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**A CONVERGENT FINITE ELEMENT-FINITE VOLUME
SCHEME FOR THE COMPRESSIBLE STOKES PROBLEM
PART I – THE ISOTHERMAL CASE**

T. GALLOUËT, R. HERBIN, AND J.-C. LATCHÉ

ABSTRACT. In this paper, we propose a discretization for the compressible Stokes problem with a linear equation of state $\rho = p$, based on Crouzeix-Raviart elements. The approximation of the momentum balance is obtained by usual finite element techniques. Since the pressure is piecewise constant, the discrete mass balance takes the form of a finite volume scheme, in which we introduce an upwinding of the density, together with two additional stabilization terms. We prove *a priori* estimates for the discrete solution, which yields its existence by a topological degree argument, and then the convergence of the scheme to a solution of the continuous problem.

1. INTRODUCTION

The problem addressed in this paper is the system of the so-called barotropic compressible Stokes equations, which reads:

$$(1.1) \quad \left\{ \begin{array}{l} -\Delta \mathbf{u} + \nabla p = \mathbf{f} \\ \operatorname{div}(\rho \mathbf{u}) = 0 \\ \rho = \varrho(p) \end{array} \right.$$

where ρ , \mathbf{u} and p stand for the density, velocity and pressure in the flow, respectively, and \mathbf{f} is a forcing term. The function $\varrho(\cdot)$ is the equation of state used for the modelling of the particular flow at hand, which may be the actual equation of state of the fluid or may result from assumptions concerning the flow; typically, laws where ρ varies linearly with $p^{1/\gamma}$, where γ is a coefficient which is specific to the considered fluid, are obtained for isentropic flows. Here, we only consider the following equation of state, which corresponds to an isothermal flow of a perfect gas:

$$(1.2) \quad \varrho(p) = \operatorname{Ma}^2 p$$

where Ma is the Mach number in the system, provided that the range of the velocity is one. The extension of the present work to the isentropic case will be the object of a further paper.

This system of equations is posed over Ω , a domain of \mathbb{R}^d , $d \leq 3$ supposed to be polygonal ($d = 2$) or polyhedral ($d = 3$). It is supplemented by homogeneous

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boundary conditions for \mathbf{u} , and by prescribing the total mass M of the fluid:

$$(1.3) \quad \int_{\Omega} \rho \, d\mathbf{x} = M$$

This paper is organized as follows. The proposed scheme is first described (section 2). Then we prove an L^2 compactness result for sequences of Crouzeix-Raviart functions with bounded broken H^1 semi-norm (section 3); this compactness result then yields the convergence of (sub-)sequences of discrete solutions to a limit, thanks to *a priori* estimates which are given in section 4. Finally, this limit is shown to be a solution to the continuous problem in section 5.

2. THE NUMERICAL SCHEME

2.1. Discrete spaces. Let \mathcal{M} be a decomposition of the domain Ω in simplices. By $\mathcal{E}(K)$, we denote the set of the edges ($d = 2$) or faces ($d = 3$) σ of the element $K \in \mathcal{M}$; for short, each edge or face will be called an edge hereafter. The set of all edges of the mesh is denoted by \mathcal{E} ; the set of edges included in the boundary of Ω is denoted by \mathcal{E}_{ext} and the set of internal ones (*i.e.* $\mathcal{E} \setminus \mathcal{E}_{\text{ext}}$) is denoted by \mathcal{E}_{int} . The decomposition \mathcal{M} is supposed to be regular in the usual sense of the finite element literature (*e.g.* [2]), and, in particular, \mathcal{M} satisfies the following properties: $\bar{\Omega} = \bigcup_{K \in \mathcal{M}} \bar{K}$; if $K, L \in \mathcal{M}$, then $\bar{K} \cap \bar{L} = \emptyset$ or $\bar{K} \cap \bar{L}$ is a common face of K and L , which is denoted by $K|L$. For each internal edge of the mesh $\sigma = K|L$, \mathbf{n}_{KL} stands for the normal vector of σ , oriented from K to L (so that $\mathbf{n}_{KL} = -\mathbf{n}_{LK}$). By $|K|$ and $|\sigma|$ we denote the measure, respectively, of the element K and of the edge σ , and h_K and h_σ stand for the diameter of K and σ , respectively. We measure the regularity of the mesh through the parameter θ defined by:

$$(2.1) \quad \theta = \inf \left\{ \frac{\xi_K}{h_K}, K \in \mathcal{M} \right\} \cup \left\{ \frac{h_L}{h_K}, \frac{h_K}{h_L}, \sigma = K|L \in \mathcal{E}_{\text{int}} \right\}$$

where ξ_K stands for the diameter of the largest ball included in K . Note that, $\forall K \in \mathcal{M}, \forall \sigma \in \mathcal{E}(K)$, the inequality $h_\sigma |\sigma| \leq 2 \theta^{-d} |K|$ holds; this relation will be used throughout this paper. Finally, as usual, we denote by h the quantity $\max_{K \in \mathcal{M}} h_K$.

The space discretization relies on the Crouzeix-Raviart element (see [3] for the seminal paper and, for instance, [4, p. 199–201] for a synthetic presentation). The reference element is the unit d -simplex and the discrete functional space is the space P_1 of affine polynomials. The degrees of freedom are determined by the following set of nodal functionals:

$$(2.2) \quad \{F_\sigma, \sigma \in \mathcal{E}(K)\}, \quad F_\sigma(v) = |\sigma|^{-1} \int_\sigma v \, d\gamma$$

The mapping from the reference element to the actual one is the standard affine mapping. Finally, the continuity of the average value of the discrete functions (*i.e.* $F_\sigma(v)$) for a discrete function v , across each face of the mesh is required, thus the discrete space V_h is defined as follows:

$$(2.3) \quad V_h = \left\{ v \in L^2(\Omega) : v|_K \in P_1(K), \forall K \in \mathcal{M}; F_\sigma(v) \text{ continuous across each edge } \sigma \in \mathcal{E}_{\text{int}}; F_\sigma(v) = 0, \forall \sigma \in \mathcal{E}_{\text{ext}} \right\}$$

The space of approximation for the velocity is the space \mathbf{W}_h of vector valued functions each component of which belongs to V_h : $\mathbf{W}_h = (V_h)^d$. The pressure is approximated by the space L_h of piecewise constant functions:

$$L_h = \{q \in L^2(\Omega) : q|_K = \text{constant}, \forall K \in \mathcal{M}\}$$

Since only the continuity of the integral over each edge of the mesh is imposed, the functions of V_h are discontinuous through each edge; the discretization is thus nonconforming in $H^1(\Omega)^d$. We then define, for $1 \leq i \leq d$ and $u \in V_h$, $\partial_{h,i} u$ as the function of $L^2(\Omega)$ which is equal to the derivative of u with respect to the i^{th} space variable almost everywhere. This notation allows to define the discrete gradient, denoted by ∇_h , for both scalar and vector valued discrete functions and the discrete divergence of vector valued discrete functions, denoted by div_h .

