, we see that the Oseen problem (1.1) can be reduced to the following scalar equation

$$-\nu\Delta u + \lambda\frac{\partial u}{\partial x_1} = f \quad \text{in } \mathbb{R}^n, \quad (1.5)$$

where $f = f_j - \frac{\partial\pi}{\partial x_j}$. Working in the L^2 spaces with anisotropic weights, Farwig [3] treated this equation in three dimensional exterior domain. The purpose of this work is the study of the potential of the scalar Oseen operator:

$$T : u \longmapsto -\Delta u + \frac{\partial u}{\partial x_1}, \quad (1.6)$$

and more precisely the boundedness of the operators $R : f \mapsto \mathcal{O} * f$, $R_j : f \mapsto \frac{\partial}{\partial x_j}(\mathcal{O} * f)$ and $R_{j,k} : f \mapsto \frac{\partial^2}{\partial x_j \partial x_k}(\mathcal{O} * f)$, for f given in $L^p(\mathbb{R}^n)$ or in $W_0^{-1,p}(\mathbb{R}^n)$. The obtained estimates can be applied to the investigation of qualitative properties of solutions of Navier-Stokes equations with a non-zero constant velocity at infinity.

2 Notations and Functional Spaces

Throughout this paper, p is a real number in the interval $]1, +\infty[$ and p' the conjugate exponent of p . A point in \mathbb{R}^n is denoted by $\mathbf{x} = (x_1, \dots, x_n)$ and we set:

$$r = |\mathbf{x}| = (x_1^2 + \dots + x_n^2)^{1/2}, \quad s = r - x_1 \quad \text{and} \quad s' = r + x_1.$$

For $R > 0$, B_R denotes the open ball of radius R centered at the origin and $B'_R = \mathbb{R}^2 \setminus \overline{B_R}$. For any $j \in \mathbb{Z}$, \mathcal{P}_j is the space of polynomials of degree lower than or equal to j . If j is a negative integer, we set by convention $\mathcal{P}_j = 0$. Given a Banach space B , with dual space B' and a closed subspace X of B , we denote by $B' \perp X$ the subspace of B' orthogonal to X :

$$B' \perp X = \{f \in B'; \forall v \in X, \langle f, v \rangle = 0\}.$$

We define the weighted Sobolev space

$$W_0^{1,p}(\mathbb{R}^n) = \left\{ u \in \mathcal{D}'(\mathbb{R}^n); \omega^{-1}u \in L^p(\mathbb{R}^n); \nabla u \in \mathbf{L}^p(\mathbb{R}^n) \right\},$$

where, $\omega = 1 + r$ if $p \neq n$ and $\omega = (1 + r)\ln(2 + r)$ if $p = n$. Equipped with it's natural norm:

$$\| u \|_{W_0^{1,p}(\mathbb{R}^n)} = \left(\| \omega^{-1}u \|_{L^p(\mathbb{R}^n)}^p + \| \nabla u \|_{\mathbf{L}^p(\mathbb{R}^n)}^p \right)^{\frac{1}{p}},$$

it is a reflexive Banach space. We denote its semi-norm by: $| u |_{W_0^{1,p}(\mathbb{R}^n)} = \| \nabla u \|_{\mathbf{L}^p(\mathbb{R}^n)}$. For more details on these spaces, see [12], [10] and [1]. However, we recall some properties and results that we use in this paper. The space $W_0^{1,p}(\mathbb{R}^n)$ contains constants when $p \geq n$ and no polynomials otherwise. The space of smooth functions with compact support $\mathcal{D}(\mathbb{R}^n)$ is dense in $W_0^{1,p}(\mathbb{R}^n)$. Then, its dual space denoted by $W_0^{-1,p'}(\mathbb{R}^n)$ is a space of distributions. We recall that there exists a constant C such that (see [1]):

$$\forall u \in W_0^{1,p}(\mathbb{R}^n), \quad \inf_{k \in \mathcal{P}_{[1-\frac{n}{p}]}} \| u + k \|_{W_0^{1,p}(\mathbb{R}^n)} \leq C \| \nabla u \|_{\mathbf{L}^p(\mathbb{R}^n)}. \quad (2.1)$$

Which implies that, in particular when $p < n$, the full norm on $W_0^{1,p}(\mathbb{R}^n)$ is equivalent to the semi norm. Inequality (2.1) permits to prove that the following gradient and divergence operators are isomorphisms (see [1]):

$$\nabla : W_0^{1,p}(\mathbb{R}^n)/\mathcal{P}_{[1-\frac{n}{p}]} \longrightarrow \mathbf{L}^p(\mathbb{R}^n) \perp \mathbf{H}_{p'}, \quad (2.2)$$

$$\text{div} : \mathbf{L}^{p'}(\mathbb{R}^n)/\mathbf{H}_{p'} \longrightarrow W_0^{1,p'}(\mathbb{R}^n) \perp \mathcal{P}_{[1-\frac{n}{p}]}, \quad (2.3)$$

where, $\mathbf{H}_p = \{\mathbf{v} \in \mathbf{L}^p(\mathbb{R}^n), \operatorname{div} \mathbf{v} = 0\}$. Inequality (2.1) also allows to have the important following result (see [1] and [2] for the expression of $k(u)$ which is given in the case $n = 3$):

Proposition 2.1 *Let u a distribution such that $\nabla u \in \mathbf{L}^p(\mathbb{R}^n)$.*

i) If $1 < p < n$, there exists a unique constant $k(u) \in \mathbb{R}$ defined by:

$$k(u) = - \lim_{|\mathbf{x}| \rightarrow \infty} \frac{1}{\omega_n} \int_{S_{n-1}} u(\sigma|\mathbf{x}|) d\sigma, \quad (2.4)$$

where ω_n denotes the area of the sphere S_{n-1} , such that $u + k(u) \in W_0^{1,p}(\mathbb{R}^n)$, and

$$\|u + k(u)\|_{W_0^{1,p}(\mathbb{R}^n)} \leq C \|\nabla u\|_{\mathbf{L}^p(\mathbb{R}^n)}. \quad (2.5)$$

ii) If $p \geq n$, then $u \in W_0^{1,p}(\mathbb{R}^n)$ and

$$\inf_{k \in \mathbb{R}} \|u + k\|_{W_0^{1,p}(\mathbb{R}^n)} \leq C \|\nabla u\|_{\mathbf{L}^p(\mathbb{R}^n)}. \quad (2.6)$$

We recall the Sobolev embeddings:

$$W_0^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n) \quad \text{if } 1 < p < n, \quad \text{with } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}, \quad (2.7)$$

$$W_0^{1,n}(\mathbb{R}^n) \hookrightarrow VMO(\mathbb{R}^n), \quad (2.8)$$

where,

$$VMO(\mathbb{R}^n) = \overline{\mathcal{D}(\mathbb{R}^n)}^{\|\cdot\|_{BMO}}.$$

The space BMO is defined as follows: A locally integrable function f belongs to BMO if

$$\|f\|_{BMO} =: \sup_Q \frac{1}{|Q|} \int_Q |f(\mathbf{x}) - f_Q| d\mathbf{x} < \infty,$$

where, the supremum is taken on all the cubes and $f_Q = \frac{1}{|Q|} \int_Q f(\mathbf{x}) d\mathbf{x}$ is the average of f on Q .

Note that if $\nabla u \in \mathbf{L}^p(\mathbb{R}^n)$, with $p > n$ and $u \in L^r(\mathbb{R}^n)$ for some $r \geq 1$ then

$$u \in L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n), \quad (2.9)$$

and if $p = n$, then u belongs to $L^q(\mathbb{R}^n)$ for any $q \geq r$.

