The scalar Oseen operator $- - \Delta + \frac{\partial}{\partial x\_\_1}$ in R^2.

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The scalar Oseen operator $-\Delta + \dfrac{\partial}{\partial x_1}$ in $\mathbb{R}^2$.

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Abstract

This paper solves the scalar Oseen equation, a linearized form of the Navier-Stokes equation. Because the fundamental solution has an anisotropic properties, the problem is set in Sobolev space with isotropic and anisotropic weights. We establish some existence results and regularities in L^p theory.

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1 Introduction

Let Ω be an exterior domain of $\mathbb{R}^2$ or the whole space $\mathbb{R}^2$. We consider the following Oseen's problem:

$$\begin{aligned} -\nu\Delta \boldsymbol{u} + \lambda \frac{\partial \boldsymbol{u}}{\partial x_1} + \nabla\pi &= \boldsymbol{f} \quad \text{in} \quad \Omega, \\ \operatorname{div} \boldsymbol{u} &= g \quad \text{in} \quad \Omega, \\ \boldsymbol{u} &= \boldsymbol{u}_* \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.1}$$

with the condition on $\boldsymbol{u}$ at infinity

$$\lim_{|x|\to+\infty} \boldsymbol{u}(x) = \boldsymbol{u}_\infty. \tag{1.2}$$

The viscosity ν, the external force $\boldsymbol{f}$, the boundary values $\boldsymbol{u}_*$ on $\partial\Omega$ and g are given. The unknown velocity field $\boldsymbol{u}$ is assumed to converge to a constant vector $\boldsymbol{u}_\infty$, and the scalar function π denotes the unknown pressure. C. W. Oseen [14] obtained (1.1) by linearizing the Navier-Stokes equations, describing the flow of a viscous and incompressible fluid. Some authors worked on this problem. We can cite Finn [6, 7], more recently Galdi [8], Farwig [3, 4], Farwig and Sohr [5] and Amrouche and Razafison [2]. When $\Omega = \mathbb{R}^2$, the system (1.1) is written as follows

$$\begin{aligned} -\nu\Delta \boldsymbol{u} + \lambda \frac{\partial \boldsymbol{u}}{\partial x_1} + \nabla\pi = \boldsymbol{f} \quad \text{in} \quad \mathbb{R}^2, \\ \operatorname{div} \boldsymbol{u} = g \quad \text{in} \quad \mathbb{R}^2, \end{aligned} \tag{1.3}$$

with the same condition at infinity. Taking the divergence of the first equation of (1.3), we obtain a decoupled set of equations

$$\Delta\pi = \boldsymbol{f} + \nu\Delta g - \lambda\frac{\partial g}{\partial x_1} \qquad \text{in} \quad \mathbb{R}^2, \tag{1.4}$$

$$-\nu\Delta\boldsymbol{u} + \lambda\,\frac{\partial \boldsymbol{u}}{\partial x_1} = \boldsymbol{f} - \nabla\,\pi \quad \text{in} \quad \mathbb{R}^2. \tag{1.5}$$

We use the results obtained in [1] for the Poisson equation to solve Equation (1.4). Now observe that each component u_j of the velocity satisfies

$$-\nu\Delta u_j + \lambda\,\frac{\partial \boldsymbol{u}_j}{\partial x_1} = f_j - \frac{\partial\pi}{\partial x_j} \quad \text{in} \quad \mathbb{R}^2. \tag{1.6}$$

Then, we see that if we solve the scalar equation

$$-\nu\Delta u + \lambda\frac{\partial u}{\partial x_1} = f \quad \text{in} \quad \mathbb{R}^2, \tag{1.7}$$

we can apply to Oseen problem the results obtained for this last equation. The aim of this paper is then to study the scalar Oseen equation (1.7). Since the fundamental solution of this equation has anisotropic decay properties, see (3.6), (3.9) we will work in Sobolev spaces with isotropic weight and anisotropic weight introduced by Farwig [3] in the particular Hilbertian case ($p = 2$). The case $\lambda = 0$ yields the Laplace's equation studied by Amrouche-Girault-Giroire [1] in weighted Sobolev spaces. This paper is divided into five sections. In section 2, we introduce the functional spaces and we recall some preliminaries results. We give also a density result of $\mathcal{D}(\mathbb{R}^2)$ in an anisotropic weighted space and characterization of homogeneous Sobolev spaces. In the section 3, by adapting a technique used by Stein, we obtained results on the Oseen's potential which we used then to solve Equation (1.7), where the left hand side f is given on the one hand in $L^p(\mathbb{R}^2)$ and on the other hand in $W_0^{-1,p}(\mathbb{R}^2)$. We also looked at the case where f belongs at the same moment to two space with different powers p and q. We considered, in the section 4, the case where f belongs to spaces L^p with anisotropic weights. Finally, in the section 5, we considered the limit case when λ tends to zero and we compared the solution obtained with the Poisson's equation. The main results of this paper are given by the below theorems.

In Theorem 1, we give (L^p, L^q) continuity properties for the Oseen opertors $f \longmapsto \mathcal{O} * f$, $\quad f \longmapsto \frac{\partial}{\partial x_i}(\mathcal{O} * f)$ and $f \longmapsto \frac{\partial^2}{\partial x_j \partial x_k}(\mathcal{O} * f)$. We observe that the continuity results obtained for the Oseen equation (1.7) are better than the classic properties of the Riesz's potential associated to the Laplacian operator, corresponding to the case $\lambda = 0$.

Theorem 1 *Let $f \in L^p(\mathbb{R}^2)$ with $1 < p < \infty$. Then,* $\frac{\partial^2}{\partial x_j \partial x_k}(\mathcal{O} * f) \in L^p(\mathbb{R}^2)$, $\frac{\partial}{\partial x_1}(\mathcal{O} * f) \in L^p(\mathbb{R}^2)$ *and satisfy the estimate*

$$\|\frac{\partial^2}{\partial x_j \partial x_k}(\mathcal{O} * f)\|_{L^p(\mathbb{R}^2)} + \|\frac{\partial}{\partial x_1}(\mathcal{O} * f)\|_{L^p(\mathbb{R}^2)} \ \leq C\|f\|_{L^p(\mathbb{R}^2)}.$$

Moreover,

1) i) *if $1 < p < 2$, $\nabla(\mathcal{O} * f) \in \boldsymbol{L}^{\frac{3p}{3-p}}(\mathbb{R}^2) \cap \boldsymbol{L}^{\frac{2p}{2-p}}(\mathbb{R}^2)$ and satisfies*

$$\|\nabla(\mathcal{O} * f)\|_{\boldsymbol{L}^{\frac{3p}{3-p}}(\mathbb{R}^2)} + \|\nabla(\mathcal{O} * f)\|_{\boldsymbol{L}^{\frac{2p}{2-p}}(\mathbb{R}^2)} \ \leq C\|f\|_{L^p(\mathbb{R}^2)}.$$

ii) *If $p = 2$, $\quad \nabla(\mathcal{O} * f) \in L^r(\mathbb{R}^2)$, for any $r \geq 6$ and the following estimate holds.*

$$\|\nabla(\mathcal{O} * f)\|_{L^r(\mathbb{R}^2)} \ \leq C\|f\|_{L^p(\mathbb{R}^2)}.$$

iii) *If* $2 < p < 3$, $\nabla(\mathcal{O} * f) \in L^{\frac{3p}{3-p}}(\mathbb{R}^2) \cap \boldsymbol{L}^{\infty}(\mathbb{R}^2)$ *and we have the estimate.*

$$\|\nabla(\mathcal{O} * f)\|_{L^{\frac{3p}{3-p}}(\mathbb{R}^2)} + \|\nabla(\mathcal{O} * f)\|_{\boldsymbol{L}^{\infty}(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}.$$

2) *if* $1 < p < \frac{3}{2}$, $\mathcal{O} * f \in L^{\frac{3p}{3-2p}}(\mathbb{R}^2) \cap L^{\infty}(\mathbb{R}^2)$ *and satisfies*

$$\|\mathcal{O} * f\|_{L^{\frac{3p}{3-2p}}(\mathbb{R}^2)} + \|\mathcal{O} * f\|_{L^{\infty}(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}.$$

In Theorem 2, we give some similar results in the case when f belongs to a negative weighted Sobolev space $W_0^{-1,p}(\mathbb{R}^2)$ and we observe again that we obtain results better than in the case $\lambda = 0$.

Theorem 2 *Let* $f \in W_0^{-1,p}(\mathbb{R}^2)$ *satisfying the compatibility condition*

$$\langle f, 1\rangle_{W_0^{-1,p}(\mathbb{R}^2)\times W_0^{1,p'}(\mathbb{R}^2)} = 0, \quad \text{when } 1 < p \leq 2.$$

i) *If* $1 < p < 3$, *then* $u = \mathcal{O} * f \in L^{\frac{3p}{3-p}}(\mathbb{R}^2)$ *is the unique solution of Equation (3.1) such that* $\nabla u \in L^p(\mathbb{R}^2)$ *and* $\frac{\partial u}{\partial x_1} \in W_0^{-1,p}(\mathbb{R}^2)$. *Moreover, we have the estimate*

$$\|u\|_{L^{\frac{3p}{3-p}}(\mathbb{R}^2)} + \|\nabla u\|_{L^p(\mathbb{R}^2)} + \|\frac{\partial u}{\partial x_1}\|_{W_0^{-1,p}(\mathbb{R}^2)} \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^2)},$$

and $u \in L^{\frac{2p}{2-p}}(\mathbb{R}^2)$ *when* $1 < p < 2$, $u \in L^r(\mathbb{R}^2)$ *for any* $r \geq 6$ *when* $p = 2$ *and* $u \in L^{\infty}(\mathbb{R}^2)$ *when* $2 < p < 3$.

ii) *If* $p \geq 3$, *then Equation (3.1) has a solution* $u \in \widetilde{W}_0^{1,p}(\mathbb{R}^2)$, *unique up to a constant and we have*

$$\inf_{k\in\mathbb{R}} \|u + k\|_{\widetilde{W}_0^{1,p}(\mathbb{R}^2)} \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^2)}.$$

Theorem 3 is concerned by the case where f belongs to L^p spaces with anisotropic weight.

Theorem 3 *Assume that* $2 < p < \frac{32}{11}$ *and* $f \in L^p_{\frac{1}{2},\frac{1}{4}}(\mathbb{R}^2)$, *we have* $u = \mathcal{O} * f \in L^p_{-\frac{1}{2},\frac{1}{4}}(\mathbb{R}^2)$, $\frac{\partial u}{\partial x_2} \in L^p_{0,\frac{1}{4}}(\mathbb{R}^2)$, $\frac{\partial u}{\partial x_1} \in L^p_{\frac{1}{2},\frac{1}{4}}(\mathbb{R}^2)$ *and* $\nabla^2 u \in L^p_{\frac{1}{2},\frac{1}{4}}(\mathbb{R}^2)$. *Moreover, we have the estimates*

$$\int_{\mathbb{R}^2}(1+r)^{-\frac{p}{2}}(1+s)^{\frac{p}{4}}|u|^p\,d\boldsymbol{x} + \int_{\mathbb{R}^2}(1+r)^{\frac{p}{2}}(1+s)^{\frac{p}{4}}(|\frac{\partial u}{\partial x_1}|^p + |\nabla^2 u|^p)\,d\boldsymbol{x}$$
$$+ \int_{\mathbb{R}^2}(1+s)^{\frac{p}{4}}|\frac{\partial u}{\partial x_2}|^p\,d\boldsymbol{x} \leq C\int_{\mathbb{R}^2}(1+r)^{\frac{p}{2}}(1+s)^{\frac{p}{4}}|f|^p\,d\boldsymbol{x}.$$

2 Functional Spaces and preliminaries

In this paper, p is a real number in the interval $]1, +\infty[$ and it's conjugate is denoted by p'. A point in $\mathbb{R}^2$ is denoted $\boldsymbol{x} = (x_1, x_2)$ and we denote:

$$r = |\boldsymbol{x}| = (x_1^2 + x_2^2)^{1/2}, \quad \rho = (1 + r^2)^{1/2}, \quad s = r - x_1,$$
$$s' = r + x_1, \quad \text{for } a, b \in \mathbb{R},\ \eta_b^a = (1+r)^a(1+s)^b.$$

For $R > 0$, B_R denotes the open ball of radius R centered at origin and $B'_R = \mathbb{R}^2 \backslash \overline{B_R}$. For any $j \in \mathbb{Z}$, $\mathcal{P}_j$ is the space of polynomials of degree lower than or equal j and if j is negative we set, by convention, $\mathcal{P}_j = 0$. Let B be a Banach space, with dual space B' and a closed subspace X of B. We denote by $B' \perp X$ the subspace of B' orthogonal to X defined by:

$$B' \perp X = \{f \in B';\ \forall v \in X,\ \langle f, v\rangle = 0\}.$$

For $m \in \mathbb{N}^*$, we set

$$k = k(m,p,\alpha) = \begin{cases} -1 & \text{if } \alpha + \frac{2}{p} \notin \{1, ..., m\} \\ m - \alpha - \frac{2}{p} & \text{if } \alpha + \frac{2}{p} \in \{1, ..., m\} \end{cases} \tag{2.1}$$

and we define the weighted Sobolev spaces

$$W^{m,p}_\alpha(\mathbb{R}^2) = \left\{ \begin{array}{l} u \in \mathcal{D}'(\mathbb{R}^2); \forall \lambda \in \mathbb{N}^2 : 0 \leq |\lambda| \leq k, \rho^{(\alpha - m + |\lambda|)}(lg\,\rho)^{-1}\partial^\lambda u \\ \in L^p(\mathbb{R}^2); \qquad k+1 \leq |\lambda| \leq m,\ \rho^{(\alpha-m+|\lambda|)}\partial^\lambda u \in L^p(\mathbb{R}^2) \end{array} \right\}.$$

where $\lg\,\rho = \ln(1+\rho)$. It is a reflexive Banach space, equipped with its natural norm:

$$\begin{aligned} \| u \|_{W^{m,p}_\alpha(\mathbb{R}^2)} &= \Big(\sum_{0 \leq |\lambda| \leq k} \| \rho^{\alpha - m + |\lambda|}(lg\,\rho)^{-1}\partial^\lambda u \|^p_{L^p(\mathbb{R}^2)} \\ &+ \sum_{k+1 \leq |\lambda| \leq m} \| \rho^{\alpha - m + |\lambda|}\partial^\lambda u \|^p_{L^p(\mathbb{R}^2)} \Big)^{\frac{1}{p}}. \end{aligned}$$

Its semi-norm is defined by

$$|u|_{W^{m,p}_\alpha(\mathbb{R}^2)} = \Big(\sum_{|\lambda| = m} \| \rho^\alpha \partial^\lambda u \|^p_{L^p(\mathbb{R}^2)} \Big)^{\frac{1}{p}}.$$

The logarithmic weight appears only when $\alpha + \frac{2}{p} \in \{1, ...m\}$. We refer to Kufner [11], Hanouzet [9], and Amrouche-Girault-Giroire [1] for a detailed study of the space $W^{m,p}_\alpha(\mathbb{R}^n)$. However, we recall some properties and results that we use in this paper. For any $\lambda \in \mathbb{N}^2$, the mapping

$$u \in W^{m,p}_\alpha(\mathbb{R}^2) \longmapsto \partial^\lambda u \in W^{m-|\lambda|,p}_\alpha(\mathbb{R}^2) \tag{2.2}$$

is continuous. When $\alpha + \frac{2}{p} \notin \{1, .., m\}$, we have the following continuous embedding and density

$$W^{m,p}_\alpha(\mathbb{R}^2) \subset W^{m-1,p}_{\alpha-1}(\mathbb{R}^2) \subset ... \subset W^{0,p}_{\alpha-m}(\mathbb{R}^2), \tag{2.3}$$

where,

$$W^{0,p}_\alpha(\mathbb{R}^2) = \big\{ u \in \mathcal{D}'(\mathbb{R}^2);\ \rho^\alpha u \in L^p(\mathbb{R}^2) \big\}$$

and note that the mapping

$$u \in W^{m,p}_\alpha(\mathbb{R}^2) \longmapsto \rho^\gamma u \in W^{m,p}_{\alpha-\gamma}(\mathbb{R}^2) \tag{2.4}$$

is continuous, what is not the case if $\alpha + \frac{2}{p} \in \{1, .., m\}$. The space $W^{m,p}_\alpha(\mathbb{R}^2)$ contains the polynomials of degree lower or equal to j, denoted $\mathcal{P}_j$, where $j \in \mathbb{N}$ is defined by

$$\begin{aligned} j &= [m - \alpha - \tfrac{2}{p}], \qquad \text{if } \alpha + \tfrac{2}{p} \notin \mathbb{Z}^- \\ j &= m - 1 - \alpha - \tfrac{2}{p}, \quad \text{otherwise.} \end{aligned} \tag{2.5}$$

The following theorem is fundamental (see [1])

Theorem 2.1 *Let $m \geq 1$ an integer and α a real number, then there exists a constant C such that*

$$\forall u \in W^{m,p}_{\alpha}(\mathbb{R}^2), \quad \inf_{\lambda \in \mathcal{P}_j} \| u + \lambda \|_{W^{m,p}_{\alpha}(\mathbb{R}^2)} \leq \; C|u|_{W^{m,p}_{\alpha}(\mathbb{R}^2)}, \tag{2.6}$$

where j is the highest degree of polynomial contained in $W^{m,p}_{\alpha}(\mathbb{R}^2)$.