The Crouzeix-Raviart pair of approximation spaces for the velocity and the pressure is *inf-sup* stable, in the usual sense for "piecewise H^1 " discrete velocities, *i.e.* there exists $c_i > 0$ independent of the mesh such that:

$$\forall p \in L_h, \quad \sup_{\mathbf{v} \in \mathbf{W}_h} \frac{\sum_{K \in \mathcal{M}} \int_K p \text{div} \mathbf{v} \, d\mathbf{x}}{\|\mathbf{v}\|_{1,b}} = \sup_{\mathbf{v} \in \mathbf{W}_h} \frac{\int_{\Omega} p \text{div}_h \mathbf{v} \, d\mathbf{x}}{\|\mathbf{v}\|_{1,b}} \geq c_i \|p - p_m\|_{L^2(\Omega)}$$

where p_m is the mean value of p over Ω and $\|\cdot\|_{1,b}$ stands for the broken Sobolev H^1 semi-norm, which is defined for scalar as well as for vector-valued functions by:

$$\|v\|_{1,b}^2 = \sum_{K \in \mathcal{M}} \int_K |\nabla v|^2 \, d\mathbf{x} = \int_{\Omega} |\nabla_h v|^2 \, d\mathbf{x}$$

We also define a discrete semi-norm on L_h , similar to the usual H^1 semi-norm used in the finite volume context, weighted by a mesh-dependent coefficient:

$$\forall p \in L_h, \quad |p|_{\mathcal{M},\beta}^2 = \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}}, \\ \sigma = K|L}} (h_K + h_L)^\beta \frac{|\sigma|}{h_\sigma} (p_K - p_L)^2$$

From the definition (2.2), each velocity degree of freedom can be univoquely associated to an element edge. Consequently, the velocity degrees of freedom are indexed by the number of the component and the associated edge, thus the set of velocity degrees of freedom reads:

$$\{v_{\sigma,i}, \sigma \in \mathcal{E}_{\text{int}}, 1 \leq i \leq d\}$$

We denote by φ_σ the usual Crouzeix-Raviart shape function associated to σ , *i.e.* the scalar function of V_h such that $F_\sigma(\varphi_\sigma) = 1$ and $F_{\sigma'}(\varphi_\sigma) = 0$, $\forall \sigma' \in \mathcal{E} \setminus \{\sigma\}$.

Similarly, each degree of freedom for the pressure is associated to a cell K , and the set of pressure degrees of freedom is denoted by $\{p_K, K \in \mathcal{M}\}$.

Finally, we define by r_h the following interpolation operator:

$$(2.4) \quad r_h : \begin{cases} H_0^1(\Omega) & \longrightarrow & V_h \\ u & \longmapsto & r_h u = \sum_{\sigma \in \mathcal{E}} F_\sigma(u) \varphi_\sigma = \sum_{\sigma \in \mathcal{E}} |\sigma|^{-1} \left(\int_\sigma v \, d\gamma \right) \varphi_\sigma \end{cases}$$

This operator naturally extends to vector-valued functions (*i.e.* to perform the interpolation from $H_0^1(\Omega)^d$ to \mathbf{W}_h), and we keep the same notation r_h for both the

scalar and vector case. The properties of r_h are gathered in the following lemma. They are proven in [3].

Lemma 2.1. *Let $\theta_0 > 0$ and let \mathcal{M} be a triangulation of the computational domain Ω such that $\theta > \theta_0$, where θ is defined by (2.1). The interpolation operator r_h enjoys the following properties:*

(1) *preservation of the divergence:*

$$\forall \mathbf{v} \in \mathbf{H}_0^1(\Omega)^d, \forall q \in L_h, \quad \int_{\Omega} q \operatorname{div}_h(r_h \mathbf{v}) \, d\mathbf{x} = \int_{\Omega} q \operatorname{div} \mathbf{v} \, d\mathbf{x}$$

(2) *stability:*

$$\forall v \in \mathbf{H}_0^1(\Omega), \quad \|v\|_{1,b} \leq c_1(\theta_0) |v|_{\mathbf{H}^1(\Omega)}$$

(3) *approximation properties:*

$$\forall v \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega), \forall K \in \mathcal{M},$$

$$\|v - r_h v\|_{\mathbf{L}^2(K)} + h_K \|\nabla_h(v - r_h v)\|_{\mathbf{L}^2(K)} \leq c_2(\theta_0) h_K^2 |v|_{\mathbf{H}^2(K)}$$

In both above inequalities, the notation $c_i(\theta_0)$ means that the real number c_i only depend on θ_0 , and, in particular, not on the parameter h characterizing the size of the cells; this notation will be kept throughout the paper.

2.2. The numerical scheme. Let ρ^* be the mean density, *i.e.* $\rho^* = M/|\Omega|$ where $|\Omega|$ stands for the measure of the domain Ω . We consider the following numerical scheme for the discretization of Problem (1.1):

$$(2.5) \quad \left\{ \begin{array}{l} \forall v \in \mathbf{W}_h, \\ \int_{\Omega} \nabla_h \mathbf{u} : \nabla_h \mathbf{v} \, d\mathbf{x} - \int_{\Omega} p \operatorname{div}_h \mathbf{v} \, d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x} \\ \forall K \in \mathcal{M}, \\ \sum_{\sigma=K|L} v_{\sigma,K}^+ \varrho(p_K) - v_{\sigma,K}^- \varrho(p_L) + \underbrace{h^\alpha |K| (\varrho(p_K) - \rho^*)}_{T_{\text{stab},1}} \\ + \underbrace{\sum_{\sigma=K|L} (h_K + h_L)^\beta \frac{|\sigma|}{h_\sigma} |\varrho(p_K) + \varrho(p_L)| (\varrho(p_K) - \varrho(p_L))}_{T_{\text{stab},2}} = 0 \end{array} \right.$$

where $v_{\sigma,K}^+$ and $v_{\sigma,K}^-$ stands respectively for $v_{\sigma,K}^+ = \max(v_{\sigma,K}, 0)$ and $v_{\sigma,K}^- = -\min(v_{\sigma,K}, 0)$ with $v_{\sigma,K} = |\sigma| \mathbf{u}_\sigma \cdot \mathbf{n}_{KL} = v_{\sigma,K}^+ - v_{\sigma,K}^-$. The first equation may be considered as the standard finite element discretization of the first relation of (1.1). Since the pressure is piecewise constant, the finite element discretization of the second relation of (1.1), *i.e.* the mass balance, is similar to a finite volume formulation, in which we introduce the standard first-order upwinding and two stabilizing terms. The first one, *i.e.* $T_{\text{stab},1}$, guarantees that the integral of the density over the computational domain is always M (this can easily be seen by summing over the second relation for $K \in \mathcal{M}$). The second one, *i.e.* $T_{\text{stab},2}$, seems to be necessary in the convergence analysis. It may be seen as a finite volume analogue of a continuous term of the form $\operatorname{div}|\rho| \nabla \rho$ weighted by a mesh-dependent coefficient tending to zero as h^β ; note, however, that h_σ is not the distance which is

usually encountered in the finite volume discretization of diffusion terms; moreover, the usual restrictions for the mesh when diffusive terms are to be approximated (namely, to satisfy a Delaunay condition) are not required here. We will suppose that $\alpha \geq 1$ and the convergence analysis will necessitate $1 < \beta < 2$.

Remark 2.2. At first glance, leaving the weight $|\rho|$ out, the stabilization term $T_{\text{stab},2}$ may look as a Brezzi-Pitkäranta regularisation [1], as used in [7] for stabilizing the collocated approximation of the Stokes problem, which would be rather puzzling since we use here an *inf-sup* stable pair of approximation spaces. However, using the equation of state (1.2), we obtain:

$$T_{\text{stab},2} = \text{Ma}^4 \sum_{\sigma=K|L} (h_K + h_L)^\beta \frac{|\sigma|}{h_\sigma} |p_K + p_L| (p_K - p_L)$$

which shows that this term rapidly vanishes when approaching the incompressible limit.