We introduce also the following space (see [2]):

$$\widetilde{W}_0^{1,p}(\mathbb{R}^n) = \left\{ v \in W_0^{1,p}(\mathbb{R}^n); \frac{\partial u}{\partial x_1} \in W_0^{-1,p}(\mathbb{R}^n) \right\},$$

which is a reflexive Banach space for the norm:

$$\|v\|_{\widetilde{W}_0^{1,p}(\mathbb{R}^n)} = \|v\|_{W_0^{1,p}(\mathbb{R}^n)} + \left\| \frac{\partial u}{\partial x_1} \right\|_{W_0^{-1,p}(\mathbb{R}^n)}.$$

3 The fundamental solution.

Following the idea of [8], we get the following fundamental solution \mathcal{O} :

i) For $n = 3$,

$$\mathcal{O}(\mathbf{x}) = \frac{1}{4\pi r} e^{-\frac{s}{2}}. \quad (3.1)$$

ii) For $n = 2$, the fundamental solution has, in a neighbourhood of the origin, the following behaviour:

$$\mathcal{O}(\mathbf{x}) = -\frac{1}{2\pi} e^{\frac{x_1}{2}} \left\{ \ln \frac{1}{r} + 2\ln 2 - \gamma + \sigma(r) \right\}, \quad (3.2)$$

where σ satisfies $\frac{d^k \sigma}{dr^k} = or^{-k}$. When r is sufficiently large, we have:

$$\mathcal{O}(\mathbf{x}) = -\frac{1}{2\sqrt{\pi r}} e^{-\frac{s}{2}} \left[1 - \frac{1}{4r} + O(r^{-2}) \right].$$

Using the inequality $e^{-s/2} \leq C_b(1+s)^b$, which holds for any real number b , we obtain the following anisotropic estimates for r sufficiently large:

$$|\mathcal{O}(\mathbf{x})| \leq C r^{-1}(1+s)^{-2}, \quad \left| \frac{\partial \mathcal{O}}{\partial x_1}(\mathbf{x}) \right| \leq C r^{-2}(1+s)^{-\frac{3}{2}}, \quad (3.3)$$

$$\left| \frac{\partial \mathcal{O}}{\partial x_j}(\mathbf{x}) \right| \leq C r^{-\frac{3}{2}}(1+s)^{-\frac{3}{2}} \left(1 + \frac{2}{r}\right), \quad j = 2, 3, \quad \text{if } n = 3,$$

$$|\mathcal{O}(\mathbf{x})| \leq C r^{-\frac{1}{2}}(1+s)^{-1}, \quad \left| \frac{\partial \mathcal{O}}{\partial x_1}(\mathbf{x}) \right| \leq C r^{-\frac{3}{2}}(1+s)^{-1}, \quad (3.4)$$

$$\left| \frac{\partial \mathcal{O}}{\partial x_2}(\mathbf{x}) \right| \leq C r^{-1}(1+s)^{-1}, \quad \text{if } n = 2.$$

In order to study the integrability properties of the fundamental solution and its derivatives, we need to estimate the following integral (for the proof, see [11]):

$$\alpha, \beta \in \mathbb{R}, \quad I_{\alpha, \beta} = \int_{|\mathbf{x}| > \mu} r^{-\alpha}(1+s)^{-\beta} d\mathbf{x}, \quad \text{with } \mu > 0.$$

Lemma 3.1 *Assume that $n - \alpha - \min(\frac{n-1}{2}, \beta) < 0$. Then, there exists a constant $C > 0$ such that, for all $\mu > 1$, we have*

$$I_{\alpha, \beta} \leq C \mu^{n - \alpha - \min(\frac{n-1}{2}, \beta)}, \quad \text{if } \beta \neq \frac{n-1}{2},$$

$$I_{\alpha, \beta} \leq C \mu^{\frac{n+1}{2} - \alpha} \ln r, \quad \text{if } \beta = \frac{n-1}{2}.$$

This lemma allows us to derive the following integrability properties of \mathcal{O} and its gradient:

$$\forall p > 3, \quad \mathcal{O} \in L^p(\mathbb{R}^2) \quad \text{and} \quad \forall p \in \left] \frac{3}{2}, 2[\right], \quad \nabla \mathcal{O} \in L^p(\mathbb{R}^2), \quad (3.5)$$

$$\forall p \in]2, 3[, \mathcal{O} \in L^p(\mathbb{R}^3) \quad \text{and} \quad \forall p \in]\frac{4}{3}, \frac{3}{2}[, \nabla \mathcal{O} \in L^p(\mathbb{R}^3). \quad (3.6)$$

Note also that

$$\mathcal{O} \in L^1_{loc}(\mathbb{R}^n) \quad \text{and} \quad \nabla \mathcal{O} \in L^1_{loc}(\mathbb{R}^n), \quad \text{for } n = 2, 3. \quad (3.7)$$

3.1 Study of the kernel

Using the Fourier's transform, the kernel of the operator T , when it is defined on the tempered distributions $\mathcal{S}'(\mathbb{R}^n)$, is given by:

Lemma 3.2 *Let $f \in \mathcal{S}'(\mathbb{R}^n)$ be a tempered distribution and let $u \in \mathcal{S}'(\mathbb{R}^n)$ be a solution of (1.6). Then u is uniquely determined up to polynomials of \mathcal{S}_k , where:*

$$\mathcal{S}_k = \{q \in \mathcal{P}_k; -\Delta q + \frac{\partial q}{\partial x_1} = 0\}. \quad (3.8)$$

Let us notice that $\mathcal{S}_0 = \mathbb{R}$ and \mathcal{S}_1 is the space of polynomials of degree lower than or equal one and not depending on x_1 .

4 Scalar Oseen Potential in three dimensional

This section is devoted to the L^p estimates of convolutions with Oseen kernels.

Theorem 4.1 *Let $f \in L^p(\mathbb{R}^3)$. Then $\frac{\partial^2}{\partial x_j \partial x_k}(\mathcal{O} * f) \in L^p(\mathbb{R}^3)$ and $\frac{\partial}{\partial x_1}(\mathcal{O} * f) \in L^p(\mathbb{R}^3)$. Moreover, the following estimate holds*

$$\left\| \frac{\partial^2}{\partial x_j \partial x_k}(\mathcal{O} * f) \right\|_{L^p(\mathbb{R}^3)} + \left\| \frac{\partial}{\partial x_1}(\mathcal{O} * f) \right\|_{L^p(\mathbb{R}^3)} \leq C \|f\|_{L^p(\mathbb{R}^3)}. \quad (4.1)$$

Moreover,

1) *if $1 < p < 2$, then $\mathcal{O} * f \in L^{\frac{2p}{2-p}}(\mathbb{R}^3)$ and satisfies*

$$\|\mathcal{O} * f\|_{L^{\frac{2p}{2-p}}(\mathbb{R}^3)} \leq C \|f\|_{L^p(\mathbb{R}^3)}. \quad (4.2)$$

2) *If $1 < p < 4$, then $\frac{\partial}{\partial x_j}(\mathcal{O} * f) \in L^{\frac{4p}{4-p}}(\mathbb{R}^3)$ and verifies the estimate*

$$\left\| \frac{\partial}{\partial x_j}(\mathcal{O} * f) \right\|_{L^{\frac{4p}{4-p}}(\mathbb{R}^3)} \leq C \|f\|_{L^p(\mathbb{R}^3)}. \quad (4.3)$$

Proof : By Fourier's transform, from Equation (1.5) we obtain:

$$\mathcal{F}\left(\frac{\partial^2}{\partial x_j \partial x_k} \mathcal{O} * f\right) = \frac{-\xi_j \xi_k}{\xi^2 + i\xi_1} \mathcal{F}(f).$$

Now, the function $\xi \mapsto m(\xi) = \frac{-\xi_j \xi_k}{\xi^2 + i\xi_1}$ is of class \mathcal{C}^2 in $\mathbb{R}^3 \setminus \{0\}$ and satisfies for every $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$

$$\left| \frac{\partial^{|\alpha|} m}{\partial \xi^\alpha}(\xi) \right| \leq C |\xi|^{-\alpha},$$

where, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ and C is a constant not depending on ξ . Then, the linear operator

$$\mathcal{A} : f \mapsto \frac{\partial^2}{\partial x_j \partial x_k} (\mathcal{O} * f)(\mathbf{x}) = \int_{\mathbb{R}^3} e^{i\mathbf{x}\xi} \frac{-\xi_j \xi_k}{\xi^2 + i\xi_1} \mathcal{F}f(\xi) d\xi$$

is continuous from $L^p(\mathbb{R}^3)$ into $L^p(\mathbb{R}^3)$ (see E. Stein [18], Thm 3.2, p.96). So, $\frac{\partial^2}{\partial x_j \partial x_k} (\mathcal{O} * f) \in L^p(\mathbb{R}^3)$ and satisfies

$$\left\| \frac{\partial^2}{\partial x_j \partial x_k} (\mathcal{O} * f) \right\|_{L^p(\mathbb{R}^3)} \leq C \|f\|_{L^p(\mathbb{R}^3)}.$$

We have also

$$\mathcal{F}\left(\frac{\partial}{\partial x_1} (\mathcal{O} * f)\right) = \frac{i\xi_1}{\xi^2 + i\xi_1} \mathcal{F}(f)$$

and since the function $\xi \mapsto m_1(\xi) = \frac{i\xi_1}{\xi^2 + i\xi_1}$ admits the same properties that those of $m(\xi)$, then $\frac{\partial}{\partial x_1} (\mathcal{O} * f) \in L^p(\mathbb{R}^3)$ and satisfies the estimate

$$\left\| \frac{\partial}{\partial x_1} (\mathcal{O} * f) \right\|_{L^p(\mathbb{R}^3)} \leq C \|f\|_{L^p(\mathbb{R}^3)},$$

which proves the first part of the proposition and Estimate (4.1). Next, to prove inequalities (4.2) and (4.3), we adapt the technique used by Stein in [18] which studied the convolution of $f \in L^p(\mathbb{R}^n)$ with the kernel $|\mathbf{x}|^{\alpha-n}$. Let us decompose the function K as $K_1 + K_\infty$ where,

$$K_1(\mathbf{x}) = K(\mathbf{x}) \quad \text{if } |\mathbf{x}| \leq \mu \quad \text{and} \quad K_1(\mathbf{x}) = 0 \quad \text{if } |\mathbf{x}| > \mu, \tag{4.4}$$

$$K_\infty(\mathbf{x}) = 0 \quad \text{if } |\mathbf{x}| \leq \mu \quad \text{and} \quad K_\infty(\mathbf{x}) = K(\mathbf{x}) \quad \text{if } |\mathbf{x}| > \mu.$$

