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Nonhomogeneous biharmonic problem in the half-space, L^p theory and generalized solutions

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Abstract

In this paper, we study the biharmonic equation in the half-space \mathbb{R}_+^N , with $N \geq 2$. We prove in L^p theory, with $1 < p < \infty$, existence and uniqueness results. We consider data and give solutions which live in weighted Sobolev spaces.

Key words: Biharmonic problem, Half-space, Weighted Sobolev spaces.
AMS Classification: 35D05, 35J35, 35J40.

1 Introduction

The purpose of this paper is the resolution of the biharmonic problem with nonhomogeneous boundary conditions

$$(\mathcal{P}) \quad \begin{cases} \Delta^2 u = f & \text{in } \mathbb{R}_+^N, \\ u = g_0 & \text{on } \Gamma = \mathbb{R}^{N-1}, \\ \partial_N u = g_1 & \text{on } \Gamma. \end{cases}$$

Since this problem is posed in the half-space, it is important to specify the behaviour at infinity for the data and solutions. We have chosen to impose such conditions by setting our problem in weighted Sobolev spaces, where the growth or decay of functions at infinity are expressed by means of weights. These weighted Sobolev spaces provide a correct functional setting for unbounded domains, in particular because the functions in these spaces satisfy

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an optimal weighted Poincaré-type inequality. The weights chosen here behave at infinity as powers to $|x|$. The reason of this choice is given by the behaviour at infinity of the fundamental solution E_N to the biharmonic operator in \mathbb{R}^N . Let's recall for instance that

$$E_3(x) = c_3 |x|, \quad E_4(x) = c_4 \ln |x|, \quad E_5(x) = \frac{c_5}{|x|},$$

and in particular if $f \in \mathcal{D}(\mathbb{R}^N)$, the convolution $E_N * f$ behaves at infinity as E_N . In this work, we shall consider more general data f ; and the solutions will have a behaviour at infinity which will naturally depend on the one of data in \mathbb{R}_+^N and on the boundary. We have also tried to give another motivation to this choice, more precisely for the biharmonic problem, in the subsection 2.4, after the definition of spaces.

Our analysis is based on the isomorphism properties of the biharmonic operator in the whole space and the resolution of the Dirichlet and Neumann problems for the Laplacian in the half-space. This last one is itself based on the isomorphism properties of the Laplace operator in the whole space and also on the reflection principle inherent in the half-space. Note here the double difficulty arising from the unboundedness of the domain in any direction and from the unboundedness of the boundary itself.

This paper is organized as follows. Section 2 is devoted to the notations and fundamental results. In Section 3, we study the biharmonic operator in the whole space and we establish isomorphism properties which we will use in the sequel. At last, Section 4 is devoted to the resolution of Problem (\mathcal{P}) . The main result is Theorem 4.1, where we obtain generalized solutions $u \in W_l^{2,p}(\mathbb{R}_+^N)$ to biharmonic problem, where l indicates the behaviour at infinity of these solutions. In a forthcoming work, we shall examine the case of regular data and the homogeneous problem with singular boundary conditions.

2 Notations, spaces, motivation and known results

2.1 Notations

For any real number $p > 1$, we always take p' to be the Hölder conjugate of p , *i.e.*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Let Ω be an open set of \mathbb{R}^N , $N \geq 2$. Writing a typical point $x \in \mathbb{R}^N$ as $x = (x', x_N)$, where $x' = (x_1, \dots, x_{N-1}) \in \mathbb{R}^{N-1}$ and $x_N \in \mathbb{R}$, we will especially look on the upper half-space $\mathbb{R}_+^N = \{x \in \mathbb{R}^N; x_N > 0\}$. We let $\overline{\mathbb{R}_+^N}$ denote

the closure of \mathbb{R}_+^N in \mathbb{R}^N and let $\Gamma = \{x \in \mathbb{R}^N; x_N = 0\} \equiv \mathbb{R}^{N-1}$ denote its boundary. Let $|x| = (x_1^2 + \dots + x_N^2)^{1/2}$ denote the Euclidean norm of x , we will use two basic weights

$$\varrho = (1 + |x|^2)^{1/2} \quad \text{and} \quad \lg \varrho = \ln(2 + |x|^2).$$

We denote by ∂_i the partial derivative $\frac{\partial}{\partial x_i}$, similarly $\partial_i^2 = \partial_i \circ \partial_i = \frac{\partial^2}{\partial x_i^2}$, $\partial_{ij}^2 = \partial_i \circ \partial_j = \frac{\partial^2}{\partial x_i \partial x_j}$, ... More generally, if $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{N}^N$ is a multi-index, then

$$\partial^\lambda = \partial_1^{\lambda_1} \dots \partial_N^{\lambda_N} = \frac{\partial^{|\lambda|}}{\partial x_1^{\lambda_1} \dots \partial x_N^{\lambda_N}} \quad , \quad \text{where } |\lambda| = \lambda_1 + \dots + \lambda_N.$$

In the sequel, for any integer q , we shall use the following polynomial spaces:

- \mathcal{P}_q is the space of polynomials of degree smaller than or equal to q ;
- \mathcal{P}_q^Δ is the subspace of harmonic polynomials of \mathcal{P}_q ;
- $\mathcal{P}_q^{\Delta^2}$ is the subspace of biharmonic polynomials of \mathcal{P}_q ;
- \mathcal{A}_q^Δ is the subspace of polynomials of \mathcal{P}_q^Δ , odd with respect to x_N , or equivalently, which satisfy the condition $\varphi(x', 0) = 0$;
- \mathcal{N}_q^Δ is the subspace of polynomials of \mathcal{P}_q^Δ , even with respect to x_N , or equivalently, which satisfy the condition $\partial_N \varphi(x', 0) = 0$;

with the convention that these spaces are reduced to $\{0\}$ if $q < 0$.

For any real number s , we denote by $[s]$ the integer part of s .

Given a Banach space B , with dual space B' and a closed subspace X of B , we denote by $B' \perp X$ the subspace of B' orthogonal to X , *i.e.*

$$B' \perp X = \{f \in B'; \forall v \in X, \langle f, v \rangle = 0\} = (B/X)'.$$

Lastly, if $k \in \mathbb{Z}$, we shall constantly use the notation $\{1, \dots, k\}$ for the set of the first k positive integers, with the convention that this set is empty if k is nonpositive.

2.2 Weighted Sobolev spaces

For any nonnegative integer m , real numbers $p > 1$, α and β , we define the following space:

$$W_{\alpha, \beta}^{m, p}(\Omega) = \left\{ u \in \mathcal{D}'(\Omega); 0 \leq |\lambda| \leq k, \varrho^{\alpha-m+|\lambda|} (\lg \varrho)^{\beta-1} \partial^\lambda u \in L^p(\Omega); \right. \\ \left. k+1 \leq |\lambda| \leq m, \varrho^{\alpha-m+|\lambda|} (\lg \varrho)^\beta \partial^\lambda u \in L^p(\Omega) \right\}, \quad (1)$$

where

$$k = \begin{cases} -1 & \text{if } \frac{N}{p} + \alpha \notin \{1, \dots, m\}, \\ m - \frac{N}{p} - \alpha & \text{if } \frac{N}{p} + \alpha \in \{1, \dots, m\}. \end{cases}$$

In the case $\beta = 0$, we simply denote the space by $W_\alpha^{m,p}(\Omega)$. Note that $W_{\alpha,\beta}^{m,p}(\Omega)$ is a reflexive Banach space equipped with its natural norm:

$$\|u\|_{W_{\alpha,\beta}^{m,p}(\Omega)} = \left(\sum_{0 \leq |\lambda| \leq k} \|\varrho^{\alpha-m+|\lambda|} (\lg \varrho)^{\beta-1} \partial^\lambda u\|_{L^p(\Omega)}^p + \sum_{k+1 \leq |\lambda| \leq m} \|\varrho^{\alpha-m+|\lambda|} (\lg \varrho)^\beta \partial^\lambda u\|_{L^p(\Omega)}^p \right)^{1/p}.$$

We also define the semi-norm:

$$|u|_{W_{\alpha,\beta}^{m,p}(\Omega)} = \left(\sum_{|\lambda|=m} \|\varrho^\alpha (\lg \varrho)^\beta \partial^\lambda u\|_{L^p(\Omega)}^p \right)^{1/p}.$$

The weights in the definition (1) are chosen so that the corresponding space satisfies two properties. On the one hand, $\mathcal{D}(\overline{\mathbb{R}_+^N})$ is dense in $W_{\alpha,\beta}^{m,p}(\mathbb{R}_+^N)$. On the other hand, the following Poincaré-type inequality holds in $W_{\alpha,\beta}^{m,p}(\mathbb{R}_+^N)$:

Theorem 2.1. *Let α and β be two real numbers and $m \geq 1$ an integer such that*

$$\frac{N}{p} + \alpha \notin \{1, \dots, m\} \quad \text{or} \quad (\beta - 1)p \neq -1.$$

Then the semi-norm $|\cdot|_{W_{\alpha,\beta}^{m,p}(\mathbb{R}_+^N)}$ defines on $W_{\alpha,\beta}^{m,p}(\mathbb{R}_+^N)/\mathcal{P}_{q'}$ a norm which is equivalent to the quotient norm, with $q' = \inf(q, m - 1)$, where q is the highest degree of the polynomials contained in $W_{\alpha,\beta}^{m,p}(\mathbb{R}_+^N)$.

Now, we define the space

$$\overset{\circ}{W}_{\alpha,\beta}^{m,p}(\mathbb{R}_+^N) = \overline{\mathcal{D}(\mathbb{R}_+^N)}^{\|\cdot\|_{W_{\alpha,\beta}^{m,p}(\mathbb{R}_+^N)}};$$

which will be characterized in Lemma 2.3 as the subspace of functions with null traces in $W_{\alpha,\beta}^{m,p}(\mathbb{R}_+^N)$. From that, we can introduce the space $W_{-\alpha,-\beta}^{-m,p'}(\mathbb{R}_+^N)$ as the dual space of $\overset{\circ}{W}_{\alpha,\beta}^{m,p}(\mathbb{R}_+^N)$. In addition, we have the following Poincaré inequality on $\overset{\circ}{W}_{\alpha,\beta}^{m,p}(\mathbb{R}_+^N)$ (cf. [4]):

Theorem 2.2. *Under the assumptions of Theorem 2.1, $|\cdot|_{W_{\alpha,\beta}^{m,p}(\mathbb{R}_+^N)}$ is a norm on $\overset{\circ}{W}_{\alpha,\beta}^{m,p}(\mathbb{R}_+^N)$ which is equivalent to the full norm $\|\cdot\|_{W_{\alpha,\beta}^{m,p}(\mathbb{R}_+^N)}$.*

We shall now recall some properties of the weighted Sobolev spaces $W_{\alpha,\beta}^{m,p}(\mathbb{R}_+^N)$.

