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Reflected Backward Doubly Stochastic Differential Equations and Application

Auguste Aman and Modeste N'Zi
UFR de Mathématiques et Informatique,
22 BP 582 Abidjan 22, Côte d'Ivoire

Abstract

This paper is devoted to study the class of reflected backward doubly stochastic differential equation (RBDSDE, for short). We first prove existence and uniqueness result under Lipschitz condition on the coefficient (drift) via the penalization method. As application we derive the obstacle problem of stochastic PDE.

Key Words: Backward doubly SDEs, Stochastic PDE, Obstacle problem.

MSC: 60H15; 60H20

1 Introduction

The first approach of linear backward stochastic differential equations (BSDE, for short) is been initiated by Bismut (see [3, 4]). But since the fundamental paper on this theory was published by Pardoux and Peng [16] in 1990, BSDEs have been taken with great interest in more and more researches. For instance, the theory of BSDE has been found to be powerful tool for mathematical economics and particular, for mathematical finance ([12] and it reference). Another main reason is due to their connection with the probabilistic interpretation of partial differential equation (PDE for short)(see [17, 18]), variational inequalities and obstacle problems ([13]) stochastic games [11], and so one.

On the other hand in order to give a probabilistic interpretation to the stochastic PDE and generalized a well-know Feynman-Kac formula, Pardoux and Peng [19] introduced a new kind of equation. They called it backward doubly stochastic differential equations (BDSDE for short). It has two different direction of stochastic integral involve with two independents standard Brownian motion: the standard (forward) dW_t and a backward stochastic integral dB_t . Since BDSDE have studied by many authors. Indeed, Buckdahn and Ma [5, 6, 7] introduced the

notion of stochastic viscosity solution to stochastic PDE under the Lipschitz coefficient of the BDSDE that coincides with the well- know viscosity solution of PDE if the coefficient in the integral which respect dB_t is null in the SPDE. But the BDSDE introduced by buckdahn and Ma is in fact a time reversal of that considered by Pardoux and Peng. However they have similar nature and called by the same name. Moreover in view of the work of Buckdahn and Ma [5, 6] Aman [1, 2] derive respectively a representation and path regularity result that generalized the one give by Ma and Zhang [14, 15] to the BSDE. We note also note that a comparison theorem of BDSDE has been given by Gu and al [20]. This result will play a important role in the proof of our main result.

In fact, the goal of this paper is to study the related obstacle problem for SPDE and extend the notion of stochastic viscosity solution to the reflected SPDE via the reflected BDSDE. The prove used penalization method (see El Karoui et al for the case of BSDE). The rest of the paper is organized as follows. Next section is devoted to claim and prove our main result. In the last section we apply the main result to study the related obstacle problem for SPDE.

2 Reflected Backward doubly stochastic differential equation

2.1 Preliminaries

Notation. The scalar product of the space \mathbb{R}^d , ($d \geq 2$) will be denote by $\langle . \rangle$ and the Euclidian norm associated by $\|.\|$.

For the remaining of the paper, let us fix a positive real number $T > 0$. First of all $\{W_t, 0 \leq t \leq T\}$ and $\{B_t, 0 \leq t \leq T\}$ are two mutually independent standard Brownian motion with values respectively in \mathbb{R}^d and \mathbb{R}^ℓ , defined respectively on the two probability space $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$. Let $\mathbf{F}^B = \{\mathcal{F}^B\}_{t \geq 0}$ denote the natural filtration generated by B , augmented by \mathbb{P}_1 - null set of \mathcal{F}_1 ; and let $\mathcal{F}^B = \mathcal{F}_\infty^B$. On the other hand we consider the following family of σ - fields:

$$\mathcal{F}_{t,T}^W = \sigma\{W_s - W_T, t \leq s \leq T\} \vee \mathcal{N}_2,$$

where \mathcal{N}_2 denotes all the \mathbb{P}_2 - null set in \mathcal{F}_2 . Denote $\mathcal{F}_T^W = \{\mathcal{F}_{t,T}^W\}_{0 \leq t \leq T}$. Next we consider the product space $(\Omega, \mathcal{F}, \mathbb{P})$ where

$$\Omega = \Omega_1 \times \Omega_2, \quad \mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2 \quad \text{and} \quad \mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2;$$

For each $t \in [0, T]$, we define

$$\mathcal{F}_t = \mathcal{F}_t^B \otimes \mathcal{F}_{t,T}^W.$$

Let us remark that the collection $F = \{\mathcal{F}_t, t \in [0, T]\}$ is neither increasing nor decreasing and it does not constitute a filtration.

Further, we assume that random variables $\xi(\omega_1)$, $\omega_1 \in \Omega_1$ and $\zeta(\omega_2)$, $\omega_2 \in \Omega_2$ are considered as random variables on Ω via the following identification:

$$\xi(\omega_1, \omega_2) = \xi(\omega_1); \quad \zeta(\omega_1, \omega_2) = \zeta(\omega_2).$$

In this paper we will always use this collection.

For any $d \geq 2$, let $M^2(0, T, \mathbb{R}^d \text{ or } \mathbb{R}^\ell)$ of (classes of $dP \otimes dt$ a.e. equal) d -dimensional jointly measurable stochastic processes $\varphi_t; t \in [0, T]$, which, satisfy :

$$(i) \|\varphi\|_{M^2}^2 = \mathbb{E} \int_0^T |\varphi_t|^2 dt < \infty;$$

(ii) φ is \mathcal{F}_t -measurable, for any $t \in [0, T]$.

Similarly, we denote by $S^2([0, T], \mathbb{R})$ the set of one dimensional continuous stochastic processes, which verify:

$$(iii) \|\varphi\|_{S^2}^2 = \mathbb{E} \left(\sup_{0 \leq t \leq T} |\varphi_t|^2 \right) < \infty;$$

(iv) φ is \mathcal{F}_t -measurable, for any $t \in [0, T]$.

As mentioned in introduction this paper studies a reflected BDSDE that is an equation of the following type:

$$Y_t = \xi + \int_0^t f(s, Y_s, Z_s) ds + \int_0^t g(s, Y_s, Z_s) dB_s - \int_0^t Z_s \downarrow dW_s + K_t, \quad 0 \leq t \leq T. \quad (2.1)$$

f and g are called generators and ξ the terminal condition.