3. A COMPACTNESS RESULT

The aim of this section is to state and prove a compactness result for sequences of discrete functions bounded with respect to the $\|\cdot\|_{1,b}$ norm. We begin by a preliminary lemma.

Lemma 3.1. *Let $\theta_0 > 0$ and let \mathcal{M} be a triangulation of the computational domain Ω such that $\theta > \theta_0$, where θ is defined by (2.1), and V_h be the space of Crouzeix-Raviart discrete functions associated to \mathcal{M} , as defined by (2.3). Then there exists a real number $c(\theta_0)$ such that the following bound holds for any $v \in V_h$:*

$$\sum_{\sigma \in \mathcal{E}} \frac{1}{h_\sigma} \int_\sigma [v]^2 d\gamma \leq c(\theta_0) \|v\|_{1,b}^2$$

where, on any $\sigma \in \mathcal{E}_{\text{int}}$, $[v]$ stands for the jump of v across σ and, on any $\sigma \in \mathcal{E}_{\text{ext}}$, $[v] = v$.

Proof. For any control volume K of the mesh, we denote by $(\nabla v)_K$ the (constant) gradient of the restriction of v to K . With this notation, using the continuity of v across σ at the mass center \mathbf{x}_σ of any internal edge and the fact that v vanishes at the mass center \mathbf{x}_σ of any external edge, we get:

$$\begin{aligned} \sum_{\sigma \in \mathcal{E}} \frac{1}{h_\sigma} \int_\sigma [v]^2 d\gamma &= \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}}, \\ \sigma=K|L}} \frac{1}{h_\sigma} \int_\sigma ((\nabla v)_K - (\nabla v)_L) \cdot (\mathbf{x} - \mathbf{x}_\sigma)^2 d\gamma \\ &\quad + \sum_{\substack{\sigma \in \mathcal{E}_{\text{ext}}, \\ \sigma \in \mathcal{E}(K)}} \frac{1}{h_\sigma} \int_\sigma ((\nabla v)_K) \cdot (\mathbf{x} - \mathbf{x}_\sigma)^2 d\gamma \end{aligned}$$

We thus have:

$$\sum_{\sigma \in \mathcal{E}} \frac{1}{h_\sigma} \int_\sigma [v]^2 d\gamma \leq 2 \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}}, \\ \sigma=K|L}} h_\sigma |\sigma| (|(\nabla v)_K|^2 + |(\nabla v)_L|^2) + \sum_{\substack{\sigma \in \mathcal{E}_{\text{ext}}, \\ \sigma \in \mathcal{E}(K)}} h_\sigma |\sigma| |(\nabla v)_K|^2$$

and the result follows by regularity of the mesh. \square

The following lemma will be useful in the subsequent developments. A sketch of its proof is given in appendix.

Lemma 3.2. *Let $\theta_0 > 0$ and let \mathcal{M} be a triangulation of the computational domain Ω such that $\theta > \theta_0$, where θ is defined by (2.1); for $\sigma \in \mathcal{E}$, let \mathcal{X}_σ be the function defined by:*

$$\mathcal{X}_\sigma : \begin{cases} \mathbb{R}^d \times \mathbb{R}^d & \longrightarrow \{0, 1\} \\ (\mathbf{x}, \mathbf{y}) & \mapsto \mathcal{X}_\sigma(\mathbf{x}, \mathbf{y}) = 1 \text{ if } [\mathbf{x}, \mathbf{y}] \cap \sigma \neq \emptyset, \mathcal{X}_\sigma(\mathbf{x}, \mathbf{y}) = 0 \text{ otherwise.} \end{cases}$$

where \mathbf{x} and \mathbf{y} are two points of \mathbb{R}^d . Then there exists a family of positive real numbers $(d_\sigma)_{\sigma \in \mathcal{E}}$ such that:

- (1) for any $\sigma \in \mathcal{E}$, $d_\sigma \geq c_1(\theta_0) h_\sigma$,
- (2) for any points \mathbf{x} and \mathbf{y} of \mathbb{R}^d (possibly located outside Ω), the following inequality holds:

$$\sum_{\sigma \in \mathcal{E}} \mathcal{X}_\sigma(\mathbf{x}, \mathbf{y}) d_\sigma \leq c_2(\theta_0) (|\mathbf{y} - \mathbf{x}| + h)$$

The following bound provides an estimate of the translates of a discrete function v as a function of $\|v\|_{1,b}$.

Lemma 3.3. *Let $\theta_0 > 0$ and let \mathcal{M} be a triangulation of the computational domain Ω such that $\theta > \theta_0$, where θ is defined by (2.1); let V_h be the space of Crouzeix-Raviart discrete functions associated to \mathcal{M} , as defined by (2.3). Let v be a function of V_h ; we denote by \tilde{v} the extension by zero of v to \mathbb{R}^d . Then the following estimate holds:*

$$\forall \boldsymbol{\eta} \in \mathbb{R}^d, \quad \|\tilde{v}(\cdot + \boldsymbol{\eta}) - \tilde{v}(\cdot)\|_{L^2(\mathbb{R}^d)}^2 \leq c(\theta_0) |\boldsymbol{\eta}| (|\boldsymbol{\eta}| + h) \|v\|_{1,b}^2$$

Proof. We follow the proof of a similar result for piecewise constant functions, namely [5, lemma 9.3, pp. 770-772]. Let $\boldsymbol{\eta} \in \mathbb{R}^d$ be given, v be a Crouzeix-Raviart discrete function and \tilde{v} its extension by zero to \mathbb{R}^d . With the definition of the function \mathcal{X}_σ of lemma 3.2, the following identity holds for any $\mathbf{x} \in \mathbb{R}^d$:

$$\tilde{v}(\mathbf{x} + \boldsymbol{\eta}) - \tilde{v}(\mathbf{x}) = \underbrace{\sum_{\sigma \in \mathcal{E}} \mathcal{X}_\sigma(\mathbf{x}, \mathbf{x} + \boldsymbol{\eta}) [v](\mathbf{y}_{\mathbf{x}, \boldsymbol{\eta}, \sigma})}_{T_1(\mathbf{x})} + \underbrace{\int_{\mathbf{x}}^{\mathbf{x} + \boldsymbol{\eta}} \nabla_h \tilde{v} \cdot \frac{\boldsymbol{\eta}}{|\boldsymbol{\eta}|} ds}_{T_2(\mathbf{x})}$$

where $\mathbf{y}_{\mathbf{x}, \boldsymbol{\eta}, \sigma}$ stands for the intersection between the line issued from \mathbf{x} and of direction $\boldsymbol{\eta}$ and the hyperplane containing σ . Defining for each edge σ of the mesh a real positive number d_σ such that lemma 3.2 holds, by the Cauchy-Schwarz inequality, we get for $T_1(\mathbf{x})$:

$$(T_1(\mathbf{x}))^2 \leq \left(\sum_{\sigma \in \mathcal{E}} \mathcal{X}_\sigma(\mathbf{x}, \mathbf{x} + \boldsymbol{\eta}) \frac{[v](\mathbf{y}_{\mathbf{x}, \boldsymbol{\eta}, \sigma})^2}{d_\sigma} \right) \left(\sum_{\sigma \in \mathcal{E}} \mathcal{X}_\sigma(\mathbf{x}, \mathbf{x} + \boldsymbol{\eta}) d_\sigma \right)$$