We define the space

$$\boldsymbol{H}_p = \{\boldsymbol{v} \in L^p(\mathbb{R}^2), \quad \mathrm{div}\boldsymbol{v} = 0\}.$$

Theorem 2.1 permits to prove that the following divergence operator is an isomorphism (see [1]):

$$\mathrm{div} \; : \; L^{p'}(\mathbb{R}^2)/\mathbf{H}_p \; \longrightarrow \; W_0^{-1,p'}(\mathbb{R}^2) \perp \mathcal{P}_{[1-\frac{2}{p}]} \tag{2.7}$$

The next result is a consequence of Theorem 2.1 (see [1]):

Proposition 2.2 *Let $m \geq 1$ be an integer and a distribution u such that*

$$\forall \lambda \in \mathbb{N}^2 \; : \; |\lambda| = m, \; \partial^{\lambda} u \in L^p(\mathbb{R}^2).$$

(i) If $1 < p < 2$, then there exists a unique polynomial $K(u) \in \mathcal{P}_{m-1}$ such that $u + K(u) \in W^{m,p}_0(\mathbb{R}^2)$, and

$$\inf_{\mu \in \mathcal{P}_{[m-\frac{2}{p}]}} \| u + K(u) + \mu \|_{W^{m,p}_0(\mathbb{R}^2)} \leq \; C|u|_{W^{m,p}_0(\mathbb{R}^2)} \tag{2.8}$$

(ii) If $p \geq 2$, then $u \in W^{m,p}_0(\mathbb{R}^2)$ and

$$\inf_{\mu \in \mathcal{P}_{[m-\frac{2}{p}]}} \| u + \mu \|_{W^{m,p}_0(\mathbb{R}^2)} \leq \; C|u|_{W^{m,p}_0(\mathbb{R}^2)}. \tag{2.9}$$

When $1 < p < 2$, we have the following charecterization of the space $W^{1,p}_0(\mathbb{R}^2)$:

$$W^{1,p}_0(\mathbb{R}^2) = \{v \in L^{\frac{2p}{2-p}}(\mathbb{R}^2); \; \nabla v \in L^p(\mathbb{R}^2)\}. \tag{2.10}$$

We recall the spaces introduced in [2]:

$$\widetilde{W}^{1,p}_0(\mathbb{R}^2) \; = \; \left\{ u \in W^{1,p}_0(\mathbb{R}^2); \; \frac{\partial u}{\partial x_1} \in W^{-1,p}_0(\mathbb{R}^2) \right\}, \tag{2.11}$$

which is a Banach space for its natural norm:

$$\| u \|_{\widetilde{W}^{1,p}_0(\mathbb{R}^2)} = \| u \|_{W^{1,p}_0(\mathbb{R}^2)} + \| \frac{\partial u}{\partial x_1} \|_{W^{-1,p}_0(\mathbb{R}^2)} .$$

Also, we define

$$\widetilde{W}^{2,p}_0(\mathbb{R}^2) \; = \; \left\{ u \in W^{2,p}_0(\mathbb{R}^2); \; \frac{\partial u}{\partial x_1} \in L^p(\mathbb{R}^2) \right\}, \tag{2.12}$$

which is a Banach space for its natural norm:

$$\| u \|_{\widetilde{W}^{2,p}_0(\mathbb{R}^2)} = \| u \|_{W^{2,p}_0(\mathbb{R}^2)} \; + \; \| \frac{\partial u}{\partial x_1} \|_{L^p(\mathbb{R}^2)} .$$

Its dual space denoted $\widetilde{W}^{-2,p'}_0(\mathbb{R}^2)$ can be characterized as follows (see also Remark 2.5).

Proposition 2.3 *Let* $f \in \widetilde{W}_0^{-2,p'}(\mathbb{R}^2)$. *Then,*
i) *if* $p \neq 2$, *there exists* $f_0 \in W_2^{0,p'}(\mathbb{R}^2)$, $\boldsymbol{F} \in (W_1^{0,p'}(\mathbb{R}^2))^2$, $\boldsymbol{H} \in (L^{p'}(\mathbb{R}^2))^{2\times 2}$ *and* $h \in L^{p'}(\mathbb{R}^2)$ *such that for all* $v \in \widetilde{W}_0^{2,p}(\mathbb{R}^2)$, *we have*

$$\begin{aligned}\langle f, v\rangle_{\widetilde{W}_0^{-2,p'}(\mathbb{R}^2)\times\widetilde{W}_0^{2,p}(\mathbb{R}^2)} &= \langle f_0, v\rangle_{W_2^{0,p'}\times W_{-2}^{0,p}} \;+\; \langle \boldsymbol{F}, \nabla v\rangle_{W_1^{0,p'}\times W_{-1}^{0,p}}\\ &+ \langle \boldsymbol{H}, \nabla^2 v\rangle_{\boldsymbol{L}^{p'}\times\boldsymbol{L}^p} \;+\; \langle h, \tfrac{\partial v}{\partial x_1}\rangle_{L^{p'}\times L^p}.\end{aligned} \tag{2.13}$$

ii) *If* $p = 2$ *we take, on a one hand, the weight* $\rho \lg\rho$ *instead of* ρ *in the definition of* $W_1^{0,p'}(\mathbb{R}^2)$ *and* $W_{-1}^{0,p}(\mathbb{R}^2)$, *on the other hand,* $\rho^2 \lg\rho$ *instead of* ρ^2 *in the definition of* $W_2^{0,p'}(\mathbb{R}^2)$ *and* $W_{-2}^{0,p}(\mathbb{R}^2)$.

Proof : **i)** Suppose $p \neq 2$. Let $\boldsymbol{E} = W_{-2}^{0,p}(\mathbb{R}^2) \times (W_{-1}^{0,p}(\mathbb{R}^2))^2 \times (L^p(\mathbb{R}^2))^{2\times 2} \times L^p(\mathbb{R}^2)$, equipped with the norm:

$$\|\boldsymbol{\psi}\|_{\boldsymbol{E}} \;=\; \|\psi_0\|_{W_{-2}^{0,p}} + \sum_{i=1}^{n} \|\psi_i\|_{W_{-1}^{0,p}} + \sum_{j,k=1}^{n} \|\psi_{j,k}\|_{L^p} + \|\Omega\|_{L^p},$$

where $\boldsymbol{\psi} = (\psi_0,\, \psi_i,\, \psi_{j,k},\, \Omega)$. It is clear that the following operator is an isometric

$$T \;:\; v \in \widetilde{W}_0^{2,p}(\mathbb{R}^2) \;\longrightarrow (v,\; \nabla v,\; \nabla^2 v,\; \frac{\partial v}{\partial x_1}) \in \boldsymbol{E}.$$

For all $f \in \widetilde{W}_0^{-2,p'}(\mathbb{R}^2)$, the operator defined by $L(h) = \langle f, T^{-1}h\rangle$ is continuous on $T(\widetilde{W}_0^{2,p}(\mathbb{R}^2))$ which is a closed subspace of $\boldsymbol{E}$. Then, by the Hahn-Banach theorem, we can extend L to an element $\widetilde{L}$ of the dual of $\boldsymbol{E}$. Now, by the Riesz theorem, there exists $f_0 \in W_2^{0,p'}(\mathbb{R}^2)$, $\boldsymbol{F} \in (W_1^{0,p'}(\mathbb{R}^2))^2$, $\boldsymbol{H} \in (L^{p'}(\mathbb{R}^2))^{2\times 2}$ and $h \in L^{p'}(\mathbb{R}^2)$ satisfying (2.13).
ii) if $p = 2$, we take $\rho \lg\rho F_i \in L^{p'}(\mathbb{R}^2)$, in the definition of $W_1^{0,p'}(\mathbb{R}^2)$, $\rho^2 \lg\rho f_0 \in L^{p'}(\mathbb{R}^2)$ in the definition of $W_2^{0,p'}(\mathbb{R}^2)$ and we proceed as the case i). Let us note that, when $1 < p < 2$, thanks to Theorem 2.1, we can take $\boldsymbol{F} = \boldsymbol{0}$. $\Diamond$

This proposition permits to prove the next result

Proposition 2.4 $\mathcal{D}(\mathbb{R}^2)$ *is dense in* $\widetilde{W}_0^{2,p}(\mathbb{R}^2)$.

Proof : Let $f \in \widetilde{W}_0^{-2,p'}(\mathbb{R}^2)$ such that

$$\forall\varphi \in \mathcal{D}(\mathbb{R}^2),\;\; \langle f, \varphi\rangle_{\widetilde{W}_0^{-2,p'}(\mathbb{R}^2)\times\widetilde{W}_0^{2,p}(\mathbb{R}^2)} = 0. \tag{2.14}$$

i) If $p' \neq 2$, by the previous proposition, there exist $f_0 \in W_2^{0,p'}(\mathbb{R}^2)$, $\boldsymbol{F} \in (W_1^{0,p'}(\mathbb{R}^2))^2$, $\boldsymbol{H} \in (L^{p'}(\mathbb{R}^2))^{2\times 2}$ and $h \in L^p(\mathbb{R}^2)$ satisfying (2.13). In particular, taking $v = \varphi \in \mathcal{D}(\mathbb{R}^2)$ in this equation, we have by (2.14):

$$f_0 - \operatorname{div}\boldsymbol{F} + \operatorname{div}(\operatorname{div}\boldsymbol{H}) - \frac{\partial h}{\partial x_1} \;= 0,$$

in distributions sense. Now, by (2.3), we have the continuous embedding and density $W_0^{1,p}(\mathbb{R}^2) \subset W_{-1}^{0,p}(\mathbb{R}^2)$. Then, by duality, we have the embedding $W_1^{0,p'}(\mathbb{R}^2) \subset W_0^{-1,p'}(\mathbb{R}^2)$, so $\boldsymbol{F} \in (W_0^{-1,p'}(\mathbb{R}^2))^2$, which implies $\operatorname{div}\boldsymbol{F} \in W_0^{-2,p'}(\mathbb{R}^2)$. By the same argument, we deduce $f_0 \in W_0^{-2,p'}(\mathbb{R}^2)$, thus the last equation yields

$$\frac{\partial h}{\partial x_1} \;=\; f_0 - \operatorname{div}\boldsymbol{F} + \operatorname{div}(\operatorname{div}\boldsymbol{H}) \in W_0^{-2,p'}(\mathbb{R}^2) \cap W_0^{-1,p'}(\mathbb{R}^2).$$

So, Equation (2.13) can be writen:

$$\langle f, v\rangle_{\widetilde{W}_0^{-2,p'}(\mathbb{R}^2)\times \widetilde{W}_0^{2,p}(\mathbb{R}^2)} = \langle f_0 - \mathrm{div}\boldsymbol{F} + \mathrm{div}(\mathrm{div}\boldsymbol{H}) - \frac{\partial h}{\partial x_1}, v\rangle_{W_0^{-2,p'}(\mathbb{R}^2)\times W_0^{2,p}(\mathbb{R}^2)}.$$

Let $v \in \widetilde{W}_0^{2,p}(\mathbb{R}^2)$. Since $\mathcal{D}(\mathbb{R}^2)$ is dense in $W_0^{2,p}(\mathbb{R}^2)$, there exists a sequence $\varphi_k \in \mathcal{D}(\mathbb{R}^2)$ such that $\varphi_k \longrightarrow v$ in $W_0^{2,p}(\mathbb{R}^2)$. We obtain then,

$$\langle f, v\rangle_{\widetilde{W}_0^{-2,p'}(\mathbb{R}^2)\times \widetilde{W}_0^{2,p}(\mathbb{R}^2)} = \lim_{k\to\infty} \langle f_0 - \mathrm{div}\boldsymbol{F} + \mathrm{div}\mathrm{div}\boldsymbol{H} - \frac{\partial h}{\partial x_1}, \varphi_k\rangle_{W_0^{-2,p'}\times W_0^{2,p}} = 0.$$

ii) If $p = 2$, we take $(\rho\, lg\, \rho)\boldsymbol{F} \in L^{p'}(\mathbb{R}^2)$ and $(\rho^2\, lg\, \rho) f_0 \in L^{p'}(\mathbb{R}^2)$ we obtain, by the previous embeddings, $\boldsymbol{F} \in (W_0^{-1,p'}(\mathbb{R}^2))^2$ and $f_0 \in W_0^{-2,p'}(\mathbb{R}^2)$. Proceeding as in case **i)**, the result holds and finishes the proof. $\diamondsuit$

Remark 2.5 Property (2.13) is equivalent to

$$\widetilde{W}_0^{-2,p'}(\mathbb{R}^2) = \left\{ f \in \mathcal{D}'(\mathbb{R}^2);\;\; f = f_0 + \mathrm{div}\boldsymbol{F} + \mathrm{div}(\mathrm{div}\boldsymbol{H}) + \frac{\partial h}{\partial x_1} \right\}, \qquad (2.15)$$

where f_0, $\boldsymbol{F}$, $\boldsymbol{H}$ and h are defined in Proposition 2.3.

Using the same technics as in the proof of the Payne-Weinberger inequality, we get the following:

Lemma 2.6 *Let $u \in \mathcal{D}'(\mathbb{R}^2)$ such that $\nabla u \in \boldsymbol{L}^p(\mathbb{R}^2)$.*
i) If $1 < p < 2$ then, there exists a unique constant u_∞, defined by

$$u_\infty \;=\; \lim_{r\to\infty} \frac{1}{2\pi}\int_0^{2\pi} u(r cos\,\theta, r sin\,\theta)\, d\theta, \qquad (2.16)$$

such that

$$u - u_\infty \in W_0^{1,p}(\mathbb{R}^2). \qquad (2.17)$$

Moreover, we have

$$u - u_\infty \in L^{\frac{2p}{2-p}}(\mathbb{R}^2), \qquad (2.18)$$

with the estimate

$$\| u - u_\infty \|_{L^{\frac{2p}{2-p}}(\mathbb{R}^2)} \;\leq\; C \,\| \nabla u \|_{L^p(\mathbb{R}^2)}, \qquad (2.19)$$

and

$$\int_0^{2\pi} |u(r cos\,\theta, r sin\,\theta) - u_\infty|^p d\theta \;\leq\; C\, r^{p-2} \int_{\{|\boldsymbol{x}|>r\}} |\nabla u|^p d\boldsymbol{x}. \qquad (2.20)$$

ii) If $p > 2$, then $u \in W_0^{1,p}(\mathbb{R}^2)$ and verifies

$$|u(\boldsymbol{x})| \leq C r^{1-\frac{2}{p}} \,\| u \|_{W_0^{1,p}(\mathbb{R}^2)} \quad \text{and} \quad r^{\frac{2}{p}-1}|u(\boldsymbol{x})| \longrightarrow 0. \qquad (2.21)$$

The next result is a corollary of the previous lemma.

Corollary 2.7 *Let $u \in \mathcal{D}'(\mathbb{R}^2)$ such that $\nabla^2 u \in L^p(\mathbb{R}^2)$. Then,*
i) if $1 < p < 2$, there exists a unique vector $\boldsymbol{A} \in \mathbb{R}^2$ such that

$$\nabla u + \boldsymbol{A} \in \boldsymbol{L}^{\frac{2p}{2-p}}(\mathbb{R}^2),$$

where $\boldsymbol{A}$ is defined by

$$\boldsymbol{A} \ = \ - \lim_{r\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \nabla\, u(rcos\,\theta, rsin\,\theta)d\theta. \tag{2.22}$$

Moreover, $u + \boldsymbol{A}.\boldsymbol{x} \in W_0^{2,p}(\mathbb{R}^2)$ and satisfies

$$\inf_{k\in\mathbb{R}} \parallel u + \boldsymbol{A}.\boldsymbol{x} + k \parallel_{W_0^{2,p}(\mathbb{R}^2)} \leq C|u|_{W_0^{2,p}(\mathbb{R}^2)}. \tag{2.23}$$

ii) If $p \geq 2$, then $u \in W_0^{2,p}(\mathbb{R}^2)$ and

$$\inf_{\lambda\in\mathcal{P}_1} \parallel u + \lambda \parallel_{W_0^{2,p}(\mathbb{R}^2)} \leq C|u|_{W_0^{2,p}(\mathbb{R}^2)}. \tag{2.24}$$

Now, with these last result, we can give precisely definition of limit at infinity.

Definition 2.8 Let $u \in \mathcal{D}'(\mathbb{R}^2)$ such that $\nabla\, u \in \boldsymbol{L}^p(\mathbb{R}^2)$. We say that u tends to $u_\infty \in \mathbb{R}$ at infinity and we denote

$$\lim_{|\boldsymbol{x}|\to\infty} u(\boldsymbol{x}) = u_\infty,$$

if

$$\lim_{r\to\infty} \int_0^{2\pi} |u(rcos\,\theta, rsin\,\theta) - u_\infty| d\,\theta \ = 0.$$

Remark 2.9 Let $u \in \mathcal{D}'(\mathbb{R}^2)$ such that $\nabla\, u \in \boldsymbol{L}^p(\mathbb{R}^2)$. If $1 < p < 2$, we have the equivalence of the following propositions
i) $u - u_\infty \in W_0^{1,p}(\mathbb{R}^2)$,
ii) $\lim_{|\boldsymbol{x}|\to\infty} u(\boldsymbol{x}) = u_\infty$ in the sense of Definition 2.8 .

Finally, we recall the following lemma

Lemma 2.10 *Let r and p two reals such that $1 < r < \infty$ and $p > 2$. Let $u \in L^r(\mathbb{R}^2)$ and $\nabla\, u \in \boldsymbol{L}^p(\mathbb{R}^2)$. Then u is a continuous function on $\mathbb{R}^2$ and*

$$\lim_{|\boldsymbol{x}|\to\infty} u(\boldsymbol{x}) = 0.$$

3 The scalar Oseen equation in $\mathbb{R}^2$.

In this section, we propose to solve the scalar Oseen equation (1.7). In order to simplify the notations, we assume without loss the generality $\lambda = \nu = 1$:

$$-\Delta u + \frac{\partial u}{\partial x_1} = f \quad \text{in} \quad \mathbb{R}^2, \tag{3.1}$$

where $f \in \mathcal{D}'(\mathbb{R}^2)$. To that end, let us define the operator

$$T: \ u \longmapsto -\Delta u + \frac{\partial u}{\partial x_1}. \tag{3.2}$$

3.1 Study of the kernel

We consider the kernel of the operator T when it is defined on the tempered distributions $\mathcal{S}'(\mathbb{R}^2)$. Let u be an element of the kernel, by Fourier's transform we can write

$$4\pi^2|\xi|^2\hat{u}(\xi) + 2i\pi\xi_1\hat{u}(\xi) = 0.$$

Setting

$$\hat{u}(\xi) = v(\xi) + iw(\xi),$$

it follows that

$$\begin{cases} 4\pi^2|\xi|^2 v(\xi) - 2\pi\xi_1 w(\xi) = 0, \\ 2\pi\xi_1 v(\xi) + 4\pi^2|\xi|^2 w(\xi) = 0. \end{cases} \tag{3.3}$$

Since the determinant of the above system is $16\pi^4|\xi|^4 + 4\pi^2|\xi|^2$, we deduce that, for $\xi \neq 0$, the support of $\hat{u}$ is included in $\{0\}$. Then we have

$$\hat{u}(\xi) = \sum c_\alpha \delta^{(\alpha)}, c_\alpha \in \mathbb{C}, \text{ with a finite sum.}$$

By the inverse Fourier's transform, we get

$$u(x) = \sum d_\alpha x^\alpha, d_\alpha \in \mathbb{C}, \text{ with a finite sum,}$$

then, u is a polynomial. Setting for all integer k

$$\mathcal{S}_k = \{q \in \mathcal{P}_k; -\Delta q + \frac{\partial q}{\partial x_1} = 0\}, \tag{3.4}$$

if T is defined on $\mathcal{S}'(\mathbb{R}^2)$, then $\ker T = \mathcal{S}_k$, and we have:

Lemma 3.1 *Let $f \in \mathcal{S}'(\mathbb{R}^2)$ be a tempered distribution and let $u \in \mathcal{S}'(\mathbb{R}^2)$ be a solution of* (3.1)*. Then u is uniquely determined up to polynomial of $\mathcal{S}_k$.*

Let us notice that $\mathcal{S}_0 = \mathbb{R}$ and $\mathcal{S}_1$ is a space of polynomials of degree less than or equal one and independent of x_1.