We have the algebraic and topological imbeddings:

$$W_{\alpha,\beta}^{m,p}(\mathbb{R}_+^N) \hookrightarrow W_{\alpha-1,\beta}^{m-1,p}(\mathbb{R}_+^N) \hookrightarrow \dots \hookrightarrow W_{\alpha-m,\beta}^{0,p}(\mathbb{R}_+^N) \quad \text{if } \frac{N}{p} + \alpha \notin \{1, \dots, m\}.$$

When $\frac{N}{p} + \alpha = j \in \{1, \dots, m\}$, then we have:

$$W_{\alpha,\beta}^{m,p} \hookrightarrow \dots \hookrightarrow W_{\alpha-j+1,\beta}^{m-j+1,p} \hookrightarrow W_{\alpha-j,\beta-1}^{m-j,p} \hookrightarrow \dots \hookrightarrow W_{\alpha-m,\beta-1}^{0,p}.$$

Note that in the first case, for any $\gamma \in \mathbb{R}$ such that $\frac{N}{p} + \alpha - \gamma \notin \{1, \dots, m\}$ and $m \in \mathbb{N}$, the mapping

$$u \in W_{\alpha,\beta}^{m,p}(\mathbb{R}_+^N) \longmapsto \varrho^\gamma u \in W_{\alpha-\gamma,\beta}^{m,p}(\mathbb{R}_+^N)$$

is an isomorphism. In both cases and for any multi-index $\lambda \in \mathbb{N}^N$, the mapping

$$u \in W_{\alpha,\beta}^{m,p}(\mathbb{R}_+^N) \longmapsto \partial^\lambda u \in W_{\alpha,\beta}^{m-|\lambda|,p}(\mathbb{R}_+^N)$$

is continuous. Finally, it can be readily checked that the highest degree q of the polynomials contained in $W_{\alpha,\beta}^{m,p}(\mathbb{R}_+^N)$ is given by

$$q = \begin{cases} m - \left(\frac{N}{p} + \alpha\right) - 1, & \text{if } \begin{cases} \frac{N}{p} + \alpha \in \{1, \dots, m\} \text{ and } (\beta - 1)p \geq -1, \\ \text{or} \\ \frac{N}{p} + \alpha \in \{j \in \mathbb{Z}; j \leq 0\} \text{ and } \beta p \geq -1, \end{cases} \\ \left[m - \left(\frac{N}{p} + \alpha\right) \right], & \text{otherwise.} \end{cases} \quad (2)$$

2.3 The spaces of traces

In order to define the traces of functions of $W_{\alpha,\beta}^{m,p}(\mathbb{R}_+^N)$ (here we don't consider the case $\beta \neq 0$), for any $\sigma \in]0, 1[$, we introduce the space:

$$W_0^{\sigma,p}(\mathbb{R}^N) = \left\{ u \in \mathcal{D}'(\mathbb{R}^N); w^{-\sigma} u \in L^p(\mathbb{R}^N) \text{ and } \forall i = 1, \dots, N, \int_0^{+\infty} t^{-1-\sigma p} dt \int_{\mathbb{R}^N} |u(x + te_i) - u(x)|^p dx < \infty \right\}, \quad (3)$$

where $w = \varrho$ if $N/p \neq \sigma$ and $w = \varrho (\lg \varrho)^{1/\sigma}$ if $N/p = \sigma$, and e_1, \dots, e_N is the canonical basis of \mathbb{R}^N . It is a reflexive Banach space equipped with its natural norm:

$$\|u\|_{W_0^{\sigma,p}(\mathbb{R}^N)} = \left(\left\| \frac{u}{w^\sigma} \right\|_{L^p(\mathbb{R}^N)}^p + \sum_{i=1}^N \int_0^{+\infty} t^{-1-\sigma p} dt \int_{\mathbb{R}^N} |u(x + te_i) - u(x)|^p dx \right)^{1/p}$$

which is equivalent to the norm

$$\left(\left\| \frac{u}{w^\sigma} \right\|_{L^p(\mathbb{R}^N)}^p + \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+\sigma p}} dx dy \right)^{1/p}.$$

Similarly, for any real number $\alpha \in \mathbb{R}$, we define the space:

$$W_\alpha^{\sigma,p}(\mathbb{R}^N) = \left\{ u \in \mathcal{D}'(\mathbb{R}^N); w^{\alpha-\sigma} u \in L^p(\mathbb{R}^N), \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\varrho^\alpha(x) u(x) - \varrho^\alpha(y) u(y)|^p}{|x - y|^{N+\sigma p}} dx dy < \infty \right\},$$

where $w = \varrho$ if $N/p + \alpha \neq \sigma$ and $w = \varrho (\lg \varrho)^{1/(\sigma-\alpha)}$ if $N/p + \alpha = \sigma$. For any $s \in \mathbb{R}^+$, we set

$$W_\alpha^{s,p}(\mathbb{R}^N) = \left\{ u \in \mathcal{D}'(\mathbb{R}^N); 0 \leq |\lambda| \leq k, \varrho^{\alpha-s+|\lambda|} (\lg \varrho)^{-1} \partial^\lambda u \in L^p(\mathbb{R}^N); k+1 \leq |\lambda| \leq [s]-1, \varrho^{\alpha-s+|\lambda|} \partial^\lambda u \in L^p(\mathbb{R}^N); |\lambda| = [s], \partial^\lambda u \in W_\alpha^{\sigma,p}(\mathbb{R}^N) \right\},$$

where $k = s - N/p - \alpha$ if $N/p + \alpha \in \{\sigma, \dots, \sigma + [s]\}$, with $\sigma = s - [s]$ and $k = -1$ otherwise. It is a reflexive Banach space equipped with the norm:

$$\|u\|_{W_\alpha^{s,p}(\mathbb{R}^N)} = \left(\sum_{0 \leq |\lambda| \leq k} \|\varrho^{\alpha-s+|\lambda|} (\lg \varrho)^{-1} \partial^\lambda u\|_{L^p(\mathbb{R}^N)}^p + \sum_{k+1 \leq |\lambda| \leq [s]-1} \|\varrho^{\alpha-s+|\lambda|} \partial^\lambda u\|_{L^p(\mathbb{R}^N)}^p \right)^{1/p} + \sum_{|\lambda|=[s]} \|\partial^\lambda u\|_{W_\alpha^{\sigma,p}(\mathbb{R}^N)}.$$

We can similarly define, for any real number β , the space:

$$W_{\alpha,\beta}^{s,p}(\mathbb{R}^N) = \left\{ v \in \mathcal{D}'(\mathbb{R}^N); (\lg \varrho)^\beta v \in W_\alpha^{s,p}(\mathbb{R}^N) \right\}.$$

We can prove some properties of the weighted Sobolev spaces $W_{\alpha,\beta}^{s,p}(\mathbb{R}^N)$. We have the algebraic and topological imbeddings in the case where $N/p + \alpha \notin \{\sigma, \dots, \sigma + [s] - 1\}$:

$$W_{\alpha,\beta}^{s,p}(\mathbb{R}^N) \hookrightarrow W_{\alpha-1,\beta}^{s-1,p}(\mathbb{R}^N) \hookrightarrow \dots \hookrightarrow W_{\alpha-[s],\beta}^{\sigma,p}(\mathbb{R}^N), \\ W_{\alpha,\beta}^{s,p}(\mathbb{R}^N) \hookrightarrow W_{\alpha+[s]-s,\beta}^{[s],p}(\mathbb{R}^N) \hookrightarrow \dots \hookrightarrow W_{\alpha-s,\beta}^{0,p}(\mathbb{R}^N).$$

When $N/p + \alpha = j \in \{\sigma, \dots, \sigma + [s] - 1\}$, then we have:

$$W_{\alpha,\beta}^{s,p} \hookrightarrow \dots \hookrightarrow W_{\alpha-j+1,\beta}^{s-j+1,p} \hookrightarrow W_{\alpha-j,\beta-1}^{s-j,p} \hookrightarrow \dots \hookrightarrow W_{\alpha-[s],\beta-1}^{\sigma,p}, \\ W_{\alpha,\beta}^{s,p} \hookrightarrow W_{\alpha+[s]-s,\beta}^{[s],p} \hookrightarrow \dots \hookrightarrow W_{\alpha-\sigma-j+1,\beta}^{[s]-j+1,p} \hookrightarrow W_{\alpha-\sigma-j,\beta-1}^{[s]-j,p} \hookrightarrow \dots \hookrightarrow W_{\alpha-s,\beta-1}^{0,p}.$$

If u is a function on \mathbb{R}_+^N , we denote its trace of order j on the hyperplane Γ by:

$$\forall j \in \mathbb{N}, \quad \gamma_j u : x' \in \mathbb{R}^{N-1} \longmapsto \partial_N^j u(x', 0).$$

Let's recall the following trace lemma due to Hanouzet (*cf.* [10]) and extended by Amrouche-Nečasová (*cf.* [4]) to this class of weighted Sobolev spaces:

Lemma 2.3. *For any integer $m \geq 1$ and real number α , the mapping*

$$\gamma = (\gamma_0, \gamma_1, \dots, \gamma_{m-1}) : \mathcal{D}(\overline{\mathbb{R}_+^N}) \longrightarrow \prod_{j=0}^{m-1} \mathcal{D}(\mathbb{R}^{N-1}),$$

can be extended to a linear continuous mapping, still denoted by γ ,

$$\gamma : W_\alpha^{m,p}(\mathbb{R}_+^N) \longrightarrow \prod_{j=0}^{m-1} W_\alpha^{m-j-1/p,p}(\mathbb{R}^{N-1}).$$

Moreover γ is surjective and $\text{Ker}\gamma = \mathring{W}_\alpha^{m,p}(\mathbb{R}_+^N)$.

2.4 Motivation

Problem (\mathcal{P}) has been investigated by Boulmezaoud (*cf.* [7]) in weighted Sobolev spaces in L^2 theory for $N \geq 3$ and without the critical cases corresponding to logarithmic factors. The aim of this work is to give results in L^p theory, with $1 < p < \infty$, to reduce critical values and especially to reach weaker solutions from more singular data.

Let's throw light on this functional framework in the L^2 case. If we consider Problem (\mathcal{P}) with homogeneous boundary conditions, *i.e.* $g_0 = g_1 = 0$, we can give the following variational formulation: For any given $f \in V'$, find $u \in V$ such that

$$\forall v \in V, \quad \int_{\mathbb{R}_+^N} \Delta u \Delta v \, dx = \langle f, v \rangle_{V' \times V}.$$

Which is the appropriate space V to use the Lax-Milgram's lemma? We must have firstly, for any $v \in V$, $\Delta v \in L^2(\mathbb{R}_+^N)$ and secondly, the coercivity condition for the bilinear form: $(u, v) \longmapsto \int_{\mathbb{R}_+^N} \Delta u \Delta v \, dx$.

By Theorem 2.2, we have:

$$\forall v \in \mathring{W}_0^{2,2}(\mathbb{R}_+^N), \quad \|v\|_{W_0^{2,2}(\mathbb{R}_+^N)} \leq C \|\nabla^2 v\|_{L^2(\mathbb{R}_+^N)^{N^2}}.$$

Moreover,

$$\forall v \in \mathring{W}_0^{2,2}(\mathbb{R}_+^N), \quad \|\nabla^2 v\|_{L^2(\mathbb{R}_+^N)^{N^2}} = \|\Delta v\|_{L^2(\mathbb{R}_+^N)},$$

hence the coercivity of the form. Consequently, Problem (\mathcal{P}) with $g_0 = g_1 = 0$ is well-posed on $V = \overset{\circ}{W}_0^{2,2}(\mathbb{R}_+^N)$. Which are the appropriate spaces of traces for the complete problem? Thanks to Lemma 2.3,

$$u \in W_0^{2,2}(\mathbb{R}_+^N) \Rightarrow (\gamma_0 u, \gamma_1 u) \in W_0^{3/2,2}(\mathbb{R}^{N-1}) \times W_0^{1/2,2}(\mathbb{R}^{N-1}),$$

consequently we must take $(g_0, g_1) \in W_0^{3/2,2}(\mathbb{R}^{N-1}) \times W_0^{1/2,2}(\mathbb{R}^{N-1})$ in the problem with nonhomogeneous boundary conditions.

Remark 2.4. If we consider the problem for the operator $I + \Delta^2$:

$$(\mathcal{Q}) \quad \begin{cases} u + \Delta^2 u = f & \text{in } \mathbb{R}_+^N, \\ u = g_0 & \text{on } \Gamma, \\ \partial_N u = g_1 & \text{on } \Gamma, \end{cases}$$

we have the following variational formulation with $g_0 = g_1 = 0$: For any given $f \in V'$, find $u \in V$ such that $\forall v \in V$,

$$\int_{\mathbb{R}_+^N} u v \, dx + \int_{\mathbb{R}_+^N} \Delta u \Delta v \, dx = \langle f, v \rangle_{V' \times V}.$$

This form satisfies naturally the coercivity condition on $V = H_0^2(\mathbb{R}_+^N)$, where $H_0^2(\mathbb{R}_+^N)$ denotes here the classical Sobolev space of functions $v \in H^2(\mathbb{R}_+^N)$ such that $v = \partial_N v = 0$ on Γ . For the nonhomogeneous problem, we must take $(g_0, g_1) \in H^{3/2}(\mathbb{R}^{N-1}) \times H^{1/2}(\mathbb{R}^{N-1})$. \square

2.5 The Laplace equation in \mathbb{R}_+^N

We shall now recall the fundamental results of the Laplace equation in the half-space, with nonhomogeneous Dirichlet or Neumann boundary conditions. These results have been proved by Boulmezaoud (*cf.* [6]) in the particular case where $p = 2$ for $N \geq 3$, then generalized by Amrouche-Nečasová (*cf.* [4]) and Amrouche (*cf.* [5]) in L^p theory for $N \geq 2$, with solutions of some critical cases by means of logarithmic factors in the weight. Let's also quote the partial results of Maz'ya-Plamenevskii-Stupyalis (*cf.* [9]) for the Stokes system in \mathbb{R}_+^3 with the velocity obtained in $W_0^{1,2}(\mathbb{R}_+^3)$ or $W_1^{2,2}(\mathbb{R}_+^3)$, and those of Tanaka (*cf.* [12]) for the same problem and the velocity vector field in $W_0^{m+2,2}(\mathbb{R}_+^3)$ with $m > 0$.