The function K will denote successively \mathcal{O} and $\frac{\partial \mathcal{O}}{\partial x_j}$ and μ is a fixed positive constant which need not be specified at this instance. Next, we shall show that the mapping $f \mapsto K * f$ is of *weak-type* (p, q) , with $q = \frac{2p}{2-p}$ when $K = \mathcal{O}$ and

$q = \frac{4p}{4-p}$ when $K = \frac{\partial \mathcal{O}}{\partial x_j}$, in the sense that:

$$\text{for all } \lambda > 0, \quad \text{mes} \{ \mathbf{x}; |(K * f)(\mathbf{x})| > \lambda \} \leq \left(C_{p,q} \frac{\|f\|_{L^p(\mathbb{R}^3)}}{\lambda} \right)^q. \quad (4.5)$$

Since $K * f = K_1 * f + K_\infty * f$, we have now:

$$\text{mes} \{ \mathbf{x}; |K * f| > 2\lambda \} \leq \text{mes} \{ \mathbf{x}; |K_1 * f| > \lambda \} + \text{mes} \{ \mathbf{x}; |K_\infty * f| > \lambda \}. \quad (4.6)$$

Note that it is enough to prove inequality (4.5) with $\|f\|_{L^p(\mathbb{R}^3)} = 1$. We have also:

$$\text{mes} \{ \mathbf{x}; |(K_1 * f)(\mathbf{x})| > \lambda \} \leq \frac{\|K_1 * f\|_{L^p(\mathbb{R}^3)}^p}{\lambda^p} \leq \frac{\|K_1\|_{L^1(\mathbb{R}^3)}^p}{\lambda^p}, \quad (4.7)$$

and

$$\|K_\infty * f\|_{L^\infty(\mathbb{R}^3)} \leq \|K_\infty\|_{L^{p'}(\mathbb{R}^3)}. \quad (4.8)$$

1) Estimate (4.2). According to (3.3), $\mathcal{O}_1 \in L^1(\mathbb{R}^3)$ and by (3.6), $\mathcal{O}_\infty \in L^{p'}(\mathbb{R}^3)$ for $1 \leq p < 2$. Then, the integral $\mathcal{O}_1 * f$ converges almost everywhere and $\mathcal{O}_\infty * f$ converges everywhere. So, $\mathcal{O} * f$ converges almost everywhere. But

$$\forall \mu > 0, \quad \|\mathcal{O}_1\|_{L^1(\mathbb{R}^3)} \leq C\mu. \quad (4.9)$$

Next, by using (3.3), we have for any $p' > 2$:

$$\forall \mu > 0, \quad \|\mathcal{O}_\infty\|_{L^{p'}(\mathbb{R}^3)} \leq C\mu^{\frac{2-p'}{p'}}. \quad (4.10)$$

Choosing now $\lambda = C\mu^{\frac{2-p'}{p'}}$ or equivalently $\mu = C'\lambda^{\frac{p}{p-2}}$. Then from (4.10) and (4.8) we have $\|\mathcal{O}_\infty * f\|_{L^\infty(\mathbb{R}^3)} < \lambda$ and so $\text{mes} \{ \mathbf{x}; |\mathcal{O}_\infty * f| > \lambda \} = 0$. Finally, for $1 \leq p < 2$, we get from inequalities (4.9), (4.6) and (4.7):

$$\text{mes} \{ \mathbf{x} \in \mathbb{R}^3; |(\mathcal{O} * f)(\mathbf{x})| > \lambda \} \leq \left(C_p \frac{1}{\lambda} \right)^{\frac{2p}{2-p}}. \quad (4.11)$$

So, for $1 \leq p < 2$, the operator $R : f \mapsto \mathcal{O} * f$ is of *weak-type* $(p, \frac{2p}{2-p})$.

2) Estimate (4.3). Here we take $K = \frac{\partial \mathcal{O}}{\partial x_j}$. First, according to (4.1), $\frac{\partial}{\partial x_1}(\mathcal{O} * f) \in W^{1,p}(\mathbb{R}^3)$ then, by the Sobolev embedding, we have in particular, $\frac{\partial}{\partial x_1}(\mathcal{O} * f) \in L^{\frac{4p}{4-p}}(\mathbb{R}^3)$. It remains to prove Estimate (4.3) for $j = 2, 3$. Firstly we have:

$$\left\| \frac{\partial \mathcal{O}}{\partial x_j} \right\|_{L^1(\mathbb{R}^3)} \leq c\mu, \quad \text{if } \mu \leq 1 \quad \text{and} \quad \left\| \frac{\partial \mathcal{O}}{\partial x_j} \right\|_{L^1(\mathbb{R}^3)} \leq c\mu^{\frac{1}{2}}, \quad \text{if } \mu > 1.$$

Furthermore, we have for $p' > \frac{4}{3}$:

$$\int_{|\mathbf{x}|>\mu} \left| \frac{\partial \mathcal{O}}{\partial x_j}(\mathbf{x}) \right|^{p'} d\mathbf{x} \leq C\mu^{4-3p'}, \quad \text{if } \mu \leq 1,$$

$$\int_{|\mathbf{x}|>\mu} \left| \frac{\partial \mathcal{O}}{\partial x_j}(\mathbf{x}) \right|^{p'} d\mathbf{x} \leq C\mu^{\frac{4-3p'}{2}}, \quad \text{if } \mu > 1.$$

In summary we have:

a) If $0 < \mu < 1$,

$$\int_{|\mathbf{x}|<\mu} \left| \frac{\partial \mathcal{O}}{\partial x_j}(\mathbf{x}) \right| d\mathbf{x} \leq c\mu \quad \text{and} \quad \int_{|\mathbf{x}|>\mu} \left| \frac{\partial \mathcal{O}}{\partial x_j}(\mathbf{x}) \right|^{p'} d\mathbf{x} \leq C\mu^{4-3p'},$$

b) if $\mu \geq 1$,

$$\int_{|\mathbf{x}|<\mu} \left| \frac{\partial \mathcal{O}}{\partial x_j}(\mathbf{x}) \right| d\mathbf{x} \leq c\mu^{\frac{1}{2}} \quad \text{and} \quad \int_{|\mathbf{x}|>\mu} \left| \frac{\partial \mathcal{O}}{\partial x_j}(\mathbf{x}) \right|^{p'} d\mathbf{x} \leq C\mu^{\frac{4-3p'}{2}}.$$

Setting $\lambda = C\mu^{\frac{4-3p'}{p'}}$ in the case a) or $\lambda = C\mu^{\frac{4-3p'}{2p'}}$ in the case b), we get in both cases:

$$\text{mes} \{ \mathbf{x} \in \mathbb{R}^3; |K * f(\mathbf{x})| > \lambda \} \leq \left(C_p \frac{1}{\lambda} \right)^{\frac{4p}{4-p}}. \quad (4.12)$$

Thus, for $1 \leq p < 4$, the operator $R_j : f \mapsto \frac{\partial}{\partial x_j}(\mathcal{O} * f)$ is of *weak-type* $(p, \frac{4p}{4-p})$. Applying now the Marcinkiewicz interpolation's theorem, we deduce that, for $1 < p < 2$, the linear operator R is continuous from $L^p(\mathbb{R}^3)$ into $L^{\frac{2p}{2-p}}(\mathbb{R}^3)$ and for $1 < p < 4$, R_j is continuous from $L^p(\mathbb{R}^3)$ into $L^{\frac{4p}{4-p}}(\mathbb{R}^3)$. \diamond

Remark 4.2 Another proof of Theorem 4.1 consists in using the Fourier's approach. Let $(f_j)_{j \in \mathbb{N}} \subset \mathcal{D}(\mathbb{R}^3)$ a sequence which converges to f in $L^p(\mathbb{R}^3)$. Then the sequence $(u_j)_{j \in \mathbb{N}}$ given by:

$$u_j = \mathcal{F}^{-1}(m_0(\boldsymbol{\xi})\mathcal{F}f_j), \quad m_0(\boldsymbol{\xi}) = (|\boldsymbol{\xi}|^2 + i\xi_1)^{-1}, \quad (4.13)$$

satisfies the equation $Tu_j = f_j$, where the operator T is defined by (1.6). Let us recall now the:

Lizorkin Theorem. Let $D = \{ \boldsymbol{\xi} \in \mathbb{R}^3; |\boldsymbol{\xi}| > 0 \}$ and $m : D \rightarrow \mathbb{C}$, a continuous function such that its derivatives $\frac{\partial^k m}{\partial \xi_1^{k_1} \partial \xi_2^{k_2} \partial \xi_3^{k_3}}$ are continuous and verify

$$|\xi_1|^{k_1+\beta} |\xi_2|^{k_2+\beta} |\xi_3|^{k_3+\beta} \left| \frac{\partial^k m}{\partial \xi_1^{k_1} \partial \xi_2^{k_2} \partial \xi_3^{k_3}} \right| \leq M, \quad (4.14)$$

where $k_1, k_2, k_3 \in \{0, 1\}$, $k = k_1 + k_2 + k_3$ and $0 \leq \beta < 1$. Then, the operator

$$\mathcal{A} : g \mapsto \mathcal{F}^{-1}(m_0 \mathcal{F}g),$$

is continuous from $L^p(\mathbb{R}^3)$ into $L^r(\mathbb{R}^3)$ with $\frac{1}{r} = \frac{1}{p} - \beta$.

Applying this continuity property with $f_j \in L^p(\mathbb{R}^3)$ and $\beta = \frac{1}{2}$, we show that

(u_j) is bounded in $L^{\frac{2p}{2-p}}(\mathbb{R}^3)$ if $1 < p < 2$. So, this sequence admits a subsequence still denoted (u_j) which converges weakly to u and satisfying $Tu = f$. For the derivative of u_j with respect to x_1 , the corresponding multiplier is on the form $m(\boldsymbol{\xi}) = i\xi_1(|\boldsymbol{\xi}|^2 + i\xi_1)^{-1}$. So that (4.14) is satisfied for $\beta = 0$ and then $\frac{\partial u}{\partial x_1} \in L^p(\mathbb{R}^3)$. The same property takes place for the second derivatives with $m(\boldsymbol{\xi}) = \xi_k \xi_l (|\boldsymbol{\xi}|^2 + i\xi_1)^{-1}$. Finally, we verify with $\beta = \frac{1}{4}$, that the first derivative of (u_j) with respect to x_k is bounded in $L^{\frac{4p}{4-p}}(\mathbb{R}^3)$, which implies $\frac{\partial u}{\partial x_k} \in L^{\frac{4p}{4-p}}(\mathbb{R}^3)$. \diamond

Theorem 4.1 states that $\frac{\partial^2}{\partial x_j \partial x_k}(\mathcal{O} * f) \in L^p(\mathbb{R}^3)$ and under conditions on p , $\frac{\partial}{\partial x_j}(\mathcal{O} * f) \in L^{\frac{4p}{4-p}}(\mathbb{R}^3)$ and $\mathcal{O} * f \in L^{\frac{2p}{2-p}}(\mathbb{R}^3)$. Now, using these results and the Sobolev embeddings (2.7)-(2.9), we have the following:

Theorem 4.3 *Let $f \in L^p(\mathbb{R}^3)$.*

1) *Assume that $1 < p < 4$. Then $\nabla(\mathcal{O} * f) \in L^{\frac{4p}{4-p}}(\mathbb{R}^3)$ with the estimate (4.3). Moreover,*

i) *if $1 < p < 3$, then $\nabla(\mathcal{O} * f) \in L^{\frac{3p}{3-p}}(\mathbb{R}^3)$ with the estimate*

$$\|\nabla(\mathcal{O} * f)\|_{L^{\frac{3p}{3-p}}(\mathbb{R}^3)} \leq C\|f\|_{L^p(\mathbb{R}^3)}. \quad (4.15)$$

ii) *If $p = 3$, then $\nabla(\mathcal{O} * f) \in L^r(\mathbb{R}^3)$ for any $r \geq 12$ and satisfies*

$$\|\nabla(\mathcal{O} * f)\|_{L^r(\mathbb{R}^3)} \leq C\|f\|_{L^p(\mathbb{R}^3)}. \quad (4.16)$$

iii) *If $3 < p < 4$, then $\nabla(\mathcal{O} * f) \in L^\infty(\mathbb{R}^3)$ and verifies the estimate*

$$\|\nabla(\mathcal{O} * f)\|_{L^\infty(\mathbb{R}^3)} \leq C\|f\|_{L^p(\mathbb{R}^3)}. \quad (4.17)$$

2) *Assume that $1 < p < 2$. Then $\mathcal{O} * f \in L^{\frac{2p}{2-p}}(\mathbb{R}^3)$ with the estimate (4.2). Moreover,*

i) *if $1 < p < \frac{3}{2}$, then $\mathcal{O} * f \in L^{\frac{3p}{3-2p}}(\mathbb{R}^3)$ and satisfies*

$$\|\mathcal{O} * f\|_{L^{\frac{3p}{3-2p}}(\mathbb{R}^3)} \leq C\|f\|_{L^p(\mathbb{R}^3)}. \quad (4.18)$$

ii) *If $p = \frac{3}{2}$, then $\mathcal{O} * f \in L^r(\mathbb{R}^3)$ for any $r \geq 6$ and*

$$\|\mathcal{O} * f\|_{L^r(\mathbb{R}^3)} \leq C\|f\|_{L^p(\mathbb{R}^3)}. \quad (4.19)$$

iii) *If $\frac{3}{2} < p < 2$, then $\mathcal{O} * f \in L^\infty(\mathbb{R}^3)$ and the following estimate holds*

$$\|\mathcal{O} * f\|_{L^\infty(\mathbb{R}^3)} \leq C\|f\|_{L^p(\mathbb{R}^3)}. \quad (4.20)$$

Proof : 1) When $1 < p < 4$, the previous theorem asserts that $\frac{\partial}{\partial x_j}(\mathcal{O} * f) \in L^{\frac{4p}{4-p}}(\mathbb{R}^3)$ and $\frac{\partial^2}{\partial x_j \partial x_k}(\mathcal{O} * f) \in L^p(\mathbb{R}^3)$. If $1 < p < 3$, according to Proposition 2.1, there exists a unique constant $k(f) \in \mathbb{R}$ such that $v = \frac{\partial}{\partial x_j}(\mathcal{O} * f) + k(f) \in W_0^{1,p}(\mathbb{R}^n)$. Then $k(f) = v - \frac{\partial}{\partial x_j}(\mathcal{O} * f) \in W_0^{1,p}(\mathbb{R}^n) + L^{\frac{4p}{4-p}}(\mathbb{R}^3)$. As none of both spaces contains constants then $k(f) = 0$, which implies that $\frac{\partial}{\partial x_j}(\mathcal{O} * f) \in W_0^{1,p}(\mathbb{R}^n)$. Now, the Sobolev embedding (2.7) yields $\frac{\partial}{\partial x_j}(\mathcal{O} * f) \in L^{\frac{3p}{3-p}}(\mathbb{R}^3)$ and estimate (4.15). If $p \geq 3$, also by the previous theorem and Proposition 2.1, we have $\frac{\partial}{\partial x_j}(\mathcal{O} * f) \in W_0^{1,p}(\mathbb{R}^n)$. The Sobolev embedding (2.8) gives $\frac{\partial}{\partial x_j}(\mathcal{O} * f) \in VMO(\mathbb{R}^3)$ if $p = 3$. Applying now the interpolation theorem between BMO and L^p (see [9]), we get $\frac{\partial}{\partial x_j}(\mathcal{O} * f) \in L^r(\mathbb{R}^3)$ for any $r \geq 12$. By Sobolev embedding (2.9), when $3 < p < 4$, we have $\frac{\partial}{\partial x_j}(\mathcal{O} * f) \in \mathbf{L}^\infty(\mathbb{R}^3)$, and the case 1) is finished.