3.2 The fundamental solution

Following the idea of [8], we look for the fundamental solution $\mathcal{O}$ of the scalar Oseen equation under the shape

$$\mathcal{O}(\boldsymbol{x}) = e^{\frac{x_1}{2}} f(\frac{r}{2}),$$

we find by a direct computations:

$$\left(-\Delta\mathcal{O} + \frac{\partial \mathcal{O}}{\partial x_1}\right) = \frac{1}{2\pi r^2} e^{\frac{x_1}{2}} \left((\frac{r}{2})^2 f''(\frac{r}{2}) + \frac{r}{2} f'(\frac{r}{2}) - (\frac{r}{2})^2 f(\frac{r}{2}) \right),$$

where, for $y = \frac{r}{2}$,

$$y^2 f''(y) + y f'(y) - y^2 f(y) = 0$$

is the modified Bessel equation. The singular solution at $y = 0$, K_0 of this equation cannot be given explicitly however, we can give an esimates in a neighborhood of zero and when y is large as follow.
(i) When y is small

$$K_0(y) = \ln\frac{1}{y} + \ln 2 - \gamma + \sigma(y), \tag{3.5}$$

where γ is the Euler constant and σ satisfies

$$\frac{d^k\sigma}{dy^k} = o(y^{-k}).$$

Then, when r is close to zero,

$$\mathcal{O}(x) = -\frac{1}{2\pi}e^{\frac{x_1}{2}}\left\{\ln\frac{1}{r} + 2\ln 2 - \gamma + \sigma(r)\right\}. \tag{3.6}$$

(ii) When $r \longrightarrow +\infty$, using the asymptotic developpement given in [10] we have

$$\begin{aligned} K_0(\frac{r}{2}) &= \left(\frac{\pi}{r}\right)^{\frac{1}{2}} e^{-\frac{r}{2}}\left[1 - \frac{1}{4r} + O(r^{-2})\right], \\ K_0'(\frac{r}{2}) &= \left(\frac{\pi}{r}\right)^{\frac{1}{2}} e^{-\frac{r}{2}}\left[-1 - \frac{3}{4r} + O(r^{-2})\right]. \end{aligned}$$

As the divatives of $\mathcal{O}$ are given by

$$\frac{\partial\mathcal{O}}{\partial x_1} = -\frac{1}{4\pi}e^{\frac{x_1}{2}}\left[K_0(\frac{r}{2}) + \frac{x_1}{r}K_0'(\frac{r}{2})\right], \tag{3.7}$$

$$\frac{\partial\mathcal{O}}{\partial x_2} = -\frac{x_2}{4\pi r}e^{\frac{x_1}{2}}K_0'(\frac{r}{2}). \tag{3.8}$$

We deduce then the behavior of the fundamental solution $\mathcal{O}$ and these derivatives when r tends to the infinity.

$$\mathcal{O}(x) = -\frac{1}{2\sqrt{\pi r}}e^{-\frac{s}{2}}\left[1 - \frac{1}{4r} + O(r^{-2})\right], \tag{3.9}$$

$$\frac{\partial\mathcal{O}}{\partial x_1} = -\frac{1}{4\sqrt{\pi r}}e^{-\frac{s}{2}}\left[\frac{s}{r} - \frac{r+3x_1}{8r^2} + O(r^{-2})\right], \tag{3.10}$$

$$\frac{\partial\mathcal{O}}{\partial x_2} = \frac{x_2}{4r\sqrt{\pi r}}e^{-\frac{s}{2}}\left[1 + \frac{3}{4r} + O(r^{-2})\right]. \tag{3.11}$$

Using the inequality:

$$\forall b \in \mathbb{R}, \ \ e^{-s/2} \le C_b(1+s)^b.$$

we obtain the following anisotropic estimates

$$\begin{aligned} &|\mathcal{O}(\boldsymbol{x})| \le C\, r^{-\frac{1}{2}}(1+s)^{-1}, \quad |\tfrac{\partial\mathcal{O}}{\partial x_1}(\boldsymbol{x})| \le C\, r^{-\frac{3}{2}}\,(1+s)^{-1}, \\ &|\tfrac{\partial\mathcal{O}}{\partial x_2}(\boldsymbol{x})| \le C\, r^{-1}\,(1+s)^{-1}. \end{aligned} \tag{3.12}$$

Let f and g two functions defined on an interval $I \subset \mathbb{R}$. We denote $f \sim g$ on $J \subset I$ if there exist two positive constants C_1 and C_2 such that $C_1 g(t) \le f(t) \le C_2 g(t)$ for all t in J.
To study the integrability properties of the fundamental solution and its derivatives, we need the following result.

Lemma 3.2 *Assume that* $2 - \alpha - \min(\frac{1}{2}, \beta) < 0$. *Then, there exists a constant* $C > 0$ *such that, for all* $\mu > 1$, *we have*

$$\int_{|x|>\mu} r^{-\alpha}(1+s)^{-\beta}\,dx \le \begin{cases} C\mu^{2-\alpha-\min(\frac{1}{2},\beta)}, & \text{if } \beta \ne \frac{1}{2}, \\ C\mu^{\frac{3}{2}-\alpha}\ln r, & \text{if } \beta = \frac{1}{2}. \end{cases} \tag{3.13}$$

Proof : First we prove the following result.

$$\int_{\partial Br} r^{-\alpha}(1+s)^{-\beta}\, d\sigma \sim \begin{cases} r^{1-\alpha-\min(\frac{1}{2},\beta)} & \text{if } \beta \neq \frac{1}{2}, \\ r^{\frac{1}{2}-\alpha}\ln r, & \text{if } \beta = \frac{1}{2}. \end{cases} \tag{3.14}$$

Using the polar coordinates, we have for $s = r(1-cos\,\theta)$:

$$I = \int_{\partial Br} r^{-\alpha}(1+s)^{-\beta}\, d\sigma = 2r^{1-\alpha}\int_0^{\pi}(1+r(1-cos\,\theta))^{-\beta}d\theta.$$

Since $r^2 sin^2\theta = 2rs - s^2$ then,

$$I = 2r^{1-\alpha}\int_0^{2r}(1+s)^{-\beta}(2rs-s^2)^{-\frac{1}{2}}\, ds.$$

i) When $0 < s \leq 1$, $1+s \sim 1$, then

$$\int_0^1 (1+s)^{-\beta}(2rs-s^2)^{-\frac{1}{2}}\, ds \;\sim\; r^{-\frac{1}{2}}\int_0^1 s^{-\frac{1}{2}}\, ds \;\sim\; r^{-\frac{1}{2}}.$$

ii) When $1 < s < r$, $1+s \sim s$ and $2rs - s^2 = s(2r-s) \sim rs$ then

$$\int_1^r (1+s)^{-\beta}(2rs-s^2)^{-\frac{1}{2}}\, ds \;\sim\; r^{-\frac{1}{2}}\int_1^r s^{-\frac{1}{2}-\beta}\, ds \;\sim\; r^{-\min(\frac{1}{2},\beta)}.$$

and, if $\beta = \frac{1}{2}$, we get

$$\int_1^r (1+s)^{-\beta}(2rs-s^2)^{-\frac{1}{2}}\, ds \sim r^{-\frac{1}{2}}\ln r.$$

iii) When $r < s < 2r$, $1+s \sim r$ and $2rs - s^2 \sim r(2r-s)$ then

$$\int_r^{2r} (1+s)^{-\beta}(2rs-s^2)^{-\frac{1}{2}}\, ds \sim r^{-\frac{1}{2}-\beta}\int_r^{2r}(2r-s)^{-\frac{1}{2}}\, ds \;\sim\; r^{-\beta}.$$

So,

$$\begin{aligned} I &\sim r^{1-\alpha-\min(\frac{1}{2},\beta)}\left(r^{\min(\frac{1}{2},\beta)-\frac{1}{2}} + 1 + r^{\min(\frac{1}{2},\beta)-\beta}\right) \\ &\sim \begin{cases} r^{1-\alpha-\min(\frac{1}{2},\beta)} & \text{if } \beta \neq \frac{1}{2}, \\ r^{\frac{1}{2}-\alpha}\ln r, & \text{if } \beta = \frac{1}{2}. \end{cases} \end{aligned}$$

By this equivalence we deduce:

$$\int_{|x|>\mu} r^{-\alpha}(1+s)^{-\beta}\, d\boldsymbol{x} < +\infty \Longleftrightarrow 2 - \alpha - \min(\frac{1}{2},\beta) < 0. \tag{3.15}$$

When this condition is satisfied we obtain our result. $\Diamond$

Using Lemma 3.2 with estimate (3.12) we deduce

$$\forall p > 3,\;\; \mathcal{O} \in L^p(\mathbb{R}^2) \quad \text{and} \quad \forall p \in]\frac{3}{2}, 2[,\;\; \nabla\,\mathcal{O} \in \boldsymbol{L}^p(\mathbb{R}^2), \tag{3.16}$$

that means that in particular $\mathcal{O} \in W_0^{1,p}(\mathbb{R}^2)$ for any $\frac{3}{2} < p < 2$. Note also that

$$\mathcal{O} \in L^1_{loc}(\mathbb{R}^2) \quad \text{and} \quad \nabla\,\mathcal{O} \in \boldsymbol{L}^1_{loc}(\mathbb{R}^2), \tag{3.17}$$

and for $\mathcal{B}^R = \mathbb{R}^2 \setminus \overline{\mathcal{B}(\boldsymbol{0}, R)}$

$$\forall p > 3, \ \ \mathcal{O} \in L^p(\mathcal{B}^R) \quad \text{and} \quad \forall p > \frac{3}{2}, \ \ \nabla \mathcal{O} \in \boldsymbol{L}^p(\mathcal{B}^R). \tag{3.18}$$

With the L^∞ weighted estimates obtained in [10] (Thms 3.5, 3.7 and 3.8), we get estimates on the convolution of $\breve{\mathcal{O}}$ with a function $\varphi \in \mathcal{D}(\mathbb{R}^2)$ as follows

Lemma 3.3 *For any $\varphi \in \mathcal{D}(\mathbb{R}^2)$ we have the estimates*

$$|\breve{\mathcal{O}} * \varphi(\boldsymbol{x})| \ \le \ C_\varphi \frac{1}{|\boldsymbol{x}|^{\frac{1}{2}}(1 + |\boldsymbol{x}| + x_1)^{\frac{1}{2}}}, \tag{3.19}$$

$$|\frac{\partial}{\partial x_1}(\breve{\mathcal{O}} * \varphi)(\boldsymbol{x})| \ \le \ C_\varphi \frac{1}{|\boldsymbol{x}|^{\frac{3}{2}}(1 + |\boldsymbol{x}| + x_1)^{\frac{1}{2}}}, \tag{3.20}$$

$$|\frac{\partial}{\partial x_2}(\breve{\mathcal{O}} * \varphi)(\boldsymbol{x})| \ \le \ C_\varphi \frac{1}{|\boldsymbol{x}|(1 + |\boldsymbol{x}| + x_1)}, \tag{3.21}$$

where C_φ depends on the support of φ and $\breve{\mathcal{O}}(\boldsymbol{x}) = \mathcal{O}(-\boldsymbol{x})$.

Remark 3.4 **1)** The behavior on $|\boldsymbol{x}|$ of $\breve{\mathcal{O}} * \varphi$ and its first derivatives is the same that that of $\breve{\mathcal{O}}$ but, the behavior on $1 + s'$ is a little bit different.
2) By Lemma 3.2 and these last estimations we find that

$$\forall q > \frac{3}{2}, \quad \breve{\mathcal{O}} * \varphi \in W_0^{1,q}(\mathbb{R}^2). \tag{3.22}$$

3.3 Oseen potential and existence results

Using the *weak-type (p,q)* operators and the Marcinkiewicz interpolation's Theorem, we have the following

Theorem 3.5 *Let f given in $L^p(\mathbb{R}^2)$. Then $\frac{\partial^2}{\partial x_j \partial x_k}(\mathcal{O} * f) \in L^p(\mathbb{R}^2)$, $\frac{\partial}{\partial x_1}(\mathcal{O} * f) \in L^p(\mathbb{R}^2)$ and satisfy the estimate*

$$\|\frac{\partial^2}{\partial x_j \partial x_k}(\mathcal{O} * f)\|_{L^p(\mathbb{R}^2)} + \|\frac{\partial}{\partial x_1}(\mathcal{O} * f)\|_{L^p(\mathbb{R}^2)} \ \le C\|f\|_{L^p(\mathbb{R}^2)}. \tag{3.23}$$

Moreover,
i) *if $1 < p < \frac{3}{2}$, then $\mathcal{O} * f \in L^{\frac{3p}{3-2p}}(\mathbb{R}^2)$ and satisfies*

$$\|\mathcal{O} * f\|_{L^{\frac{3p}{3-2p}}(\mathbb{R}^2)} \ \le C\|f\|_{L^p(\mathbb{R}^2)}. \tag{3.24}$$

ii) *If $1 < p < 3$, then $\frac{\partial}{\partial x_i}(\mathcal{O} * f) \in L^{\frac{3p}{3-p}}(\mathbb{R}^2)$ and verifies the estimate*

$$\|\frac{\partial}{\partial x_i}(\mathcal{O} * f)\|_{L^{\frac{3p}{3-p}}(\mathbb{R}^2)} \ \le C\|f\|_{L^p(\mathbb{R}^2)}. \tag{3.25}$$

Proof : By the Fourier's transform, we obtain from Equation (3.1):

$$\mathcal{F}(\frac{\partial^2}{\partial x_j \partial x_k}(\mathcal{O} * f)) = \frac{-4\pi^2 \xi_j \xi_k}{4\pi^2|\boldsymbol{\xi}|^2 + 2\pi i \xi_1}\mathcal{F}(f).$$

Since the function $\xi \mapsto m(\xi) = \frac{-4\pi^2 \xi_j \xi_k}{4\pi^2|\boldsymbol{\xi}|^2 + 2\pi i \xi_1}$ is of class $\mathcal{C}^2$ in $\mathbb{R}^2 \setminus \{0\}$ and satisfies for every $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$

$$|\frac{\partial^{|\alpha|} m}{\partial \xi^\alpha}(\xi)| \le B|\xi|^{-\alpha},$$

where, $|\alpha| = \alpha_1 + \alpha_2$ and B is a constant. Then, the linear operator

$$T : f \mapsto \frac{\partial^2}{\partial x_j \partial x_k}(\mathcal{O} * f)(\boldsymbol{x}) = \int_{\mathbb{R}^2} e^{2\pi i \boldsymbol{x}\boldsymbol{\xi}} \frac{-4\pi^2 \xi_j \xi_k}{4\pi^2|\boldsymbol{\xi}|^2 + 2\pi i \xi_1} \mathcal{F}(f)(\boldsymbol{\xi})\, d\boldsymbol{\xi}$$

is continous from $L^p(\mathbb{R}^2)$ to $L^p(\mathbb{R}^2)$. So, $\frac{\partial^2}{\partial x_j \partial x_k}(\mathcal{O} * f) \in L^p(\mathbb{R}^2)$ and satisfies

$$\|\frac{\partial^2}{\partial x_j \partial x_k}(\mathcal{O} * f)\|_{L^p(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}. \tag{3.26}$$

(see E. Stein [17], Thm 3.2, p.96.) Now, from Equation (3.1), we deduce that $\frac{\partial}{\partial x_1}(\mathcal{O} * f) \in L^p(\mathbb{R}^2)$ and the estimate

$$\|\frac{\partial}{\partial x_1}(\mathcal{O} * f)\|_{L^p(\mathbb{R}^2)} \leq C\left(\|\Delta(\mathcal{O} * f)\|_{L^p(\mathbb{R}^2)} + \|f\|_{L^p(\mathbb{R}^2)}\right), \tag{3.27}$$

which proves the first part of proposition and Estimate (3.23). Next, to prove i) and ii), we adapt the technic used by Stein in [17] which studied the convolution of $f \in L^p(\mathbb{R}^n)$ with the kernel $|\boldsymbol{x}|^{\alpha-n}$. We split the function K into $K_1 + K_\infty$ where,

$$K_1(\boldsymbol{x}) = K(\boldsymbol{x}) \quad \text{if} \quad |\boldsymbol{x}| \leq \mu \quad \text{and} \quad K_1(\boldsymbol{x}) = 0 \quad \text{if} \quad |\boldsymbol{x}| > \mu,$$

$$K_\infty(\boldsymbol{x}) = 0 \qquad \text{if} \quad |\boldsymbol{x}| \leq \mu \quad \text{and} \quad K_\infty(\boldsymbol{x}) = K(\boldsymbol{x}) \;\; \text{if} \;\; |\boldsymbol{x}| > \mu.$$

The function K denotes successively $\mathcal{O}$ and $\frac{\partial \mathcal{O}}{\partial x_i}$ and the positive number μ will be fixed in the sequel.