Integrating now over \mathbb{R}^d , we thus obtain:

$$\int_{\mathbb{R}^d} (T_1(\mathbf{x}))^2 d\mathbf{x} \leq c_1(\theta_0) (|\boldsymbol{\eta}| + h) \left(\sum_{\sigma \in \mathcal{E}} \int_{\mathbb{R}^d} \mathcal{X}_\sigma(\mathbf{x}, \mathbf{x} + \boldsymbol{\eta}) \frac{[v](\mathbf{y}_{\mathbf{x}, \boldsymbol{\eta}, \sigma})^2}{d_\sigma} d\mathbf{x} \right)$$

The function $s \mapsto \mathcal{X}_\sigma(\mathbf{x}, \mathbf{x} + \boldsymbol{\eta})$ is in fact the characteristic function of a parallelepiped $Q_{\sigma, \boldsymbol{\eta}}$, two opposite sides of which are σ and $\sigma - \boldsymbol{\eta} = \{\mathbf{x} \in \mathbb{R}^d \text{ such that } \mathbf{x} +$

$\boldsymbol{\eta} \in \sigma\}$, and the function $\boldsymbol{x} \mapsto [v](\boldsymbol{y}_{\boldsymbol{x}, \boldsymbol{\eta}, \sigma})$ takes over each segment $\{\boldsymbol{y}_\sigma - t\boldsymbol{\eta}, \boldsymbol{y}_\sigma \in \sigma, t \in [0, 1]\}$ the constant value $[v](\boldsymbol{y}_\sigma)$. Hence the following equality holds:

$$\int_{\mathbb{R}^d} \mathcal{X}_\sigma(\boldsymbol{x}, \boldsymbol{x} + \boldsymbol{\eta}) [v](\boldsymbol{y}_{\boldsymbol{x}, \boldsymbol{\eta}, \sigma})^2 d\boldsymbol{x} = |\boldsymbol{\eta}| c_\sigma \int_\sigma [v]^2 d\gamma$$

where c_σ stands for:

$$c_\sigma = \left| \boldsymbol{n}_\sigma \cdot \frac{\boldsymbol{\eta}}{|\boldsymbol{\eta}|} \right|$$

and \boldsymbol{n}_σ is a vector normal to σ . Finally, we thus get:

$$\int_{\mathbb{R}^d} (T_1(\boldsymbol{x}))^2 d\boldsymbol{x} \leq c_3(\theta_0) (|\boldsymbol{\eta}| + h) |\boldsymbol{\eta}| \sum_{\sigma \in \mathcal{E}} \frac{c_\sigma}{d_\sigma} \int_\sigma [v]^2 d\gamma$$

and thus, by choice of d_σ :

$$(3.1) \quad \int_{\mathbb{R}^d} (T_1(\boldsymbol{x}))^2 d\boldsymbol{x} \leq c_4(\theta_0) (|\boldsymbol{\eta}| + h) |\boldsymbol{\eta}| \sum_{\sigma \in \mathcal{E}} \frac{1}{h_\sigma} \int_\sigma [v]^2 d\gamma$$

On the other hand, by the Cauchy-Schwarz inequality, we have for T_2 :

$$(T_2(\boldsymbol{x}))^2 \leq |\boldsymbol{\eta}| \int_{\boldsymbol{x}}^{\boldsymbol{x} + \boldsymbol{\eta}} |\nabla_h \tilde{v}|^2 ds$$

and thus:

$$\int_{\mathbb{R}^d} (T_2(\boldsymbol{x}))^2 d\boldsymbol{x} \leq |\boldsymbol{\eta}| \int_{\mathbb{R}^d} \left[\int_{\boldsymbol{x}}^{\boldsymbol{x} + \boldsymbol{\eta}} |\nabla_h \tilde{v}|^2 \right] d\boldsymbol{x}$$

Using the Fubini theorem and remarking that $\nabla_h \tilde{v}$ vanishes outside Ω , we thus get:

$$(3.2) \quad \int_{\mathbb{R}^d} (T_2(\boldsymbol{x}))^2 d\boldsymbol{x} \leq |\boldsymbol{\eta}|^2 \|v\|_{1,b}^2$$

The results then follows by using $|\tilde{v}(\boldsymbol{x} + \boldsymbol{\eta}) - \tilde{v}(\boldsymbol{x})|^2 \leq 2(T_1(\boldsymbol{x}))^2 + 2(T_2(\boldsymbol{x}))^2$, collecting the bounds (3.1) and (3.2) and invoking lemma 3.1. \square

We are now in position to state the following compactness result.

Theorem 3.4. *Let $(v^{(m)})_{m \in \mathbb{N}}$ be a sequence of functions satisfying the following assumptions:*

- (1) $\forall m \in \mathbb{N}$, there exists a triangulation of the domain $\mathcal{M}^{(m)}$ such that $v^{(m)} \in V_h^{(m)}$, where $V_h^{(m)}$ is the space of Crouzeix-Raviart discrete functions associated to $\mathcal{M}^{(m)}$, as defined by (2.3), and the parameter $\theta^{(m)}$ characterizing the regularity of $\mathcal{M}^{(m)}$ is bounded away from zero independently of m ,
- (2) the sequence $(v^{(m)})_{m \in \mathbb{N}}$ is uniformly bounded with respect to the broken Sobolev H^1 semi-norm, i.e.:

$$\forall m \in \mathbb{N}, \quad \|v^{(m)}\|_{1,b} \leq C$$

where C is a constant real number and $\|\cdot\|_{1,b}$ stands for the broken Sobolev H^1 semi-norm associated to $\mathcal{M}^{(m)}$ (with a slight abuse of notation, namely dropping, for short, the index $^{(m)}$ pointing the dependence of the norm with respect to the mesh).

Then, possibly up to the extraction of a subsequence, the sequence $(v^{(m)})_{m \in \mathbb{N}}$ converges strongly in $L^2(\Omega)$ to a limit \bar{v} such that $\bar{v} \in H_0^1(\Omega)$.

Proof. The result follows from the translates estimates of lemma 3.3. The compactness in $L^2(\Omega)$ of the sequence is a consequence of the Kolmogorov theorem (see e.g. [5, theorem 14.1, p. 833] for a statement of this result). The fact that the limits belong to $H_0^1(\Omega)$ follows from the particular expression for the bound of the translates and is proven in [5, theorem 14.2, pp. 833-834]. \square

4. EXISTENCE OF A SOLUTION AND A PRIORI ESTIMATES

The existence of a solution to (2.5) follows, with minor changes to cope with the diffusion stabilization term, from the theory developed in [9, section 2]. In this latter paper, it is obtained for fairly general equations of state by a topological degree argument. We only give here the obtained result, together with a proof of the *a priori* estimates verified by the solution, and we refer to [9] for the proof of existence.

Theorem 4.1. *Let $\theta_0 > 0$ and let \mathcal{M} be a triangulation of the computational domain Ω such that $\theta > \theta_0$, where θ is defined by (2.1). The problem (2.5) admits at least one solution $(\mathbf{u}, p) \in \mathbf{W}_h \times L_h$, and any possible solution is such that $p_K > 0, \forall K \in \mathcal{M}$, and satisfies:*

$$(4.1) \quad \|\mathbf{u}\|_{1,b} + \|p\|_{L^2(\Omega)} + \|\rho\|_{L^2(\Omega)} + |\rho|_{\mathcal{M},\beta} \leq C$$

where the real number C only depends on the data of the problem Ω , Ma , \mathbf{f} , M and, in a non-decreasing way, on θ_0 .