2) By the previous theorem, when $1 < p < 2$, we have $\mathcal{O} * f \in L^{\frac{2p}{2-p}}(\mathbb{R}^3)$ and $\nabla(\mathcal{O} * f) \in \mathbf{L}^{\frac{3p}{3-p}}(\mathbb{R}^3)$. Now by Sobolev embedding (2.7), $\mathcal{O} * f \in L^{p^*}(\mathbb{R}^3)$, where $\frac{1}{p^*} = \frac{3-p}{3p} - \frac{1}{3} = \frac{1}{p} - \frac{2}{3}$ if $1 < p < \frac{3}{2}$, which gives (4.15). For the remainder of the proof, we use the same arguments that in the previous case with $\mathcal{O} * f$ instead of $\frac{\partial}{\partial x_j}(\mathcal{O} * f)$ and $\frac{\partial}{\partial x_j}(\mathcal{O} * f)$ instead of $\frac{\partial^2}{\partial x_j \partial x_k}(\mathcal{O} * f)$. \diamond

Remark 4.4 i) We can also find this result by showing that $\mathcal{O} \in L^{2,\infty}(\mathbb{R}^3)$, *i.e.*

$$\sup_{\mu > 0} \mu^2 \text{mes} \{ \mathbf{x} \in \mathbb{R}^3; \mathcal{O}(\mathbf{x}) > \mu \} < +\infty. \quad (4.21)$$

So that, for any $1 < q < 2$, according to weak Young inequality (*cf.* [17], chap. IX.4), we obtain:

$$\| \mathcal{O} * f \|_{L^{\frac{2q}{2-q},\infty}(\mathbb{R}^3)} \leq C \| \mathcal{O} \|_{L^{2,\infty}(\mathbb{R}^3)} \| f \|_{L^q(\mathbb{R}^3)}. \quad (4.22)$$

Let now $p \in]1, 2[$. There exist p_0 and p_1 such that $1 < p_0 < p < p_1 < 2$ and such that the operator $R : f \mapsto \mathcal{O} * f$ is continuous from $L^{p_0}(\mathbb{R}^3)$ into $L^{\frac{2p_0}{2-p_0},\infty}(\mathbb{R}^3)$ and from $L^{p_1}(\mathbb{R}^3)$ into $L^{\frac{2p_1}{2-p_1},\infty}(\mathbb{R}^3)$. The Marcinkiewicz theorem allows again to conclude that the operator R is continuous from $L^p(\mathbb{R}^3)$ into $L^{\frac{2p}{2-p}}(\mathbb{R}^3)$

ii) The same remark is true for $\nabla \mathcal{O}$ which belongs to $\mathbf{L}^{\frac{4}{3},\infty}(\mathbb{R}^3)$. \diamond

Using the Young inequality with the relations (3.6) and (3.7), we get the following result:

Proposition 4.5 *Let $f \in L^1(\mathbb{R}^3)$. Then*

1) $\mathcal{O} * f \in L^p(\mathbb{R}^3)$ for any $p \in]2, 3[$ and satisfies the estimate

$$\| \mathcal{O} * f \|_{L^p(\mathbb{R}^3)} \leq C \| f \|_{L^1(\mathbb{R}^3)}, \quad (4.23)$$

2) $\nabla(\mathcal{O} * f) \in \mathbf{L}^p(\mathbb{R}^3)$ for all $p \in]\frac{4}{3}, \frac{3}{2}[$ and the following estimate holds

$$\|\nabla(\mathcal{O} * f)\|_{\mathbf{L}^p(\mathbb{R}^3)} \leq C\|f\|_{L^1(\mathbb{R}^3)}. \quad (4.24)$$

Remark 4.6 Taking "formally" $p = 1$ in Theorem 4.3, we find that $\mathcal{O} * f \in L^q(\mathbb{R}^3)$ for any $q \in]2, 3[$ and $\nabla(\mathcal{O} * f) \in L^q(\mathbb{R}^3)$ for any $q \in]\frac{4}{3}, \frac{3}{2}[$. We notice that they are the same results obtained in Theorem 4.5 by using the Young inequality.

Now, we are going to study the Oseen potential $\mathcal{O} * f$ when f is given in $W_0^{-1,p}(\mathbb{R}^3)$. For that purpose, we give the following definition of the convolution of f with the fundamental solution \mathcal{O} :

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^3), \quad \langle \mathcal{O} * f, \varphi \rangle =: \langle f, \check{\mathcal{O}} * \varphi \rangle_{W_0^{-1,p}(\mathbb{R}^3) \times W_0^{1,p'}(\mathbb{R}^3)}, \quad (4.25)$$

where $\check{\mathcal{O}}(\mathbf{x}) = \mathcal{O}(-\mathbf{x})$. With the L^∞ weighted estimates obtained in [11] (Thms 3.1 and 3.2), we get an estimate on the convolution of $\check{\mathcal{O}}$ with a function $\varphi \in \mathcal{D}(\mathbb{R}^3)$ which we shall use afterward as follows

Lemma 4.7 For any $\varphi \in \mathcal{D}(\mathbb{R}^3)$ we have the estimates

$$|\check{\mathcal{O}} * \varphi(\mathbf{x})| \leq C_\varphi \frac{1}{|\mathbf{x}|(1 + |\mathbf{x}| + x_1)}, \quad (4.26)$$

$$|\nabla(\check{\mathcal{O}} * \varphi)(\mathbf{x})| \leq C_\varphi \frac{1}{|\mathbf{x}|^{\frac{3}{2}}(1 + |\mathbf{x}| + x_1)^{\frac{3}{2}}}, \quad (4.27)$$

where C_φ depends on the support of φ .

Remark 4.8 1) The behaviour on $|\mathbf{x}|$ of $\check{\mathcal{O}} * \varphi$ and its first derivatives is the same that of $\check{\mathcal{O}}$, but the behaviour on $1 + s'$ is a little bit different (see (3.3)).
2) By Lemma 3.1 and estimates (4.26)-(4.27) we find that

$$\forall q > \frac{4}{3}, \quad \check{\mathcal{O}} * \varphi \in W_0^{1,q}(\mathbb{R}^3). \quad (4.28)$$

3) In (4.26) and (4.27), when φ tends to zero in $\mathcal{D}(\mathbb{R}^3)$, then C_φ tends to zero in \mathbb{R} .

The next theorem studies the continuity of the operators R and R_j when f belongs to $W_0^{-1,p}(\mathbb{R}^3)$.

Theorem 4.9 Assume that $1 < p < 4$ and let $f \in W_0^{-1,p}(\mathbb{R}^3)$ satisfying the compatibility condition

$$\langle f, 1 \rangle_{W_0^{-1,p}(\mathbb{R}^3) \times W_0^{1,p'}(\mathbb{R}^3)} = 0, \quad \text{when } 1 < p \leq \frac{3}{2}. \quad (4.29)$$

Then $\mathcal{O} * f \in L^{\frac{4p}{4-p}}(\mathbb{R}^3)$ and $\nabla(\mathcal{O} * f) \in \mathbf{L}^p(\mathbb{R}^3)$ with the following estimate

$$\|\mathcal{O} * f\|_{L^{\frac{4p}{4-p}}(\mathbb{R}^3)} + \|\nabla(\mathcal{O} * f)\|_{\mathbf{L}^p(\mathbb{R}^3)} \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^3)}. \quad (4.30)$$

Moreover,

i) if $1 < p < 3$, then $\mathcal{O} * f \in L^{\frac{3p}{3-p}}(\mathbb{R}^3)$ and the following estimate holds

$$\|\mathcal{O} * f\|_{L^{\frac{3p}{3-p}}(\mathbb{R}^3)} \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^3)}. \quad (4.31)$$

ii) If $p = 3$, then $\mathcal{O} * f \in L^r(\mathbb{R}^3)$ for any $r \geq 12$ and satisfies

$$\|\mathcal{O} * f\|_{L^r(\mathbb{R}^3)} \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^3)}. \quad (4.32)$$

iii) If $3 < p < 4$, then $\mathcal{O} * f \in L^\infty(\mathbb{R}^3)$ and we have the estimate

$$\|\mathcal{O} * f\|_{L^\infty(\mathbb{R}^3)} \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^3)}. \quad (4.33)$$

Proof : Let $1 < p < 4$. By Lemma 4.7 and Remark 4.8 point 3), if $\varphi \rightarrow 0$ in $\mathcal{D}(\mathbb{R}^3)$, then $C_\varphi \rightarrow 0$ where C_φ is defined by (4.26). Thus, $\check{\mathcal{O}} * \varphi \rightarrow 0$ in $W_0^{1,p'}(\mathbb{R}^3)$ for all $p \in]1, 4[$, what implies that $\mathcal{O} * f \in \mathcal{D}'(\mathbb{R}^3)$. Next, by Isomorphism (2.3), there exists $\mathbf{F} \in \mathbf{L}^p(\mathbb{R}^3)$ such that

$$f = \operatorname{div} \mathbf{F} \quad \text{and} \quad \|\mathbf{F}\|_{\mathbf{L}^p(\mathbb{R}^3)} \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^3)}. \quad (4.34)$$

According to (4.1), we have for any $\varphi \in \mathcal{D}(\mathbb{R}^3)$,

$$\begin{aligned} |\langle \frac{\partial}{\partial x_j}(\mathcal{O} * f), \varphi \rangle_{\mathcal{D}'(\mathbb{R}^3) \times \mathcal{D}(\mathbb{R}^3)}| &= |\langle \mathbf{F}, \nabla \frac{\partial}{\partial x_j}(\check{\mathcal{O}} * \varphi) \rangle_{\mathbf{L}^p(\mathbb{R}^3) \times \mathbf{L}^{p'}(\mathbb{R}^3)}| \\ &\leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^3)} \|\varphi\|_{L^{p'}(\mathbb{R}^3)}. \end{aligned}$$

Then we deduce the second part of (4.30). We also have for all $\varphi \in \mathcal{D}(\mathbb{R}^3)$:

$$\langle \mathcal{O} * f, \varphi \rangle_{\mathcal{D}'(\mathbb{R}^3) \times \mathcal{D}(\mathbb{R}^3)} = -\langle \mathbf{F}, \nabla(\check{\mathcal{O}} * \varphi) \rangle_{\mathbf{L}^p(\mathbb{R}^3) \times \mathbf{L}^{p'}(\mathbb{R}^3)},$$

and by (4.3): $|\langle \mathcal{O} * f, \varphi \rangle_{\mathcal{D}'(\mathbb{R}^3) \times \mathcal{D}(\mathbb{R}^3)}| \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^3)} \|\varphi\|_{L^{\frac{4p}{5p-4}}(\mathbb{R}^3)}$. Note that $1 < p < 4 \iff 1 < \frac{4p}{5p-4} < 4$. Consequently, we have the first part of (4.30).