Let's first recall the main result of the Dirichlet problem

$$(\mathcal{P}_D) \quad \begin{cases} \Delta u = f & \text{in } \mathbb{R}_+^N, \\ u = g & \text{on } \Gamma, \end{cases}$$

with a different behaviour at infinity according to l .

Theorem 2.5 (Amrouche-Nečasová). *Let $l \in \mathbb{Z}$ such that*

$$\frac{N}{p'} \notin \{1, \dots, l\} \quad \text{and} \quad \frac{N}{p} \notin \{1, \dots, -l\}. \quad (4)$$

For any $f \in W_l^{-1,p}(\mathbb{R}_+^N)$ and $g \in W_l^{-1/p,p}(\Gamma)$ satisfying the compatibility condition

$$\forall \varphi \in \mathcal{A}_{[1+l-N/p]}^\Delta, \quad \langle f, \varphi \rangle_{W_l^{-1,p}(\mathbb{R}_+^N) \times \mathring{W}_{-l}^{1,p'}(\mathbb{R}_+^N)} = \langle g, \partial_N \varphi \rangle_\Gamma, \quad (5)$$

where $\langle \cdot, \cdot \rangle_\Gamma$ denotes the duality between $W_l^{1/p',p}(\Gamma)$ and $W_{-l}^{-1/p',p'}(\Gamma)$, problem (\mathcal{P}_D) has a solution $u \in W_l^{1,p}(\mathbb{R}_+^N)$, unique up to an element of $\mathcal{A}_{[1-l-N/p]}^\Delta$.

The second recall deals with the problem with more regular data.

Theorem 2.6 (Amrouche-Nečasová). *Let $l \in \mathbb{Z}$ and $m \geq 1$ be two integers such that*

$$\frac{N}{p'} \notin \{1, \dots, l+1\} \quad \text{and} \quad \frac{N}{p} \notin \{1, \dots, -l-m\}. \quad (6)$$

For any $f \in W_{m+l}^{m-1,p}(\mathbb{R}_+^N)$ and $g \in W_{m+l}^{m+1-1/p,p}(\Gamma)$, satisfying the compatibility condition (5), problem (\mathcal{P}_D) has a solution $u \in W_{m+l}^{m+1,p}(\mathbb{R}_+^N)$, unique up to an element of $\mathcal{A}_{[1-l-N/p]}^\Delta$.

Concerning the Neumann problem

$$(\mathcal{P}_N) \quad \begin{cases} \Delta u = f & \text{in } \mathbb{R}_+^N, \\ \partial_N u = g & \text{on } \Gamma, \end{cases}$$

let's first recall the existence and unicity result with the weakest hypotheses.

Theorem 2.7 (Amrouche). *Let $l \in \mathbb{Z}$ such that*

$$\frac{N}{p'} \notin \{1, \dots, l\} \quad \text{and} \quad \frac{N}{p} \notin \{1, \dots, -l+1\}. \quad (7)$$

For any $f \in W_l^{0,p}(\mathbb{R}_+^N)$ and $g \in W_{l-1}^{-1/p,p}(\Gamma)$ satisfying the compatibility condition

$$\forall \varphi \in \mathcal{N}_{[l-N/p]}^\Delta, \quad \langle f, \varphi \rangle_{W_l^{0,p}(\mathbb{R}_+^N) \times W_{-l}^{0,p'}(\mathbb{R}_+^N)} + \langle g, \varphi \rangle_\Gamma = 0, \quad (8)$$

where $\langle \cdot, \cdot \rangle_\Gamma$ denotes the duality between $W_{l-1}^{-1/p,p}(\Gamma)$ and $W_{-l+1}^{1-1/p',p'}(\Gamma)$, problem (\mathcal{P}_N) has a solution $u \in W_{l-1}^{1,p}(\mathbb{R}_+^N)$, unique up to an element of $\mathcal{N}_{[2-l-N/p]}^\Delta$.

As for the Dirichlet problem, we can prove the following result:

Theorem 2.8. Let $l \in \mathbb{Z}$ and $m \geq 0$ be two integers such that

$$\frac{N}{p'} \notin \{1, \dots, l\} \quad \text{and} \quad \frac{N}{p} \notin \{1, \dots, -l - m\}. \quad (9)$$

For any $f \in W_{m+l}^{m,p}(\mathbb{R}_+^N)$ and $g \in W_{m+l}^{m+1-1/p,p}(\Gamma)$ satisfying the compatibility condition (8), problem (\mathcal{P}_N) has a solution $u \in W_{m+l}^{m+2,p}(\mathbb{R}_+^N)$, unique up to an element of $\mathcal{N}_{[2-l-N/p]}^\Delta$.

Remark 2.9. Note that for these four theorems, the solutions continuously depend on the data with respect to the quotient norm. \square

3 Biharmonic operator in \mathbb{R}^N

In this section, we shall give some isomorphism results relative to the biharmonic operator in the whole space. We shall rest on these for our investigation in the half-space. At first, we characterize the kernel

$$K = \left\{ v \in W_l^{2,p}(\mathbb{R}^N); \Delta^2 v = 0 \text{ in } \mathbb{R}^N \right\}.$$

Lemma 3.1. Let $l \in \mathbb{Z}$.

- i) If $\frac{N}{p} \notin \{1, \dots, -l\}$, then $K = \mathcal{P}_{[2-l-N/p]}^{\Delta^2}$.
- ii) If $\frac{N}{p} \in \{1, \dots, -l\}$, then $K = \mathcal{P}_{1-l-N/p}^{\Delta^2}$.

Proof. Let $u \in K$. As we know that $\Delta^2 u = 0$ and moreover $u \in W_l^{2,p}(\mathbb{R}^N) \subset \mathcal{S}'(\mathbb{R}^N)$, the space of tempered distributions, we can deduce that u is a polynomial on \mathbb{R}^N . But according to (2), we know that the highest degree q of the polynomials contained in $W_l^{2,p}(\mathbb{R}^N)$ is given by:

$$q = \begin{cases} 1 - l - N/p & \text{if } \frac{N}{p} + l \in \{j \in \mathbb{Z}; j \leq 0\}, \\ [2 - l - N/p] & \text{otherwise.} \end{cases}$$

We can thus see the conditions of the statement appear precisely. \square

More generally, for any integer $m \in \mathbb{N}$, we define the kernel

$$K^m = \left\{ v \in W_{m+l}^{m+2,p}(\mathbb{R}^N); \Delta^2 v = 0 \text{ in } \mathbb{R}^N \right\}.$$

The same arguments lead us to a result which includes the precedent, corresponding then to case $m = 0$.

Lemma 3.2. *Let $l \in \mathbb{Z}$ and $m \in \mathbb{N}$ such that*

- i) $\frac{N}{p} \notin \{1, \dots, -l - m\}$, then $K^m = \mathcal{P}_{[2-l-N/p]}^{\Delta^2}$.
- ii) $\frac{N}{p} \in \{1, \dots, -l - m\}$, then $K^m = \mathcal{P}_{1-l-N/p}^{\Delta^2}$.

We can now formulate the first result of isomorphism in \mathbb{R}^N :

Theorem 3.3. *Let $l \in \mathbb{Z}$. Under hypothesis (4), the following operator is an isomorphism:*

$$\Delta^2 : W_l^{2,p}(\mathbb{R}^N) / \mathcal{P}_{[2-l-N/p]}^{\Delta^2} \longrightarrow W_l^{-2,p}(\mathbb{R}^N) \perp \mathcal{P}_{[2+l-N/p]}^{\Delta^2}.$$

Proof. Let us recall (cf. [2]) that under assumption (4), the operator

$$\Delta : W_l^{2,p}(\mathbb{R}^N) / \mathcal{P}_{[2-l-N/p]}^{\Delta} \longrightarrow W_l^{0,p}(\mathbb{R}^N) \perp \mathcal{P}_{[l-N/p]}^{\Delta} \quad (10)$$

is an isomorphism. By duality, we can deduce that it is the same for the operator

$$\Delta : W_l^{0,p}(\mathbb{R}^N) / \mathcal{P}_{[l-N/p]}^{\Delta} \longrightarrow W_l^{-2,p}(\mathbb{R}^N) \perp \mathcal{P}_{[2-l-N/p]}^{\Delta}. \quad (11)$$

If we suppose now that $l - N/p' < 0$, we can compose isomorphisms (10) and (11) to deduce that the operator

$$\Delta^2 : W_l^{2,p}(\mathbb{R}^N) / \mathcal{P}_{[2-l-N/p]}^{\Delta^2} \longrightarrow W_l^{-2,p}(\mathbb{R}^N) \perp \mathcal{P}_{[2+l-N/p]}^{\Delta} \quad (12)$$

is an isomorphism. By duality, we can deduce that the operator

$$\Delta^2 : W_l^{2,p}(\mathbb{R}^N) / \mathcal{P}_{[2-l-N/p]}^{\Delta} \longrightarrow W_l^{-2,p}(\mathbb{R}^N) \perp \mathcal{P}_{[2+l-N/p]}^{\Delta^2} \quad (13)$$

is an isomorphism provided that we have $-l - N/p < 0$.

To combine (12) and (13), it remains to be noted that if $l - N/p' < 0$, then we have $\mathcal{P}_{[2+l-N/p]}^{\Delta^2} = \mathcal{P}_{[2+l-N/p]}^{\Delta} = \mathcal{P}_{[2+l-N/p]}$; and symmetrically, if $-l - N/p < 0$, we have $\mathcal{P}_{[2-l-N/p]}^{\Delta^2} = \mathcal{P}_{[2-l-N/p]}^{\Delta} = \mathcal{P}_{[2-l-N/p]}$. Moreover, if we note that the reunion of those two cases covers all integers $l \in \mathbb{Z}$, we can deduce that for any $l \in \mathbb{Z}$ satisfying (4), the operator

$$\Delta^2 : W_l^{2,p}(\mathbb{R}^N) / \mathcal{P}_{[2-l-N/p]}^{\Delta^2} \longrightarrow W_l^{-2,p}(\mathbb{R}^N) \perp \mathcal{P}_{[2+l-N/p]}^{\Delta^2} \quad (14)$$

is an isomorphism. □

We can establish now a result for more regular data, with two preliminary lemmas.