Definition 2.1 *By solution of the reflected BDSDE we mean a triplet*

$(Y, Z, K) = \{Y_t, Z_t, K_t\}_{t \in [0, T]} \in S^2([0, T]; \mathbb{R}) \times M^2(0, T; \mathbb{R}^d) \times S^2([0, T]; \mathbb{R})$ such that for a given barrier process $S = \{S_t\}_{t \geq 0}$, the following holds \mathbb{P} - a.s

$$(i) Y_t \geq S_t, \quad 0 \leq t \leq T,$$

$$(ii) K \text{ is an increasing process such that } K_0 = 0 \text{ and } \int_0^T (Y_t - S_t) dK_t = 0;$$

(iii) processes (Y, Z, K) verify equation (2.1).

In the sequel we will use the notation reflected BDSDE (ξ, f, g, S) to say that we consider the reflected BDSDE whose generator is f and g , with terminal condition ξ and $S = \{S_t\}_{t \geq 0}$ a barrier process.

2.2 Existence and Uniqueness result

In this subsection, we consider reflected BDSDE (ξ, f, g, S) when generators, terminal condition and barrier process satisfy those assumptions:

(A1) $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$; $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^\ell$ be jointly measurable such that for any $(y, z) \in \mathbb{R} \times \mathbb{R}^d$

$$f(\cdot, y, z) \in M^2(0, T, \mathbb{R}), \quad g(\cdot, y, z) \in M^2(0, T, \mathbb{R}^\ell)$$

Moreover, we assume that there exist constants $C > 0$ and $0 < \alpha < 1$ such that for any $(\omega, t) \in \Omega \times [0, T]$; $(y_1, z_1), (y_2, z_2) \in \mathbb{R} \times \mathbb{R}^d$,

$$\mathbf{H1} \begin{cases} |f(t, y_1, z_1) - f(t, y_2, z_2)|^2 \leq C(|y_1 - y_2|^2 + \|z_1 - z_2\|^2) \\ \|g(t, y_1, z_1) - g(t, y_2, z_2)\|^2 \leq C|y_1 - y_2|^2 + \alpha\|z_1 - z_2\|^2 \end{cases}$$

(A2) ξ is a \mathcal{F}_0 -measurable variable such that $\mathbb{E}(|\xi|^2) < +\infty$

(A3) $(S_t)_{t \geq 0}$ is a continuous progressively measurable real-valued process satisfying:

(i) $\mathbb{E}(\sup_{0 \leq t \leq T} (S_t^+)^2) < +\infty$

(ii) $S_T \leq \xi$ \mathbb{P} a.s.

Our objectif is to construct a solution of reflected BDSDE (ξ, f, g, S) . As we explained in introduction, our method (penalization method) relies heavily on a priori estimate and the comparison theorem.

For each $n \in \mathbb{N}^*$ we set

$$f_n(s, y, z) = f(s, y, z) - n(S_s - y)^-$$

and considered the following BDSDE

$$Y_t^n = \xi + \int_0^t f_n(s, Y_s^n, Z_s^n) ds + \int_0^t g(s, Y_s^n, Z_s^n) dB_s - \int_0^t Z_s^n \downarrow dW_s \quad (2.2)$$

that is the penalized version of reflected BDSDE (ξ, f, g, S) . Then it is known from Pardoux and Peng in [19], that for $t \in [0, T]$ the BDSDE (2.2) has a unique solution $(Y^n, Z^n) \in S^2([0, T]; \mathbb{R}) \times M^2(0, T; \mathbb{R}^d)$.

Lemma 2.2 *Let us consider $(Y^n, Z^n) \in S^2([0, T]; \mathbb{R}) \times M^2(0, T; \mathbb{R}^d)$ solution of BDSDE (2.2). Then there exists $C > 0$ such that for each $n \in \mathbb{N}$*

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} |Y_t^n|^2 + \int_0^T \|Z_s^n\|^2 ds + |K_T^n|^2 \right) < C$$

where

$$K_t^n = n \int_0^t (Y_s^n - S_s)^- ds, \quad 0 \leq t \leq T.$$

Proof.

From Itô's formula, it follows that

$$\begin{aligned} |Y_t^n|^2 + \int_0^t |Z_s^n|^2 ds &= |\xi|^2 + 2 \int_0^t Y_s^n f(s, Y_s^n, Z_s^n) ds + 2 \int_0^t \langle Y_s^n g(s, Y_s^n, Z_s^n), dB_s \rangle \\ &\quad + \int_0^t \|g(s, Y_s^n, Z_s^n)\|^2 ds + 2 \int_0^t Y_s^n dK_s^n - 2 \int_0^t \langle Y_s^n Z_s^n, \downarrow dW_s \rangle. \end{aligned} \quad (2.3)$$

By using **(H1)** we have

$$\begin{aligned} 2Y_s^n f(s, Y_s^n, Z_s^n) &\leq \beta |Y_s^n| + \frac{1}{\beta} |f(s, Y_s^n, Z_s^n)|^2 \\ &\leq \left(\beta + \frac{C}{\beta}\right) |Y_s^n|^2 + \frac{C}{\beta} \|Z_s^n\|^2 + \frac{1}{\beta} |f(s, 0, 0)|^2 \end{aligned} \quad (2.4)$$

and

$$\begin{aligned} \|g(s, Y_s^n, Z_s^n)\|^2 &\leq (1 + \beta') \|g(s, Y_s^n, Z_s^n) - g(s, 0, 0)\|^2 + \left(\frac{1}{\beta'} + 1\right) \|g(s, 0, 0)\|^2 \\ &\leq C(1 + \beta') |Y_s^n|^2 + \alpha(1 + \beta') \|Z_s^n\|^2 + \left(\frac{1}{\beta'} + 1\right) \|g(s, 0, 0)\|^2. \end{aligned} \quad (2.5)$$

Moreover since $\int_0^t (Y_s^n - S_s) dK_s^n \leq 0$ we obtain

$$\begin{aligned} \int_0^t Y_s^n dK_s^n &= \int_0^t (Y_s^n - S_s) dK_s^n + \int_0^t S_s dK_s^n \\ &\leq \int_0^t S_s dK_s^n. \end{aligned} \quad (2.6)$$

If we choose β and β' such that $\frac{C}{\beta} + \alpha(1 + \beta') \leq 1$ it follows from (2.3) – (2.6) that

$$\begin{aligned} \mathbb{E} \left\{ |Y_t^n|^2 + \int_0^T \|Z_s^n\|^2 ds \right\} &\leq C \mathbb{E} \left\{ |\xi|^2 + \int_0^T |Y_s^n|^2 ds + \int_0^T |f(s, 0, 0)|^2 ds \right. \\ &\quad \left. + \int_0^T \|g(s, 0, 0)\|^2 ds + \int_0^T S_s dK_s^n \right\}. \end{aligned}$$