Moreover, by Sobolev embeddings (2.7)-(2.9), $\mathcal{O} * f \in L^{\frac{3p}{3-p}}(\mathbb{R}^3)$ if $1 < p < 3$, $\mathcal{O} * f$ belongs to $L^r(\mathbb{R}^2)$ for all $r \geq 12$ if $p = 3$ and belongs to $L^\infty(\mathbb{R}^2)$ if $3 < p < 4$. Thus, we showed that if $1 < p < 4$, the operators R and R_j are continuous. \diamond

Corollary 4.10 *Assume that $1 < p < 4$. If u is a distribution such that $\nabla u \in \mathbf{L}^p(\mathbb{R}^3)$ and $\frac{\partial u}{\partial x_1} \in W_0^{-1,p}(\mathbb{R}^3)$, then there exists a unique constant $k(u)$*

such that $u + k(u) \in L^{\frac{4p}{4-p}}(\mathbb{R}^3)$ and

$$\|u + k(u)\|_{L^{\frac{4p}{4-p}}(\mathbb{R}^3)} \leq C(\|\nabla u\|_{L^p(\mathbb{R}^3)} + \|\frac{\partial u}{\partial x_1}\|_{W_0^{-1,p}(\mathbb{R}^3)}). \quad (4.35)$$

Moreover, if $1 < p < 3$, then $u + k(u) \in L^{\frac{3p}{3-p}}(\mathbb{R}^3)$, where $k(u)$ is defined by:

$$k(u) = - \lim_{|\mathbf{x}| \rightarrow \infty} \frac{1}{\omega_3} \int_{S_2} u(\sigma|\mathbf{x}|) d\sigma, \quad (4.36)$$

where, ω_3 denotes the area of the sphere S_2 and u tends to the constant $-k(u)$ as \mathbf{x} tends to infinity in the following sense:

$$\lim_{|\mathbf{x}| \rightarrow \infty} \int_{S_2} |u(\sigma|\mathbf{x}|) + k(u)| d\sigma = 0. \quad (4.37)$$

If $p = 3$, then $u + k(u)$ belongs to $L^r(\mathbb{R}^3)$ for any $r \geq 12$. If $3 < p < 4$, then u belongs to $L^\infty(\mathbb{R}^3)$, is continuous in \mathbb{R}^3 and tends to $-k(u)$ pointwise.

Proof : Setting $g = -\Delta u + \frac{\partial u}{\partial x_1} \in W_0^{-1,p}(\mathbb{R}^3)$. Since $\mathcal{P}_{[1-\frac{3}{p}]}$ contains at most constants and according to the density of $\mathcal{D}(\mathbb{R}^3)$ in $\widetilde{W}_0^{1,p}(\mathbb{R}^3)$ (see [2]), then g satisfies the compatibility condition (4.29). By the previous theorem, there exists a unique $v = \mathcal{O} * g \in L^{\frac{4p}{4-p}}(\mathbb{R}^3)$ such that $\nabla v \in L^p(\mathbb{R}^3)$ and $\frac{\partial v}{\partial x_1} \in L^p(\mathbb{R}^3)$, satisfying $T(u - v) = 0$, where T is the Oseen operator, with the estimate:

$$\|v\|_{L^{\frac{4p}{4-p}}(\mathbb{R}^3)} \leq C(\|\nabla u\|_{L^p(\mathbb{R}^3)} + \|\frac{\partial u}{\partial x_1}\|_{W_0^{-1,p}(\mathbb{R}^3)}). \quad (4.38)$$

Setting $w = u - v$, we have for all $i = 1, 2, 3$, $\frac{\partial w}{\partial x_i} \in L^p(\mathbb{R}^3)$ and satisfies $T(\frac{\partial w}{\partial x_i}) = 0$. We deduce then by Lemma 3.2 that $\nabla u = \nabla v$ and consequently there exists a unique constant $k(u)$, defined by (4.36), such that $u + k(u) = v$. The last properties are consequence of (2.8) and (2.9). \diamond

Remark 4.11 Let $u \in \mathcal{D}'(\mathbb{R}^3)$ such that $\nabla u \in L^p(\mathbb{R}^3)$.

i) When $1 < p < 3$, according to Proposition 2.1, we know that there exists a unique constant $k(u)$ such that $u + k(u) \in L^{\frac{3p}{3-p}}(\mathbb{R}^3)$. Here, the fact that in addition $\frac{\partial u}{\partial x_1} \in W_0^{-1,p}(\mathbb{R}^3)$ we have moreover $u + k(u) \in L^{\frac{4p}{4-p}}(\mathbb{R}^2)$, with $\frac{4p}{4-p} < \frac{3p}{3-p}$.

ii) When $3 \leq p < 4$, by Proposition 2.1, for any constant k , $u + k$ belongs only to $W_0^{1,p}(\mathbb{R}^3)$ but not to the space $L^r(\mathbb{R}^3)$. But, if moreover $\frac{\partial u}{\partial x_1} \in W_0^{-1,p}(\mathbb{R}^3)$ then, $u + k(u) \in L^{\frac{4p}{4-p}}(\mathbb{R}^3)$ for some unique constant $k(u)$. Moreover $u + k(u) \in L^r(\mathbb{R}^3)$ for any $r \geq \frac{4p}{4-p}$ and $u \in L^\infty(\mathbb{R}^3)$ if $p > 3$.

5 Scalar Oseen potential in two dimensional.

In this section we study also the continuity of the operators R , R_j and $R_{j,k}$ when f is given in $L^p(\mathbb{R}^2)$ or in $W_0^{-1,p}(\mathbb{R}^2)$. We begin by the case where f belongs to $L^p(\mathbb{R}^2)$.

Theorem 5.1 *Let $f \in L^p(\mathbb{R}^2)$. Then $\frac{\partial^2}{\partial x_1 \partial x_2}(\mathcal{O} * f) \in L^p(\mathbb{R}^2)$, $\frac{\partial}{\partial x_1}(\mathcal{O} * f) \in L^p(\mathbb{R}^2)$ and satisfy the estimate*

$$\left\| \frac{\partial^2}{\partial x_1 \partial x_2}(\mathcal{O} * f) \right\|_{L^p(\mathbb{R}^2)} + \left\| \frac{\partial}{\partial x_1}(\mathcal{O} * f) \right\|_{L^p(\mathbb{R}^2)} \leq C \|f\|_{L^p(\mathbb{R}^2)}. \quad (5.1)$$

Moreover,

1) if $1 < p < \frac{3}{2}$, then $\mathcal{O} * f \in L^{\frac{3p}{3-2p}}(\mathbb{R}^2)$ and satisfies

$$\|\mathcal{O} * f\|_{L^{\frac{3p}{3-2p}}(\mathbb{R}^2)} \leq C \|f\|_{L^p(\mathbb{R}^2)}. \quad (5.2)$$

2) If $1 < p < 3$, then $\frac{\partial}{\partial x_j}(\mathcal{O} * f) \in L^{\frac{3p}{3-p}}(\mathbb{R}^2)$ and verifies the estimate

$$\left\| \frac{\partial}{\partial x_j}(\mathcal{O} * f) \right\|_{L^{\frac{3p}{3-p}}(\mathbb{R}^2)} \leq C \|f\|_{L^p(\mathbb{R}^2)}. \quad (5.3)$$