Lemma 3.4. *Let $m \geq 1$ and $l \leq -2$ be two integers such that*

$$\frac{N}{p} \notin \{1, \dots, -l - m\}, \quad (15)$$

then the following operator is an isomorphism:

$$\Delta^2 : W_{m+l}^{m+2,p}(\mathbb{R}^N) / \mathcal{P}_{[2-l-N/p]}^{\Delta^2} \longrightarrow W_{m+l}^{m-2,p}(\mathbb{R}^N).$$

Proof. We use here another isomorphism result (cf. [3]). Let $m \geq 1$ and $l \leq -1$ be two integers. Under hypothesis (15), the Laplace operator

$$\Delta : W_{m+l}^{m+1,p}(\mathbb{R}^N) / \mathcal{P}_{[1-l-N/p]}^{\Delta} \longrightarrow W_{m+l}^{m-1,p}(\mathbb{R}^N), \quad (16)$$

is an isomorphism. Then, replacing m by $m - 1$ and l by $l + 1$, we can obtain that for $m \geq 2$ and $l \leq -2$, under hypothesis (15), the operator

$$\Delta : W_{m+l}^{m,p}(\mathbb{R}^N) / \mathcal{P}_{[-l-N/p]}^{\Delta} \longrightarrow W_{m+l}^{m-2,p}(\mathbb{R}^N), \quad (17)$$

is an isomorphism. Moreover (cf. [2]), for $l \leq -2$, the operator

$$\Delta : W_{1+l}^{1,p}(\mathbb{R}^N) / \mathcal{P}_{[-l-N/p]}^{\Delta} \longrightarrow W_{1+l}^{-1,p}(\mathbb{R}^N) \\ \text{if } N/p \notin \{1, \dots, -l - 1\}, \quad (18)$$

is an isomorphism. Then, combining (17) and (18), we can deduce that for $m \geq 1$ and $l \leq -2$, under hypothesis (15), the operator

$$\Delta : W_{m+l}^{m,p}(\mathbb{R}^N) / \mathcal{P}_{[-l-N/p]}^{\Delta} \longrightarrow W_{m+l}^{m-2,p}(\mathbb{R}^N), \quad (19)$$

is an isomorphism. Replacing now m by $m + 1$ and l by $l - 1$ in (16), we obtain that for $m \geq 0$ and $l \leq 0$, under hypothesis (15), the operator

$$\Delta : W_{m+l}^{m+2,p}(\mathbb{R}^N) / \mathcal{P}_{[2-l-N/p]}^{\Delta} \longrightarrow W_{m+l}^{m,p}(\mathbb{R}^N), \quad (20)$$

is an isomorphism. The lemma follows from the composition of isomorphisms (19) and (20). \square

Lemma 3.5. *Let $m \geq 1$ an integer such that*

$$\frac{N}{p'} \neq 1 \quad \text{or} \quad m = 1,$$

then the biharmonic operator

$$\Delta^2 : W_{m-1}^{m+2,p}(\mathbb{R}^N) / \mathcal{P}_{[3-N/p]}^{\Delta^2} \longrightarrow W_{m-1}^{m-2,p}(\mathbb{R}^N) \perp \mathcal{P}_{[1-N/p']}$$

is an isomorphism.

Proof. Let's note that it suffices to prove that the operator is surjective. Here again, we compose two Laplace operators. We have the following isomorphism (cf. [2]): for $m \in \mathbb{N}$,

$$\Delta : W_m^{1+m,p}(\mathbb{R}^N) / \mathcal{P}_{[1-N/p]}^\Delta \longrightarrow W_m^{-1+m,p}(\mathbb{R}^N) \perp \mathcal{P}_{[1-N/p']} \quad (21)$$

if $\frac{N}{p'} \neq 1$ or $m = 0$.

Replacing m by $m - 1$, we obtain that for $m \geq 1$, the operator

$$\Delta : W_{m-1}^{m,p}(\mathbb{R}^N) / \mathcal{P}_{[1-N/p]}^\Delta \longrightarrow W_{m-1}^{m-2,p}(\mathbb{R}^N) \perp \mathcal{P}_{[1-N/p']} \quad (22)$$

if $\frac{N}{p'} \neq 1$ or $m = 1$,

is an isomorphism. Composing with (20), for $l = -1$, we obtain the result. \square

We can now give a global result for the biharmonic operator.

Theorem 3.6. *i) Let $l \in \mathbb{Z}$ such that*

$$\frac{N}{p'} \notin \{1, \dots, l+1\} \quad \text{and} \quad \frac{N}{p} \notin \{1, \dots, -l-1\},$$

then the biharmonic operator

$$\Delta^2 : W_{l+1}^{3,p}(\mathbb{R}^N) / \mathcal{P}_{[2-l-N/p]}^{\Delta^2} \longrightarrow W_{l+1}^{-1,p}(\mathbb{R}^N) \perp \mathcal{P}_{[2+l-N/p']}^{\Delta^2}$$

is an isomorphism.

ii) Let $l \in \mathbb{Z}$ and $m \geq 2$ be two integers such that

$$\frac{N}{p'} \notin \{1, \dots, l+2\} \quad \text{and} \quad \frac{N}{p} \notin \{1, \dots, -l-m\},$$

then the biharmonic operator

$$\Delta^2 : W_{m+l}^{m+2,p}(\mathbb{R}^N) / \mathcal{P}_{[2-l-N/p]}^{\Delta^2} \longrightarrow W_{m+l}^{m-2,p}(\mathbb{R}^N) \perp \mathcal{P}_{[2+l-N/p']}^{\Delta^2}$$

is an isomorphism.

Proof. For $l \leq -1$, it's clear that lemmas 3.4 and 3.5 exactly cover points *i)* and *ii)*. It remains to establish the theorem for $l \geq 0$.

According to [2], for $l \geq 0$, the following operator is an isomorphism:

$$\Delta : W_{l+1}^{1,p}(\mathbb{R}^N) \longrightarrow W_{l+1}^{-1,p}(\mathbb{R}^N) \perp \mathcal{P}_{[2+l-N/p']}^\Delta \quad (23)$$

if $N/p' \notin \{1, \dots, l+1\}$.

For $m \geq 1$ and $l \geq 1$, we also have the isomorphism:

$$\Delta : W_{m+l}^{m+1,p}(\mathbb{R}^N) \longrightarrow W_{m+l}^{m-1,p}(\mathbb{R}^N) \perp \mathcal{P}_{[1+l-N/p']}^\Delta \quad (24)$$

if $N/p' \notin \{1, \dots, l+1\}$.

Replacing m by $m-1$ and l by $l+1$, we deduce for $m \geq 2$ and $l \geq 0$, the isomorphism:

$$\Delta : W_{m+l}^{m,p}(\mathbb{R}^N) \longrightarrow W_{m+l}^{m-2,p}(\mathbb{R}^N) \perp \mathcal{P}_{[2+l-N/p']}^\Delta \quad (25)$$

if $N/p' \notin \{1, \dots, l+2\}$.

Replacing m by $m+1$ and l by $l-1$ in (24), we obtain for $m \geq 1$ and $l \geq 2$, the isomorphism:

$$\Delta : W_{m+l}^{m+2,p}(\mathbb{R}^N) \longrightarrow W_{m+l}^{m,p}(\mathbb{R}^N) \perp \mathcal{P}_{[l-N/p']}^\Delta \quad (26)$$

if $N/p' \notin \{1, \dots, l\}$.

And now replacing m by $m+1$ in (21), we obtain for $m \geq 1$, the isomorphism:

$$\Delta : W_{m+1}^{m+2,p}(\mathbb{R}^N) / \mathcal{P}_{[1-N/p]}^\Delta \longrightarrow W_{m+1}^{m,p}(\mathbb{R}^N) \perp \mathcal{P}_{[1-N/p']} \quad (27)$$

if $N/p' \neq 1$.

Finally, if we return to (16) with $l = -1$ and $m+1$ instead of m , we have for $m \geq 1$, the isomorphism:

$$\Delta : W_m^{m+2,p}(\mathbb{R}^N) / \mathcal{P}_{[2-N/p]} \longrightarrow W_m^{m,p}(\mathbb{R}^N). \quad (28)$$

Then, combining (26), (27) and (28), we obtain for $m \geq 1$ and $l \geq 0$, the isomorphism:

$$\Delta : W_{m+l}^{m+2,p}(\mathbb{R}^N) / \mathcal{P}_{[2+l-N/p]} \longrightarrow W_{m+l}^{m,p}(\mathbb{R}^N) \perp \mathcal{P}_{[l-N/p']}^\Delta \quad (29)$$

if $N/p' \notin \{1, \dots, l\}$.

It remains to justify orthogonality conditions to compose (29) with (23) and (25), which will give us respectively the isomorphisms of points *i*) and *ii*).

Let $f \in W_{m+l}^{m-2,p}(\mathbb{R}^N) \perp \mathcal{P}_{[2+l-N/p']}^{\Delta^2}$ with $m \geq 1$, then we have $f \perp \mathcal{P}_{[2+l-N/p']}^\Delta$ and according to (23) or (25), there exists $u \in W_{m+l}^{m,p}(\mathbb{R}^N)$ such that $\Delta u = f$. We will show that $u \perp \mathcal{P}_{[l-N/p']}^\Delta$. Let $\psi \in \mathcal{P}_{[l-N/p']}^\Delta$, we know that there exists $\varphi \in \mathcal{P}_{[2+l-N/p']}^{\Delta^2}$ such that $\psi = \Delta\varphi$, *i.e.* $\varphi \in \mathcal{P}_{[2+l-N/p']}^{\Delta^2}$.

a) Case $m = 1$: $u \in W_{l+1}^{1,p}(\mathbb{R}^N)$, $f \in W_{l+1}^{-1,p}(\mathbb{R}^N) \perp \mathcal{P}_{[2+l-N/p']}^{\Delta^2}$.

Let us note that $\psi \in W_{-l}^{0,p'}(\mathbb{R}^N)$ and $\varphi \in W_{-l}^{2,p'}(\mathbb{R}^N)$, since $\frac{N}{p'} \notin \{1, \dots, l\}$. We

also have the imbedding $W_{-l}^{2,p'}(\mathbb{R}^N) \hookrightarrow W_{-l-1}^{1,p'}(\mathbb{R}^N)$, since $\frac{N}{p'} \neq l+1$. Then,

we have $\psi = \Delta\varphi \in W_{-l-1}^{-1,p'}(\mathbb{R}^N)$. This implies

$$\begin{aligned}
\langle u, \psi \rangle_{W_{l+1}^{1,p}(\mathbb{R}^N) \times W_{-l-1}^{-1,p'}(\mathbb{R}^N)} &= \langle u, \Delta \varphi \rangle_{W_{l+1}^{1,p}(\mathbb{R}^N) \times W_{-l-1}^{-1,p'}(\mathbb{R}^N)} \\
&= \langle \Delta u, \varphi \rangle_{W_{l+1}^{-1,p}(\mathbb{R}^N) \times W_{-l-1}^{1,p'}(\mathbb{R}^N)} \\
&= \langle f, \varphi \rangle_{W_{l+1}^{-1,p}(\mathbb{R}^N) \times W_{-l-1}^{1,p'}(\mathbb{R}^N)} \\
&= 0.
\end{aligned}$$

b) Case $m \geq 2$: $u \in W_{m+l}^{m,p}(\mathbb{R}^N)$, $f \in W_{m+l}^{m-2,p}(\mathbb{R}^N) \perp \mathcal{P}_{[2+l-N/p']}^{\Delta^2}$.

Since $l \geq 0$, we have $\frac{N}{p} + m + l \notin \{1, \dots, m\}$, therefore we can deduce the chain of imbeddings $W_{m+l}^{m,p}(\mathbb{R}^N) \hookrightarrow \dots \hookrightarrow W_{l+1}^{1,p}(\mathbb{R}^N)$. Moreover $\frac{N}{p'} \neq l + 2$, then we also have $W_{m+l}^{m-2,p}(\mathbb{R}^N) \hookrightarrow \dots \hookrightarrow W_{l+2}^{0,p}(\mathbb{R}^N) \hookrightarrow W_{l+1}^{-1,p}(\mathbb{R}^N)$. After that, we repeat the reasoning of case $m = 1$.

Then, we have $u \in W_{m+l}^{m,p}(\mathbb{R}^N) \perp \mathcal{P}_{[l-N/p']}^{\Delta}$, and (29) shows us that there exists $z \in W_m^{m+2,p}(\mathbb{R}^N)$ such that $\Delta z = u$. Thus it follows that the operator

$$\Delta^2 : W_{m+l}^{m+2,p}(\mathbb{R}^N) / \mathcal{P}_{[2+l-N/p]}^{\Delta^2} \longrightarrow W_{m+l}^{m-2,p}(\mathbb{R}^N) \perp \mathcal{P}_{[2+l-N/p']}^{\Delta^2}$$

is an isomorphism. □

4 Generalized solutions of Δ^2 in \mathbb{R}_+^N

In this section, we shall deal with Problem (\mathcal{P}) in the half-space.