Proof. Let $(\mathbf{u}, p) \in \mathbf{W}_h \times L_h$ be a solution to (2.5). Let $\rho_K = \varrho(p_K)$, and let ρ denote the vector $(\rho_K)_{K \in \mathcal{M}}$. Reordering equations and unknowns, the second set of equations of (2.5) leads to a linear system of the form $M\rho = c$, where $c \in \mathbb{R}^N$, N is the number discretization cells, $c \in \mathbb{R}^N$, $c > 0$, and where M is an M -matrix (in particular $M^{-1} \geq 0$ and $M^{-t} \geq 0$). Therefore the i -th component of ρ reads $\rho_i = M^{-1}c \cdot e_i = c \cdot M^{-t}e_i$ where e_i is the i -th canonical basis vector of \mathbb{R}^N . Since $M^{-t} \geq 0$, we get $M^{-t}e_i \geq 0$, and since $M^{-t}e_i \neq 0$, this proves that $\rho_i > 0$, which, in turns, yields $p_K > 0, \forall K \in \mathcal{M}$. Let us then prove the estimate (4.1). To this end, we take $\mathbf{v} = \mathbf{u}$ in the first equation of the discrete system (2.5) and obtain:

$$(4.2) \quad \int_{\Omega} |\nabla_h \mathbf{u}|^2 \, d\mathbf{x} - \int_{\Omega} p \, \text{div}_h \mathbf{u} \, d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \mathbf{u} \, d\mathbf{x}$$

Then we multiply the second equation by $\text{Ma}^{-2}[1 + \log(\varrho(p_K))]$ and we sum over each control volume of the mesh; dropping the terms which vanish by conservativity, we then obtain:

$$T_1 + T_2 + T_3 = 0 \quad \text{with:}$$

$$\left| \begin{array}{l} T_1 = \text{Ma}^{-2} \sum_{K \in \mathcal{M}} \log(\varrho(p_K)) \sum_{\sigma=K|L} v_{\sigma,K}^+ \varrho(p_K) - v_{\sigma,K}^- \varrho(p_L) \\ T_2 = \text{Ma}^{-2} h^\alpha \sum_{K \in \mathcal{M}} |K| [1 + \log(\varrho(p_K))] [\varrho(p_K) - \rho^*] \\ T_3 = \text{Ma}^{-2} \sum_{K \in \mathcal{M}} \log(\varrho(p_K)) \sum_{\sigma=K|L} (h_K + h_L)^\beta \frac{|\sigma|}{h_\sigma} \\ \qquad \qquad \qquad (\varrho(p_K) + \varrho(p_L)) (\varrho(p_K) - \varrho(p_L)) \end{array} \right.$$

where, from the formulation of the scheme, the term $|\varrho(p_K) + \varrho(p_L)|$ has been changed to $\varrho(p_K) + \varrho(p_L)$ thanks to the positivity of the pressure. Let us first write T_1 as:

$$T_1 = \text{Ma}^{-2} \sum_{K \in \mathcal{M}} \log(\rho_K) \sum_{\sigma=K|L} v_{\sigma,K} \rho_\sigma$$

where $\rho_K = \varrho(p_K)$ and ρ_σ is either ρ_K (if $v_{\sigma,K} \geq 0$) or ρ_L (if $v_{\sigma,K} < 0$). Adding and subtracting the same quantity, T_1 equivalently reads:

$$\begin{aligned} T_1 = \text{Ma}^{-2} \sum_{K \in \mathcal{M}} \rho_K \sum_{\sigma=K|L} v_{\sigma,K} + \text{Ma}^{-2} \sum_{K \in \mathcal{M}} \sum_{\sigma=K|L} v_{\sigma,K} (-\rho_K) \\ + \text{Ma}^{-2} \sum_{K \in \mathcal{M}} \log(\rho_K) \sum_{\sigma=K|L} v_{\sigma,K} \rho_\sigma \end{aligned}$$

In the first summation, we recognize $\int_{\Omega} p \operatorname{div}_h \mathbf{u} \, d\mathbf{x}$. Reordering the other terms, we get:

$$T_1 = \int_{\Omega} p \operatorname{div}_h \mathbf{u} \, d\mathbf{x} + \text{Ma}^{-2} \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}}, \\ \sigma=K|L}} v_{\sigma,K} [(\rho_\sigma \log(\rho_K) - \rho_K) - (\rho_\sigma \log(\rho_L) - \rho_L)]$$

Let $\bar{\rho}_\sigma$ be defined as follows:

$$\left| \begin{array}{ll} \text{If } \rho_K = \rho_L : & \bar{\rho}_\sigma = \rho_K = \rho_L \\ \text{Otherwise, } \bar{\rho}_\sigma \text{ is given by:} & \bar{\rho}_\sigma \log(\rho_K) - \rho_K = \bar{\rho}_\sigma \log(\rho_L) - \rho_L \end{array} \right.$$

It may be shown that $\min(\rho_K, \rho_L) \leq \bar{\rho}_\sigma \leq \max(\rho_K, \rho_L)$ (see remark 4.2 below). With this notation, we get:

$$T_1 = \int_{\Omega} p \operatorname{div}_h \mathbf{u} \, d\mathbf{x} + \text{Ma}^{-2} \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}}, \\ \sigma=K|L}} v_{\sigma,K} (\rho_\sigma - \bar{\rho}_\sigma) (\log(\rho_K) - \log(\rho_L))$$

In the last summation, we can, without loss of generality, choose the orientation of each edge in such a way that $v_{\sigma,K} \geq 0$. With this convention, the term in the summation reads $v_{\sigma,K} (\rho_\sigma - \bar{\rho}_\sigma) (\log(\rho_K) - \log(\rho_L))$, and is seen to be non-negative, since $\rho_\sigma \in [\min(\rho_K, \rho_L), \max(\rho_K, \rho_L)]$ and the function $\log(\cdot)$ is increasing. We thus finally obtain:

$$(4.3) \quad T_1 \geq \int_{\Omega} p \operatorname{div}_h \mathbf{u} \, d\mathbf{x}$$

Let us now turn to the estimate of T_2 . As the function $z \mapsto z \log(z)$ is convex for z positive and its derivative is $z \mapsto 1 + \log(z)$, we simply have:

$$(4.4) \quad T_2 \geq \text{Ma}^{-2} h^\alpha \sum_{K \in \mathcal{M}} |K| [\rho_K \log(\rho_K) - \rho^* \log(\rho^*)]$$

Finally, reordering the sums, the term T_3 reads:

$$T_3 = \text{Ma}^{-2} \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}}, \\ \sigma=K|L}} (h_K + h_L)^\beta \frac{|\sigma|}{h_\sigma} (\rho_K + \rho_L) (\rho_K - \rho_L) (\log(\rho_K) - \log(\rho_L))$$

By concavity of the $\log(\cdot)$ function, we have:

$$|\log(\rho_K) - \log(\rho_L)| \geq \frac{1}{\max(\rho_K, \rho_L)} |\rho_K - \rho_L|$$

and thus:

$$(4.5) \quad T_3 \geq \text{Ma}^{-2} \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}}, \\ \sigma = K|L}} (h_K + h_L)^\beta \frac{|\sigma|}{h_\sigma} (\rho_K - \rho_L)^2$$

Summing equations (4.2)–(4.5) and using Young’s inequality, we obtain:

$$\|\mathbf{u}\|_{1,b} + \text{Ma}^{-1} |\rho|_{\mathcal{M},\beta} \leq C(\mathbf{f}, M)$$

In addition, summing the second relation of (2.5) over the control volumes of the mesh, the mean value of the pressure p_m is given by:

$$p_m = \frac{1}{|\Omega|} \int_{\Omega} p \, d\mathbf{x} = \text{Ma}^{-2} \rho^*$$