Proof : As in three-dimensional case, since the operator

$$\mathcal{A} : f \mapsto \frac{\partial^2}{\partial x_1 \partial x_2}(\mathcal{O} * f)(\mathbf{x}) = \int_{\mathbb{R}^2} e^{i\mathbf{x}\boldsymbol{\xi}} \frac{-\xi_1 \xi_2}{\xi^2 + i\xi_1} \widehat{f}(\boldsymbol{\xi}) d\boldsymbol{\xi}$$

is continuous from $L^p(\mathbb{R}^2)$ into $L^p(\mathbb{R}^2)$, we get $\frac{\partial^2}{\partial x_1 \partial x_2}(\mathcal{O} * f) \in L^p(\mathbb{R}^2)$. We have also $\frac{\partial}{\partial x_1}(\mathcal{O} * f) \in L^p(\mathbb{R}^2)$ and the estimate:

$$\left\| \frac{\partial^2}{\partial x_1 \partial x_2}(\mathcal{O} * f) \right\|_{L^p(\mathbb{R}^2)} + \left\| \frac{\partial}{\partial x_1}(\mathcal{O} * f) \right\|_{L^p(\mathbb{R}^2)} \leq C \|f\|_{L^p(\mathbb{R}^2)}, \quad (5.4)$$

which proves the first part of theorem and Estimate (5.1). Now, as in the three-dimensional case, we will show that the operators $R : f \mapsto \mathcal{O} * f$ and $R_j : f \mapsto \frac{\partial}{\partial x_j}(\mathcal{O} * f)$ are *weak-type* $(p, \frac{3p}{3-2p})$ if $1 \leq p < \frac{3}{2}$ and *weak-type* $(p, \frac{3p}{3-p})$ if $1 \leq p < 3$ respectively. Using the decomposition (4.4), according to (3.7) and Estimate (3.3), the integral $K * f = K_1 * f + K_\infty * f$ converges almost everywhere, where K denotes \mathcal{O} and $\frac{\partial \mathcal{O}}{\partial x_j}$ respectively.

1) **Estimate (5.2).** We observe that:

$$\forall \mu > 0, \quad \|\mathcal{O}_1\|_{L^1(\mathbb{R}^2)} \leq C\mu, \quad (5.5)$$

and for all $p' > 3$:

$$\forall \mu > 0, \quad \|\mathcal{O}_\infty\|_{p'} \leq C\mu^{\frac{3-p'}{2p'}}. \quad (5.6)$$

Setting $\lambda = C\mu^{\frac{3-p'}{2p'}}$ or equivalently $\mu = C'\lambda^{\frac{2p'}{3-p'}} = C'\lambda^{\frac{2p}{2p-3}}$, we get from this last inequality that $\|\mathcal{O}_\infty * f\|_{L^\infty(\mathbb{R}^2)} < \lambda$. Then by Estimate (4.8), we have $\text{mes}\{\mathbf{x} \in \mathbb{R}^2; |\mathcal{O}_\infty * f(\mathbf{x})| > \lambda\} = 0$. So, for any $1 < p < \frac{3}{2}$, we have from (4.6) and (4.7):

$$\text{mes}\{\mathbf{x} \in \mathbb{R}^2; |\mathcal{O} * f(\mathbf{x})| > 2\lambda\} \leq C\frac{\mu^p}{\lambda^p} \leq C\left(\frac{1}{\lambda}\right)^{\frac{3p}{3-2p}}, \quad (5.7)$$

which proves that the operator R is of *weak type* $(p, \frac{3p}{3-2p})$.

2) Estimate (5.3). According to (5.1) and the Sobolev embedding, we get that $\frac{\partial}{\partial x_1}(\mathcal{O} * f)$ belongs in particular to $L^{\frac{3p}{3-p}}(\mathbb{R}^2)$. It remains then to show (5.3) for $i = 2$. As previously, we have:

$$\forall \mu > 0, \quad \left\| \frac{\partial \mathcal{O}}{\partial x_2} \right\|_{L^1(\mathbb{R}^2)} \leq C\mu^{\frac{1}{2}}, \quad (5.8)$$

and for any $p' > \frac{3}{2}$,

$$\left\| \frac{\partial \mathcal{O}}{\partial x_2} \right\|_{L^{p'}(\mathbb{R}^2)} \leq C\mu^{\frac{3-2p'}{p'}}. \quad (5.9)$$

We have also, for $1 < p < 3$ and any $\lambda > 0$:

$$\text{mes}\{\mathbf{x} \in \mathbb{R}^2; \left| \frac{\partial}{\partial x_2}(\mathcal{O} * f)(\mathbf{x}) \right| > 2\lambda\} \leq C\left(\frac{1}{\lambda}\right)^{\frac{3p}{3-p}}. \quad (5.10)$$

Thus, the operator R_2 is of *weak-type* $(p, \frac{3p}{3-p})$. Now, from inequalities (5.7), (5.10) and by Marcinkiewicz interpolation's Theorem, the operator $R : f \mapsto \mathcal{O} * f$ is continuous from $L^p(\mathbb{R}^2)$ into $L^{\frac{3p}{3-2p}}(\mathbb{R}^2)$ and $R_2 : f \mapsto \frac{\partial}{\partial x_2}(\mathcal{O} * f)$ is continuous from $L^p(\mathbb{R}^2)$ into $L^{\frac{3p}{3-p}}(\mathbb{R}^2)$. \diamond

Remark 5.2 i) We can also prove this result as follows. We observe first that $\mathcal{O} \in L^{3,\infty}(\mathbb{R}^2)$, *i.e.*

$$\sup_{\mu > 0} \mu^3 \text{mes}\{\mathbf{x} \in \mathbb{R}^2; |\mathcal{O}(\mathbf{x})| > \mu\} < +\infty. \quad (5.11)$$

So that, when $1 < q < \frac{3}{2}$ and using weak Young inequality (*cf.* [17], chap. IX.4), we obtain:

$$\|\mathcal{O} * f\|_{L^{\frac{3q}{3-2q},\infty}(\mathbb{R}^2)} \leq C\|\mathcal{O}\|_{L^{3,\infty}(\mathbb{R}^2)}\|f\|_{L^{q,\infty}(\mathbb{R}^2)}. \quad (5.12)$$

Now, let $1 < p < \frac{3}{2}$. This last estimate shows that there exist p_0 and p_1 such that $1 < p_0 < p < p_1 < \frac{3}{2}$ and such that the operator $R : f \mapsto \mathcal{O} * f$ is continuous from $L^{p_0}(\mathbb{R}^2)$ into $L^{\frac{3p_0}{3-2p_0},\infty}(\mathbb{R}^2)$ and from $L^{p_1}(\mathbb{R}^2)$ into $L^{\frac{3p_1}{3-2p_1},\infty}(\mathbb{R}^2)$. The Marcinkiewicz theorem allows again to conclude that

$R : L^p(\mathbb{R}^2) \longrightarrow L^{\frac{3p}{3-2p}}(\mathbb{R}^2)$ is bounded.

ii) The same remark is true for $\nabla \mathcal{O}$ which belongs to $\mathbf{L}^{\frac{3}{2}, \infty}(\mathbb{R}^2)$.

Theorem 5.1 and the Sobolev embedding yield the following result.

Theorem 5.3 *Let $f \in L^p(\mathbb{R}^2)$.*

1) *Assume that $1 < p < 3$. Then, $\nabla(\mathcal{O} * f) \in \mathbf{L}^{\frac{3p}{3-p}}(\mathbb{R}^2)$ and satisfies Estimate (5.3). Moreover,*

i) *if $1 < p < 2$, then $\nabla(\mathcal{O} * f) \in \mathbf{L}^{\frac{2p}{2-p}}(\mathbb{R}^2)$ with*

$$\|\nabla(\mathcal{O} * f)\|_{\mathbf{L}^{\frac{2p}{2-p}}(\mathbb{R}^2)} \leq C \|f\|_{L^p(\mathbb{R}^2)}. \quad (5.13)$$

ii) *When $p = 2$, then $\nabla(\mathcal{O} * f) \in \mathbf{L}^r(\mathbb{R}^2)$, $r \geq 6$ and the following estimate holds:*

$$\|\nabla(\mathcal{O} * f)\|_{\mathbf{L}^r(\mathbb{R}^2)} \leq C \|f\|_{L^p(\mathbb{R}^2)}. \quad (5.14)$$

iii) *Finally, if $2 < p < 3$, then $\nabla(\mathcal{O} * f) \in \mathbf{L}^\infty(\mathbb{R}^2)$ and we have the inequality:*

$$\|\nabla(\mathcal{O} * f)\|_{\mathbf{L}^\infty(\mathbb{R}^2)} \leq C \|f\|_{L^p(\mathbb{R}^2)}. \quad (5.15)$$

2) *Assume that $1 < p < \frac{3}{2}$. Then, besides (5.2), $\mathcal{O} * f \in L^\infty(\mathbb{R}^2)$ and satisfies the estimate:*

$$\|\mathcal{O} * f\|_{L^\infty(\mathbb{R}^2)} \leq C \|f\|_{L^p(\mathbb{R}^2)}. \quad (5.16)$$

The proof of this theorem is the same that of Theorem 4.3. However, in the case 2, we have $\frac{2p}{2-p} > 2$ which gives the result by using the Sobolev embedding (2.9).