For any $q \in \mathbb{Z}$, we introduce the space \mathcal{B}_q as a subspace of $\mathcal{P}_q^{\Delta^2}$:

$$\mathcal{B}_q = \left\{ u \in \mathcal{P}_q^{\Delta^2}; u = \partial_N u = 0 \text{ on } \Gamma \right\}.$$

We shall establish the main theorem:

Theorem 4.1. *Let $l \in \mathbb{Z}$ such that*

$$\frac{N}{p'} \notin \{1, \dots, l\} \quad \text{and} \quad \frac{N}{p} \notin \{1, \dots, -l\}.$$

For any $f \in W_l^{-2,p}(\mathbb{R}_+^N)$, $g_0 \in W_l^{2-1/p,p}(\Gamma)$ and $g_1 \in W_l^{1-1/p,p}(\Gamma)$ satisfying the compatibility condition

$$\begin{aligned}
\forall \varphi \in \mathcal{B}_{[2+l-N/p]}, \\
\langle f, \varphi \rangle_{W_l^{-2,p}(\mathbb{R}_+^N) \times \overset{\circ}{W}_{-l}^{2,p'}(\mathbb{R}_+^N)} + \langle g_1, \Delta \varphi \rangle_{\Gamma} - \langle g_0, \partial_N \Delta \varphi \rangle_{\Gamma} = 0,
\end{aligned} \tag{30}$$

problem (\mathcal{P}) admits a solution $u \in W_l^{2,p}(\mathbb{R}_+^N)$, unique up to an element of

$\mathcal{B}_{[2-l-N/p]}$, and there exists a constant C such that

$$\inf_{q \in \mathcal{B}_{[2-l-N/p]}} \|u + q\|_{W_l^{2,p}(\mathbb{R}_+^N)} \leq C \left(\|f\|_{W_l^{-2,p}(\mathbb{R}_+^N)} + \|g_0\|_{W_l^{2-1/p,p}(\Gamma)} + \|g_1\|_{W_l^{1-1/p,p}(\Gamma)} \right).$$

NB: $\langle g_1, \Delta\varphi \rangle_\Gamma$ denotes the duality bracket $\langle g_1, \Delta\varphi \rangle_{W_l^{1-1/p,p}(\Gamma) \times W_l^{-1/p',p'}(\Gamma)}$, and $\langle g_0, \partial_N \Delta\varphi \rangle_\Gamma$ the duality bracket $\langle g_0, \partial_N \Delta\varphi \rangle_{W_l^{2-1/p,p}(\Gamma) \times W_l^{-1-1/p',p'}(\Gamma)}$.

4.1 Characterization of the kernel

Let us denote by \mathcal{K} the kernel of the operator

$$(\Delta^2, \gamma_0, \gamma_1) : W_l^{2,p}(\mathbb{R}_+^N) \longrightarrow W_l^{-2,p}(\mathbb{R}_+^N) \times W_l^{2-1/p,p}(\Gamma) \times W_l^{1-1/p,p}(\Gamma),$$

i.e.

$$\mathcal{K} = \left\{ u \in W_l^{2,p}(\mathbb{R}_+^N); \Delta^2 u = 0 \text{ in } \mathbb{R}_+^N, u = \partial_N u = 0 \text{ on } \Gamma \right\}.$$

The following characterization uses the reflection principle (*cf.* Farwig [8]).

Lemma 4.2. *Let $l \in \mathbb{Z}$.*

- i)* If $\frac{N}{p} \notin \{1, \dots, -l\}$, then $\mathcal{K} = \mathcal{B}_{[2-l-N/p]}$.
- ii)* If $\frac{N}{p} \in \{1, \dots, -l\}$, then $\mathcal{K} = \mathcal{B}_{1-l-N/p}$.

Proof. Given $u \in \mathcal{K}$, we set

$$\tilde{u}(x', x_N) = \begin{cases} u(x', x_N) & \text{if } x_N \geq 0, \\ (-u - 2x_N \partial_N u - x_N^2 \Delta u)(x', -x_N) & \text{if } x_N < 0. \end{cases}$$

Then we have $\tilde{u} \in \mathcal{S}'(\mathbb{R}^N)$ and we show that $\Delta^2 \tilde{u} = 0$ in \mathbb{R}^N . We can deduce that \tilde{u} , and consequently u , is a polynomial. Furthermore, $u \in W_l^{2,p}(\mathbb{R}_+^N)$ implies that its maximum degree is the same as in Lemma 3.1. \square

More generally, for any $m \in \mathbb{N}$, we denote by \mathcal{K}^m the kernel of the operator

$$(\Delta^2, \gamma_0, \gamma_1) : W_{m+l}^{m+2,p}(\mathbb{R}_+^N) \longrightarrow W_{m+l}^{m-2,p}(\mathbb{R}_+^N) \times W_{m+l}^{m+2-1/p,p}(\Gamma) \times W_{m+l}^{m+1-1/p,p}(\Gamma),$$

i.e.

$$\mathcal{K}^m = \left\{ u \in W_{m+l}^{m+2,p}(\mathbb{R}_+^N); \Delta^2 u = 0 \text{ in } \mathbb{R}_+^N, u = \partial_N u = 0 \text{ on } \Gamma \right\}.$$

Identical arguments lead us to the following result:

Lemma 4.3. *Let $l \in \mathbb{Z}$ and $m \in \mathbb{N}$.*

- i) *If $\frac{N}{p} \notin \{1, \dots, -l - m\}$, then $\mathcal{K}^m = \mathcal{B}_{[2-l-N/p]}$.*
- ii) *If $\frac{N}{p} \in \{1, \dots, -l - m\}$, then $\mathcal{K}^m = \mathcal{B}_{1-l-N/p}$.*

We now introduce the two operators Π_D and Π_N , defined by:

$$\begin{aligned} \forall r \in \mathcal{A}_k^\Delta, \quad \Pi_D r &= \frac{1}{2} \int_0^{x_N} t r(x', t) dt, \\ \forall s \in \mathcal{N}_k^\Delta, \quad \Pi_N s &= \frac{1}{2} x_N \int_0^{x_N} s(x', t) dt. \end{aligned}$$

So we obtain the second characterization of \mathcal{K}^m :

Lemma 4.4. *Let $l \in \mathbb{Z}$ and $m \in \mathbb{N}$. Under hypothesis (4), we have*

$$\mathcal{K}^m = \mathcal{B}_{[2-l-N/p]} = \Pi_D \mathcal{A}_{[-l-N/p]}^\Delta \oplus \Pi_N \mathcal{N}_{[-l-N/p]}^\Delta. \quad (31)$$

Proof. A direct calculation with these operators yields the following formulas:

$$\forall r \in \mathcal{A}_k^\Delta, \quad \begin{cases} \Delta \Pi_D r = r & \text{in } \mathbb{R}_+^N, \\ \partial_N \Pi_D r = \frac{1}{2} x_N r & \text{in } \mathbb{R}_+^N, \\ \Pi_D r = \partial_N \Pi_D r = 0 & \text{on } \Gamma, \end{cases} \quad (32)$$

and

$$\forall s \in \mathcal{N}_k^\Delta, \quad \begin{cases} \Delta \Pi_N s = s & \text{in } \mathbb{R}_+^N, \\ \partial_N \Pi_N s = \frac{1}{2} \left(x_N s + \int_0^{x_N} s(x', t) dt \right) & \text{in } \mathbb{R}_+^N, \\ \Pi_N s = \partial_N \Pi_N s = 0 & \text{on } \Gamma. \end{cases} \quad (33)$$

Moreover, for any $r \in \mathcal{A}_k^\Delta$ and $s \in \mathcal{N}_k^\Delta$, $\Pi_D r \in \mathcal{P}_{k+2}$ and $\Pi_N s \in \mathcal{P}_{k+2}$. Thus, if $r \in \mathcal{A}_{[-l-N/p]}^\Delta$ and $s \in \mathcal{N}_{[-l-N/p]}^\Delta$, we can deduce that $\Pi_D r \in \mathcal{B}_{[2-l-N/p]}$ and $\Pi_N s \in \mathcal{B}_{[2-l-N/p]}$.

Conversely, if we consider $u \in \mathcal{B}_{[2-l-N/p]}$, then we have $\Delta u \in \mathcal{P}_{[-l-N/p]}^\Delta$. Since $\mathcal{P}_{[-l-N/p]}^\Delta = \mathcal{A}_{[-l-N/p]}^\Delta \oplus \mathcal{N}_{[-l-N/p]}^\Delta$, there exists $(r, s) \in \mathcal{A}_{[-l-N/p]}^\Delta \times \mathcal{N}_{[-l-N/p]}^\Delta$ such that $\Delta u = r + s$ in \mathbb{R}_+^N . According to formulas (32) and (33), the function $z = u - \Pi_D r - \Pi_N s$ satisfies: $\Delta z = 0$ in \mathbb{R}_+^N and $z = \partial_N z = 0$ on Γ . The function z belonging to $\mathcal{A}_{[2-l-N/p]}^\Delta \cap \mathcal{N}_{[2-l-N/p]}^\Delta = \{0\}$, then $u = \Pi_D r + \Pi_N s$. Furthermore, the sum (31) is direct, because if $(r, s) \in \mathcal{A}_{[-l-N/p]}^\Delta \times \mathcal{N}_{[-l-N/p]}^\Delta$ such that $\Pi_D r = \Pi_N s = u$, then $\Delta u = r = s$. That implies $\Delta u = 0$ in \mathbb{R}_+^N with $u = \partial_N u = 0$ on Γ , hence $u = 0$ in \mathbb{R}_+^N . \square

The following proposition clarifies the kernel $\mathcal{B}_{[2-l-N/p]}$ in the simplest cases.

Proposition 4.5. *Let $l \in \mathbb{Z}$ such that $\frac{N}{p} \notin \{1, \dots, -l\}$.*

- i) If $-l - N/p < 0$, then $\mathcal{B}_{[2-l-N/p]} = \{0\}$.*
- ii) If $0 < -l - N/p < 1$, then $\mathcal{B}_{[2-l-N/p]} = \mathbb{R}x_N^2$.*

Proof. If $-l - N/p < 0$, then we have $\mathcal{B}_{[2-l-N/p]} \subset \mathcal{P}_1$. Now, if $\varphi \in \mathcal{P}_1$ with $\varphi = \partial_N \varphi = 0$ on Γ , we necessarily have $\varphi = 0$. If $0 < -l - N/p < 1$, then $\mathcal{B}_{[2-l-N/p]} = \mathcal{B}_2 = \{\varphi \in \mathcal{P}_2^{\Delta^2}; \varphi = \partial_N \varphi = 0 \text{ on } \Gamma\}$. Now, if $\varphi \in \mathcal{P}_2$ with $\varphi = \partial_N \varphi = 0$ on Γ , a direct calculation shows that $\varphi(x) = cx_N^2$, where $c \in \mathbb{R}$. \square

Remark 4.6. This proposition yields an answer to important particular cases:

- i) If $l \geq 0$ or ($l = -1$ and $N/p > 1$), then $\mathcal{B}_{[2-l-N/p]} = \{0\}$.*
- ii) If $l = -1$ and $N/p < 1$, then $\mathcal{B}_{[3-N/p]} = \mathcal{B}_2 = \mathbb{R}x_N^2$.* \square

4.2 The compatibility condition

We shall now show the necessity of condition (30) in Theorem 4.1.