1) Estimate (3.24). According to (3.6), we have $\mathcal{O}_1 \in L^1(\mathbb{R}^2)$ and by (3.16), $\mathcal{O}_\infty \in L^{p'}(\mathbb{R}^2)$, then, $\mathcal{O}_1 * f$ exists almost everywhere and $\mathcal{O}_\infty * f$ exists everywhere so, $\mathcal{O} * f = \mathcal{O}_1 * f + \mathcal{O}_\infty * f$ exists almost everywhere. Next, we shall show that $f \mapsto \mathcal{O} * f$ is of *weak-type* (p, q) with $q = \frac{3p}{3-2p}$ in the sense that:

$$\text{mes}\,\{\boldsymbol{x}; |(\mathcal{O} * f)(\boldsymbol{x})| > \lambda\} \leq \left(C_{p,q}\frac{\|f\|_{L^p(\mathbb{R}^2)}}{\lambda}\right)^q, \quad \text{for all } \lambda > 0. \tag{3.28}$$

We have:

$$\text{mes}\,\{\boldsymbol{x}; |\mathcal{O} * f| > 2\lambda\} \leq \text{mes}\,\{\boldsymbol{x}; |\mathcal{O}_1 * f| > \lambda\} + \text{mes}\,\{\boldsymbol{x}; |\mathcal{O}_\infty * f| > \lambda\},$$

and

$$\text{mes}\,\{\boldsymbol{x}; |(\mathcal{O}_1 * f)(\boldsymbol{x})| > \lambda\} \leq \frac{\|\mathcal{O}_1\|^p_{L^1(\mathbb{R}^2)}\ \|f\|^p_{L^p(\mathbb{R}^2)}}{\lambda^p},$$

$$\|\mathcal{O}_\infty * f\|_{L^\infty(\mathbb{R}^2)} \leq \|\mathcal{O}_\infty\|_{L^{p'}(\mathbb{R}^2)}\ \|f\|_{L^p(\mathbb{R}^2)}.$$

Note that it is enough to prove inequality (3.28) for $\|f\|_{L^p(\mathbb{R}^2)} = 1$.

i) Estimate of $I = \int_{|\boldsymbol{x}|<\mu} |\mathcal{O}(\boldsymbol{x})|\, d\boldsymbol{x}$.

If $0 < \mu \leq 1$, then by (3.6), $I \leq C\mu$.

If $\mu > 1$,

$$I = \int_{|\boldsymbol{x}|<1} |\mathcal{O}(\boldsymbol{x})|\, d\boldsymbol{x} + \int_{1<|\boldsymbol{x}|\leq\mu} |\mathcal{O}(\boldsymbol{x})|\, d\boldsymbol{x}.$$

Since $\mathcal{O} \in L^1_{loc}(\mathbb{R}^2)$, then

$$\int_{|\boldsymbol{x}|<1} |\mathcal{O}(\boldsymbol{x})|\, d\boldsymbol{x} \leq C \leq C\mu.$$

Further, from estimate (3.12) and using lemma 3.2, we have

$$\int_{1<|\boldsymbol{x}|<\mu} |\mathcal{O}(\boldsymbol{x})|\, d\boldsymbol{x} \leq C \int_{1<|\boldsymbol{x}|<\mu} r^{-\frac{1}{2}}(1+s)^{-1}\, d\boldsymbol{x} \leq C\mu,$$

then,

$$\forall \mu > 0, \;\; \|\mathcal{O}_1\|_{L^1(\mathbb{R}^2)} \leq C\mu. \tag{3.29}$$

ii) Estimate of $J = \int_{|\boldsymbol{x}|>\mu} |\mathcal{O}(\boldsymbol{x})|^{p'}\, d\boldsymbol{x}$**.**
If $\mu > 1$, $|\mathcal{O}(\boldsymbol{x})|^{p'} \sim e^{-\frac{p's}{2}} r^{-\frac{p'}{2}} \leq Cr^{-\frac{p'}{2}}(1+s)^{-p'}$. Then by Lemma 3.2, for $p' > 3$, we have $J \leq C\mu^{\frac{3}{2}-\frac{p'}{2}}$.
If $0 < \mu \leq 1$,

$$J = \int_{\mu<|x|<1} |\mathcal{O}(x)|^{p'}\, dx + \int_{|x|>1} |\mathcal{O}(x)|^{p'}\, dx = J_1 + J_2.$$

Proceeding as previously, we get $J_2 \leq C \leq C\mu^{\frac{3}{2}-\frac{p'}{2}}$. We have also

$$J_1 = \int_{\mu<|x|\leq 1} e^{\frac{p'x_1}{2}} |-\ln r + 2\ln 2 + \gamma + o(r)|^{p'}\, dx \leq C \leq C\mu^{\frac{3}{2}-\frac{p'}{2}}.$$

Then,

$$\text{for } p' > 3 \;\text{ and }\; \mu > 0, \;\; \|\mathcal{O}_\infty\|_{L^{p'}(\mathbb{R}^2)} \leq C\mu^{\frac{3-p'}{2p'}}. \tag{3.30}$$

Setting $\lambda = C\mu^{\frac{3-p'}{2p'}}$ which implies $\mu = C'\lambda^{\frac{2p'}{3-p'}} = C'\lambda^{\frac{2p}{2p-3}}$, we get

$$\text{mes}\,\{x \in \mathbb{R}^2; |(\mathcal{O}_\infty * f)(\boldsymbol{x})| > \lambda\} = 0.$$

So, for $1 < p < \frac{3}{2}$, we have

$$\text{mes}\,\{\boldsymbol{x} \in \mathbb{R}^2; (|\mathcal{O} * f)(\boldsymbol{x})| > 2\lambda\} \leq C\frac{\|\mathcal{O}_1\|^p_{L^1(\mathbb{R}^2)}}{\lambda^p} \leq C\frac{\mu^p}{\lambda^p} \leq C\left(\frac{1}{\lambda}\right)^{\frac{3p}{3-2p}},$$

which proves inequality (3.28).

2) Estimate (3.25). We have also $K_1 \in L^1(\mathbb{R}^2)$ and $K_\infty \in L^{p'}(\mathbb{R}^2)$ where, $K = \frac{\partial \mathcal{O}}{\partial x_i}, i = 1, 2$.
i) Estimate of $\int_{|\boldsymbol{x}|>\mu} |\frac{\partial \mathcal{O}}{\partial x_i}(\boldsymbol{x})|^{p'}\, d\boldsymbol{x}$**.**
Using estimate (3.12), we get for $\mu \geq 1$ and $p < 3$:

$$\int_{|\boldsymbol{x}|>\mu} |\frac{\partial \mathcal{O}}{\partial x_i}(\boldsymbol{x})|^{p'}\, d\boldsymbol{x} \leq C\mu^{\frac{3}{2}-\frac{3p'}{2}} \leq C\mu^{\frac{3}{2}-p'}. \tag{3.31}$$

For $\mu < 1$,

$$\int_{|\boldsymbol{x}|>\mu} |\frac{\partial \mathcal{O}}{\partial x_i}(\boldsymbol{x})|^{p'}\, d\boldsymbol{x} = \int_{\mu<|\boldsymbol{x}|<1} |\frac{\partial \mathcal{O}}{\partial x_i}(\boldsymbol{x})|^{p'}\, d\boldsymbol{x} + \int_{|\boldsymbol{x}|>1} |\frac{\partial \mathcal{O}}{\partial x_i}(\boldsymbol{x})|^{p'}\, d\boldsymbol{x}.$$

The case $\mu \geq 1$ yields

$$\int_{|\boldsymbol{x}|>1} |\frac{\partial \mathcal{O}}{\partial x_i}(\boldsymbol{x})|^{p'}\, d\boldsymbol{x} \leq C \leq C\mu^{\frac{3}{2}-p'}.$$

We have also

$$\begin{aligned}
\int_{\mu<|\boldsymbol{x}|<1} |\frac{\partial \mathcal{O}}{\partial x_i}(\boldsymbol{x})|^{p'}\, d\boldsymbol{x} \;\; &\leq \;\; \int_\mu^1 r^{1-q} dr \int_0^\pi e^{\frac{p'}{2} r\cos\theta} |\sin\theta + C'|^{p'}\, d\theta \\
&\leq \;\; C\int_\mu^1 r^{\frac{1}{2}-q} dr \leq C\mu^{\frac{3}{2}-p'}.
\end{aligned}$$

So, by these two inequalities and (3.31), we get

$$\|\frac{\partial \mathcal{O}}{\partial x_i}\|_{L^{p'}(\mathbb{R}^2)} \leq C\mu^{\frac{3-2p'}{p'}}. \tag{3.32}$$

ii) Estimate of $J = \int_{|\boldsymbol{x}|<\mu} |\frac{\partial \mathcal{O}}{\partial x_i}(\boldsymbol{x})|\, d\boldsymbol{x}$.
If $0 < \mu < 1$,

$$\begin{aligned} J = \int_{|\boldsymbol{x}|<\mu} |e^{\frac{x_1}{2}}|\frac{x_2}{r^2} + o(\frac{1}{r})|\, d\boldsymbol{x} &= \int_0^{\mu}\int_{-\pi}^{\pi} e^{\frac{r}{2}cos\,\theta}|sin\,\theta + C'|drd\theta \\ &\leq C\int_0^{\mu} dr \ \leq C\mu \ \leq C\mu^{\frac{1}{2}}. \end{aligned}$$

If $\mu \geq 1$,

$$J = \int_{|\boldsymbol{x}|<1} |\frac{\partial \mathcal{O}}{\partial x_i}|d\boldsymbol{x} + \int_{1<|\boldsymbol{x}|<\mu} |\frac{\partial \mathcal{O}}{\partial x_i}|d\boldsymbol{x} \ = \ J_1 + J_2.$$

The preceding case yields $J_1 \leq C \leq C\mu^{\frac{1}{2}}$. By Estimate (3.12) and Lemma 3.2 we have

$$J_2 \leq C\int_{|\boldsymbol{x}|<\mu} \frac{d\boldsymbol{x}}{r(1+s)} \leq C\int_0^{\mu} r^{-\frac{1}{2}}dr \leq C\mu^{\frac{1}{2}}.$$

We obtain then

$$\|\frac{\partial \mathcal{O}}{\partial x_i}\|_{L^1(\mathbb{R}^2)} \leq C\mu^{\frac{1}{2}}. \tag{3.33}$$

Since $\mathcal{O} \in L^1_{loc}(\mathbb{R}^2)$ and $\frac{\partial \mathcal{O}}{\partial x_i} \in L^1_{loc}(\mathbb{R}^2)$, then $\frac{\partial}{\partial x_i}(\mathcal{O}*f) = \frac{\partial \mathcal{O}}{\partial x_i}*f$. As previously, we have, for $1 < p < 3$ and all $\lambda > 0$:

$$\text{mes}\,\{\boldsymbol{x} \in \mathbb{R}^2; |\frac{\partial}{\partial x_i}(\mathcal{O}*f)(\boldsymbol{x})| > 2\lambda\} \leq C\left(\frac{1}{\lambda}\right)^{\frac{3p}{3-p}}.$$

Now, using the Marcinkiewicz Theorem, the operator $R : f \mapsto \mathcal{O}*f$ is continuous from $L^p(\mathbb{R}^2)$ into $L^{\frac{3p}{3-2p}}(\mathbb{R}^2)$ and $R_i : f \mapsto \frac{\partial}{\partial x_i}(\mathcal{O}*f)$ is continuous from $L^p(\mathbb{R}^2)$ into $L^{\frac{3p}{3-p}}(\mathbb{R}^2)$. $\Diamond$

Remark 3.6 i) We can prove that $\mathcal{O} \in L^{3,\infty}(\mathbb{R}^2)$, *i.e*

$$\sup_{\mu>0} \mu^3 \text{ mes}\,\{\boldsymbol{x} \in \mathbb{R}^2;\ |\mathcal{O}(\boldsymbol{x})| > \mu\} < +\infty. \tag{3.34}$$

So that, thanks to the weak Young inequality (*cf.* Reed-Simon [16]):

$$\|\mathcal{O}*f\|_{L^{\frac{3p}{3-2p},\infty}(\mathbb{R}^2)} \leq C\|\mathcal{O}\|_{L^{3,\infty}(\mathbb{R}^2)}\|f\|_{L^{p,\infty}(\mathbb{R}^2)}. \tag{3.35}$$

This estimate shows that if $1 < p < \frac{3}{2}$, there exist p_0 and p_1 such that $1 < p_0 < p < p_1 < \frac{3}{2}$ and such that the operator

$$T\ : f \longmapsto \mathcal{O}*f$$

is continuous from $L^{p_0}(\mathbb{R}^2)$ into $L^{\frac{3p_0}{3-2p_0},\infty}(\mathbb{R}^2)$ and from $L^{p_1}(\mathbb{R}^2)$ into $L^{\frac{3p_1}{3-2p_1},\infty}(\mathbb{R}^2)$. The interpolation Marcinkiewicz theorem allows again to conclude that the operator $T : L^p(\mathbb{R}^2) \longrightarrow L^{\frac{3p}{3-2p}}(\mathbb{R}^2)$ is continuous.
ii) The same remark is true for $\nabla\,\mathcal{O}$ which belongs to $L^{\frac{3}{2},\infty}(\mathbb{R}^2)$.

By Theorem 3.5 and the Sobolev embedding we easily obtain the following result.

Theorem 3.7 *Let $f \in L^p(\mathbb{R}^2)$ with $1 < p < \infty$. Then,* $\frac{\partial^2}{\partial x_j \partial x_k}(\mathcal{O} * f) \in L^p(\mathbb{R}^2)$, $\frac{\partial}{\partial x_1}(\mathcal{O} * f) \in L^p(\mathbb{R}^2)$ *and satisfy the estimate*

$$\|\frac{\partial^2}{\partial x_j \partial x_k}(\mathcal{O} * f)\|_{L^p(\mathbb{R}^2)} + \|\frac{\partial}{\partial x_1}(\mathcal{O} * f)\|_{L^p(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}. \tag{3.36}$$

Moreover,
1) i) *if $1 < p < 2$, $\nabla(\mathcal{O} * f) \in \boldsymbol{L}^{\frac{3p}{3-p}}(\mathbb{R}^2) \cap \boldsymbol{L}^{\frac{2p}{2-p}}(\mathbb{R}^2)$ and satisfies*

$$\|\nabla(\mathcal{O} * f)\|_{\boldsymbol{L}^{\frac{3p}{3-p}}(\mathbb{R}^2)} + \|\nabla(\mathcal{O} * f)\|_{\boldsymbol{L}^{\frac{2p}{2-p}}(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}. \tag{3.37}$$

ii) *If $p = 2$, $\nabla(\mathcal{O} * f) \in L^r(\mathbb{R}^2)$ for any $r \geq 6$ and the following estimate holds.*

$$\|\nabla(\mathcal{O} * f)\|_{L^r(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}. \tag{3.38}$$

iii) *If $2 < p < 3$, $\nabla(\mathcal{O} * f) \in L^{\frac{3p}{3-p}}(\mathbb{R}^2) \cap \boldsymbol{L}^{\infty}(\mathbb{R}^2)$ and we have the estimate.*

$$\|\nabla(\mathcal{O} * f)\|_{L^{\frac{3p}{3-p}}(\mathbb{R}^2)} + \|\nabla(\mathcal{O} * f)\|_{\boldsymbol{L}^{\infty}(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}. \tag{3.39}$$

2) *if $1 < p < \frac{3}{2}$, $\mathcal{O} * f \in L^{\frac{3p}{3-2p}}(\mathbb{R}^2) \cap L^{\infty}(\mathbb{R}^2)$ and satisfies*

$$\|\mathcal{O} * f\|_{L^{\frac{3p}{3-2p}}(\mathbb{R}^2)} + \|\mathcal{O} * f\|_{L^{\infty}(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}. \tag{3.40}$$

Remark 3.8 i) Applying Young Inequality and (3.16) we verify that if $f \in L^p(\mathbb{R}^2)$ with $1 < p < \frac{3}{2}$, then $\mathcal{O} * f \in L^q(\mathbb{R}^2)$ for all $q \in]\frac{3p}{3-2p}, +\infty[$, property a little weaker than (3.40).
ii) the same remark is true for $\nabla(\mathcal{O} * f)$.

By using Theorem 3.7 and Lemma 3.1 it is clear that if $f \in L^p(\mathbb{R}^2)$ then the solutions of Equation (3.1) are of the form:

$$u = \mathcal{O} * f + Q, \quad \text{with } Q \in \mathcal{S}_{[2-\frac{3}{p}]}. \tag{3.41}$$

What means that $\mathcal{O} * f$ is the unique solution of Equation (3.1) if $1 < p < \frac{3}{2}$, up to a constants if $\frac{3}{2} \leq p < 3$ and up to an elements of $\mathcal{S}_1$ if $p \geq 3$.

By Theorem 3.7, we have the following result for a given $f \in L^p(\mathbb{R}^2)$.