Gronwall's lemma applied to Y^n gives

$$\begin{aligned} \mathbb{E} \{ |Y_t^n|^2 \} &\leq C \mathbb{E} \left\{ |\xi|^2 + \int_0^T |f(s, 0, 0)|^2 ds \right. \\ &\quad \left. + \int_0^T \|g(s, 0, 0)\|^2 ds + \int_0^T S_s dK_s^n \right\}. \end{aligned} \quad (2.7)$$

Then we have

$$\begin{aligned} \mathbb{E} \left\{ \int_t^T \|Z_s^n\|^2 ds \right\} &\leq C' \mathbb{E} \left\{ |\xi|^2 + \int_0^T |f(s, 0, 0)|^2 ds \right. \\ &\quad \left. + \int_0^T \|g(s, 0, 0)\|^2 ds + \int_0^T S_s dK_s^n \right\}. \end{aligned} \quad (2.8)$$

We now give an estimation of $\mathbb{E}(|K_T^n|^2)$. From the equation

$$K_T^n = Y_T - \xi - \int_0^T f(s, Y_s^n, Z_s^n) ds - \int_0^T g(s, Y_s^n, Z_s^n) dB_s + \int_0^T Z_s^n \downarrow dW_s \quad (2.9)$$

and inequalities (2.7) and (2.8), we show the following inequalities

$$\begin{aligned} \mathbb{E} \{ |K_T^n|^2 \} &\leq C \mathbb{E} \left\{ |\xi|^2 + \int_0^T |f(s, 0, 0)|^2 ds \right. \\ &\quad \left. + \int_0^T \|g(s, 0, 0)\|^2 ds + \int_0^T S_s dK_s^n \right\} \\ &\leq C \mathbb{E} \left\{ |\xi|^2 + \int_0^T |f(s, 0, 0)|^2 ds \right. \\ &\quad \left. + \int_0^T \|g(s, 0, 0)\|^2 ds + \gamma \sup_{0 \leq t \leq T} (S_t^+)^2 \right\} + \frac{1}{\gamma} \mathbb{E}(|K_T^n|^2). \end{aligned}$$

Choosing γ such $\frac{C}{\gamma} < 1$ we get

$$\begin{aligned} \mathbb{E} \{ |K_T^n|^2 \} &\leq C \mathbb{E} \left\{ |\xi|^2 + \int_0^T |f(s, 0, 0)|^2 ds \right. \\ &\quad \left. + \int_0^T \|g(s, 0, 0)\|^2 ds + \sup_{0 \leq t \leq T} (S_t^+)^2 \right\}. \end{aligned} \quad (2.10)$$

Combining (2.7), (2.8) and (2.10), it follows that

$$\begin{aligned} \mathbb{E} \left\{ |Y_t^n|^2 + \int_0^T \|Z_s^n\|^2 ds + |K_T^n|^2 \right\} &\leq C \mathbb{E} \left\{ |\xi|^2 + \int_0^T |f(s, 0, 0)|^2 ds \right. \\ &\quad \left. + \int_0^T \|g(s, 0, 0)\|^2 ds + \sup_{0 \leq t \leq T} (S_t^+)^2 \right\}. \end{aligned} \quad (2.11)$$

Finally applied Burkholder-Gundy-Davis inequality we have

$$\mathbb{E} \left\{ \sup_{0 \leq t \leq T} |Y_t^n|^2 + \int_0^T \|Z_s^n\|^2 ds + |K_T^n|^2 \right\} \leq C \mathbb{E} \left\{ |\xi|^2 + \int_0^T |f(s, 0, 0)|^2 ds + \int_0^T \|g(s, 0, 0)\|^2 ds + \sup_{0 \leq t \leq T} (S_t^+)^2 \right\} \quad (2.12)$$

which end the prove of this Lemma ■

The main result of this section is the following.

Theorem 2.3 *Assume that (A1), (A2), (A3) hold. Moreover if the obstacle process (S_t) is the Itô process in the form $dS_t = m_t dt + n_t dB_t + v_t dW_t$,*

with $\mathbb{E} \left(\int_0^T (|m_s|^2 + |v_s|^2 + |n_s|^2) ds \right) < +\infty$. Then there exists a unique triple (Y, Z, K) solution of the reflected BDSDE (ξ, f, g, S) . Further,

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left(\sup_{0 \leq t \leq \tau} |K_t^n - K_t|^2 \right) = 0.$$

Proof.

This proof will be spilt in two steps. In the sequel, C denotes a positive constant which may vary from one line to another.

Existence

According it definition we have $f_n(t, y, z) \leq f_{n+1}(t, y, z)$. Thanks to the comparison theorem (see Th.3.1 in Yufeng Shi et al [20]) we deduce that the sequence $(Y^n)_{n>0}$ is non-decreasing. Hence there exists a progressively measurable process Y such that $Y_t^n \nearrow Y_t$ a.s. Then, according to Fatou's Lemma and Lemma 2.2, we deduce that

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} |Y_t|^2 \right) < +\infty.$$

Moreover, the Lebeque's dominated convergence theorem ensures that

$$\mathbb{E} \left(\int_0^T |Y_s^n - Y_s|^2 ds \right) \longrightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (2.13)$$

On the other hand, for any $n \geq p$, it follows by Itô's formula that

$$\begin{aligned} & |Y_t^n - Y_t^p|^2 + \int_0^t |Z_s^n - Z_s^p|^2 ds \\ = & 2 \int_t^T (Y_s^n - Y_s^p) [f(s, Y_s^n, Z_s^n) - f(s, Y_s^p, Z_s^p)] ds + \int_0^t \|g(s, Y_s^n, Z_s^n) - g(s, Y_s^p, Z_s^p)\|^2 ds \\ & + 2 \int_0^t (Y_s^n - Y_s^p) \langle (g(s, Y_s^n, Z_s^n) - g(s, Y_s^p, Z_s^p)), dB_s \rangle - 2 \int_0^t (Y_s^n - Y_s^p) \langle (Z_s^n - Z_s^p), \downarrow dW_s \rangle \\ & + 2 \int_0^t (Y_s^n - Y_s^p) (dK_s^n - dK_s^p). \end{aligned} \quad (2.14)$$

Following the same computation as in Lemma 2.1, taking the expectation and in virtue of assumptions (H1) we obtain

$$E \left\{ |Y_s^n - Y_s^p|^2 + \int_0^T \|Z_s^n - Z_s^p\|^2 ds \right\} \leq c \mathbb{E} \left\{ \int_0^T |Y_s^n - Y_s^p|^2 ds + \int_0^T (Y_s^n - Y_s^p)(dK_s^n - dK_s^p) \right\}. \quad (2.15)$$