Lemma 4.7. *Let $l \in \mathbb{Z}$ such that*

$$\frac{N}{p'} \notin \{1, \dots, l\}. \quad (34)$$

Let $f \in W_l^{-2,p}(\mathbb{R}_+^N)$, $g_0 \in W_l^{2-1/p,p}(\Gamma)$ and $g_1 \in W_l^{1-1/p,p}(\Gamma)$. If problem (\mathcal{P}) admits a solution in $W_l^{2,p}(\mathbb{R}_+^N)$, then we have the compatibility condition:

$$\forall \varphi \in \mathcal{B}_{[2+l-N/p]}, \langle f, \varphi \rangle_{W_l^{-2,p}(\mathbb{R}_+^N) \times W_l^{2,p}(\mathbb{R}_+^N)} + \langle g_1, \Delta \varphi \rangle_{\Gamma} - \langle g_0, \partial_N \Delta \varphi \rangle_{\Gamma} = 0,$$

where $\langle g_1, \Delta \varphi \rangle_{\Gamma}$ denotes the duality bracket $\langle g_1, \Delta \varphi \rangle_{W_l^{1-1/p,p}(\Gamma) \times W_l^{-1/p',p'}(\Gamma)}$ and $\langle g_0, \partial_N \Delta \varphi \rangle_{\Gamma}$ denotes the duality bracket $\langle g_0, \partial_N \Delta \varphi \rangle_{W_l^{2-1/p,p}(\Gamma) \times W_l^{-1-1/p',p'}(\Gamma)}$.

Remark 4.8. By Proposition 4.5, if $l - N/p' < 0$ and particularly if $l \leq 0$, we have $\mathcal{B}_{[2+l-N/p']} = \{0\}$. Thus there is no compatibility condition in these cases. \square

Proof. So we assume that $l \geq 1$. The first point is to justify the dualities in the spaces of traces. Noting that under hypothesis (34), for any $\varphi \in \mathcal{B}_{[2+l-N/p']}$, we have $\varphi \in W_{-l+1}^{3,p'}(\mathbb{R}_+^N)$ and also $\varphi \in W_{-l+2}^{4,p'}(\mathbb{R}_+^N)$, we can deduce that $\Delta \varphi|_{\Gamma} \in W_{-l+1}^{1-1/p',p'}(\Gamma)$ and $\partial_N \Delta \varphi|_{\Gamma} \in W_{-l+2}^{1-1/p',p'}(\Gamma)$. It remains to verify the imbeddings

$$W_{-l+1}^{1-1/p', p'}(\Gamma) \hookrightarrow W_{-l}^{-1/p', p'}(\Gamma), \quad (35)$$

$$W_{-l+2}^{1-1/p', p'}(\Gamma) \hookrightarrow W_{-l}^{-1-1/p', p'}(\Gamma). \quad (36)$$

i) To show (35), we break down this imbedding into

$$W_{-l+1}^{1-1/p', p'}(\mathbb{R}^{N-1}) \hookrightarrow W_{-l+1/p'}^{0, p'}(\mathbb{R}^{N-1}), \quad (37)$$

$$W_{-l+1/p'}^{0, p'}(\mathbb{R}^{N-1}) \hookrightarrow W_{-l}^{-1/p', p'}(\mathbb{R}^{N-1}), \quad (38)$$

where (38) is equivalent by duality to

$$W_l^{1/p', p}(\mathbb{R}^{N-1}) \hookrightarrow W_{l-1/p'}^{0, p}(\mathbb{R}^{N-1}). \quad (39)$$

Observe that (37) holds if and only if $\frac{N-1}{p'} - l + 1 \neq 1 - \frac{1}{p'}$, i.e. $\frac{N}{p'} \neq l$, which is included in (34). Likewise (39) is satisfied if and only if $\frac{N-1}{p} + l \neq \frac{1}{p'}$, i.e. $\frac{N}{p} \neq -l + 1$, which can't happen for $l \geq 1$.

ii) Similarly, the imbedding (36) is equivalent to

$$W_{-l+2}^{1-1/p', p'}(\mathbb{R}^{N-1}) \hookrightarrow W_{-l+1+1/p'}^{0, p'}(\mathbb{R}^{N-1}) \quad (40)$$

$$W_l^{1+1/p', p}(\mathbb{R}^{N-1}) \hookrightarrow W_{l-1-1/p'}^{0, p}(\mathbb{R}^{N-1}). \quad (41)$$

The imbedding (40) holds if and only if $\frac{N}{p'} \neq l - 1$, which is included in (34).

The imbedding (41) is satisfied if and only if $\frac{N}{p} \notin \{-l + 1, -l + 2\}$. Since $l \geq 1$, it suffices that $\frac{N}{p} \neq 1$ for $l = 1$. Assume that $l = 1$ and $\frac{N}{p} = 1$, then we have $\frac{N}{p'} = N - 1$ and thus $\mathcal{B}_{[2+l-N/p']} = \mathcal{B}_{[4-N]}$. If $N \geq 3$, there is no compatibility condition because $\mathcal{B}_{[4-N]} = \{0\}$. If $N = 2$, then we have $p = p' = 2$ and $\frac{N}{p} = 1$, but that is excluded by (34).

Now it is clear that for any $u \in \mathcal{D}(\overline{\mathbb{R}_+^N})$ we have

$$\forall \varphi \in \mathcal{B}_{[2+l-N/p]}, \quad \int_{\mathbb{R}_+^N} \varphi \Delta^2 u \, dx = \int_{\Gamma} u \Delta \partial_N \varphi \, dx' - \int_{\Gamma} \partial_N u \Delta \varphi \, dx'.$$

Let $u \in W_l^{2, p}(\mathbb{R}_+^N)$ and $\varphi \in \mathcal{B}_{[2+l-N/p']}$. Thanks to the density of $\mathcal{D}(\overline{\mathbb{R}_+^N})$ in $W_l^{2, p}(\mathbb{R}_+^N)$, there exists a sequence $(u_k)_{k \in \mathbb{N}} \subset \mathcal{D}(\overline{\mathbb{R}_+^N})$ such that $u_k \rightarrow u$ in $W_l^{2, p}(\mathbb{R}_+^N)$. Therefore $\Delta^2 u_k \rightarrow \Delta^2 u$ in $W_l^{-2, p}(\mathbb{R}_+^N)$, $u_k \rightarrow u$ in $W_l^{2-1/p, p}(\Gamma)$

and $\partial_N u_k \rightarrow \partial_N u$ in $W_l^{1-1/p,p}(\Gamma)$. Writing the previous formula for any u_k , we obtain by passing to the limit as $k \rightarrow \infty$

$$\forall \varphi \in \mathcal{B}_{[2+l-N/p']}, \quad \langle \Delta^2 u, \varphi \rangle_{W_l^{-2,p}(\mathbb{R}_+^N) \times \mathring{W}_{-l}^{2,p'}(\mathbb{R}_+^N)} = \langle u, \partial_N \Delta \varphi \rangle_\Gamma - \langle \partial_N u, \Delta \varphi \rangle_\Gamma.$$

This proves the necessity of condition (30). \square

4.3 The homogeneous problem

Here we consider the homogeneous problem in \mathbb{R}_+^N , *i.e.* $f = 0$, with standard boundary conditions. Let the problem

$$(\mathcal{P}^0) \quad \begin{cases} \Delta^2 u = 0 & \text{in } \mathbb{R}_+^N, \\ u = g_0 & \text{on } \Gamma, \\ \partial_N u = g_1 & \text{on } \Gamma, \end{cases}$$

with $g_0 \in W_l^{2-1/p,p}(\Gamma)$ and $g_1 \in W_l^{1-1/p,p}(\Gamma)$.

Lemma 4.9. *Let $l \in \mathbb{Z}$. Under hypothesis (4), for any $g_0 \in W_l^{2-1/p,p}(\Gamma)$ and $g_1 \in W_l^{1-1/p,p}(\Gamma)$ satisfying the compatibility condition*

$$\forall \varphi \in \mathcal{B}_{[2+l-N/p']}, \quad \langle g_1, \Delta \varphi \rangle_\Gamma - \langle g_0, \partial_N \Delta \varphi \rangle_\Gamma = 0, \quad (42)$$

problem (\mathcal{P}^0) admits a solution $u \in W_l^{2,p}(\mathbb{R}_+^N)$, unique up to an element of $\mathcal{B}_{[2-l-N/p]}$, with the estimate

$$\inf_{q \in \mathcal{B}_{[2-l-N/p]}} \|u + q\|_{W_l^{2,p}(\mathbb{R}_+^N)} \leq C \left(\|g_0\|_{W_l^{2-1/p,p}(\Gamma)} + \|g_1\|_{W_l^{1-1/p,p}(\Gamma)} \right).$$

Proof. Firstly, thanks to Lemma 4.4, note that condition (42) is equivalent to both conditions

$$\forall r \in \mathcal{A}_{[l-N/p']}^\Delta, \quad \langle g_0, \partial_N r \rangle_\Gamma = 0, \quad (43)$$

$$\forall s \in \mathcal{N}_{[l-N/p']}^\Delta, \quad \langle g_1, s \rangle_\Gamma = 0. \quad (44)$$

Consider the Dirichlet problem:

$$(\mathcal{R}^0) \quad \begin{cases} \Delta \vartheta = 0 & \text{in } \mathbb{R}_+^N, \\ \vartheta = g_0 & \text{on } \Gamma. \end{cases}$$

Since $g_0 \in W_l^{2-1/p,p}(\Gamma) = W_{1+(l-1)}^{1+1-1/p,p}(\Gamma)$, Theorem 2.6 holds with $m = 1$ and $l - 1$ instead of l . Then hypothesis (6) becomes $\frac{N}{p'} \notin \{1, \dots, l\}$ and

$\frac{N}{p} \notin \{1, \dots, -l\}$. Moreover compatibility condition (5) corresponds precisely to (43). We can deduce that problem (\mathcal{R}^0) admits a solution $\vartheta \in W_l^{2,p}(\mathbb{R}_+^N)$.

Consider now the Neumann problem:

$$(\mathcal{S}^0) \begin{cases} \Delta \zeta = 0 & \text{in } \mathbb{R}_+^N, \\ \partial_N \zeta = g_1 & \text{on } \Gamma. \end{cases}$$

Theorem 2.8 holds with $m = 0$. Moreover compatibility condition (8) corresponds precisely to (44). We can deduce that problem (\mathcal{S}^0) admits a solution $\zeta \in W_l^{2,p}(\mathbb{R}_+^N)$. So we can readily verify that the function defined by

$$u = x_N \partial_N (\zeta - \vartheta) + \vartheta \quad (45)$$

is a solution to (\mathcal{P}^0) . However we must show that $u \in W_l^{2,p}(\mathbb{R}_+^N)$. For this, we remark that u satisfies

$$(\mathcal{T}) \begin{cases} \Delta u = 2 \partial_N^2 (\zeta - \vartheta) & \text{in } \mathbb{R}_+^N, \\ u = g_0 & \text{on } \Gamma, \end{cases}$$

with $2 \partial_N^2 (\zeta - \vartheta) \in W_l^{0,p}(\mathbb{R}_+^N)$ and $g_0 \in W_l^{2-1/p,p}(\Gamma)$.

i) If $\frac{N}{p} \neq -l + 1$, then we have the imbedding $W_l^{2,p}(\mathbb{R}_+^N) \hookrightarrow W_{l-1}^{1,p}(\mathbb{R}_+^N)$. By (45), we deduce that $u \in W_{l-1}^{1,p}(\mathbb{R}_+^N)$. Furthermore we have the following Green formula:

$$\forall r \in \mathcal{A}_{[l-N/p]}^\Delta, \quad \langle \Delta u, r \rangle_{W_{l-1}^{-1,p}(\mathbb{R}_+^N) \times \mathring{W}_{-l+1}^{1,p'}(\mathbb{R}_+^N)} = \langle u, \partial_N r \rangle_{W_{l-1}^{-1/p,p}(\Gamma) \times W_{-l+1}^{-1/p',p'}(\Gamma)},$$

i.e.

$$\forall r \in \mathcal{A}_{[l-N/p]}^\Delta, \quad \langle 2 \partial_N^2 (\zeta - \vartheta), r \rangle_{W_{l-1}^{-1,p}(\mathbb{R}_+^N) \times \mathring{W}_{-l+1}^{1,p'}(\mathbb{R}_+^N)} = \langle g_0, \partial_N r \rangle_\Gamma.$$

Thus the compatibility condition of problem (\mathcal{T}) is satisfied and thanks to Theorem 2.8, it admits a solution $y \in W_l^{2,p}(\mathbb{R}_+^N)$, unique up to an element of $\mathcal{A}_{[2-l-N/p]}^\Delta$. So the function $z = u - y \in W_{l-1}^{1,p}(\mathbb{R}_+^N)$ and satisfies

$$(\mathcal{K}) \begin{cases} \Delta z = 0 & \text{in } \mathbb{R}_+^N, \\ z = 0 & \text{on } \Gamma. \end{cases}$$

We can deduce that $z \in \mathcal{A}_{[2-l-N/p]}^\Delta$, *i.e.* $u = y + r$ with $r \in \mathcal{A}_{[2-l-N/p]}^\Delta \subset W_l^{2,p}(\mathbb{R}_+^N)$, which shows that $u \in W_l^{2,p}(\mathbb{R}_+^N)$.

ii) If $\frac{N}{p} = -l + 1$, the previous imbedding does not hold. Then we only have $W_l^{2,p}(\mathbb{R}_+^N) \hookrightarrow W_{l-1,-1}^{1,p}(\mathbb{R}_+^N)$, with the introduction of a logarithmic weight in the second space. By (45), we can deduce that $u \in W_{l-1,-1}^{1,p}(\mathbb{R}_+^N)$. Furthermore we have $l - \frac{N}{p'} < 0$, thus there is no compatibility condition for (\mathcal{T}) which admits consequently a solution $y \in W_l^{2,p}(\mathbb{R}_+^N)$, unique up to an element of $\mathcal{A}_1^\Delta = \mathbb{R}x_N$ which is included in $W_l^{2,p}(\mathbb{R}_+^N)$. The end of the proof is similar to the previous case. \square

We can now extend this result to more regular data.