Using the *inf-sup* stability of the discretization, we get on the other hand:

$$\begin{aligned} \|p - p_m\|_{L^2(\Omega)} &\leq \frac{1}{c_i} \sup_{\mathbf{v} \in \mathcal{W}_h} \frac{1}{\|\mathbf{v}\|_{1,b}} \int_{\Omega} p \, \text{div}_h \mathbf{v} \, d\mathbf{x} \\ &= \sup_{\mathbf{v} \in \mathcal{W}_h} \frac{1}{\|\mathbf{v}\|_{1,b}} \int_{\Omega} (\nabla_h \mathbf{u} : \nabla_h \mathbf{v} - \mathbf{f} \cdot \mathbf{v}) \, d\mathbf{x} \end{aligned}$$

and the control of $\|p\|_{L^2(\Omega)}$ (or, equivalently, $\|\rho\|_{L^2(\Omega)}$) follows from the estimate for $\|\mathbf{u}\|_{1,b}$. \square

Remark 4.2 (On the choice of $(\log(\rho_K))_{K \in \mathcal{M}}$ as test function). At first look, the choice of $\log(\rho_K)$ to multiply the second equation of (2.5) in the preceding proof may seem rather puzzling. In fact, this development is a particular case of the so-called “elastic potential identity”, wellknown at the continuous level and central in *a priori* estimates for compressible Navier-Stokes equations [10, 11, 8]. An analog of this identity is proven at the discrete level, for the same discretizations as here, in [9, theorem 2.1].

For the particular case under consideration, an elementary explanation of this choice may be given. Indeed, it is crucial in the above proof that the quantity $\bar{\rho}_\sigma$ lies in the interval $[\min(\rho_K, \rho_L), \max(\rho_K, \rho_L)]$. Let us suppose, without loss of generality, that $0 < \rho_K < \rho_L$ and that, instead of the function $\log(\cdot)$, the computation is performed with a non-specified increasing and continuously differentiable function $f(\cdot)$; then we get:

$$\bar{\rho}_\sigma = \frac{\rho_L - \rho_K}{f(\rho_L) - f(\rho_K)}$$

The condition $\bar{\rho}_\sigma \geq \rho_K$ is equivalent to:

$$\frac{1}{\rho_K} \geq \frac{f(\rho_L) - f(\rho_K)}{\rho_L - \rho_K}$$

which is verified for $f(\cdot) = \log(\cdot)$ by concavity of the latter and, letting ρ_L tend to ρ_K , yields $f'(x) \leq 1/x$. Conversely, the condition $\bar{\rho}_\sigma \leq \rho_L$ yields:

$$\frac{1}{\rho_L} \leq \frac{f(\rho_L) - f(\rho_K)}{\rho_L - \rho_K}$$

which, once again, is verified by the function $\log(\cdot)$, and now implies $f'(x) \geq 1/x$.

This limitation for the choice of the test function is the reason for the expression of the stabilizing diffusion term.

5. CONVERGENCE ANALYSIS

We first recall a trace lemma, the proof of which can be found in [13, section 3].

Lemma 5.1. *Let \mathcal{M} be a given triangulation of Ω and K be a control volume of \mathcal{M} , h_K its diameter and σ one of its edges. Let u be a function of $H^1(K)$. Then the following inequality holds:*

$$\|u\|_{L^2(\sigma)} \leq \left(d \frac{|\sigma|}{|K|} \right)^{1/2} \left(\|u\|_{L^2(K)} + h_K \|\nabla u\|_{L^2(K)} \right)$$

We will also need the following Poincaré inequality:

$$(5.1) \quad \forall K \in \mathcal{M}, \forall u \in H^1(K), \quad \|u - u_{m,K}\|_{L^2(K)} \leq \frac{1}{\pi} h_K \|\nabla u\|_{L^2(K)}$$

where $u_{m,K}$ stands for the mean value of u over K . This relation is a consequence of the more general result for any convex domain proven in [12].

We are now in position to prove the following technical result.

Lemma 5.2. *Let $\theta_0 > 0$ and let \mathcal{M} be a triangulation of the computational domain Ω such that $\theta > \theta_0$, where θ is defined by (2.1); let u be a function of the Crouzeix-Raviart space V_h associated to \mathcal{M} and f be a function of $H_0^1(\Omega)$. Then the following bound holds:*

$$\sum_{\sigma \in \mathcal{E}_{\text{int}}} \left| \int_{\sigma} [u] f \, d\gamma \right| \leq c(\theta_0) h \|u\|_{1,b} \|f\|_{H^1(\Omega)}$$

Proof. Since the integral of the jump across any edge of the mesh of a function of V_h is zero, we have:

$$\sum_{\sigma \in \mathcal{E}_{\text{int}}} \int_{\sigma} [u] f \, d\gamma = \sum_{\sigma \in \mathcal{E}_{\text{int}}} \int_{\sigma} [u] (f - f_{\sigma}) \, d\gamma$$

where $(f_{\sigma})_{\sigma \in \mathcal{E}_{\text{int}}}$ is any family of real numbers. Using the Cauchy-Schwarz inequality, first in $L^2(\sigma)$ then in $\mathbb{R}^{\text{card}(\mathcal{E})}$ we thus get:

$$\begin{aligned} \sum_{\sigma \in \mathcal{E}_{\text{int}}} \left| \int_{\sigma} [u] f \, d\gamma \right| &\leq \sum_{\sigma \in \mathcal{E}_{\text{int}}} \left[\int_{\sigma} [u]^2 \, d\gamma \right]^{1/2} \left[\int_{\sigma} (f - f_{\sigma})^2 \, d\gamma \right]^{1/2} \\ &\leq \left[\sum_{\sigma \in \mathcal{E}_{\text{int}}} \frac{1}{h_{\sigma}} \int_{\sigma} [u]^2 \, d\gamma \right]^{1/2} \underbrace{\left[\sum_{\sigma \in \mathcal{E}_{\text{int}}} h_{\sigma} \int_{\sigma} (f - f_{\sigma})^2 \, d\gamma \right]^{1/2}}_{T_1} \end{aligned}$$

By lemma 3.1, the first term of the latter product is bounded by $c(\theta_0) \|u\|_{1,b}^2$. For the second one, choosing arbitrarily one adjacent simplex to each edge and applying the above trace lemma 5.1, we get:

$$T_1^2 \leq \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ (\sigma \in \mathcal{E}(K))}} 2d h_{\sigma} \frac{|\sigma|}{|K|} \left(\|f - f_{\sigma}\|_{L^2(K)}^2 + h_K^2 \|\nabla f\|_{L^2(K)}^2 \right)$$

Choosing for f_σ the mean value of f on K and using (5.1), we thus get:

$$T_1^2 \leq \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ (\sigma \in \mathcal{E}(K))}} 2d \left(1 + \frac{1}{\pi^2}\right) h_\sigma \frac{|\sigma|}{|K|} h_K^2 \|\nabla f\|_{L^2(K)}^2$$

and the result follows by observing that the H^1 semi-norm of f on K appears at most $(d+1)$ times in the summation and using the regularity of the mesh. \square

We then have the following convergence result.