But it clearly seen that

$$\mathbb{E} \left\{ \int_t^T (Y_s^n - Y_s^p)(dK_s^n - dK_s^p) \right\} \leq \mathbb{E} \left\{ \sup_{0 \leq s \leq T} (Y_s^n - S_s)^- K_T^n + \sup_{0 \leq s \leq T} (Y_s^p - S_s)^- K_T^p \right\}$$

and it follows from (2.15), the Hölder inequality and Lemma 2.1 that

$$E \left\{ |Y_s^n - Y_s^p|^2 + \int_0^T \|Z_s^n - Z_s^p\|^2 ds \right\} \leq c \mathbb{E} \left\{ \int_0^T |Y_s^n - Y_s^p|^2 ds + \sup_{0 \leq s \leq T} |(Y_s^n - S_s)^-|^2 + \sup_{0 \leq s \leq T} |(Y_s^p - S_s)^-|^2 \right\}. \quad (2.16)$$

If we admit for a moment the following

Lemma 2.4

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} |(Y_t^n - S_t)^-|^2 \right) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty;$$

then we can conclude now. Indeed by passing of limit in (2.16) and using Lemma 2.2, we deduce that

$$\mathbb{E} \left\{ |Y_s^n - Y_s^p|^2 + \int_0^T |Z_s^n - Z_s^p|^2 ds \right\} \longrightarrow 0, \quad \text{as } n, p \longrightarrow \infty. \quad (2.17)$$

Taking $\sup_{0 \leq s \leq T} (\cdot)$ in (2.14) and using Bürkhölder-Davis-Gundy's inequality, it holds that

$$\mathbb{E} \left(\sup_{0 \leq s \leq T} |Y_s^n - Y_s^p|^2 \right) \longrightarrow 0, \quad \text{as } n, p \longrightarrow \infty, \quad (2.18)$$

and then from (2.9), we derive easily

$$\mathbb{E} \left(\sup_{0 \leq s \leq T} |K_s^n - K_s^p|^2 \right) \longrightarrow 0, \quad \text{as } n, p \longrightarrow \infty. \quad (2.19)$$

According (2.17) and (2.19) we deduce that the sequence of processes (Y^n, Z^n, K^n) is a Cauchy sequence in the Banach space $S^2([0, T]; \mathbb{R}) \times M^2(0, T; \mathbb{R}^d) \times S^2([0, T]; \mathbb{R})$. Consequently there exists a triplet $(Y, Z, K) \in S^2([0, T]; \mathbb{R}) \times M^2(0, T; \mathbb{R}^d) \times S^2([0, T]; \mathbb{R})$ such that

$$\mathbb{E} \left\{ \sup_{0 \leq s \leq T} |Y_s^n - Y_s|^2 + \int_0^T |Z_s^n - Z_s|^2 ds + \sup_{0 \leq s \leq T} |K_s^n - K_s|^2 \right\} \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (2.20)$$

In view of Lemma 2.2 we have $Y_t \geq S_t$ a.s. Now, to conclude the proof of the existence, it remains to show (ii) of Definition 2.1. For instance let us remark that by (2.13) and (2.19) we get

$$\int_0^T (Y_s^n - S_s) dK_s^n \longrightarrow \int_0^T (Y_s - S_s) dK_s$$

in $L^1(\Omega)$, as $n \rightarrow \infty$. But since $Y_t \geq S_t$ a.s. and

$$\int_0^T (Y_s^n - S_s) dK_s^n = -n \int_0^T |(Y_s^n - S_s)^-|^2 ds \leq 0 \text{ a.s.}$$

we obtain

$$\int_0^T (Y_s - S_s) dK_s = 0, \text{ a.s.}$$

Finally passing to the limit in (2.2) we proved that (Y, Z, K) solves the reflected BDSDE associated to data (ξ, f, g, S) . Now to end let us return to the proof of Lemma 2.2.

Proof of Lemma 3.1

Since $Y_t^n \geq Y_t^0$, we can w.l.o.g. replace S_t by $S_t \vee Y_t^0$, i.e. we may assume that $\mathbb{E}(\sup_{0 \leq t \leq T} S_t^2) < \infty$.

Let $\{(\tilde{Y}_t^n, \tilde{Z}_t^n), 0 \leq t \leq T\}$ be the unique solution of the BDSDE

$$\begin{aligned} \tilde{Y}_t^n &= \xi + \int_0^t f(s, \tilde{Y}_s^n, \tilde{Z}_s^n) ds + \int_0^t g(s, \tilde{Y}_s^n, \tilde{Z}_s^n) dB_s \\ &\quad + n \int_0^t (S_s - \tilde{Y}_s^n) ds - \int_0^t \tilde{Z}_s^n \downarrow dW_s, \quad 0 \leq t \leq T \end{aligned}$$

From the comparison theorem for BDSDE (see [20]), we have for every $n \in \mathbb{N}^*$,

$$\tilde{Y}_t^n \leq Y_t^n, \text{ a.s.}$$

On the other hand, let ν be a stopping time such that $0 \leq \nu \leq T$. Hence it follows from Itô's formula applying to $e^{-n(\nu-t)} Y_t$ that

$$\begin{aligned} \tilde{Y}_\nu^n &= e^{-n\nu} \xi + \int_0^\nu e^{-n(\nu-s)} f(s, \tilde{Y}_s^n, \tilde{Z}_s^n) ds \\ &\quad + \int_0^\nu e^{-n(\nu-s)} g(s, \tilde{Y}_s^n, \tilde{Z}_s^n) dB_s + n \int_0^\nu e^{-n(\nu-s)} S_s ds \\ &\quad - \int_0^\nu e^{-n(\nu-s)} \tilde{Z}_s^n dW_s \end{aligned} \quad (2.21)$$

Taking the conditional expectation of both side of (2.21) we get

$$\begin{aligned}\tilde{Y}_\nu^n &= \mathbb{E} \left\{ e^{-n(\nu)}\xi + \int_0^\nu e^{-n(\nu-s)} f(s, \tilde{Y}_s^n, \tilde{Z}_s^n) ds \right. \\ &\quad \left. + \int_0^\nu e^{-n(\nu-s)} g(s, \tilde{Y}_s^n, \tilde{Z}_s^n) dB_s \right. \\ &\quad \left. + n \int_0^\nu e^{-n(\nu-s)} S_s ds \mid \mathcal{F}_\nu \right\}\end{aligned}\tag{2.22}$$