Lemma 4.10. *Let $l \in \mathbb{Z}$ and $m \geq 1$. Under hypothesis (9), for any $g_0 \in W_{m+l}^{m+2-1/p,p}(\Gamma)$ and $g_1 \in W_{m+l}^{m+1-1/p,p}(\Gamma)$, satisfying the compatibility condition (42), problem (\mathcal{P}^0) admits a solution $u \in W_{m+l}^{m+2,p}(\mathbb{R}_+^N)$, unique up to an element of $\mathcal{B}_{[2-l-N/p]}$ with the estimate*

$$\inf_{q \in \mathcal{B}_{[2-l-N/p]}} \|u + q\|_{W_{m+l}^{m+2,p}(\mathbb{R}_+^N)} \leq C \left(\|g_0\|_{W_{m+l}^{m+2-1/p,p}(\Gamma)} + \|g_1\|_{W_{m+l}^{m+1-1/p,p}(\Gamma)} \right).$$

Proof. We strictly resume the proof of Lemma 4.9. In this case, we note that $g_0 \in W_{(m+1)+(l-1)}^{(m+1)+1-1/p,p}(\Gamma)$ and $g_1 \in W_{m+l}^{m+1-1/p,p}(\Gamma)$, then we use Theorems 2.6 and 2.8. To show that $u \in W_{m+l}^{m+2,p}(\mathbb{R}_+^N)$, we must distinguish two cases. If $\frac{N}{p} \neq -l - m + 1$, then we have the imbedding $W_{m+l}^{m+2,p}(\mathbb{R}_+^N) \hookrightarrow W_{m+l-1}^{m+1,p}(\mathbb{R}_+^N)$. If $\frac{N}{p} = -l - m + 1$, then we have $W_{m+l}^{m+2,p}(\mathbb{R}_+^N) \hookrightarrow W_{m+l-1,-1}^{m+1,p}(\mathbb{R}_+^N)$. In the second case, we must remark that $l - \frac{N}{p'} < 0$, so there is again no compatibility condition for (\mathcal{T}) . \square

Note that we have the chain of imbeddings $W_{m+l}^{m+2,p}(\mathbb{R}_+^N) \hookrightarrow W_{m+l-1}^{m+1,p}(\mathbb{R}_+^N) \hookrightarrow \dots \hookrightarrow W_l^{2,p}(\mathbb{R}_+^N)$ if and only if $\frac{N}{p} \notin \{-l - m + 1, \dots, -l\}$, and then Lemma 4.10 is a regularity result with respect to Lemma 4.9.

4.4 Existence of a solution to problem (\mathcal{P})

We come back to the general problem (\mathcal{P}) and Theorem 4.1. By Lemma 2.3, there exists a lifting function $u_g \in W_l^{2,p}(\mathbb{R}_+^N)$ of (g_0, g_1) , i.e. $u_g = g_0$ on Γ and

$\partial_N u_g = g_1$ on Γ , such that

$$\|u_g\|_{W_l^{2,p}(\mathbb{R}_+^N)} \leq C \left(\|g_0\|_{W_l^{2-1/p,p}(\Gamma)} + \|g_1\|_{W_l^{1-1/p,p}(\Gamma)} \right).$$

Set $h = f - \Delta^2 u_g \in W_l^{-2,p}(\mathbb{R}_+^N)$ and $v = u - u_g$, then problem (\mathcal{P}) is equivalent to the following with homogeneous boundary conditions:

$$\Delta^2 v = h \quad \text{in } \mathbb{R}_+^N, \quad v = \partial_N v = 0 \quad \text{on } \Gamma.$$

Then, the compatibility condition (30) for Problem (\mathcal{P}) becomes:

$$\forall \varphi \in \mathcal{B}_{[2+l-N/p']}, \quad \langle h, \varphi \rangle_{W_l^{-2,p}(\mathbb{R}_+^N) \times \mathring{W}_{-l}^{2,p'}(\mathbb{R}_+^N)} = 0. \quad (46)$$

So, we can consider now the lifted problem

$$(\mathcal{P}^*) \quad \begin{cases} \Delta^2 u = f & \text{in } \mathbb{R}_+^N, \\ u = 0 & \text{on } \Gamma, \\ \partial_N u = 0 & \text{on } \Gamma, \end{cases}$$

where $f \in W_l^{-2,p}(\mathbb{R}_+^N)$ and $f \perp \mathcal{B}_{[2+l-N/p']}$.

Give at first a characterization of $W_l^{-2,p}(\mathbb{R}_+^N)$:

Lemma 4.11. *For any $f \in W_l^{-2,p}(\mathbb{R}_+^N)$, there exists $F = (F_{ij})_{1 \leq i, j \leq N} \in W_l^{0,p}(\mathbb{R}_+^N)^{N^2}$ such that*

$$f = \operatorname{div} \operatorname{div} F = \sum_{i,j=1}^N \partial_{ij}^2 F_{ij},$$

with the estimate

$$\sum_{i,j=1}^N \|F_{ij}\|_{W_l^{0,p}(\mathbb{R}_+^N)} \leq C \|f\|_{W_l^{-2,p}(\mathbb{R}_+^N)}.$$

Proof. We know by Hardy's inequality that the norm and the semi-norm in $\mathring{W}_{-l}^{2,p'}(\mathbb{R}_+^N)$ are equivalent, i.e. there exists a constant C such that

$$\forall u \in \mathring{W}_{-l}^{2,p'}(\mathbb{R}_+^N), \quad \|\nabla^2 u\|_{W_{-l}^{0,p'}(\mathbb{R}_+^N)^{N^2}} \leq \|u\|_{\mathring{W}_{-l}^{2,p'}(\mathbb{R}_+^N)} \leq C \|\nabla^2 u\|_{W_{-l}^{0,p'}(\mathbb{R}_+^N)^{N^2}}.$$

Let

$$T : \begin{aligned} & \mathring{W}_{-l}^{2,p'}(\mathbb{R}_+^N) \longrightarrow W_{-l}^{0,p'}(\mathbb{R}_+^N)^{N^2} \\ & u \longmapsto \nabla^2 u. \end{aligned}$$

By the previous inequalities, T is a linear continuous injective mapping. We set $G = T\left(\mathring{W}_{-l}^{2,p'}(\mathbb{R}_+^N)\right)$, equipped with the norm of $W_{-l}^{0,p'}(\mathbb{R}_+^N)^{N^2}$, and $S = T^{-1} : G \longrightarrow \mathring{W}_{-l}^{2,p'}(\mathbb{R}_+^N)$. The mapping $h \in G \longmapsto \langle f, Sh \rangle_{W_l^{-2,p}(\mathbb{R}_+^N) \times \mathring{W}_{-l}^{2,p'}(\mathbb{R}_+^N)}$ is a linear functional on G . Thanks to Hahn-Banach theorem, we can extend it to a linear functional on $W_{-l}^{0,p'}(\mathbb{R}_+^N)^{N^2}$ denoted by Φ . Thanks to Riesz representation theorem, we know that there exists $F = (F_{ij}) \in W_l^{0,p}(\mathbb{R}_+^N)^{N^2}$ such that

$$\forall h = (h_{ij}) \in W_{-l}^{0,p'}(\mathbb{R}_+^N)^{N^2}, \quad \langle \Phi, h \rangle = \int_{\mathbb{R}_+^N} F_{ij} h_{ij} dx,$$

with Einstein convention of summation on repeated indices. Particularly, if $h \in G$, we have

$$\langle f, Sh \rangle = \int_{\mathbb{R}_+^N} F_{ij} h_{ij} dx,$$

i.e.

$$\forall u \in \mathring{W}_{-l}^{2,p'}(\mathbb{R}_+^N), \quad \langle f, u \rangle = \int_{\mathbb{R}_+^N} F_{ij} \partial_{ij}^2 u dx.$$

We can deduce that

$$\forall u \in \mathcal{D}(\mathbb{R}_+^N), \quad \langle f, u \rangle = \langle \partial_{ij}^2 F_{ij}, u \rangle,$$

$$i.e. \quad f = \operatorname{div} \operatorname{div} F = \partial_{ij}^2 F_{ij}. \quad \square$$

Now we can establish a first isomorphism result in the half-space:

Proposition 4.12. *Let $l \in \mathbb{Z}$. Under hypothesis (4), with $2 + l - N/p' < 0$ or $2 - l - N/p < 0$, the biharmonic operator*

$$\Delta^2 : \mathring{W}_l^{2,p}(\mathbb{R}_+^N) / \mathcal{B}_{[2-l-N/p]} \longrightarrow W_l^{-2,p}(\mathbb{R}_+^N) \perp \mathcal{B}_{[2+l-N/p]}$$

is an isomorphism.

Proof. Let's first assume that $2 + l - N/p' < 0$. Let $f \in W_l^{-2,p}(\mathbb{R}_+^N)$. Then by Lemma 4.11, we can write $f = \partial_{ij}^2 F_{ij}$ with $(F_{ij})_{1 \leq i, j \leq N} \in W_l^{0,p}(\mathbb{R}_+^N)^{N^2}$. If we extend F_{ij} to \mathbb{R}^N by 0, we obtain $(\tilde{F}_{ij})_{1 \leq i, j \leq N} \in W_l^{0,p}(\mathbb{R}^N)^{N^2}$, and thus $\tilde{f} = \partial_{ij}^2 \tilde{F}_{ij} \in W_l^{-2,p}(\mathbb{R}^N)$ as extension of f such that $\|\tilde{f}\|_{W_l^{-2,p}(\mathbb{R}^N)} \leq C \|f\|_{W_l^{-2,p}(\mathbb{R}_+^N)}$. By Theorem 3.3, there exists $\tilde{z} \in W_l^{2,p}(\mathbb{R}^N)$ such that $\tilde{f} = \Delta^2 \tilde{z}$ in \mathbb{R}^N and writing $z = \tilde{z}|_{\mathbb{R}_+^N}$, we have $f = \Delta^2 z$ in \mathbb{R}_+^N , with $z \in W_l^{2,p}(\mathbb{R}_+^N)$, $z|_{\Gamma} \in W_l^{2-1/p,p}(\Gamma)$ and $\partial_N z|_{\Gamma} \in W_l^{1-1/p,p}(\Gamma)$. Since $\mathcal{B}_{[2+l-N/p]} = \{0\}$, there

is no compatibility condition for Lemma 4.9 which asserts the existence of a solution $v \in W_l^{2,p}(\mathbb{R}_+^N)$ to the homogeneous problem

$$\Delta^2 v = 0 \quad \text{in } \mathbb{R}_+^N, \quad v = z \quad \text{and} \quad \partial_N v = \partial_N z \quad \text{on } \Gamma. \quad (47)$$

The function $u = z - v$ answers to problem (\mathcal{P}^*) in this case.