Theorem 3.9 *Let $f \in L^p(\mathbb{R}^2)$, then Equation (3.1) has at least a solution u of the form (3.41) such that $\nabla^2 u \in L^p(\mathbb{R}^2)$, $\frac{\partial u}{\partial x_1} \in L^p(\mathbb{R}^2)$ and verify the estimate*

$$\|\nabla^2 u\|_{L^p(\mathbb{R}^2)} + \|\frac{\partial u}{\partial x_1}\|_{L^p(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}. \tag{3.42}$$

Moreover,
1) *if $1 < p < \frac{3}{2}$, then $u \in L^{\frac{3p}{3-2p}}(\mathbb{R}^2) \cap L^{\infty}(\mathbb{R}^2)$, $\nabla u \in L^{\frac{3p}{3-p}}(\mathbb{R}^2) \cap L^{\frac{2p}{2-p}}(\mathbb{R}^2)$ and satisfy*

$$\|u\|_{L^{\frac{3p}{3-2p}}(\mathbb{R}^2)} + \|u\|_{L^{\infty}(\mathbb{R}^2)} + \|\nabla u\|_{L^{\frac{3p}{3-p}}(\mathbb{R}^2)} + \|\nabla u\|_{L^{\frac{2p}{2-p}}(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}. \tag{3.43}$$

2) i) *if $\frac{3}{2} \leq p < 2$, then $\nabla u \in L^{\frac{3p}{3-p}}(\mathbb{R}^2) \cap L^{\frac{2p}{2-p}}(\mathbb{R}^2)$ and satisfies*

$$\|\nabla u\|_{L^{\frac{3p}{3-p}}(\mathbb{R}^2)} + \|\nabla u\|_{L^{\frac{2p}{2-p}}(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}, \tag{3.44}$$

ii) *if $p = 2$, $\nabla u \in L^r(\mathbb{R}^2)$ for any $r \geq 6$ and the following estimate holds.*

$$\|\nabla u\|_{L^r(\mathbb{R}^2)} \ \leq C\|f\|_{L^p(\mathbb{R}^2)}. \tag{3.45}$$

iii) *if $2 < p < 3$, then $\nabla u \in L^{\frac{3p}{3-p}}(\mathbb{R}^2) \cap \boldsymbol{L}^\infty(\mathbb{R}^2)$ and:*

$$\|\nabla u\|_{L^{\frac{3p}{3-p}}(\mathbb{R}^2)} + \|\nabla u\|_{\boldsymbol{L}^\infty(\mathbb{R}^2)} \ \leq C\|f\|_{L^p(\mathbb{R}^2)}. \tag{3.46}$$

3) *If $p \geq 3$, then $u \in W_0^{2,p}(\mathbb{R}^2)$ and we have the estimate*

$$\inf_{\lambda \in \mathcal{S}_1} \|u + \lambda\|_{W_0^{2,p}(\mathbb{R}^2)} \ \leq C\|f\|_{L^p(\mathbb{R}^2)}. \tag{3.47}$$

Remark 3.10 Another demonstration of Theorem 3.9 consists in using the Fourier's approach. Let $(f_j)_{j\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^2)$ a sequence converging to f in $L^p(\mathbb{R}^2)$. Then the sequence (u_j) given by:

$$u_j \ = \mathcal{F}^{-1}(m_0(\xi)\hat{f}_j), \quad m_0(\xi) = (4\pi|\xi|^2 + 2i\pi\xi_1)^{-1}, \tag{3.48}$$

is a solution of Equation (3.1) with the right-hand side f_j. Let us recall now the:

Lizorkin Theorem *Let $D = \{\boldsymbol{\xi} \in \mathbb{R}^2;\ |\xi_1| > 0,\ |\xi_2| > 0\}$ and $m : D \longrightarrow \mathbb{C}$, a continuous fonction such that its derivatives $\frac{\partial^k m}{\partial\xi_1^{k_1}\partial\xi_2^{k_2}}$ are continuous and verify*

$$|\xi_1|^{k_1+\beta}|\xi_2|^{k_2+\beta}\left|\frac{\partial^k m}{\partial\xi_1^{k_1}\partial\xi_2^{k_2}}\right| \leq M, \tag{3.49}$$

where $k_1, k_2 \in \{0,1\}$, $k = k_1 + k_2$ and $0 \leq \beta < 1$. Then, the operator

$$T : g \longmapsto \mathcal{F}^{-1}(m_0\, \mathcal{F}(g)), \quad m_0(\boldsymbol{\xi}) = \frac{1}{4\pi^2|\boldsymbol{\xi}|^2 + 2i\pi\xi_1},$$

is continuous from $L^p(\mathbb{R}^2)$ into $L^r(\mathbb{R}^2)$ with $\frac{1}{r} = \frac{1}{p} - \beta$.

Applying this continuous property with $f_j \in L^p(\mathbb{R}^2)$, $\beta = \frac{2}{3}$ we show that (u_j) is bounded in $L^{\frac{3p}{3-2p}}(\mathbb{R}^2)$ if $1 < p < \frac{3}{2}$ so, this sequence admits a subsequence still denoted (u_j) which converges weakly to u solution of Equation (3.1) with right-hand side f. For the derivative of u_j with respect to x_1, the multiplier which intervenes is on the form $m(\boldsymbol{\xi}) = 2i\pi\xi_1(4\pi^2|\boldsymbol{\xi}|^2 + 2i\pi\xi_1)^{-1}$, so that (3.49) is satisfied for $\beta = 0$, so $r = p$. The same property takes place for the second derivatives with $m(\boldsymbol{\xi}) = -4\pi^2\xi_1\xi_2(4\pi^2|\boldsymbol{\xi}|^2 + 2i\pi\xi_1)^{-1}$. We verify finally, with $\beta = \frac{1}{3}$, that the first derivative of (u_j) with respect to x_2 is bounded in $L^{\frac{3p}{3-p}}(\mathbb{R}^2)$ which implie $\frac{\partial u}{\partial x_2} \in L^{\frac{3p}{3-p}}(\mathbb{R}^2)$. $\diamondsuit$

In order to study Equation (3.1) with a right-hand side $f \in W_0^{-1,p}(\mathbb{R}^2)$, we give the following definition of the convolution of f with the fundamental solution $\mathcal{O}$:

$$\forall\varphi \in \mathcal{D}(\mathbb{R}^2),\ \ \langle \mathcal{O} * f, \varphi\rangle =: \langle f, \breve{\mathcal{O}} * \varphi\rangle_{W_0^{-1,p}(\mathbb{R}^2)\times W_0^{1,p'}(\mathbb{R}^2)},$$

where $\breve{\mathcal{O}}(\boldsymbol{x}) = \mathcal{O}(-\boldsymbol{x})$.

Theorem 3.11 *Let $f \in W_0^{-1,p}(\mathbb{R}^2)$ satisfying the compatibility condition*

$$\langle f, 1\rangle_{W_0^{-1,p}(\mathbb{R}^2)\times W_0^{1,p'}(\mathbb{R}^2)} \ = \ 0, \quad \text{when } 1 < p \leq 2. \tag{3.50}$$

i) *If $1 < p < 3$, then $u = \mathcal{O} * f \in L^{\frac{3p}{3-p}}(\mathbb{R}^2)$ is the unique solution of Equation (3.1) such that $\nabla u \in L^p(\mathbb{R}^2)$ and $\frac{\partial u}{\partial x_1} \in W_0^{-1,p}(\mathbb{R}^2)$. Moreover, we have the estimate*

$$\|u\|_{L^{\frac{3p}{3-p}}(\mathbb{R}^2)} + \|\nabla u\|_{L^p(\mathbb{R}^2)} + \|\frac{\partial u}{\partial x_1}\|_{W_0^{-1,p}(\mathbb{R}^2)} \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^2)}, \tag{3.51}$$

and $u \in L^{\frac{2p}{2-p}}(\mathbb{R}^2)$ When $1 < p < 2$, $u \in L^r(\mathbb{R}^2)$ for any $r \geq 6$ when $p = 2$ and $u \in L^\infty(\mathbb{R}^2)$ when $2 < p < 3$.
ii) *If $p \geq 3$, then Equation (3.1) has a solution $u \in \widetilde{W}_0^{1,p}(\mathbb{R}^2)$, unique up to a constant and we have*

$$\inf_{k\in\mathbb{R}} \|u + k\|_{\widetilde{W}_0^{1,p}(\mathbb{R}^2)} \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^2)}. \tag{3.52}$$

Proof : Let $f \in W_0^{-1,p}(\mathbb{R}^2)$ satisfying the condition (3.50). Thanks to Lemma 3.3 and Remark 3.4, if $\varphi \to 0$ in $\mathcal{D}(\mathbb{R}^2)$, we have $\breve{\mathcal{O}} * \varphi \to 0$ in $W_0^{1,p'}(\mathbb{R}^2)$ for all $p \in]1, 3[$ which implies that $\mathcal{O} * f \in \mathcal{D}'(\mathbb{R}^2)$. We know also, by Isomorphism (2.7), that there exists $\boldsymbol{F} \in \boldsymbol{L}^p(\mathbb{R}^2)$ such that

$$f = \mathrm{div}\boldsymbol{F} \quad \text{and} \quad \|\boldsymbol{F}\|_{\boldsymbol{L}^p(\mathbb{R}^2)} \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^2)}. \tag{3.53}$$

i) Suppose now that $1 < p < 3$. Then,

$$\begin{aligned}
\langle \frac{\partial}{\partial x_j}(\mathcal{O} * f), \varphi\rangle_{\mathcal{D}'(\mathbb{R}^2)\times\mathcal{D}(\mathbb{R}^2)} &= -\langle \mathcal{O} * f, \frac{\partial \varphi}{\partial x_j}\rangle_{\mathcal{D}'(\mathbb{R}^2)\times\mathcal{D}(\mathbb{R}^2)} \\
&= \langle \boldsymbol{F}, \nabla (\breve{\mathcal{O}} * \frac{\partial \varphi}{\partial x_j})\rangle_{\boldsymbol{L}^p(\mathbb{R}^2)\times\boldsymbol{L}^{p'}(\mathbb{R}^2)} \\
&= \langle \boldsymbol{F}, \nabla \frac{\partial}{\partial x_j}(\breve{\mathcal{O}} * \varphi)\rangle_{\boldsymbol{L}^p(\mathbb{R}^2)\times\boldsymbol{L}^{p'}(\mathbb{R}^2)}.
\end{aligned}$$

Moreover, by (3.23),

$$\begin{aligned}
|\langle \frac{\partial}{\partial x_j}(\mathcal{O} * f), \varphi\rangle_{\mathcal{D}'(\mathbb{R}^2)\times\mathcal{D}(\mathbb{R}^2)}| &\leq \|\boldsymbol{F}\|_{\boldsymbol{L}^p(\mathbb{R}^2)} \|\nabla \frac{\partial}{\partial x_j}(\breve{\mathcal{O}} * \varphi)\|_{\boldsymbol{L}^{p'}(\mathbb{R}^2)} \\
&\leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^2)}\|\varphi\|_{L^{p'}(\mathbb{R}^2)}.
\end{aligned}$$

There is

$$\|\frac{\partial}{\partial x_j}(\mathcal{O} * f)\|_{L^p(\mathbb{R}^2)} \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^2)}.$$

With the same condition on p as in the previous case, for all $\varphi \in \mathcal{D}(\mathbb{R}^2)$, we have

$$\langle \mathcal{O} * f, \varphi\rangle_{\mathcal{D}'(\mathbb{R}^2)\times\mathcal{D}(\mathbb{R}^2)} = -\langle \boldsymbol{F}, \nabla (\breve{\mathcal{O}} * \varphi)\rangle_{\boldsymbol{L}^p(\mathbb{R}^2)\times\boldsymbol{L}^{p'}(\mathbb{R}^2)},$$

and by (3.25)

$$\begin{aligned}
|\langle \mathcal{O} * f, \varphi\rangle_{\mathcal{D}'(\mathbb{R}^2)\times\mathcal{D}(\mathbb{R}^2)}| &\leq \|\boldsymbol{F}\|_{\boldsymbol{L}^p(\mathbb{R}^2)} \|\frac{\partial}{\partial x_j}(\breve{\mathcal{O}} * \varphi)\|_{\boldsymbol{L}^{p'}(\mathbb{R}^2)} \\
&\leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^2)}\|\varphi\|_{L^{\frac{3p}{4p-3}}(\mathbb{R}^2)}.
\end{aligned}$$

Note that $1 < p < 3 \Longleftrightarrow 1 < \frac{3p}{4p-3} < 3$. Consequently, $\mathcal{O} * f \in L^{\frac{3p}{3-p}}(\mathbb{R}^2)$ and

$$\|\mathcal{O} * f\|_{L^{\frac{3p}{3-p}}(\mathbb{R}^2)} \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^2)}.$$

Moreover, by the Sobolev embedding, $\mathcal{O} * f \in L^{\frac{2p}{2-p}}(\mathbb{R}^2)$ if $1 < p < 2$, $\mathcal{O} * f$ belongs to $L^r(\mathbb{R}^2)$ for all $r \geq 6$ if $p = 2$ and belongs to $L^\infty(\mathbb{R}^2)$ if $2 < p < 3$. We thus showed that if $1 < p < 3$, the operator

$$\begin{aligned} R : W_0^{-1,p}(\mathbb{R}^2) \perp \mathcal{P}_{[1-\frac{2}{p'}]} &\longrightarrow W_0^{1,p}(\mathbb{R}^2) \cap L^{\frac{3p}{3-p}}(\mathbb{R}^2), \\ f &\longmapsto \mathcal{O} * f, \end{aligned} \tag{3.54}$$

is continuous.
ii) Suppose now that $p \geq 3$ and let $f \in W_0^{-1,p}(\mathbb{R}^2)$. Then we have the relation (3.53). Now, since $\boldsymbol{\mathcal{D}}(\mathbb{R}^2)$ is dense in $\boldsymbol{L}^p(\mathbb{R}^2)$, there exists a sequence $\boldsymbol{F}_m \in \boldsymbol{\mathcal{D}}(\mathbb{R}^2)$ such that $\boldsymbol{F}_m \to \boldsymbol{F}$ in $\boldsymbol{L}^p(\mathbb{R}^2)$. Set $f_m = \mathrm{div}\boldsymbol{F}_m$ and $\psi_m = \mathcal{O} * f_m$. For all $\varphi \in \mathcal{D}(\mathbb{R}^2)$, we have

$$\langle \frac{\partial \psi_m}{\partial x_j}, \varphi \rangle = \langle \boldsymbol{F}_m, \nabla \frac{\partial}{\partial x_j}(\check{\mathcal{O}} * \varphi) \rangle.$$

Then, according to inequality (3.36), we have

$$\begin{aligned} |\langle \frac{\partial \psi_m}{\partial x_j}, \varphi \rangle| &\leq C \|\boldsymbol{F}_m\|_{\boldsymbol{L}^p(\mathbb{R}^2)} \|\varphi\|_{L^{p'}(\mathbb{R}^2)}, \\ &\leq C \|f\|_{W_0^{-1,p}(\mathbb{R}^2)} \|\varphi\|_{L^{p'}(\mathbb{R}^2)}. \end{aligned} \tag{3.55}$$

So that, $\nabla \psi_m$ is bounded in $\boldsymbol{L}^p(\mathbb{R}^2)$. We can apply Theorem 2.1: for each m, there exists a constant C_m such that $\psi_m + C_m \in W_0^{1,p}(\mathbb{R}^2)$ and

$$\|\psi_m + C_m\|_{W_0^{1,p}(\mathbb{R}^2)} \leq C \|f\|_{W_0^{-1,p}(\mathbb{R}^2)}.$$

From this follows that $\psi_m + C_m$ converges weakly to some function $u \in W_0^{1,p}(\mathbb{R}^2)$ and

$$-\Delta u + \frac{\partial u}{\partial x_1} = f,$$

so that Equation (3.1) admits a solution u and moreover $u \in \widetilde{W}_0^{1,p}(\mathbb{R}^2)$. $\Diamond$

Remark 3.12 i) If $1 < p < 2$, as the solution u of Equation (3.1) given by Theorem 3.11 belongs in particular to $W_0^{1,p}(\mathbb{R}^2)$, we deduce that

$$\lim_{|x| \to \infty} u(x) \;=\; 0,$$

in the sense of Definition 2.8. Consequently, for any given constant u_∞, the distribution $v = u + u_\infty$ is the unique solution of Equation (3.1) which is such that $\nabla v \in \boldsymbol{L}^p(\mathbb{R}^2)$, $\frac{\partial v}{\partial x_1} \in W_0^{-1,p}(\mathbb{R}^2)$ and satisfying the condition at infinity

$$\lim_{|x| \to \infty} v(x) \;=\; u_\infty.$$

ii) If $2 < p < 3$, by Lemma 2.10, the same result holds with a pointwise convergence.

Corollary 3.13 *Assume $1 < p < 3$. If u is a distribution such that $\nabla u \in L^p(\mathbb{R}^2)$ and $\frac{\partial u}{\partial x_1} \in W_0^{-1,p}(\mathbb{R}^2)$. Then, there exists a unique constant k such that $u + k \in L^{\frac{3p}{3-p}}(\mathbb{R}^2)$ and*

$$\| u + k \|_{L^{\frac{3p}{3-p}}(\mathbb{R}^2)} \;\leq\; C(\| \nabla u \|_{L^p(\mathbb{R}^2)} + \| \frac{\partial u}{\partial x_1} \|_{W_0^{-1,p}(\mathbb{R}^2)}). \tag{3.56}$$

Moreover, if $1 < p < 2$, then $u + k \in L^{\frac{2p}{2-p}}(\mathbb{R}^2)$ and $u(\boldsymbol{x})$ tends to the constant $-k$ when $|\boldsymbol{x}|$ tends to infinity in the sense of Definition 2.8. If $p = 2$, then $u + k$ belongs to $L^r(\mathbb{R}^2)$ for any $r \geq 6$. If $2 < p < 3$, then u belongs to $L^\infty(\mathbb{R}^2)$, is continuous in $\mathbb{R}^2$ and tends to $-k$ pointwise.

Proof : Setting $g = -\Delta u + \frac{\partial u}{\partial x_1} \in W_0^{-1,p}(\mathbb{R}^2)$. Since $\mathcal{P}_{[1-\frac{2}{p'}]}$ contains at most a constants and according to the density of $\mathcal{D}(\mathbb{R}^2)$ in $\widetilde{W}_0^{1,p}(\mathbb{R}^2)$, then g satisfies the compatibility condition (3.50). By the previous theorem, there exists a unique $v \in L^{\frac{3p}{3-p}}(\mathbb{R}^2)$ such that $\nabla v \in L^p(\mathbb{R}^2)$ and $\frac{\partial v}{\partial x_1} \in L^p(\mathbb{R}^2)$, satisfying $T(u-v) = 0$ (T is the Oseen operator, see (3.2)), with the estimate

$$\begin{aligned}\| v \|_{L^{\frac{3p}{3-p}}(\mathbb{R}^2)} &\leq C(\| \Delta u \|_{W_0^{-1,p}(\mathbb{R}^2)} + \| \frac{\partial u}{\partial x_1} \|_{W_0^{-1,p}(\mathbb{R}^2)}) \\ &\leq C(\| \nabla u \|_{L^p(\mathbb{R}^2)} + \| \frac{\partial u}{\partial x_1} \|_{W_0^{-1,p}(\mathbb{R}^2)}).\end{aligned} \tag{3.57}$$

Setting $w = u-v$, we have for all $i = 1, 2$, $\frac{\partial w}{\partial x_i} \in L^p(\mathbb{R}^2)$ and satisfies $T(\frac{\partial w}{\partial x_i}) = 0$. We deduce then by Lemma 3.1 that $\nabla u = \nabla v$, then there exists a unique constant k such that $v = u + k$. The last properties are consequence of Lemma 2.6 and Lemma 2.10.