It easily seen that

$$e^{-n(\nu)}\xi + n \int_0^\nu e^{-n(s-\nu)} S_s ds \longrightarrow \xi \mathbf{1}_{\{\nu=0\}} + S_\nu \mathbf{1}_{\{\nu>0\}}\tag{2.23}$$

a.s. and in $L^2(\Omega)$, the conditional expectation converge also in $L^2(\Omega)$. Moreover, since

$$\left| \int_0^\nu e^{-n(s-\nu)} f(s, \tilde{Y}_s^n, \tilde{Z}_s^n) ds \right| \leq \frac{1}{\sqrt{2n}} \left(\int_0^\nu |f(s, \tilde{Y}_s^n, \tilde{Z}_s^n)|^2 ds \right)^{\frac{1}{2}}$$

and

$$\mathbb{E} \left(\int_0^\nu e^{-n(\nu-s)} g(s, \tilde{Y}_s^n, \tilde{Z}_s^n) dB_s \right) = 0$$

$$\begin{aligned}\tilde{Y}_\nu^n &= \mathbb{E} \left\{ \int_0^\nu e^{-n(\nu-s)} f(s, \tilde{Y}_s^n, \tilde{Z}_s^n) ds \right. \\ &\quad \left. + \int_0^\nu e^{-n(\nu-s)} g(s, \tilde{Y}_s^n, \tilde{Z}_s^n) dB_s \mid \mathcal{F}_\nu \right\} \longrightarrow 0\end{aligned}$$

in $L(\Omega)$, as $n \longrightarrow \infty$. Consequently,

$$\tilde{Y}_\nu^n \longrightarrow \xi \mathbf{1}_{\{\nu=0\}} + S_\nu \mathbf{1}_{\{\nu>0\}} \text{ in } L^2(\Omega)$$

and further

$$Y_\nu \geq S_\nu \text{ a.s.}\tag{2.24}$$

From (2.24) and the section theorem in Dellacherie and Meyer [10], (page 220), we get that

$$Y_t \geq S_t \text{ a.s.}$$

Hence $(Y_t^n - S_t)^- \downarrow 0$, *a.s.* and according Dini's theorem, the convergence is uniform in t . Finally, since $(Y_t^n - S_t)^- \leq (S_t - Y_t^0)^+ \leq |S_t| + |Y_t^0|$, the dominated convergence theorem ensures that

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left(\sup_{0 \leq t \leq T} |(Y_t^n - S_t)^-|^2 \right) = 0.$$

Uniqueness

Let us define

$$\{(\Delta Y_t, \Delta Z_t, \Delta K_t), 0 \leq t \leq T\} = \{(Y_t - Y'_t, Z_t - Z'_t, K_t - K'_t), 0 \leq t \leq T\}$$

where $\{(Y_t, Z_t, K_t), 0 \leq t \leq \tau\}$ and $\{(Y'_t, Z'_t, K'_t), 0 \leq t \leq T\}$ denote two solutions of the reflected BDSDE associated to the data (ξ, f, g, S) . Let us first note that

$$\int_t^T \Delta Y_s \Delta dK_s \leq 0. \quad (2.25)$$

Moreover, Itô formula yields that for every $0 \leq t \leq T$

$$\begin{aligned} & |\Delta Y_t|^2 + \int_0^t \|\Delta Z_s\|^2 ds \\ = & 2 \int_0^t \Delta Y_s (f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s)) ds + \int_0^t \|g(s, Y_s, Z_s) - g(s, Y'_s, Z'_s)\|^2 ds \\ & + \int_0^t \langle \Delta Y_s, (g(s, Y_s, Z_s) - g(s, Y'_s, Z'_s)) \rangle dB_s - 2 \int_0^t \langle \Delta Y_s, \Delta Z_s dW_s \rangle + 2 \int_0^t \Delta Y_s \Delta dK_s. \end{aligned}$$

Then, by using similar computation as the proof of existence and using (2.25) one have

$$\mathbb{E} \left\{ |\Delta Y_t|^2 + \int_0^T \|\Delta Z_s\|^2 ds \right\} \leq C \mathbb{E} \int_0^T |\Delta Y_s| ds$$

from which, we deduce that $\Delta Y_t = 0$ and further $\Delta Z_t = 0$. On the other hand since

$$\begin{aligned} \Delta K_t &= \Delta Y_t - \int_0^t f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s) ds \\ &\quad - \int_0^t (g(s, Y_s, Z_s) - g(s, Y'_s, Z'_s)) dB_s + \int_0^t \Delta Z_s dW_s, \end{aligned}$$

we have $\Delta K_t = 0$ which end the proof of the theorem.

3 Reflected BDSDE and obstacle problem for a non linear parabolic stochastic PDE

3.1 Preliminaries and definitions

According notations of subsection 2.1, let recall $\mathbf{F}^B = \{\mathcal{F}_t^B\}_{0 \leq t \leq T}$ and $\mathcal{M}_{0,T}^B$ denote all the \mathbf{F}^B -stopping times τ such $0 \leq \tau \leq T$, a.s. For generic Euclidean spaces \mathbf{E} and \mathbf{E}_1 we introduce the

following vector spaces of functions:

- for integers k, n the symbol $C^{k,n}([0, T] \times \mathbf{E}; \mathbf{E}_1)$ stands for the space of all \mathbf{E}_1 -valued functions defined on $[0, T] \times \mathbf{E}$ which are k -times continuously differentiable in t and n -times continuously differentiable in x , and $C_b^{k,n}([0, T] \times \mathbf{E}; \mathbf{E}_1)$ denotes the subspace of $C^{k,n}([0, T] \times \mathbf{E}; \mathbf{E}_1)$ in which all functions have uniformly bounded partial derivatives.
- For any sub- σ -field $\mathcal{G} \subseteq \mathcal{F}_T^B$, $C^{k,n}(\mathcal{G}, [0, T] \times \mathbf{E}; \mathbf{E}_1)$ (resp. $C_b^{k,n}(\mathcal{G}, [0, T] \times \mathbf{E}; \mathbf{E}_1)$) denote the space of all $C^{k,n}([0, T] \times \mathbf{E}; \mathbf{E}_1)$ (resp. $C_b^{k,n}([0, T] \times \mathbf{E}; \mathbf{E}_1)$)-valued random variable that are $\otimes \mathcal{B}([0, T] \times \mathbf{E})$ -measurable;
- $C^{k,n}(\mathbf{F}^B, [0, T] \times \mathbf{E}; \mathbf{E}_1)$ (resp. $C_b^{k,n}(\mathbf{F}^B, [0, T] \times \mathbf{E}; \mathbf{E}_1)$) is the space of all random fields $\phi \in C^{k,n}(\mathcal{F}_T, [0, T] \times \mathbf{E}; \mathbf{E}_1)$ (resp. $C_b^{k,n}(\mathcal{F}_T, [0, T] \times \mathbf{E}; \mathbf{E}_1)$), such that for fixed $x \in \mathbb{E}$, the mapping $(t, \omega_1) \rightarrow \phi(t, \omega_1, x)$ is \mathbf{F}^B -progressively measurable.
- For any sub- σ -field $\mathcal{G} \subseteq \mathcal{F}^B$ and a real number $p \geq 0$, $L^p(\mathcal{G}; \mathbf{E})$ to be all \mathbf{E} -valued \mathcal{G} -measurable random variable ξ such that $\mathbb{E}|\xi|^p < \infty$. Furthermore, for $(t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}$, we denote $D_x = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d})$, $D_{xx} = (\partial_{x_i x_j}^2)_{i,j=1}^d$, $D_y = \frac{\partial}{\partial y}$, $D_t = \frac{\partial}{\partial t}$. The meaning of D_{xy} and D_{yy} is then self-explanatory. In this section, we consider the continuous coefficients f, g as the following:

$$\begin{aligned} f & : \Omega_1 \times [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R} \\ g & : \Omega_1 \times [0, T] \times \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}^\ell \\ h & : \Omega_1 \times [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R} \end{aligned}$$

with the property that for all $x \in \mathbb{R}^d$, $f(\cdot, x, \cdot, \cdot)$ and $g(\cdot, x, \cdot)$ are Lipschitz continuous in x and satisfy the condition **(H1)** uniformly in x where for some constant $K > 0$. Moreover h is jointly continuous in (t, x) such that

$$|h(t, x)| \leq K(1 + \|x\|). \quad \text{(H2)}$$

Furthermore, we shall make use the following assumptions;

- (A4)** The function $\sigma : \mathbb{R}^d \longrightarrow \mathbb{R}^{d \times d}$ and $b : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ are uniformly Lipschitz continuous, with common Lipschitz constant $K > 0$.
- (A5)** The function $l : \mathbb{R}^d \longrightarrow \mathbb{R}$ is continuous, such that for some constant $K > 0$,

$$|l(x)| \leq K(1 + \|x\|) \text{ and } h(0, x) \leq l(x), \quad x \in \mathbb{R}^d.$$

- (A6)** The function $g \in \mathcal{C}_b^{0,2,3}([0, T] \times \mathbb{R}^d \times \mathbb{R}; \mathbb{R}^\ell)$.

Let us consider the related obstacle problem for stochastic partial differential equation. Roughly speaking, a solution of a stochastic obstacle problem of SPDE (f, g, h, l) is a random

field $u : \Omega_1 \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ which satisfies

$$(f, g, h, l) \begin{cases} \min \{u(t, x) - h(t, x), du(t, x) + [\mathbf{L}u(t, x) + f(t, x, u(t, x), \sigma^*(x)D_x u(t, x))]dt \\ \quad + g(t, x, u(t, x)) dB_s\} = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^d \\ u(0, x) = l(x), \quad x \in \mathbb{R}^d \end{cases}$$

where

$$\mathbf{L} = \frac{1}{2} \sum_{i,j}^d (\sigma(x)\sigma^*(x))_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i}.$$

More precisely, we shall consider the solution of SPDE associated to the data (f, g, f, h, l) in the two stochastic viscosity sense, inspired respectively by the work of Buckdahn and Ma [5, 6] and [7]. It will convenient for the sequel to define this two notion of stochastic viscosity for SPDE (f, g, h, l) . Let us remark that the second stochastic viscosity notion use stochastic g -superjet and g -subjet notion. So Before give that definition, it is reasonable to recall the notion of stochastic g -superjet or subjet.

Definition 3.1 *Let $\tau \in \mathcal{M}_{0,T}^B$ and $\xi \in L^0(\mathcal{F}_\tau^B; \mathbb{R}^d)$. We say that a sequence of random variables (τ_k, ξ_k) is a (τ, ξ) -approximating sequence if $(\tau_k, \xi_k) \in \mathcal{M}_{0,T}^B \times L^2(\mathcal{F}_\tau^B; \mathbb{R}^d), \forall k$ such that*

- (i) $\xi_k \rightarrow \xi$ in probability
- (ii) either $\tau_k \uparrow \tau$ a.s. and $\tau_k < \tau$ on $\{\tau > 0\}$.

If $\{\xi_k\}$ is a sequence of random variable that converge to 0 in probability, then we shall denote $\theta_k = o(\xi_k), k = 1, 2, \dots, \dots$ to be any sequence of random variables such that $[o(\xi_k)/\xi_k]I_{\{\xi_k \neq 0\}} \rightarrow 0$ as $k \rightarrow \infty$ in probability.

Definition 3.2 *Assume $g \in C^{0,1}([0, T] \times \mathbb{R}^{d+1}; \mathbb{R}^\ell)$. Let $(\tau, \xi) \in \mathcal{M}_{0,T}^B \times L^2(\mathcal{F}_\tau, \mathbb{R})$, and $u \in C(\mathbf{F}^B; [0, T] \times \mathbb{R}^d)$. A triplet (a, p, X) is called a stochastic g -superjet of u at (τ, ξ) if the following hold*

- (i) (a, b, c, p, q, X) is an $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}(n)$ -valued \mathcal{F}_τ -measurable random vector;
- (ii) denoting $\begin{cases} b = g(\tau, \xi, u(\tau, \xi)) & c = (g\partial_u g)(\tau, \xi, u(\tau, \xi)) \\ q = \partial_x g(\tau, \xi, u(\tau, \xi)) + \partial_u g(\tau, \xi, u(\tau, \xi))p \end{cases}$

then for any (τ, ξ) -approximation sequence (τ_k, ξ_k) it holds that

$$\begin{aligned} u(\tau_k, \xi_k) &\leq u(\tau, \xi) + a(\tau_k - \tau) + b(B_k - B_\tau) + \frac{c}{2}(B_k - B_\tau)^2 \\ &\quad + \langle p, \xi_k - \xi \rangle + \langle q, \xi_k - \xi \rangle (B_k - B_\tau) \\ &\quad + \frac{1}{2} \langle X(\xi_k - \xi), \xi_k - \xi \rangle + o(|\tau_k - \tau|) + o(|\xi_k - \xi|^2). \end{aligned} \quad (3.1)$$

We denote the set of all stochastic g -superjet of u at (τ, ξ) by $\mathcal{J}^{1,2+}u(\tau, \xi)$. Similarly, we say that the triple (a, p, X) is a stochastic g -subjet of u at (τ, ξ) if (i) holds and the inequality in (3.1) is reversed; and we denote the set of stochastic g -subjet by $\mathcal{J}^{1,2-}u(\tau, \xi)$.