Using the Young inequality with the relations (3.5) and (3.7), we get the following:

Proposition 5.4 *Let $f \in L^1(\mathbb{R}^2)$. Then*

1) *$\mathcal{O} * f \in L^p(\mathbb{R}^2)$ for any $p > 3$ and satisfies the estimate*

$$\|\mathcal{O} * f\|_{L^p(\mathbb{R}^2)} \leq C \|f\|_{L^1(\mathbb{R}^2)}, \quad (5.17)$$

2) *$\nabla(\mathcal{O} * f) \in \mathbf{L}^p(\mathbb{R}^2)$ for all $p \in]\frac{3}{2}, 2[$ and the following estimate holds*

$$\|\nabla(\mathcal{O} * f)\|_{\mathbf{L}^p(\mathbb{R}^2)} \leq C \|f\|_{L^1(\mathbb{R}^2)}. \quad (5.18)$$

Proof : Since by (3.7) $\mathcal{O} \in L^1_{loc}(\mathbb{R}^2)$ and $\nabla \mathcal{O} \in \mathbf{L}^1_{loc}(\mathbb{R}^2)$, then $\nabla(\mathcal{O} * f) = (\nabla \mathcal{O}) * f$. According to the Young inequality and the relation (3.5), this last term belongs to $L^p(\mathbb{R}^3)$ if $\frac{4}{3} < p < \frac{3}{2}$. With the same argument we get the case 1). \diamond

Remark 5.5 Taking "formally" $p = 1$ in Theorem 5.3, we find that $\nabla(\mathcal{O} * f) \in \mathbf{L}^q(\mathbb{R}^3)$ for any $q \in]\frac{3}{2}, 2[$. We find also $\mathcal{O} * f \in L^3(\mathbb{R}^2)$ and

$\nabla(\mathcal{O} * f) \in L^2(\mathbb{R}^2)$. The Sobolev embedding (2.8) gives then $\mathcal{O} * f \in L^q(\mathbb{R}^2)$ for any $q > 3$ and we notice that they are the same results obtained by Theorem 5.4 by using the Young inequality.

In order to study the case when f is given in $W_0^{-1,p}(\mathbb{R}^2)$, we use the L^∞ weighted estimates obtained in [11] (Thms 3.5, 3.7 and 3.8) and we get an estimates on the convolution of $\check{\mathcal{O}}$ with a function $\varphi \in \mathcal{D}(\mathbb{R}^2)$ as follows:

Lemma 5.6 *For any $\varphi \in \mathcal{D}(\mathbb{R}^2)$ we have the estimates*

$$|\check{\mathcal{O}} * \varphi(\mathbf{x})| \leq C_\varphi \frac{1}{|\mathbf{x}|^{\frac{1}{2}}(1 + |\mathbf{x}| + x_1)^{\frac{1}{2}}}, \quad (5.19)$$

$$\left| \frac{\partial}{\partial x_1}(\check{\mathcal{O}} * \varphi)(\mathbf{x}) \right| \leq C_\varphi \frac{1}{|\mathbf{x}|^{\frac{3}{2}}(1 + |\mathbf{x}| + x_1)^{\frac{1}{2}}}, \quad (5.20)$$

$$\left| \frac{\partial}{\partial x_2}(\check{\mathcal{O}} * \varphi)(\mathbf{x}) \right| \leq C_\varphi \frac{1}{|\mathbf{x}|(1 + |\mathbf{x}| + x_1)}, \quad (5.21)$$

where C_φ depends on the support of φ .

Remark 5.7 1) The behaviour on $|\mathbf{x}|$ of $\check{\mathcal{O}} * \varphi$ and its first derivatives is the same that that of $\check{\mathcal{O}}$ but, the behaviour on $1 + s'$ is a little bit different.

2) By Lemma 3.1 and this last estimates, we find that

$$\forall q > \frac{3}{2}, \quad \check{\mathcal{O}} * \varphi \in W_0^{1,q}(\mathbb{R}^2). \quad (5.22)$$

3) In (5.19)-(5.21), when φ tends to zero in $\mathcal{D}(\mathbb{R}^3)$, then C_φ tends to zero in \mathbb{R} .

With the definition (4.25), when f is given in $W_0^{-1,p}(\mathbb{R}^2)$, we have a similar result to Theorem 4.9.

Theorem 5.8 *Assume that $1 < p < 3$ and let $f \in W_0^{-1,p}(\mathbb{R}^2)$ satisfying the compatibility condition*

$$\langle f, 1 \rangle_{W_0^{-1,p}(\mathbb{R}^2) \times W_0^{1,p'}(\mathbb{R}^2)} = 0, \quad \text{when } 1 < p \leq 2. \quad (5.23)$$

Then, $\mathcal{O} * f \in L^{\frac{3p}{3-p}}(\mathbb{R}^2)$ and $\nabla(\mathcal{O} * f) \in \mathbf{L}^p(\mathbb{R}^2)$ with the following estimate:

$$\|\mathcal{O} * f\|_{L^{\frac{3p}{3-p}}(\mathbb{R}^2)} + \|\nabla(\mathcal{O} * f)\|_{\mathbf{L}^p(\mathbb{R}^2)} \leq C \|f\|_{W_0^{-1,p}(\mathbb{R}^2)}. \quad (5.24)$$

Moreover,

i) if $1 < p < 2$, then $\mathcal{O} * f \in L^{\frac{2p}{2-p}}(\mathbb{R}^2)$ and satisfies the following inequality:

$$\|\mathcal{O} * f\|_{L^{\frac{2p}{2-p}}(\mathbb{R}^2)} \leq C \|f\|_{W_0^{-1,p}(\mathbb{R}^2)}. \quad (5.25)$$

ii) If $p = 2$, then $\mathcal{O} * f \in L^r(\mathbb{R}^2)$ for any $r \geq 6$ and

$$\|\mathcal{O} * f\|_{L^r(\mathbb{R}^2)} \leq C \|f\|_{W_0^{-1,p}(\mathbb{R}^2)}. \quad (5.26)$$

iii) If $2 < p < 3$, then $\mathcal{O} * f \in L^\infty(\mathbb{R}^2)$ and we have the estimate

$$\|\mathcal{O} * f\|_{L^\infty(\mathbb{R}^2)} \leq C \|f\|_{W_0^{-1,p}(\mathbb{R}^2)}. \quad (5.27)$$

Proof : Because the proof is the same that that of Theorem 4.9, then we give it briefly. Let $f \in W_0^{-1,p}(\mathbb{R}^2)$ satisfying condition (5.23). As in three-dimensional case, we get $\mathcal{O} * f \in \mathcal{D}'(\mathbb{R}^2)$ if $1 < p < 3$. By Isomorphism (2.3) and (5.1), we have

$$\begin{aligned} |\langle \mathcal{O} * f, \varphi \rangle_{\mathcal{D}'(\mathbb{R}^2) \times \mathcal{D}(\mathbb{R}^2)}| &\leq \|\mathbf{F}\|_{L^p(\mathbb{R}^2)} \left\| \frac{\partial}{\partial x_j} (\check{\mathcal{O}} * \varphi) \right\|_{L^{p'}(\mathbb{R}^2)} \\ &\leq C \|f\|_{W_0^{-1,p}(\mathbb{R}^2)} \|\varphi\|_{L^{\frac{3p}{4p-3}}(\mathbb{R}^2)}. \end{aligned}$$

Note that $1 < p < 3 \iff 1 < \frac{3p}{4p-3} < 3$. Then, $\mathcal{O} * f \in L^{\frac{3p}{3-p}}(\mathbb{R}^2)$ and

$$\|\mathcal{O} * f\|_{L^{\frac{3p}{3-p}}(\mathbb{R}^2)} \leq C \|f\|_{W_0^{-1,p}(\mathbb{R}^2)}.$$

Moreover, by the Sobolev embedding, $\mathcal{O} * f \in L^{\frac{2p}{2-p}}(\mathbb{R}^2)$ if $1 < p < 2$, $\mathcal{O} * f$ belongs to $L^r(\mathbb{R}^2)$ for all $r \geq 6$ if $p = 2$ and belongs to $L^\infty(\mathbb{R}^2)$ if $2 < p < 3$. We thus showed that if $1 < p < 3$, the following operator is continuous:

$$\begin{aligned} R : W_0^{-1,p}(\mathbb{R}^2) \perp \mathcal{P}_{[1-\frac{2}{p}]} &\longrightarrow W_0^{1,p}(\mathbb{R}^2) \cap L^{\frac{3p}{3-p}}(\mathbb{R}^2), \\ f &\longmapsto \mathcal{O} * f. \end{aligned}$$

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