Remark 3.14 Let $u \in \mathcal{D}'(\mathbb{R}^2)$ such that $\nabla u \in \boldsymbol{L}^p(\mathbb{R}^2)$.
i) When $1 < p < 2$, thanks to Proposition 2.2, we know that there exists a unique constant k such that $u + k \in L^{\frac{2p}{2-p}}(\mathbb{R}^2)$. Here, the fact that in add $\frac{\partial u}{\partial x_1} \in W_0^{-1,p}(\mathbb{R}^2)$ we have, in more $u + k \in L^{\frac{3p}{3-p}}(\mathbb{R}^2)$.
ii) When $2 \leq p < 3$, u is only in $W_0^{1,p}(\mathbb{R}^2)$ but is in no space $L^r(\mathbb{R}^2)$. But, if moreover $\frac{\partial u}{\partial x_1} \in W_0^{-1,p}(\mathbb{R}^2)$ then, $u + k \in L^{\frac{3p}{3-p}}(\mathbb{R}^2)$ for some unique constant k and $u \in L^r(\mathbb{R}^2)$ for any $r \geq 6$ if $p = 2$ and $u \in L^\infty(\mathbb{R}^2)$ otherwise.

As consequence of Theorems 3.9 and 3.11 we solve Equation (3.1) when the data f belongs to intersection of two weighted spaces. We have then the two following results.

Proposition 3.15 *Suppose that $f \in W_0^{-1,p}(\mathbb{R}^2) \cap W_0^{-1,q}(\mathbb{R}^2)$ with $1 < p < q < \infty$ and satisfies the compatibility condition (3.50). Then, Equation* (3.1) *has a solution $u \in \widetilde{W}_0^{1,p}(\mathbb{R}^2) \cap \widetilde{W}_0^{1,q}(\mathbb{R}^2)$ satisfying*

$$\begin{aligned}&\|\nabla u\|_{L^p(\mathbb{R}^2)} + \|\nabla u\|_{L^q(\mathbb{R}^2)} + \|\frac{\partial u}{\partial x_1}\|_{W_0^{-1,p}} + \|\frac{\partial u}{\partial x_1}\|_{W_0^{-1,q}(\mathbb{R}^2)} \\ &\qquad \leq C(\|f\|_{W_0^{-1,p}(\mathbb{R}^2)} + \|f\|_{W_0^{-1,q}(\mathbb{R}^2)}).\end{aligned} \tag{3.58}$$

Moreover:
i) *The solution u is unique if $p < 3$ and up to a constant if $p \geq 3$. It is equal to $\mathcal{O} * f$ if $p < 3$.*
ii) *If $p < q < 2$, then $u \in L^{\frac{3p}{3-p}}(\mathbb{R}^2) \cap L^{\frac{2q}{2-q}}(\mathbb{R}^2)$ and verifies*

$$\|u\|_{L^{\frac{3p}{3-p}}(\mathbb{R}^2)} + \|u\|_{L^{\frac{2q}{2-q}}(\mathbb{R}^2)} \leq C(\|f\|_{W_0^{-1,p}(\mathbb{R}^2)} + \|f\|_{W_0^{-1,q}(\mathbb{R}^2)}). \tag{3.59}$$

iii) *If $p < q = 2$, then $u \in L^r(\mathbb{R}^2)$ for any $r \geq \frac{3p}{3-p}$ and*

$$\|u\|_{L^r(\mathbb{R}^2)} \leq C(\|f\|_{W_0^{-1,p}(\mathbb{R}^2)} + \|f\|_{W_0^{-1,q}(\mathbb{R}^2)}). \tag{3.60}$$

iv) *If $p < 3$ and $q > 2$ then $u \in L^{\frac{3p}{3-p}}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ with the estimate*

$$\|u\|_{L^{\frac{3p}{3-p}}(\mathbb{R}^2)} + \|u\|_{L^\infty(\mathbb{R}^2)} \leq C(\|f\|_{W_0^{-1,p}(\mathbb{R}^2)} + \|f\|_{W_0^{-1,q}(\mathbb{R}^2)}). \tag{3.61}$$

Proof : Let $f \in W_0^{-1,p}(\mathbb{R}^2) \cap W_0^{-1,q}(\mathbb{R}^2)$ and satisfies the compatibility condition (3.50) with $1 < p < q < \infty$. We know that there exists $u \in \widetilde{W}_0^{1,p}(\mathbb{R}^2)$ and

$v \in \widetilde{W}_0^{1,q}(\mathbb{R}^2)$ solutions of (3.1). Moreover, by a uniqueness argument we have necessary $\nabla u = \nabla v$ and Estimate (3.58) comes from (3.51).

i) Now, in the one hand, if $p \geq 3$, then $u - v = k$, where k is an arbitrary constant, so $u = v + k \in \widetilde{W}_0^{1,p}(\mathbb{R}^2) \cap \widetilde{W}_0^{1,q}(\mathbb{R}^2)$. If on the other hand $p < 3$, then $u = \mathcal{O} * f$.

ii) Suppose that $q < 2$, we know that $u = \mathcal{O} * f \in L^{\frac{3p}{3-p}}(\mathbb{R}^2)$ and $u = v \in L^{\frac{2p}{2-p}}(\mathbb{R}^2)$ and satisfies Estimate (3.59).

iii) If $q = 2$, then, by Theorem 3.11, $u = \mathcal{O} * f \in L^{\frac{3p}{3-p}}(\mathbb{R}^2)$ and $u = v \in L^r(\mathbb{R}^2)$ for any $r \geq \frac{3p}{3-p}$.

iv) If $p < 3$ and $q > 2$, we know that $u = \mathcal{O} * f \in L^{\frac{3p}{3-p}}(\mathbb{R}^2)$. Since $\nabla u \in \boldsymbol{L}^q(\mathbb{R}^2)$ with $q > 2$, then $u \in L^\infty(\mathbb{R}^2)$ and we have Estimate (3.61).

Remark 3.16 When $f \in W_0^{-1,q}(\mathbb{R}^2)$ with $q \geq 3$, we saw that $\mathcal{O} * f$ is not necessary defined. But if moreover $f \in W_0^{-1,p}(\mathbb{R}^2)$ with $p < 3$, and satisfying the compatibility condition (3.50), then $\mathcal{O} * f$ has a sense in $\widetilde{W}_0^{1,p}(\mathbb{R}^2)$ and belongs to $\widetilde{W}_0^{1,q}(\mathbb{R}^2)$.

Proposition 3.17 *Let $f \in L^p(\mathbb{R}^2) \cap W_0^{-1,p}(\mathbb{R}^2)$ satisfying the compatibility condition (3.50). Then Equation* (3.1) *has a solution $u = \mathcal{O} * f$ such that $\nabla u \in W^{1,p}(\mathbb{R}^2)$, $\frac{\partial u}{\partial x_1} \in W^{1,p}(\mathbb{R}^2) \cap W_0^{-1,p}(\mathbb{R}^2)$ and satisfies the estimate*

$$\|\nabla u\|_{W^{1,p}} + \|\frac{\partial u}{\partial x_1}\|_{W^{1,p}} + \|\frac{\partial u}{\partial x_1}\|_{W^{-1,p}} \leq C(\|f\|_{L^p} + \|f\|_{W_0^{-1,p}}). \quad (3.62)$$

Moreover:

i) *if $p < \frac{3}{2}$, u is unique, belongs to $L^{\frac{3p}{3-2p}}(\mathbb{R}^2) \cap W^{1,\frac{3p}{3-p}}(\mathbb{R}^2)$ and verifies the estimate*

$$\|u\|_{L^{\frac{3p}{3-2p}}(\mathbb{R}^2)} + \|u\|_{W^{1,\frac{3p}{3-p}}(\mathbb{R}^2)} \leq C(\|f\|_{L^p} + \|f\|_{W_0^{-1,p}}). \quad (3.63)$$

ii) *If $\frac{3}{2} \leq p < 3$, u is unique in $W^{1,\frac{3p}{3-p}}(\mathbb{R}^2)$ and satisfies the estimate*

$$\|u\|_{W^{1,\frac{3p}{3-p}}(\mathbb{R}^2)} \leq C(\|f\|_{L^p(\mathbb{R}^2)} + \|f\|_{W_0^{-1,p}(\mathbb{R}^2)}). \quad (3.64)$$

iii) *If $p \geq 3$, u belongs to $\in W_0^{2,p}(\mathbb{R}^2) \cap \widetilde{W}_0^{1,p}(\mathbb{R}^2)$, unique up to a constant, and*

$$\inf_{k\in\mathbb{R}}(\|u + k\|_{W_0^{2,p}(\mathbb{R}^2)} + \|u + k\|_{\widetilde{W}_0^{1,p}(\mathbb{R}^2)}) \leq C(\|f\|_{L^p} + \|f\|_{W_0^{-1,p}}). \quad (3.65)$$

Proof : The proof is the same that that given in the previous proposition.

Now we take f more regular, for example $f \in W_0^{-1,p}(\mathbb{R}^2) \cap W_1^{0,q}(\mathbb{R}^2)$ and we look which regularity we obtain for the solution u.

Proposition 3.18 *Let p and q two reals numbers such that $1 < p < \infty$, $q > 2$ and $\frac{1}{p} = \frac{1}{q} + \frac{1}{2}$. Suppose that $f \in W_0^{-1,p}(\mathbb{R}^2) \cap W_1^{0,q}(\mathbb{R}^2)$ and satisfies the compatibility condition (3.50) Then the unique solution of Equation* (3.1) *given by Proposition* 3.15 *satisfies the complementary properties*

$$\nabla^2 u \in W_1^{0,q}(\mathbb{R}^2) \quad \text{and} \quad \frac{\partial u}{\partial x_1} \in W_1^{0,q}(\mathbb{R}^2)$$

Proof : From the relation $\frac{1}{p} = \frac{1}{q} + \frac{1}{2}$ we have $1 < p < 2$ and since $q > 2$, then

$$\mathcal{P}_{[1-\frac{2}{q'}]} = \mathcal{P}_{[1-\frac{2}{p}]} = \{0\}.$$

Since $W_1^{0,q}(\mathbb{R}^2) \subset W_0^{-1,q}(\mathbb{R}^2)$ then $f \in W_0^{-1,p}(\mathbb{R}^2) \cap W_0^{-1,q}(\mathbb{R}^2)$ and satisfies the compatibility condition (3.50) for p and q.

i) If $2 < q < 3$, Equation (3.1) has a unique solution $u \in L^{\frac{3p}{3-p}}(\mathbb{R}^2) \cap L^{\frac{3q}{3-q}}(\mathbb{R}^2)$ such that $\nabla u \in L^p(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$ and $\frac{\partial u}{\partial x_1} \in W_0^{-1,p}(\mathbb{R}^2) \cap W_0^{-1,q}(\mathbb{R}^2)$. Further,

$$-\Delta(\rho\frac{\partial u}{\partial x_j}) + \frac{\partial}{\partial x_1}(\rho\frac{\partial u}{\partial x_j}) = \rho\frac{\partial f}{\partial x_j} - 2\nabla\rho\,\nabla(\frac{\partial u}{\partial x_j}) - \frac{\partial u}{\partial x_j}\Delta\rho + \frac{\partial u}{\partial x_j}\frac{\partial \rho}{\partial x_1} =: F$$

In the one hand, since $\nabla u \in L^q(\mathbb{R}^2)$, in view of (2.3), (2.2) and (2.4), the terms $\rho\frac{\partial f}{\partial x_j}$, $\nabla\rho\nabla(\frac{\partial u}{\partial x_j})$ and $\frac{\partial u}{\partial x_j}\Delta\rho$ belong to $W_0^{-1,q}(\mathbb{R}^2)$. In the other hand, since $\nabla u \in L^p(\mathbb{R}^2)$ the term $\frac{\partial u}{\partial x_j}\frac{\partial \rho}{\partial x_1}$ belongs to $L^p(\mathbb{R}^2)$. By the Sobolev embedding and the relation between p and q, $L^p(\mathbb{R}^2) \subset W_0^{-1,q}(\mathbb{R}^2)$ because $W_0^{1,q'}(\mathbb{R}^2) \subset L^{p'}(\mathbb{R}^n)$ and we deduce that $F \in W_0^{-1,q}(\mathbb{R}^2)$. Then there exists by Theorem 3.11 a unique $v_j \in L^{\frac{3q}{3-q}}(\mathbb{R}^2)$ such that $\nabla v_j \in L^q(\mathbb{R}^2)$ and $\frac{\partial v_j}{\partial x_1} \in W_0^{-1,q}(\mathbb{R}^2)$ satisfying

$$-\Delta(v_j - \rho\tfrac{\partial u}{\partial x_j}) + \tfrac{\partial}{\partial x_1}(v_j - \rho\tfrac{\partial u}{\partial x_j}) = 0.$$

We deduce that $w_j = v_j - \rho\frac{\partial u}{\partial x_j}$ is a polynomial. Since $\nabla v_j \in L^q(\mathbb{R}^2)$ and $q > 2$ we have, by Proposition 2.2, $v_j \in W_0^{1,q}(\mathbb{R}^2) \subset W_{-1}^{0,q}(\mathbb{R}^2)$. We have also $\rho\frac{\partial u}{\partial x_j} \in W_{-1}^{0,q}(\mathbb{R}^2)$, so $w_j \in \mathcal{P}_{[1-\frac{2}{q}]} = \mathcal{P}_0$. Then, there exists a constant k such that $\rho\frac{\partial u}{\partial x_j} = v_j + k \in W_0^{1,q}(\mathbb{R}^2)$, which imply $\frac{\partial u}{\partial x_j} \in W_1^{1,q}(\mathbb{R}^2)$ and so $\nabla^2 u \in W_1^{0,q}(\mathbb{R}^2)$. The same argument prove that $\frac{\partial u}{\partial x_1} \in W_1^{0,q}(\mathbb{R}^2)$.

ii) If $q \geq 3$, Equation (3.1) has in view of Proposition 3.15 ii) a unique solution $u \in \widetilde{W}_0^{1,q}(\mathbb{R}^2) \cap \widetilde{W}_0^{1,p}(\mathbb{R}^2)$. The right-hand side $\boldsymbol{F}$ also belongs to $W_0^{-1,q}(\mathbb{R}^2)$ and we proceed as previously. $\Diamond$

4 Study in anisotropic weighted spaces.

In this section we consider the case where the weight is anisotropic, in the form $r^\alpha\,(1+s)^\beta$ or $\eta_\beta^\alpha = (1+r)^\alpha\,(1+s)^\beta$. Note that the behavior at infinity of these weights is not uniform. In fact, in the parabola $s = 1$ we have $r^\alpha\,(1+s)^\beta \sim \eta_\beta^\alpha \sim r^\alpha$ and out of a sector $\mathcal{S}_{\lambda,R} = \{x \in \mathbb{R}^2;\ x_1 > \lambda r,\ 0 < \lambda < 1\}$, we have $r^\alpha\,(1+s)^\beta \sim \eta_\beta^\alpha \sim r^{\alpha+\beta}$. It's for this reason these functions are called anisotropic weights. For $R > 0$ we denote by B_R the ball centered at origin with the radius R, $B'_R = \mathbb{R}^2 \setminus \overline{B_R}$ and we define the space

$$L^p_{\alpha,\beta}(\Omega) = \Big\{v \in \mathcal{D}'(\Omega);\ \eta_\beta^\alpha v \in L^p(\Omega)\Big\},$$

where $\Omega = \mathbb{R}^2$ or any open domain of $\mathbb{R}^2$. We begin by study the problem

$$\begin{aligned} -\Delta z + \tfrac{\partial z}{\partial x_1} + a_0 z &= g \quad \text{in} \quad B'_R, \\ z &= 0 \quad \text{on} \quad \partial B'_R, \end{aligned} \tag{4.1}$$

where $g \in L^p_{\frac{1}{2},0}(B'_R)$ and

$$a_0 = \frac{1}{8r}\frac{2s^2+s+2}{(1+s)^2}. \tag{4.2}$$

First we have the following

Lemma 4.1 *Let p such that $2 < p < \frac{32}{11}$ and let $g \in L^p_{\frac{1}{2},0}(B'_R)$. There exists $R^* > 0$ such that, if $R > R^*$, Problem (4.1) has a unique solution $z \in L^p_{-\frac{1}{2},0}(B'_R)$, such that $\nabla^2 z \in L^p(B'_R)$ and $\frac{\partial z}{\partial x_1} \in L^p(B'_R)$. Moreover there exists $C > 0$ such that*

$$\|z\|_{L^p_{-\frac{1}{2},0}(B'_R)} + \|\frac{\partial z}{\partial x_1}\|_{L^p(B'_R)} + \|\nabla^2 z\|_{L^p(B'_R)} \ \leq C\|g\|_{L^p_{\frac{1}{2},0}(B'_R)}. \tag{4.3}$$

Proof : For all $\varepsilon > 0$, since $g \in L^p_{\frac{1}{2},0}(B'_R)$ and $a_0 > 0$, the problem

$$\begin{aligned} -\Delta z_\varepsilon + \tfrac{\partial z_\varepsilon}{\partial x_1} + a_0 z_\varepsilon + \varepsilon\, z_\varepsilon = \ & g \ \text{ in } \ B'_R, \\ z_\varepsilon = \ & 0 \ \text{ on } \ \partial B'_R. \end{aligned} \tag{4.4}$$

has a unique solution $z_\varepsilon \in W^{2,p}(B'_R)$. Multiplying the first equation of problem (4.4) by $r^{1-\frac{p}{2}}\, |z_\varepsilon|^{p-2}\, z_\varepsilon$, since in two dimensional, $\Delta(r^{1-\frac{p}{2}}) = (1-\frac{p}{2})^2\, r^{-1-\frac{p}{2}}$, we get after integration by part in B'_R

$$(p-1)\int_{B'_R} r^{1-\frac{p}{2}} |\, z_\varepsilon|^{p-2} |\,\nabla z_\varepsilon|^2 \ + \int_{B'_R} a_0\, r^{1-\frac{p}{2}}\, |z_\varepsilon|^p \ + \varepsilon \int_{B'_R} r^{1-\frac{p}{2}}\, |z_\varepsilon|^p \ = \frac{1}{p}(1-\frac{p}{2})^2 \int_{B'_R} r^{-1-\frac{p}{2}}\, |z_\varepsilon|^p \ + (\frac{1}{p}-\frac{1}{2}) \int_{B'_R} |z_\varepsilon|^p \frac{x_1}{r}\, r^{-\frac{p}{2}} \ + \int_{B'_R} r^{1-\frac{p}{2}}\, |z_\varepsilon|^{p-2}\, z_\varepsilon\, g\ .$$