Now we will be able to give the two definitions of stochastic viscosity solution of the reflected SPDE (f, g, h, l) . For this end let us recall some notations in [7]. Indeed, we consider process $\eta \in \mathcal{C}(\mathbf{F}^B, [0, T] \times \mathbb{R}^d \times \mathbb{R})$ as the solution to the equation

$$\eta(t, x, y) = y + \frac{1}{2} \int_0^t (gD_y g)(s, x, \eta(s, x, y)) ds + \int_0^t g(s, x, \eta(s, x, y)) ds.. \quad (3.2)$$

Further the mapping $y \mapsto \eta(s, x, y)$ defines a diffeomorphism for all t, x a.s. (see Protter, 1990 [?]). Denote the y -inverse of $\eta(s, x, y)$ by $\varepsilon(s, x, y)$. Then since $\varepsilon(s, x, \eta(s, x, y)) = y$, one can show that (see Buckdhan and Ma [5, 6])

$$\varepsilon(t, x, y) = y - \int_0^t D_y \varepsilon(s, x, y) g(s, x, y) \circ dB_s. \quad (3.3)$$

To simplify the notation in the sequel we denote

$$A_{f,g}(\varphi(t, x)) = \mathbf{L}\varphi(t, x) + f(t, x, \varphi(t, x), \sigma^* D_x \varphi(t, x)) - \frac{1}{2}(g, D_y g)(t, x, \varphi(t, x)).$$

Definition 3.3 (a) A random field $u \in \mathcal{C}(\mathbf{F}^B, [0, T] \times \mathbb{R}^d)$ is called a stochastic viscosity subsolution of the SPDE (f, g, h, l) if $u(0, x) \leq l(x)$, $x \in \mathbb{R}^d$ and if for any stopping time $\tau \in \mathcal{M}_{0,T}^B$, any state variable $\xi \in L^0(\mathcal{F}_\tau^B; \mathbb{R}^d)$, any random field $\varphi \in \mathcal{C}^{1,2}(\mathcal{F}_\tau^B, [0, T] \times \mathbb{R})$ with the property that for almost all $\omega \in \{0 < \tau < T\}$, $u(t, x) - \eta(t, x, \varphi(t, x)) \leq 0 = u(\tau(\omega), \xi(\omega)) - \eta(\tau(\omega), \xi(\omega), \varphi(\tau(\omega), \xi(\omega)))$ is fulfilled for all (t, x) in some neighborhood $\mathcal{V}(\omega, \tau(\omega), \xi(\omega))$ of $(\tau(\omega), \xi(\omega))$, the following condition is satisfied:

$$\min(u(\tau, \xi) - h(\tau, \xi), A_{f,g}(\psi(\tau, \xi)) - D_y \psi(\tau, \xi) D_t \varphi(\tau, \xi)) \leq 0$$

hold almost surely.

(b) A random field $u \in \mathcal{C}(\mathbf{F}^B, [0, T] \times \mathbb{R}^d)$ is called a stochastic viscosity supersolution of the SPDE (f, g, h, l) if $u(T, x) \geq l(x)$, $x \in \mathbb{R}^d$ and if for any stopping time $\tau \in \mathcal{M}_{0,t}^B$, any state variable $\xi \in L^0(\mathcal{F}_\tau^B; \mathbb{R}^d)$, any random field $\varphi \in \mathcal{C}^{1,2}(\mathcal{F}_\tau^B, [0, T] \times \mathbb{R})$ with the property that for almost all $\omega \in \{0 < \tau < T\}$, $u(t, x) - \eta(t, x, \varphi(t, x)) \geq 0 = u(\tau(\omega), \xi(\omega)) - \eta(\tau(\omega), \xi(\omega), \varphi(\tau(\omega), \xi(\omega)))$ is fulfilled for all (t, x) in some neighborhood $\mathcal{V}(\omega, \tau(\omega), \xi(\omega))$ of $(\tau(\omega), \xi(\omega))$, the following condition is satisfied:

$$\min(u(\tau, \xi) - h(\tau, \xi), A_{f,g}(\psi(\tau, \xi)) - D_y \psi(\tau, \xi) D_t \varphi(\tau, \xi)) \geq 0$$

hold almost surely.

(c) $u \in \mathcal{C}(\mathbf{F}^B, [0, T] \times \mathbb{R}^d)$ is said to be a stochastic solution of SPDE (f, g, h, l) if it both a stochastic viscosity subsolution and a stochastic viscosity supersolution.

Definition 3.4 (a) A random field $u \in \mathcal{C}(\mathbf{F}^B, [0, T] \times \mathbb{R}^d)$ is called a stochastic viscosity subsolution of SPDE (f, g, h, l) if $u(0, x) \leq l(x)$, $\forall x \in \mathbb{R}^d$ and for any $(\tau, \xi) \in \mathcal{M}_{0,T}^B \times L^2(\mathcal{F}_\tau^B; \mathbb{R}^d)$ and any $(a, p, X) \in \mathcal{J}^{1,2+}u(\tau, \xi)$ it holds that

$$\min \left(u(\tau, \xi) - h(\tau, \xi), -a - \frac{1}{2} \text{Tra}(\sigma \sigma^* X) - \langle b(x), p \rangle - f(\tau, \xi, u(\tau, \xi), p \sigma(\tau, \xi)) + \frac{1}{2} (g \partial_u g)(\tau, \xi, u(\tau, \xi)) \right) \leq 0. \quad (3.4)$$

In the other words at any $(\tau, \xi) \in \mathcal{M}_{0,T}^B \times L^2(\mathcal{F}_\tau^B; \mathbb{R}^d)$ where $u(\tau, \xi) > h(\tau, \xi)$,

$$-a - \frac{1}{2} \text{Tra}(\sigma \sigma^* X) - \langle b(x), p \rangle - f(\tau, \xi, u(\tau, \xi), p \sigma(\tau, \xi)) + \frac{1}{2} (g \partial_u g)(\tau, \xi, u(\tau, \xi)) \leq 0.$$

(b) u is said to be a stochastic viscosity supersolution if $u(0, x) \geq l(x)$, $\forall x \in \mathbb{R}^d$, and for any $(\tau, \xi) \in \mathcal{M}_{0,T}^B \times L^2(\mathcal{F}_\tau^B; \mathbb{R}^d)$ and any $(a, p, X) \in \mathcal{J}^{1,2-}u(\tau, \xi)$, (3.4) holds with the inequality being reversed. In the other words, at each $(\tau, \xi) \in \mathcal{M}_{0,T}^B \times L^2(\mathcal{F}_\tau^B; \mathbb{R}^d)$, we have both $u(\tau, \xi) \geq h(\tau, \xi)$ and

$$-a - \frac{1}{2} \text{Tra}(\sigma \sigma^* X) - \langle b(x), p \rangle - f(\tau, \xi, u(\tau, \xi), p \sigma(\tau, \xi)) + \frac{1}{2} (g \partial_u g)(\tau, \xi, u(\tau, \xi)) \geq 0.$$

(c) If u is both a stochastic viscosity subsolution and supersolution, we say that u is a stochastic viscosity solution of SPDE (f, g, h, l) .