Theorem 5.3. *Let a sequence of triangulations $(\mathcal{M}^{(m)})_{m \in \mathbb{N}}$ of Ω be given. We suppose that $h^{(m)}$ tends to zero when m tends to $+\infty$. In addition, we assume that the sequence of discretizations is regular, in the sense that there exists $\theta_0 > 0$ such that $\theta^{(m)} \geq \theta_0$, $\forall m \in \mathbb{N}$. For $m \in \mathbb{N}$, we denote by $\mathbf{W}_h^{(m)}$ and $L_h^{(m)}$ respectively the discrete spaces for the velocity and the pressure associated to $\mathcal{M}^{(m)}$ and by $(\mathbf{u}^{(m)}, p^{(m)}) \in \mathbf{W}_h^{(m)} \times L_h^{(m)}$ the corresponding solution to the discrete problem (2.5), with $\alpha \geq 1$ and $1 < \beta < 2$. Then, up to the extraction of a subsequence, the sequence $(\mathbf{u}^{(m)})_{m \in \mathbb{N}}$ strongly converges to a limit $\bar{\mathbf{u}}$ in $L^2(\Omega)^d$ and $(p^{(m)})_{m \in \mathbb{N}}$ converges to \bar{p} weakly in $L^2(\Omega)$, where the pair $(\bar{\mathbf{u}}, \bar{p})$ is solution to the continuous problem (1.1) in the following weak sense:*

$\bar{\mathbf{u}} \in H_0^1(\Omega)^d$, $\bar{p} \in L^2(\Omega)$ and :

$$\left\{ \begin{array}{l} \int_{\Omega} \nabla \bar{\mathbf{u}} : \nabla \psi \, d\mathbf{x} - \int_{\Omega} \bar{p} \operatorname{div} \psi \, d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \psi \, d\mathbf{x} \quad \forall \psi \in C_c^\infty(\Omega)^d \\ \int_{\Omega} \bar{p} \bar{\mathbf{u}} \cdot \nabla \psi \, d\mathbf{x} = 0 \quad \forall \psi \in C_c^\infty(\Omega) \\ \int_{\Omega} \varrho(\bar{p}) = M \end{array} \right.$$

Proof. The proof is divided in three steps: we first show the existence of the limits $\bar{\mathbf{u}}$ and \bar{p} , then we pass to the limit in the first equation of the scheme and, finally, in the second one. Since the equation of state is linear, the last equation is then a straightforward consequence of the weak convergence in $L^2(\Omega)$ of the (sub)sequence $(p^{(m)})_{m \in \mathbb{N}}$ to \bar{p} .

Step 1: existence of a limit.

By the *a priori* estimates of theorem 4.1, we know that:

$$\forall m \in \mathbb{N}, \quad \|\mathbf{u}^{(m)}\|_{1,b} \leq C(\mathbf{f}, M)$$

The compactness in $L^2(\Omega)^d$ of the sequence $(\mathbf{u}^{(m)})_{m \in \mathbb{N}}$ thus follows by applying theorem 3.4 to each component $u_i^{(m)}$, $1 \leq i \leq d$, together with the fact that the limit $\bar{\mathbf{u}}$ lies in $H_0^1(\Omega)^d$. Once again by theorem 4.1, we have:

$$\forall m \in \mathbb{N}, \quad \|p^{(m)}\|_{L^2(\Omega)} \leq C(\mathbf{f}, M)$$

which is sufficient to ensure a weak convergence in $L^2(\Omega)$ of the sequence $(p^{(m)})_{m \in \mathbb{N}}$ to $\bar{p} \in L^2(\Omega)$.

Step 2: passing to the limit in the first equation.

Let $\boldsymbol{\psi}$ be a function of $C_c^\infty(\Omega)^d$. We denote by $\boldsymbol{\psi}^{(m)}$ the interpolation of $\boldsymbol{\psi}$ in $\mathbf{W}_h^{(m)}$, i.e. $\boldsymbol{\psi}^{(m)} = r_h^{(m)}\boldsymbol{\psi}$. Taking $v = \boldsymbol{\psi}^{(m)}$ in the first equation of (2.5), we get, $\forall m \in \mathbb{N}$:

$$\int_{\Omega} \nabla_h \mathbf{u}^{(m)} : \nabla_h \boldsymbol{\psi}^{(m)} \, d\mathbf{x} - \int_{\Omega} p^{(m)} \operatorname{div}_h \boldsymbol{\psi}^{(m)} \, d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\psi}^{(m)} \, d\mathbf{x}$$

Since the considered interpolation operator preserves the divergence (first relation of lemma 2.1), we have:

$$\int_{\Omega} p^{(m)} \operatorname{div}_h \boldsymbol{\psi}^{(m)} \, d\mathbf{x} = \int_{\Omega} p^{(m)} \operatorname{div} \boldsymbol{\psi} \, d\mathbf{x} \longrightarrow \int_{\Omega} \bar{p} \operatorname{div} \boldsymbol{\psi} \, d\mathbf{x} \quad \text{as } m \longrightarrow +\infty$$

By the approximation properties of the interpolation operator (third estimate of lemma 2.1) invoked component by component, we have:

$$\int_{\Omega} \mathbf{f} \cdot \boldsymbol{\psi}^{(m)} \, d\mathbf{x} \longrightarrow \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\psi} \, d\mathbf{x} \quad \text{as } m \longrightarrow \infty$$

Finally, we have:

$$\begin{aligned} \int_{\Omega} \nabla_h \mathbf{u}^{(m)} : \nabla_h \boldsymbol{\psi}^{(m)} \, d\mathbf{x} = \\ \underbrace{\int_{\Omega} \nabla_h \mathbf{u}^{(m)} : \nabla_h (\boldsymbol{\psi}^{(m)} - \boldsymbol{\psi}) \, d\mathbf{x}}_{T_1} + \underbrace{\int_{\Omega} \nabla_h \mathbf{u}^{(m)} : \nabla_h \boldsymbol{\psi} \, d\mathbf{x}}_{T_2} \end{aligned}$$

Once again by lemma 2.1 (third relation), the term T_1 obeys the following estimate:

$$|T_1| \leq \|\mathbf{u}^{(m)}\|_{1,b} \|\boldsymbol{\psi}^{(m)} - \boldsymbol{\psi}\|_{1,b} \leq c(\theta_0) h^{(m)} \|\mathbf{u}^{(m)}\|_{1,b} |\boldsymbol{\psi}|_{H^2(\Omega)}$$

and thus tends to zero as m tends to $+\infty$. Integrating by parts over each control volume, the term T_2 reads:

$$T_2 = - \int_{\Omega} \mathbf{u}^{(m)} \cdot \Delta \boldsymbol{\psi} \, d\mathbf{x} + \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(m)}} \int_{\sigma} [\mathbf{u}^{(m)}] \nabla \boldsymbol{\psi} \cdot \mathbf{n}_{\sigma} \, d\gamma$$

where \mathbf{n}_{σ} is a normal vector to σ , with the same orientation as that of the jump through σ . Applying lemma 5.2 component by component, the last term of this equation can be estimated as follows:

$$\sum_{\sigma \in \mathcal{E}_{\text{int}}^{(m)}} \int_{\sigma} [\mathbf{u}^{(m)}] \nabla \boldsymbol{\psi} \cdot \mathbf{n}_{\sigma} \, d\gamma \leq c(\theta_0) h^{(m)} \|\mathbf{u}^{(m)}\|_{1,b} |\boldsymbol{\psi}|_{H^2(\Omega)}$$

and thus tends to zero, while the first one tends to $-\int_{\Omega} \bar{\mathbf{u}} \cdot \Delta \boldsymbol{\psi} \, d\mathbf{x}$ as m tends to $+\infty$. Since $\bar{\mathbf{u}} \in H_0^1(\Omega)^d$, we may integrate by parts, and collecting the limits, we obtain:

$$\int_{\Omega} \nabla \bar{\mathbf{u}} : \nabla \boldsymbol{\psi} \, d\mathbf{x} - \int_{\Omega} \bar{p} \operatorname{div} \boldsymbol{\psi} \, d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\psi} \, d\mathbf{x}$$

which is the relation we are seeking.