Note that $a_0 \geq \frac{5}{32r}$, then

$$(\frac{5}{32} - |\frac{1}{p} - \frac{1}{2}|) \int_{B'_R} r^{-\frac{p}{2}} |z_\varepsilon|^p \leq \frac{1}{p}(1-\frac{p}{2})^2 \int_{B'_R} r^{-1-\frac{p}{2}} |z_\varepsilon|^p \ + \int_{B'_R} r^{1-\frac{p}{2}} |z_\varepsilon|^{p-1} |g|\ . \tag{4.5}$$

Moreover, since $r > R$,

$$\frac{1}{p}(1-\frac{p}{2})^2 \int_{B'_R} r^{-1-\frac{p}{2}}\, |z_\varepsilon|^p \leq \frac{1}{pR}(1-\frac{p}{2})^2 \int_{B'_R} r^{-\frac{p}{2}}\, |z_\varepsilon|^p\ . \tag{4.6}$$

Inequalities (4.5) and (4.6) give

$$(\frac{5}{32} - |\frac{1}{p} - \frac{1}{2}| - \frac{1}{pR}(1-\frac{p}{2})^2) \int_{B'_R} r^{-\frac{p}{2}} |z_\varepsilon|^p \leq \int_{B'_R} r^{1-\frac{p}{2}} |z_\varepsilon|^{p-1} |g|.$$

Since $2 < p < \frac{32}{11}$, we have $\frac{5}{32} - |\frac{1}{p} - \frac{1}{2}| - \frac{1}{pR}(1-\frac{p}{2})^2 > 0$, if $R > R^*$, with R^* sufficiently large. Thus, from the previous inequality we obtain

$$\int_{B'_R} r^{-\frac{p}{2}}\, |z_\varepsilon|^p \ \leq C_1 \int_{B'_R} r^{1-\frac{p}{2}}\, |z_\varepsilon|^{p-1} |g| \ \leq C_1 (\int_{B'_R} r^{\frac{p}{2}}\, |g|^p)^{\frac{1}{p}} (\int_{B'_R} r^{-\frac{p}{2}}\, |z_\varepsilon|^p)^{\frac{p-1}{p}}.$$

Then

$$\int_{B'_R} r^{-\frac{p}{2}} |z_\varepsilon|^p \ \leq \ C \int_{B'_R} r^{\frac{p}{2}} |g|^p,$$

where the constant C is independent of R and ε. The sequence (z_ε) is then bounded in $L^p_{-\frac{1}{2},0}(B'_R)$, which is a reflexive space, so $z_\varepsilon \rightharpoonup z$ in $L^p_{-\frac{1}{2},0}(B'_R)$, and

$$\|\, z\,\|_{L^p_{-\frac{1}{2},0}(B'_R)} \ \leq \ \liminf_{\varepsilon \to 0} \|\, z_\varepsilon\,\|_{L^p_{-\frac{1}{2},0}(B'_R)} \ \leq \ C \,\|\, g\,\|_{L^p_{\frac{1}{2},0}(B'_R)},$$

where z satisfies the equation

$$-\Delta z + \frac{\partial z}{\partial x_1} = g - a_0 z \quad \text{in} \quad B'_R.$$

Let us show that $\nabla^2 z \in L^p(B'_R)$ and $\frac{\partial z}{\partial x_1} \in L^p(B'_R)$. Now, the fact that the function $g - a_0 z_\varepsilon$ is bounded in $L^p_{\frac{1}{2},0}(B'_R)$ implies that it is bounded in $L^p(B'_R)$. Since, $\nabla^2 z_\varepsilon$ remains bounded in $L^p(B'_R)$, then $\nabla^2 z \in L^p(B'_R)$ and

$$\|\nabla^2 z\|_{L^p(B'_R)} \leq \liminf_{\varepsilon\to 0} \|\nabla^2 z_\varepsilon\|_{L^p(B'_R)} \leq C\|g\|_{L^p_{\frac{1}{2},0}(B'_R)}. \tag{4.7}$$

Thus, $\frac{\partial z}{\partial x_1} \in L^p(B'_R)$ and we have Estimate (4.3). It remains to prove that $z = 0$ on $\partial B'_R$. Since $\nabla^2 z_e$ is bounded in $L^p(B'_R)$ Then, if Ω is a bounded domain such that $\overline{B}_R \subset \Omega$, setting $\Omega = \Omega \cap B'_R$, we have

$$z_\varepsilon \rightharpoonup v \quad \text{in} \quad W^{2,p}(\Omega).$$

Since $z_\varepsilon = 0$ on $\partial B'_R$, then $v = 0$ on $\partial B'_R$. Moreover, since $z_\varepsilon \rightharpoonup z$ in $L^p_{-\frac{1}{2},0}(B'_R)$, then $v = z\,|_{\overline{\Omega}}$ and so $z = 0$ on $\partial B'_R$. $\diamondsuit$

We know, according to Proposition 3.18, that for f given in $W_0^{-1,p}(\mathbb{R}^2) \cap W_1^{0,q}(\mathbb{R}^2)$, where p and q verify relation $\frac{1}{p} = \frac{1}{q} + \frac{1}{2}$, we obtain that $\nabla^2 u$ and $\frac{\partial u}{\partial x_1}$ belong to $W_1^{0,q}(\mathbb{R}^2)$. But if f is only given in $W_1^{0,p}(\mathbb{R}^2)$, we cannot find the same regularity on $\nabla^2 u$ and $\frac{\partial u}{\partial x_1}$. Then we look at f in $L^p_{\alpha,\beta}(\mathbb{R}^2)$, with $\alpha + \beta$ close to 1. Moreover, taking account of the conditions put by Pokorny in [10] on α and β one takes $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{4}$.

Theorem 4.2 *Assume* $2 < p < \frac{32}{11}$ *and* $f \in L^p_{\frac{1}{2},\frac{1}{4}}(\mathbb{R}^2)$. *Then,* $\mathcal{O} * f \in L^p_{-\frac{1}{2},\frac{1}{4}}(\mathbb{R}^2)$, $\frac{\partial}{\partial x_2}(\mathcal{O} * f) \in L^p_{0,\frac{1}{4}}(\mathbb{R}^2)$, $\frac{\partial}{\partial x_1}(\mathcal{O} * f) \in L^p_{\frac{1}{2},\frac{1}{4}}(\mathbb{R}^2)$ *and* $\nabla^2 \mathcal{O} * f \in L^p_{\frac{1}{2},\frac{1}{4}}(\mathbb{R}^2)$. *Moreover, we have the estimate*

$$\begin{aligned}\|\mathcal{O} * f\|_{L^p_{-\frac{1}{2},\frac{1}{4}}(\mathbb{R}^2)} + \|\tfrac{\partial}{\partial x_2}(\mathcal{O} * f)\|_{L^p_{0,\frac{1}{4}}(\mathbb{R}^2)} + \|\tfrac{\partial}{\partial x_1}(\mathcal{O} * f)\|_{L^p_{\frac{1}{2},\frac{1}{4}}(\mathbb{R}^2)} + \\ \|\nabla^2 (\mathcal{O} * f)\|_{L^p_{\frac{1}{2},\frac{1}{4}}(\mathbb{R}^2)} \leq C\|f\|_{L^p_{\frac{1}{2},\frac{1}{4}}(\mathbb{R}^2)}.\end{aligned} \tag{4.8}$$

Proof : From [10], we have $\mathcal{O} * f \in L^p_{-\frac{1}{2}-\varepsilon,\frac{1}{4}}(\mathbb{R}^2)$, $\frac{\partial}{\partial x_2}(\mathcal{O} * f) \in L^p_{0,\frac{1}{4}}(\mathbb{R}^2)$, $\frac{\partial}{\partial x_1}(\mathcal{O} * f) \in L^p_{\frac{1}{2}-\varepsilon,\frac{1}{4}}(\mathbb{R}^2)$, for all $\varepsilon > 0$. It remains to prove that $\mathcal{O} * f \in L^p_{-\frac{1}{2},\frac{1}{4}}(\mathbb{R}^2)$, and $\frac{\partial}{\partial x_1}(\mathcal{O} * f) \in L^p_{\frac{1}{2},\frac{1}{4}}(\mathbb{R}^2)$. For $R > R^*$, we use the following partition of unity

$$\begin{gathered}\varphi_1, \varphi_2 \in \mathcal{C}^\infty(\mathbb{R}^2),\ 0 \leq \varphi_1, \varphi_2 \leq 1,\ \varphi_1 + \varphi_2 = 1 \quad \text{in} \quad \mathbb{R}^2 \\ \varphi_1 = 1 \quad \text{in} \quad B_R \quad \text{and} \quad \text{Supp}\,\varphi_1 \subset \mathrm{B_{R+1}}.\end{gathered}$$

We set $u = \mathcal{O} * f$ and we split u into $u = u_1 + u_2$, where $u_1 = \varphi_1.u$ and $u_2 = \varphi_2.u$. Since $\mathrm{Supp\ u_1} \subset \mathrm{B_{R+1}}$ so, $u_1 \in L^p_{-\frac{1}{2},\frac{1}{4}}(\mathbb{R}^2)$ and satisfies

$$\|u_1\|_{L^p_{-\frac{1}{2},\frac{1}{4}}(\mathbb{R}^2)} \leq C\|f\|_{L^p_{\frac{1}{2},\frac{1}{4}}(\mathbb{R}^2)}.$$

Furthermore, u_2 is solution of the following problem

$$-\Delta u_2 + \tfrac{\partial u_2}{\partial x_1} = \widetilde{f} \quad \text{in} \quad \mathbb{R}^2,$$

where $\widetilde{f} = \varphi_2\, f + u\Delta\varphi_1 + 2\nabla\, u \nabla\, \varphi_1 - u\frac{\partial \varphi_1}{\partial x_1}$. Since the regularity of $\varphi_2\, f$ determines that of $\widetilde{f}$, then $\widetilde{f} \in L^p_{\frac{1}{2},\frac{1}{4}}(\mathbb{R}^2)$. Setting $v = (1+s)^{\frac{1}{4}}\, u_2$, we have $v \in L^p_{-\frac{1}{2}-\varepsilon,0}(\mathbb{R}^2)$, and satisfies Equation

$$-\Delta v + \frac{\partial v}{\partial x_1} = (1+s)^{\frac{1}{4}}\widetilde{f} - 2\nabla\, u_2\,.\nabla\,(1+s)^{\frac{1}{4}} - u_2[\Delta(1+s)^{\frac{1}{4}} - \frac{\partial}{\partial x_1}(1+s)^{\frac{1}{4}}].$$

A simple calculation yields

$$(\Delta - \tfrac{\partial}{\partial x_1})(1+s)^{\frac14} = \tfrac{1}{8r}(2s^2+s+2)(1+s)^{-\frac74},$$

then, $u_2[\Delta(1+s)^{\frac14} - \frac{\partial}{\partial x_1}(1+s)^{\frac14}] = a_0 v$, where a_0 is defined in (4.2). Hence, v satisfies problem (4.1), where $g = (1+s)^{\frac14}\widetilde{f} - 2\nabla u_2 .\nabla (1+s)^{\frac14} \in L_{\frac12,0}(B'_R)$. Applying Lemma 4.1, there exists a unique $w \in L^p_{-\frac12,0}(B'_R)$ solution of this problem. Setting $z = v - w$, we have $z \in L^p_{-\frac12-\varepsilon,0}(\mathbb{R}^2)$, and satisfies

$$-\Delta z + \tfrac{\partial z}{\partial x_1} + a_0 z = 0 \quad \text{in } \mathbb{R}^2.$$

Then $z = 0$, which implies that $v \in L^p_{-\frac12,0}(\mathbb{R}^2)$ and

$$\|v\|_{L^p_{-\frac12,0}(\mathbb{R}^2)} \leq C\|g\|_{L^p_{\frac12,0}(B'_R)} \leq C\|f\|_{L^p_{\frac12,\frac14}(\mathbb{R}^2)}.$$

Hence $u_2 \in L^p_{-\frac12,\frac14}(\mathbb{R}^2)$ and

$$\|u_2\|_{L^p_{-\frac12,\frac14}(\mathbb{R}^2)} \leq C \parallel f \parallel_{L^p_{\frac12,\frac14}(\mathbb{R}^2)},$$

which proves that $u \in L^p_{-\frac12,\frac14}(\mathbb{R}^2)$ and satisfies

$$\|u\|_{L^p_{-\frac12,\frac14}(\mathbb{R}^2)} \leq C\|f\|_{L^p_{\frac12,\frac14}(\mathbb{R}^2)}. \tag{4.9}$$

Now, using the fact that u_2 satisfies

$$-\Delta(\eta_{1/4}^{1/2}\, u_2) + \tfrac{\partial}{\partial x_1}(\eta_{1/4}^{1/2}\, u_2) =: F,$$

where

$$F = \eta_{1/4}^{1/2}\, f - u\Delta(\eta_{1/4}^{1/2}\,\varphi_2) - 2\nabla u.\nabla\,(\eta_{1/4}^{1/2}\,\varphi_2) + u\frac{\partial}{\partial x_1}(\eta_{1/4}^{1/2}\,\varphi_2) \in L^p(\mathbb{R}^2).$$

From Theorem 3.9, there exists a function v such that $\nabla^2 v \in L^p(\mathbb{R}^2)$ and $\frac{\partial v}{\partial x_1} \in L^p(\mathbb{R}^2)$ satisfying

$$-\Delta v + \tfrac{\partial v}{\partial x_1} = -\Delta(\eta_{1/4}^{1/2}\, u_2) + \tfrac{\partial}{\partial x_1}(\eta_{1/4}^{1/2}\, u_2).$$

Moreover,

$$\|\nabla^2 v\|_{L^p(\mathbb{R}^2)} + \|\frac{\partial v}{\partial x_1}\|_{L^p(\mathbb{R}^2)} \leq C\|F\|_{L^p(\mathbb{R}^2)} \leq C\|f\|_{L^p_{\frac12,\frac14}(\mathbb{R}^2)}. \tag{4.10}$$

We set $w = \nabla^2 v - \nabla^2(\eta_{1/4}^{1/2}\, u_2)$, since $\nabla^2 u \in \bigcap_{\varepsilon>0} L^p_{\frac12-\varepsilon,\frac14}(\mathbb{R}^2)$, then $w \in \bigcap_{\varepsilon>0} L^p_{-\varepsilon,0}(\mathbb{R}^2)$ and satisfies

$$-\Delta w + \tfrac{\partial w}{\partial x_1} = 0, \quad \text{in} \quad \mathbb{R}^2.$$

Then $w = 0$, which imply that

$$\nabla^2\,(\eta_{1/4}^{1/2}\, u) \in L^p(\mathbb{R}^2).$$

We obtain then

$$\nabla^2 u \in L^p_{\frac12,\frac14}(\mathbb{R}^2), \qquad \tfrac{\partial u}{\partial x_1} \in L^p_{\frac12,\frac14}(\mathbb{R}^2),$$

and the estimate

$$\|\nabla^2 u\|_{L^p_{\frac12,\frac14}(\mathbb{R}^2)} + \|\frac{\partial u}{\partial x_1}\|_{L^p_{\frac12,\frac14}(\mathbb{R}^2)} \;\leq C\|f\|_{L^p_{\frac12,\frac14}(\mathbb{R}^2)}. \tag{4.11}$$

This finishes the proof. $\diamondsuit$

Setting

$$K^p_{\alpha,\beta}(\Omega) = \;\left\{v\in\mathcal{D}'(\Omega);\;\; r^\alpha(1+s)^\beta \in L^p(\Omega)\right\},$$

which is a reflexive Banach space when it is equipped by its natural norm. With the same arguments we prove the following result. The case $\beta=\frac14$ corresponds to Theorem 4.2.