So we have shown that if $2 + l - N/p' < 0$, the operator

$$\Delta^2 : \mathring{W}_l^{2,p}(\mathbb{R}_+^N) / \mathcal{B}_{[2-l-N/p]} \longrightarrow W_l^{-2,p}(\mathbb{R}_+^N) \quad (48)$$

is an isomorphism. Thus by duality we obtain the isomorphism

$$\Delta^2 : \mathring{W}_l^{2,p}(\mathbb{R}_+^N) \longrightarrow W_l^{-2,p}(\mathbb{R}_+^N) \perp \mathcal{B}_{[2+l-N/p]}, \quad (49)$$

if $2 - l - N/p < 0$. □

It remains to solve (\mathcal{P}^*) if

$$2 + l - N/p' \geq 0 \quad \text{and} \quad 2 - l - N/p \geq 0. \quad (50)$$

It suffices to check the cases $l \in \{-1, 0, 1\}$, outside which condition (50) does not hold. For that, we establish a preliminary proposition:

Proposition 4.13. *Let $l \in \{-1, 0\}$ such that $N/p \neq 1$ if $l = -1$. For any $f \in W_l^{0,p}(\mathbb{R}_+^N)$, there exists $z \in W_l^{4,p}(\mathbb{R}_+^N)$ such that $\Delta^2 z = f$.*

Proof. Under these hypotheses, consider the extension \tilde{f} of f to \mathbb{R}^N by 0, so $\tilde{f} \in W_l^{0,p}(\mathbb{R}^N)$. Show at first that there exists $\tilde{z} \in W_l^{4,p}(\mathbb{R}^N)$ such that $\Delta^2 \tilde{z} = \tilde{f}$.

- a) If $l = -1$, then $\tilde{f} \in W_{-1}^{0,p}(\mathbb{R}^N)$ and we have $N/p \neq 1$. Thus Lemma 3.4 of isomorphism in \mathbb{R}^N holds with $m = 2$ and $l = -3$, hence the existence of $\tilde{z} \in W_{-1}^{4,p}(\mathbb{R}^N)$ such that $\Delta^2 \tilde{z} = \tilde{f}$.
- b) If $l = 0$, then $\tilde{f} \in L^p(\mathbb{R}^N)$. Here again Lemma 3.4 holds with $m = 2$ and $l = -2$, hence the existence of $\tilde{z} \in W_0^{4,p}(\mathbb{R}^N)$ such that $\Delta^2 \tilde{z} = \tilde{f}$.

Then we come back to the restriction $z = \tilde{z}|_{\mathbb{R}_+^N}$ for which we have naturally $\Delta^2 z = f$ in \mathbb{R}_+^N . □

Now we can fill the gap of Proposition 4.12:

Proposition 4.14. *Let $l \in \{-1, 0, 1\}$ such that*

$$\frac{N}{p'} \neq 1 \quad \text{if } l = 1 \quad \text{and} \quad \frac{N}{p} \neq 1 \quad \text{if } l = -1.$$

Then the biharmonic operator

$$\Delta^2 : W_l^{\circ,2,p}(\mathbb{R}_+^N) / \mathcal{B}_{[2-l-N/p]} \longrightarrow W_l^{-2,p}(\mathbb{R}_+^N) \perp \mathcal{B}_{[2+l-N/p']}$$

is an isomorphism.

Proof. At first we will use Lemma 4.11 and Proposition 4.13 to solve (\mathcal{P}^*) for $l \in \{-1, 0\}$.

Let $f \in W_l^{-2,p}(\mathbb{R}_+^N)$ with $l \in \{-1, 0\}$ verifying (4). By Lemma 4.11, there exists $F = (F_{ij})_{1 \leq i, j \leq N} \in W_l^{0,p}(\mathbb{R}_+^N)^{N^2}$ such that $f = \operatorname{div} \operatorname{div} F$. It suffices to apply Proposition 4.13 to all the components F_{ij} of F to find $Z = (Z_{ij})_{1 \leq i, j \leq N} \in W_l^{4,p}(\mathbb{R}_+^N)^{N^2}$ such that $\Delta^2 Z = F$ in \mathbb{R}_+^N . Setting $z = \operatorname{div} \operatorname{div} Z$, we obtain $z \in W_l^{2,p}(\mathbb{R}_+^N)$ such that $\Delta^2 z = f$ in \mathbb{R}_+^N because the operators div and Δ commute. Thus we have $z|_{\Gamma} \in W_l^{2-1/p,p}(\Gamma)$ and $\partial_N z|_{\Gamma} \in W_l^{1-1/p,p}(\Gamma)$, and Lemma 4.9 asserts the existence of a solution $v \in W_l^{2,p}(\mathbb{R}_+^N)$ to problem (47), since we have still $\mathcal{B}_{[2+l-N/p]} = \{0\}$ (cf. Remark 4.8). Then the function $u = z - v$ answer again to problem (\mathcal{P}^*) for $l \in \{-1, 0\}$.

Finally to solve the case $l = 1$, we proceed by duality from the case $l = -1$. We have the isomorphism

$$\Delta^2 : W_{-1}^{\circ,2,p}(\mathbb{R}_+^N) / \mathcal{B}_{[3-N/p]} \longrightarrow W_{-1}^{-2,p}(\mathbb{R}_+^N) \quad \text{if } \frac{N}{p} \neq 1, \quad (51)$$

therefore by duality, the isomorphism

$$\Delta^2 : W_1^{\circ,2,p}(\mathbb{R}_+^N) \longrightarrow W_1^{-2,p}(\mathbb{R}_+^N) \perp \mathcal{B}_{[3-N/p']} \quad \text{if } \frac{N}{p'} \neq 1. \quad (52)$$

□

Remark 4.15. It is also possible to solve directly the case $l = 1$. The first step is to extend Proposition 4.13 to $l = 1$ with $N/p' \neq 1$. Here we consider the extension $\tilde{f} \in W_1^{0,p}(\mathbb{R}^N)$ of $f \in W_1^{0,p}(\mathbb{R}_+^N)$ defined by:

$$\tilde{f}(x', x_N) = \begin{cases} f(x', x_N) & \text{if } x_N > 0, \\ 0 & \text{if } x_N = 0, \\ -f(x', -x_N) & \text{if } x_N < 0. \end{cases}$$

Then we use Lemma 3.5 with $m = 2$, which asserts the existence of a function $\tilde{z} \in W_1^{4,p}(\mathbb{R}^N)$ such that $\Delta^2 \tilde{z} = \tilde{f}$ in \mathbb{R}^N , if $N/p' \neq 1$ and $\tilde{f} \perp \mathcal{P}_{[1-N/p']}^{\Delta}$. There are two cases: either $N/p' > 1$, then $\mathcal{P}_{[1-N/p']}^{\Delta} = \{0\}$ and there is no condition on \tilde{f} ; or $N/p' < 1$, then $\mathcal{P}_{[1-N/p']}^{\Delta} = \mathcal{P}_0$ and we must have $\tilde{f} \perp \mathcal{P}_0$. But $N/p' < 1$ implies that $W_1^{0,p}(\mathbb{R}^N) \hookrightarrow L^1(\mathbb{R}^N)$ and we have

$$\int_{\mathbb{R}^N} \tilde{f} dx = 0,$$

as a straightforward consequence of this extension of f . That exactly means that $\tilde{f} \perp \mathcal{P}_0$. Thus $z = \tilde{z}|_{\mathbb{R}_+^N} \in W_l^{4,p}(\mathbb{R}_+^N)$ satisfies $\Delta^2 z = f$ in \mathbb{R}_+^N .

The second step is to resume the proof of Proposition 4.14 for $l = 1$ with $N/p' \neq 1$. If $N/p' > 1$, we have still $\mathcal{B}_{[3-N/p']} = \{0\}$, so the same reasoning holds; if $N/p' < 1$, we know that $\mathcal{B}_{[3-N/p']} = \mathbb{R} x_N^2$ and Lemma 4.9 requires the following compatibility condition for problem (47):

$$\forall \varphi \in \mathbb{R} x_N^2, \quad \langle \partial_N z, \Delta \varphi \rangle_\Gamma - \langle z, \partial_N \Delta \varphi \rangle_\Gamma = 0,$$

which boils down to

$$\langle \partial_N z, 1 \rangle_\Gamma = 0. \quad (53)$$

But remember that f must satisfy the orthogonality condition for (\mathcal{P}^*) , *i.e.* $\langle f, x_N^2 \rangle_{W_1^{-2,p}(\mathbb{R}_+^N) \times \mathring{W}_{-1}^{2,p'}(\mathbb{R}_+^N)} = 0$ and moreover we have $f = \Delta^2 z$ in \mathbb{R}_+^N ; thus $\langle \Delta^2 z, x_N^2 \rangle_{W_1^{-2,p}(\mathbb{R}_+^N) \times \mathring{W}_{-1}^{2,p'}(\mathbb{R}_+^N)} = 0$. It suffices to write the Green formula

$$\langle \Delta^2 z, x_N^2 \rangle_{W_1^{-2,p}(\mathbb{R}_+^N) \times \mathring{W}_{-1}^{2,p'}(\mathbb{R}_+^N)} = - \langle \partial_N z, \Delta x_N^2 \rangle_\Gamma = -2 \langle \partial_N z, 1 \rangle_\Gamma,$$

to see that (53) holds. □

To finish the proof of Theorem 4.1, it remains to combine Propositions 4.12 and 4.14, which provides the isomorphism

$$\Delta^2 : \mathring{W}_l^{2,p}(\mathbb{R}_+^N) / \mathcal{B}_{[2-l-N/p]} \longrightarrow W_l^{-2,p}(\mathbb{R}_+^N) \perp \mathcal{B}_{[2+l-N/p]}, \quad (54)$$

for any $l \in \mathbb{Z}$ verifying (4). This answers globally to problem (\mathcal{P}^*) and thus to general problem (\mathcal{P}) by means of the lifting function mentioned above.

References

- [1] R. A. ADAMS. *Sobolev Spaces*. Academic Press, New York, 1975.
- [2] C. AMROUCHE, V. GIRAULT, J. GIROIRE. Weighted Sobolev spaces for Laplace's equation in \mathbb{R}^N . *J. Math. Pures Appl.* **73**, **6** (1994), 579–606.
- [3] C. AMROUCHE, V. GIRAULT, J. GIROIRE. Equation de Poisson et espaces de Sobolev avec poids critiques. *Publications du Laboratoire de Mathématiques Appliquées de l'Université de Pau* **06/25**.
- [4] C. AMROUCHE, S. NEČASOVÁ. Laplace equation in half space with a nonhomogeneous Dirichlet boundary condition. *Mathematica Bohemica* **126**, **2** (2001), 265–274.
- [5] C. AMROUCHE. The Neumann problem in the half space. *C. R. Acad. Sci. Paris, Série I* **335** (2002), 151–156.

- [6] T. Z. BOULMEZAOUD. Espaces de Sobolev avec poids pour l'équation de Laplace dans le demi-espace. *C. R. Acad. Sci. Paris, Série I* **328** (1999), 221–226.
- [7] T. Z. BOULMEZAOUD. On the Stokes system and the biharmonic equation in the half-space: an approach via weighted Sobolev spaces. *Mathematical Methods in the Applied Sciences* **25** (2002), 373–398.
- [8] R. FARWIG. A Note on the Reflection Principle for the Biharmonic Equation and the Stokes system. *Acta applicandae Mathematicae* **25** (1994), 41–51.
- [9] V. G. MAZ'YA, B. A. PLAMENEVSKIĀ, L. I. STUPYALIS. The three-dimensional problem of steady-state motion of a fluid with a free surface. *Amer. Math. Soc. Transl.* **123** (1984), 171–268.
- [10] B. HANOZET. Espaces de Sobolev avec poids. Application au problème de Dirichlet dans un demi-espace. *Rend. Sem. Univ. Padova* **46** (1971), 227–272.
- [11] L. SCHWARTZ. *Théorie des distributions*. Hermann, Paris, 1966.
- [12] N. TANAKA. On the boundary value problem for the stationary Stokes system in the half-space. *Journal of Differential Equations* **115** (1995), 70–74.