Remark 3.5 Observe that if f, h are deterministic and $g \equiv 0$ the Definition 3.3 coincide with the deterministic case (see [13]).

3.2 Existence of stochastic viscosity solutions

In this subsection we apply the result of the previous section to prove the existence of stochastic viscosity solutions with respect Definition 3.4 to obstacle problem of quasi-linear SPDE. Let consider u a random field defined on $\Omega_1 \times [0, T] \times \mathbb{R}^d$ by

$$u(t, x) = Y_t^{t,x}, \quad (t, x) \in [0, T] \times \mathbb{R}^d, \quad (3.5)$$

which is \mathcal{F}_t^B -measurable for each $t \in [0, T]$; in the other words, $u \in C(\mathbf{F}^B, [0, T] \times \mathbb{R}^d)$.

Theorem 3.6 *u defined by (3.5), is the stochastic viscosity solution with respect Definition 3.4 of the obstacle problem of SPDE (f, g, l, h) .*

Proof. For each $(t, x) \in [0, T] \times \mathbb{R}^n, n \geq 1$, let $\{^n Y_s^{t,x}, ^n Z_s^{t,x}, 0 \leq s \leq t\}$ denote the solution of the BDSDE

$$\begin{aligned} ^n Y_s^{t,x} &= l(X_T^{t,x}) = \int_0^s f(r, X_r^{t,x}, ^n Y_r^{t,x}, ^n Z_r^{t,x}) dr + n \int_0^s (^n Y_r^{t,x} - h(r, X_r^{t,x}))^- dr \\ &\quad + \int_0^s g(r, X_r^{t,x}, ^n Y_r^{t,x}) dB_r - \int_0^s ^n Z_r^{t,x} dW_r. \end{aligned}$$

It is know from Buckdahn and Ma [7] that

$$u_n(t, x) = ^n Y_t^{t,x}, \quad (t, x) \in [0, T] \times \mathbb{R}^n,$$

is the stochastic viscosity solution of the parabolic SPDE:

$$\begin{cases} du_n(t, x) + [Lu_n(t, x) + f_n(t, x, u_n(t, x), \sigma^* D_x u_n(t, x))] dt \\ \quad + g_n(t, x, u_n(t, x)) dB_t = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^d \\ u_n(0, x) = l(x), \quad x \in \mathbb{R}^d, \end{cases}$$

where $f_n(t, x, y, z) = f(t, x, y, z) + n(y - h(t, x))^-$. But from the proof of theorem 2.1, for each $(t, x) \in [0, T] \times \mathbb{R}^d$ we have

$$u_n(t, x) \uparrow u(t, x) \quad a.s. \quad \text{as } n \rightarrow \infty.$$

Since u_n and u are continuous, it follows from Dini's theorem that the above convergence is uniform on the compacts. We now show that u is a stochastic viscosity subsolution of obstacle problem of SPDE (f, g, l, h) . Let $(\tau, \xi) \in \mathcal{M}_{0,T}^B \times L^2(\mathcal{F}_\tau^B; \mathbb{R}^d)$ at which $u(\tau, \xi) > h(\tau, \xi)$, and let $(a, p, X) \in \mathcal{J}^{1,2+} u(\tau, \xi)$. The stochastic version of Lemma 6.1 in Crandall and al [8], proved there exists sequences

$$\begin{aligned} n_k &\rightarrow +\infty \\ (\tau_k, \xi_k) &\rightarrow (\tau, \xi) \\ (a_k, p_k, X_k) &\in \mathcal{J}^{1,2+} u_{n_k}(\tau_k, \xi_k) \end{aligned}$$

such that

$$(a_k, p_k, X_k) \rightarrow (a, p, X).$$

But for any k , since $(a_k, p_k, X_k) \in \mathcal{J}^{1,2+}u_{n_k}(\tau k, \xi_k)$ and u_{n_k} is a stochastic viscosity solution of SPDE (f_n, g, l) we have

$$-a_k - \frac{1}{2}Tra(\sigma\sigma^*X_k) - \langle b, p \rangle - f_{n_k}(\tau_k, \xi_k, u_{n_k}(\tau_k, \xi_k), p_k\sigma(\tau_k, \xi_k)) + \frac{1}{2}(g\partial_u g)(\tau_k, \xi_k, u(\tau_k, \xi_k)) \leq 0. \quad (3.6)$$

On the other hand since $u(\tau, \xi) > h(\tau, \xi)$ and the uniform convergence of u_n , it follows that for k large enough $u(\tau_k, \xi_k) > h(\tau_k, \xi_k)$, hence by passing to the limit as $k \rightarrow \infty$ in (3.6) we get

$$-a - \frac{1}{2}Tra(\sigma\sigma^*X) - \langle b, p \rangle - f(\tau, \xi, u(\tau, \xi), p\sigma(\tau, \xi)) + \frac{1}{2}(g\partial_u g)(\tau, \xi, u(\tau, \xi)) \leq 0,$$

and we have proved that u given by (3.5) is a stochastic viscosity subsolution of SPDE (f, g, h, l) . Using the similarly argument with replace $\mathcal{J}^{1,2+}u_{n_k}(\tau k, \xi_k)$ by $\mathcal{J}^{1,2-}u_{n_k}(\tau k, \xi_k)$ it not difficult to prove that u given by (3.5) is a stochastic viscosity supersolution of SPDE (f, g, h, l) . So we conclude that u is a stochastic viscosity of SPDE (f, g, h, l) and finish the proof. ■

Remark 3.7 *If the function g takes a simpler form: $g(t, x, u) = g(t, x)u$ then according of Buckdahn and Ma work's [7], it follows easily that the two previous definitions of stochastic viscosity solution of refected SPDE (f, g, h, l) coincide.*

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