Theorem 4.3 *Assume* $2\leq p<\frac{8}{3-\beta}$ *and* $0<\beta<\frac14$. *Then, for* $f\in K^p_{\frac12,\beta}(\mathbb{R}^2)$, *we have* $\mathcal{O}*f\in K^p_{-\frac12,\beta}(\mathbb{R}^2)$, $\frac{\partial}{\partial x_2}(\mathcal{O}*f)\in K^p_{0,\beta}(\mathbb{R}^2)$, $\frac{\partial}{\partial x_1}(\mathcal{O}*f)\in K^p_{\frac12,\beta}(\mathbb{R}^2)$ *and* $\nabla^2(\mathcal{O}*f)\in K^p_{\frac12,\beta}(\mathbb{R}^2)$. *Moreover, we have the estimates*

$$\begin{aligned}\|\mathcal{O}*f\|_{K^p_{-\frac12,\beta}(\mathbb{R}^2)} + \|\tfrac{\partial}{\partial x_2}(\mathcal{O}*f)\|_{K^p_{0,\beta}(\mathbb{R}^2)} + \|\tfrac{\partial}{\partial x_1}(\mathcal{O}*f)\|_{K^p_{\frac12,\beta}(\mathbb{R}^2)} +\\ \|\nabla^2(\mathcal{O}*f)\|_{K^p_{\frac12,\beta}(\mathbb{R}^2)} \;\leq C\|f\|_{K^p_{\frac12,\beta}(\mathbb{R}^2)}.\end{aligned} \tag{4.12}$$

For $\alpha,\beta\in\mathbb{R}$ we denote by

$$L^p_{\alpha,\beta(s')}(\mathbb{R}^2) = \;\left\{v\in\mathcal{D}'(\Omega);\;\; \rho^\alpha(1+s')^\beta\, v\in L^p(\mathbb{R}^2)\right\},$$

which is a reflexive Banach space when it is equipped by its natural norm

$$\|v\|_{L^p_{\alpha,\beta(s')}(\mathbb{R}^2)} = \|\rho^\alpha(1+s')^\beta\, v\|_{L^p(\mathbb{R}^2)}.$$

Proposition 4.4 *For all given* $f\in L^2_{\frac12,\frac{\delta-1}{2}(s')}(\mathbb{R}^2)$, *with* $\delta>0$ *close to zero, Equation (3.1) has a unique solution* $u\in K^2_{\frac{\delta}{2}-1,0}(\mathbb{R}^2)$, *such that* $\nabla u\in\mathbb{L}^2_{\frac{\delta}{4}-\frac12,0}(\mathbb{R}^2)$. *Moreover, there exists a constant* $C>0$ *such that*

$$\|u\|_{K^2_{\frac{\delta}{2}-1,0}(\mathbb{R}^2)} + \|\nabla u\|_{\mathbb{L}^2_{\frac{\delta}{4}-\frac12,0}(\mathbb{R}^2)} \;\leq C\|f\|_{L^2_{\frac12,\frac{\delta-1}{2}(s')}(\mathbb{R}^2)}. \tag{4.13}$$

Proof : By the density of $\mathcal{D}(\mathbb{R}^2)$ in $L^2_{\frac12,\frac{\delta-1}{2}(s')}(\mathbb{R}^2)$ (see [2]), there exists a sequence (f_k) of $\mathcal{D}(\mathbb{R}^2)$ such that $f_k\to f$ in $L^2_{\frac12,\frac{\delta-1}{2}(s')}(\mathbb{R}^2)$. Since $f_k\in\mathcal{D}(\mathbb{R}^2)$, so $f_k\in K^2_{\frac12,\beta}(\mathbb{R}^2)$, $0<\beta<\frac14$. Then, from Theorem 4.2, Equation

$$-\Delta u_k + \frac{\partial u_k}{\partial x_1} \;=\; f_k \quad\text{in}\quad \mathbb{R}^2, \tag{4.14}$$

has a solution $u_k=\mathcal{O}*f_k\in K^2_{-\frac12,0}(\mathbb{R}^2)$ such that $\nabla u_k\in K^2_{0,\beta}(\mathbb{R}^2)$, $\nabla^2 u_k\in K^2_{\frac12,\beta}(\mathbb{R}^2)$ and $\frac{\partial u_k}{\partial x_1}\in K^2_{\frac12,\beta}(\mathbb{R}^2)$. Multiply Equation (4.14) by hu_k where $h=\breve{\mathcal{O}}*r^{\delta-2}$, $(\delta>0)$, $\breve{\mathcal{O}}$ is the fundamental solution of the operator $-\Delta-\frac{\partial}{\partial x_1}$, we obtain after two integrations by part

$$\int_{\mathbb{R}^2}|\nabla u_k|^2\, h\, dx \;+\; \frac12\int_{\mathbb{R}^2}u_k^2\,(-\Delta h-\frac{\partial h}{\partial x_1})\,dx \;=\; \int_{\mathbb{R}^2}f_k\, h\, u_k\, dx. \tag{4.15}$$

Since $-\Delta h-\frac{\partial h}{\partial x_1}=r^{\delta-2}$, we have

$$\int_{\mathbb{R}^2}|\nabla u_k|^2\, h\, dx \;+\; \frac12\int_{\mathbb{R}^2}u_k^2\, r^{\delta-2} = \;\int_{\mathbb{R}^2}f_k\, h\, u_k\, dx\,,$$

and as $h \geq 0$ we get then the two inequalities

$$\int_{\mathbb{R}^2} u_k^2\, r^{\delta-2} \;\leq\; 2\int_{\mathbb{R}^2} f_k\, h\, u_k\, dx, \tag{4.16}$$

$$\int_{\mathbb{R}^2} |\nabla u_k|^2\, h\, dx \;\leq\; \int_{\mathbb{R}^2} f_k\, h\, u_k\, dx. \tag{4.17}$$

A simple calculation yields

$$(-\Delta - \frac{\partial}{\partial x_1})(1+r)^{\frac{\delta}{2}-1} = \frac{2-\delta}{4}\,(1+r)^{\frac{\delta}{2}-2}\left(\frac{4-\delta}{1+r} - \frac{1}{r} - \frac{x_1}{r}\right),$$

then

$$(-\Delta - \frac{\partial}{\partial x_1})\left(h - M(1+r)^{\frac{\delta}{2}-1}\right) \geq \frac{1}{r^{2-\delta}} - M\frac{2-\delta}{2r}\,(1+r)^{\frac{\delta}{2}-1} \;\geq 0,$$

for $0 < M \leq \frac{2^{2+\delta/2}}{2-\delta}\left(\frac{1-\delta}{2+\delta}\right)^{1+\delta/2}$. Then, there exists $M > 0$ such that $h(x) \geq M\,(1+r)^{\frac{\delta}{2}-1}$, so from inequality (4.17), we obtain

$$M\int_{\mathbb{R}^2} (1+r)^{\frac{\delta}{2}-1}\,|\nabla u_k|^2\, dx \;\leq \int_{\mathbb{R}^2} f_k\, h\, u_k\, dx. \tag{4.18}$$

The Cauchy-Schwarz inequality gives

$$\int_{\mathbb{R}^2} f_k\, h\, u_k\, dx \;\leq \left(\int_{\mathbb{R}^2} f_k^2\, h^2\, r^{2-\delta}\, dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2} r^{\delta-2}\, u_k^2\, dx\right)^{\frac{1}{2}}$$

So, from inequalities (4.16) we get

$$\begin{aligned}\int_{\mathbb{R}^2} r^{\delta-2}\, u_k^2\, dx \;&\leq\; 4\int_{\mathbb{R}^2} f_k^2\, h^2\, r^{2-\delta}\, dx\\ &=\; 4\int_{\mathbb{R}^2} f_k^2\,\frac{1+r}{(1+s')^{1-\delta}}\, h^2\, r^{1-\delta}\,(1+s')^{1-\delta}\, dx,\end{aligned}$$

We adapt the result of Theorem 3.5 obtained in [10], we have $h^2\, r^{1-\delta}\,(1+s')^{1-\delta} \in L^\infty(\mathbb{R}^2)$, then $u_k \in K^2_{\frac{\delta}{2}-1,0}(\mathbb{R}^2)$ and there exists $C > 0$ such that

$$\|u_k\|_{K^2_{\frac{\delta}{2}-1,0}(\mathbb{R}^2)} \;\leq C\|f_k\|_{L^2_{\frac{1}{2},\frac{\delta}{2}-\frac{1}{2}(s')}} \leq C\|f\|_{L^2_{\frac{1}{2},\frac{\delta}{2}-\frac{1}{2}(s')}}. \tag{4.19}$$

Now, using inequalities (4.18) and (4.19), we deduce that $\nabla u_k \in L^2_{\frac{\delta}{4}-\frac{1}{2},0}(\mathbb{R}^2)$ and

$$\|\nabla u_k\|_{L^2_{\frac{\delta}{4}-\frac{1}{2},0}(\mathbb{R}^2)} \;\leq C\|f_k\|_{L^2_{\frac{1}{2},\frac{\delta}{2}-\frac{1}{2}(s')}} \leq C\|f\|_{L^2_{\frac{1}{2},\frac{\delta}{2}-\frac{1}{2}(s')}}. \tag{4.20}$$

So, the sequences u_k and $v_k = \nabla u_k$ remain bounded in $K^2_{\frac{\delta}{2}-1,0}(\mathbb{R}^2)$ and in $L^2_{\frac{\delta}{4}-\frac{1}{2},0}(\mathbb{R}^2)$ respectively. These spaces are reflexives, therefore extracting a subsequence if necessary, we have

$$u_k \rightharpoonup u \;\text{ in }\; K^2_{\frac{\delta}{2}-1,0}(\mathbb{R}^2) \;\text{ and }\; \nabla u_k \rightharpoonup \nabla u \;\text{ in }\; L^2_{\frac{\delta}{4}-\frac{1}{2},0}(\mathbb{R}^2),$$

with the estimates

$$\|u\|_{K^2_{\frac{\delta}{2}-1,0}(\mathbb{R}^2)} \;\leq \liminf_{k\to\infty}\|u_k\|_{K^2_{\frac{\delta}{2}-1,0}(\mathbb{R}^2)} \;\leq C\|f\|_{L^2_{\frac{1}{2},\frac{\delta}{2}-\frac{1}{2}(s')}}, \tag{4.21}$$

$$\|\nabla u\|_{L^2_{\frac{\delta}{4}-\frac{1}{2},0}(\mathbb{R}^2)} \;\leq \liminf_{k\to\infty}\|\nabla u_k\|_{L^2_{\frac{\delta}{4}-\frac{1}{2},0}(\mathbb{R}^2)} \;\leq C\|f\|_{L^2_{\frac{1}{2},\frac{\delta}{2}-\frac{1}{2}(s')}}. \tag{4.22}$$

We get then Estimate (4.13) and we verify easily that u is a solution of Equation 3.1. Uniqueness is given by the fact that the space $K^2_{\frac{\delta}{2}-1,0}$ contains no polynomials. $\Diamond$

5 Behaviour of u_λ when $\lambda \to 0$

Assume $1 < p < 2$, $f \in L^p(\mathbb{R}^2)$ and, for $\lambda > 0$, consider the equation

$$-\Delta u_\lambda + \lambda \frac{\partial u_\lambda}{\partial x_1} = f \ \text{ in } \ \mathbb{R}^2. \tag{5.1}$$

Setting

$$y = \lambda x, \ \ u_\lambda(x) = v(y) \ \text{ and } \ f(x) = \lambda^2 \, g(y),$$

then v satisfies the equation

$$-\Delta v(y) + \frac{\partial v}{\partial y_1}(y) = g(y) \ \text{ in } \ \mathbb{R}^2, \tag{5.2}$$

where clearly, $g \in L^p(\mathbb{R}^2)$. We know by Theorem 3.9, that, if $1 < p < 2$, Equation (5.2) has a solution v such that, in particular, $\nabla v \in L^{\frac{2p}{2-p}}(\mathbb{R}^2)$, $\nabla^2 v \in L^p(\mathbb{R}^2)$, $\frac{\partial v}{\partial x_1} \in L^p(\mathbb{R}^2)$ and satisfies

$$\|\nabla v\|_{L^{\frac{2p}{2-p}}(\mathbb{R}^2)} + \|\nabla^2 v\|_{L^p(\mathbb{R}^2)} \ \leq C \|g\|_{L^p(\mathbb{R}^2)}. \tag{5.3}$$

By a simple calculation we obtain from Inequality (5.3) the estimate

$$\|\nabla u_\lambda\|_{L^{\frac{2p}{2-p}}(\mathbb{R}^2)} + \|\nabla^2 u_\lambda\|_{L^p(\mathbb{R}^2)} \ \leq C \|f\|_{L^p(\mathbb{R}^2)}, \tag{5.4}$$

where C does not depend on λ. We deduce that the sequences ∇u_λ and $\nabla^2 u_\lambda$ remain bounded in $L^p(\mathbb{R}^2)$ and $L^{p^*}(\mathbb{R}^2)$, with $p^* = \frac{2p}{2-p}$ respectively. Now, setting

$$-\Delta u_\lambda = f_\lambda \quad \text{in} \quad \mathbb{R}^2, \tag{5.5}$$

then, the sequence f_λ is bounded in $L^p(\mathbb{R}^2) \cap W_0^{-1,p^*}(\mathbb{R}^2)$. These spaces are reflexives, extracting a subsequence if necessary, also denoted f_λ, we have

$$f_\lambda \ \rightharpoonup \ f \text{ in } L^p(\mathbb{R}^2) \ \text{ and } \ f_\lambda \ \rightharpoonup \ f \text{ in } W_0^{-1,p^*}(\mathbb{R}^2).$$

Further, note that $p^* > 2$, then there exists $z \in W_0^{1,p^*}(\mathbb{R}^2)$ and $w \in W_0^{2,p}(\mathbb{R}^2)$ such that

$$-\Delta z = -\Delta w = f \ \text{ in } \ \mathbb{R}^2.$$

Since $\nabla z \in L^{p^*}(\mathbb{R}^2)$, $\nabla w \in L^{p^*}(\mathbb{R}^2)$ by Sobolev embedding and $\nabla z - \nabla w$ is harmonic so $\nabla z - \nabla w = 0$ in $\mathbb{R}^2$ then, there exists $k \in \mathbb{R} \subset W_0^{2,p}(\mathbb{R}^2)$ such that $z = w + k$, thus $z \in W_0^{2,p}(\mathbb{R}^2) \cap W_0^{1,p^*}(\mathbb{R}^2)$. Now, since the norm on $W_0^{2,p}(\mathbb{R}^2)/\mathbb{R}$ is equivalent to its semi-norm, we deduce from inequality (5.4), that there exists $k_\lambda \in \mathbb{R}$ and $u \in W_0^{2,p}(\mathbb{R}^2) \cap W_0^{1,p^*}(\mathbb{R}^2)$ such that

$$u_\lambda + k_\lambda \ \rightharpoonup \ u \ \text{ in } \ W_0^{2,p}(\mathbb{R}^2) \ \text{ and in } \ W_0^{1,p^*}(\mathbb{R}^2).$$

Since $-\Delta u = f$ in $\mathbb{R}^2$, there exists $k \in \mathbb{R}$ such that $z = u + k$. We refind thus the result obtained by Amrouche, Girault and Giroire in [1] for $f \in L^p(\mathbb{R}^2)$. The following proposition is then aquired.

Proposition 5.1 *Assume that $1 < p < 2$ and let $f \in L^p(\mathbb{R}^2)$. Then Equation (5.1) has at least a solution u_λ of the form (3.41) such that $\nabla u_\lambda \in L^{\frac{3p}{3-p}}(\mathbb{R}^2) \cap L^{\frac{2p}{2-p}}(\mathbb{R}^2)$, $\nabla^2 u_\lambda \in L^p(\mathbb{R}^2)$, and $\frac{\partial u_\lambda}{\partial x_1} \in L^p(\mathbb{R}^2)$. Moreover, if $1 < p < \frac{3}{2}$, then*

$u_\lambda \in L^{\frac{3p}{3-2p}}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. *Furthermore, there exists* $k_\lambda \in \mathbb{R}$ *such that, when* $\lambda \to 0$,

$$u_\lambda + k_\lambda \rightharpoonup u \text{ in } W_0^{2,p}(\mathbb{R}^2) \text{ and in } W_0^{1,p^*}(\mathbb{R}^2),$$

where u *is the unique solution of Poisson's Equation*

$$-\Delta u = f \qquad \text{in } \mathbb{R}^2, \tag{5.6}$$

with the estimate

$$\|\nabla u\|_{L^{p^*}(\mathbb{R}^2)} + \|\nabla^2 u\|_{L^p(\mathbb{R}^2)} \le C\|f\|_{L^p(\mathbb{R}^2)}. \tag{5.7}$$

For $f \in W_0^{-1,p}(\mathbb{R}^2)$ we have the following result

Proposition 5.2 *Assume* $1 < p < 2$ *and let* $f \in W_0^{-1,p}(\mathbb{R}^2)$, *satisfying the compatibility condition*

$$\langle f, 1\rangle_{W_0^{-1,p}(\mathbb{R}^2)\times W_0^{1,p'}(\mathbb{R}^2)} = 0. \tag{5.8}$$

Then, Equation (5.1) has a unique solution $u_\lambda \in L^{\frac{3p}{3-p}}(\mathbb{R}^2) \cap L^{p^*}(\mathbb{R}^2)$ *such that* $\nabla u_\lambda \in L^p(\mathbb{R}^2)$ *and* $\frac{\partial u_\lambda}{\partial x_1} \in W_0^{-1,p}(\mathbb{R}^2)$. *Moreover,*

$$u_\lambda \rightharpoonup u \quad \text{in} \quad W_0^{1,p}(\mathbb{R}^2) \quad \text{as} \quad \lambda \to 0,$$

where u *is the unique solution of Poisson's Equation*

$$-\Delta u = f \qquad \text{in } \mathbb{R}^2, \tag{5.9}$$

and the following estimate holds.

$$\|u\|_{L^{p^*}(\mathbb{R}^2)} + \|\nabla u\|_{L^p(\mathbb{R}^2)} \le C\|f\|_{W_0^{-1,p}(\mathbb{R}^2)}. \tag{5.10}$$

Proof : By Isomorphism (2.7), there exists $\boldsymbol{F} \in L^p(\mathbb{R}^2)$ such that $f = \mathrm{div}\boldsymbol{F}$ and

$$\|\boldsymbol{F}\|_{L^p(\mathbb{R}^2)} \le C\|f\|_{W_0^{-1,p}(\mathbb{R}^2)}. \tag{5.11}$$

Setting

$$y = \lambda x, \quad u_\lambda(x) = v(y), \quad \boldsymbol{F}(x) = \lambda \boldsymbol{G}(y) \text{ and } g = \mathrm{div}\boldsymbol{G},$$

v satisfies Equation (5.2) where $g \in W_0^{-1,p}(\mathbb{R}^2) \perp \mathbb{R}$. By Theorem 3.11, this equation has a unique solution $v \in L^{\frac{3p}{3-p}}(\mathbb{R}^2) \cap L^{p^*}(\mathbb{R}^2)$ such that $\nabla v \in L^p(\mathbb{R}^2)$ and $\frac{\partial v}{\partial x_1} \in W_0^{-1,p}(\mathbb{R}^2)$, with the estimate

$$\|v\|_{L^{p^*}(\mathbb{R}^2)} + \|\nabla v\|_{L^p(\mathbb{R}^2)} \le C\|g\|_{W_0^{-1,p}(\mathbb{R}^2)} \le C\|\boldsymbol{G}\|_{L^p(\mathbb{R}^2)}. \tag{5.12}$$

As previously we get the estimate

$$\|u_\lambda\|_{L^{p^*}(\mathbb{R}^2)} + \|\nabla u_\lambda\|_{L^p(\mathbb{R}^2)} \le C\|\boldsymbol{F}\|_{L^p(\mathbb{R}^2)}. \tag{5.13}$$

The sequences u_λ and ∇u_λ remain bounded in $L^{p^*}(\mathbb{R}^2)$ and $L^p(\mathbb{R}^2)$ respectively. These spaces are reflexives, there exists $u \in L^{p^*}(\mathbb{R}^2)$ such that $u_\lambda \rightharpoonup u$ in $L^{p^*}(\mathbb{R}^2)$ and $\nabla u_\lambda \rightharpoonup \nabla u$ in $L^p(\mathbb{R}^2)$. We verify easily that u is a solution of Poisson's Equation (5.9) and satisfies Estimate (5.10). Uniqueness follows by the fact that the space $L^{p^*}(\mathbb{R}^2)$ contains no polynomials. We deduce that $u \in W_0^{1,p}(\mathbb{R}^2)$ and we refind also the result obtained in [1] for $f \in W_0^{-1,p}(\mathbb{R}^2)$. $\diamondsuit$

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