Step 3: passing to the limit in the second equation.

Let ψ be a function of $C_c^\infty(\Omega)$. Multiplying the second equation of (2.5) by $1/|K| \psi$ and integrating over Ω yields $T_3^{(m)} + T_4^{(m)} + T_5^{(m)} = 0$, $\forall m \in \mathbb{N}$, with:

$$\begin{aligned} T_3^{(m)} &= \sum_{K \in \mathcal{M}^{(m)}} \frac{1}{|K|} \left(\sum_{\sigma=K|L} v_{\sigma,K}^{(m)} \rho_\sigma^{(m)} \right) \int_K \psi \, d\mathbf{x} \\ T_4^{(m)} &= (h^{(m)})^\alpha \sum_{K \in \mathcal{M}^{(m)}} \left(\rho_K^{(m)} - \rho^* \right) \int_K \psi \, d\mathbf{x} \\ T_5^{(m)} &= \sum_{K \in \mathcal{M}^{(m)}} \frac{1}{|K|} \\ &\quad \left(\sum_{\sigma=K|L} (h_K + h_L)^\beta \frac{|\sigma|}{h_\sigma} \left(\rho_K^{(m)} + \rho_L^{(m)} \right) \left(\rho_K^{(m)} - \rho_L^{(m)} \right) \right) \int_K \psi \, d\mathbf{x} \end{aligned}$$

Let $\mathbf{q}^{(m)} \in \mathbf{W}_h$ be defined as $\mathbf{q}^{(m)}(\mathbf{x}) = \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(m)}} \mathbf{u}_\sigma^{(m)} \rho_\sigma^{(m)} \varphi_\sigma(\mathbf{x})$. The divergence of $\mathbf{q}^{(m)}$ is a piecewise constant function and reads:

$$\forall K \in \mathcal{M}^{(m)}, \quad \operatorname{div} \mathbf{q}^{(m)} = \frac{1}{|K|} \sum_{\sigma=K|L} v_{\sigma,K}^{(m)} \rho_\sigma^{(m)} \quad \text{a.e. in } K$$

We thus have for $T_3^{(m)}$:

$$T_3^{(m)} = \sum_{K \in \mathcal{M}^{(m)}} \int_K \psi \operatorname{div} \mathbf{q}^{(m)} \, d\mathbf{x}$$

Integrating by parts over each control volume, we get:

$$\begin{aligned} T_3^{(m)} &= - \int_\Omega \nabla \psi \cdot \mathbf{q}^{(m)} \, d\mathbf{x} + \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(m)}} \int_\sigma \psi [\mathbf{q}^{(m)}] \cdot \mathbf{n}_\sigma \, d\gamma \\ &= - \int_\Omega \nabla \psi \cdot (\rho^{(m)} \mathbf{u}^{(m)}) \, d\mathbf{x} \\ &\quad + \underbrace{\sum_{\sigma \in \mathcal{E}_{\text{int}}^{(m)}} \int_\sigma \psi [\mathbf{q}^{(m)}] \cdot \mathbf{n}_\sigma \, d\gamma}_{T_6^{(m)}} + \underbrace{\int_\Omega \nabla \psi \cdot (\mathbf{q}^{(m)} - \rho^{(m)} \mathbf{u}^{(m)}) \, d\mathbf{x}}_{T_7^{(m)}} \end{aligned}$$

By the weak and strong convergence respectively of $(\rho^{(m)})_{m \in \mathbb{N}}$ and $(\mathbf{u}^{(m)})_{m \in \mathbb{N}}$ to $\bar{\rho}$ and $\bar{\mathbf{u}}$ in $L^2(\Omega)$ and $L^2(\Omega)^d$ respectively, we have:

$$\int_\Omega \nabla \psi \cdot (\rho^{(m)} \mathbf{u}^{(m)}) \, d\mathbf{x} \longrightarrow \int_\Omega \nabla \psi \cdot (\bar{\rho} \bar{\mathbf{u}}) \, d\mathbf{x} \quad \text{as } m \longrightarrow +\infty$$

The term $T_6^{(m)}$ can be estimated as follows:

$$|T_6^{(m)}| \leq c_\psi h^{(m)} \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(m)}} \rho_\sigma^{(m)} \int_\sigma [\mathbf{u}^{(m)}] \cdot \mathbf{n}_\sigma \, d\gamma$$

where c_ψ only depends on ψ . Using the Cauchy-Schwarz inequality then yields:

$$\begin{aligned} |T_6^{(m)}| &\leq c_\psi h^{(m)} \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(m)}} |\sigma|^{1/2} \rho_\sigma^{(m)} \int_\sigma |[\mathbf{u}^{(m)}]|^2 d\gamma \\ &\leq c_\psi h^{(m)} \left[\sum_{\sigma \in \mathcal{E}_{\text{int}}^{(m)}} h_\sigma |\sigma| (\rho_\sigma^{(m)})^2 \right]^{1/2} \left[\sum_{\sigma \in \mathcal{E}_{\text{int}}^{(m)}} \frac{1}{h_\sigma} \int_\sigma |[\mathbf{u}^{(m)}]|^2 d\gamma \right]^{1/2} \end{aligned}$$

By the regularity of the mesh, the first summation is bounded by $\|\rho^{(m)}\|_{L^2(\Omega)}$ while, by lemma 3.1, the second one is bounded by $c(\theta_0) \|\mathbf{u}^{(m)}\|_{1,b}^2$. Consequently $T_6^{(m)}$ tends to zero as m tends to $+\infty$. On the other side, we have for $T_7^{(m)}$:

$$T_7^{(m)} = \sum_{K \in \mathcal{M}^{(m)}} \int_K \sum_{\sigma=K|L} (\rho_\sigma^{(m)} - \rho_K^{(m)}) \varphi_\sigma(\mathbf{x}) \mathbf{u}_\sigma^{(m)} \cdot \nabla \psi(\mathbf{x}) d\mathbf{x}$$

Since $\nabla \psi$ is bounded in $L^\infty(\Omega)^d$, and since the functions φ_σ are bounded, we get:

$$|T_7^{(m)}| \leq c_\psi \sum_{K \in \mathcal{M}^{(m)}} |K| \sum_{\sigma=K|L} |\rho_\sigma^{(m)} - \rho_K^{(m)}| |\mathbf{u}_\sigma^{(m)}|$$

Reordering the summations and using the Cauchy-Schwarz inequality yields:

$$|T_7^{(m)}| \leq c_\psi \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}}^{(m)} \\ (\sigma=K|L)}} (|K| + |L|) |\rho_K^{(m)} - \rho_L^{(m)}| |\mathbf{u}_\sigma^{(m)}| \leq c_\psi \left(T_8^{(m)}\right)^{1/2} \left(T_9^{(m)}\right)^{1/2}$$

with:

$$\begin{aligned} T_8^{(m)} &= \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}}^{(m)} \\ (\sigma=K|L)}} (|K| + |L|) |\mathbf{u}_\sigma^{(m)}|^2 \\ T_9^{(m)} &= \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}}^{(m)} \\ (\sigma=K|L)}} h_\sigma (h_K + h_L)^{(1-\gamma)} \frac{|K| + |L|}{|\sigma| (h_K + h_L)} (h_K + h_L)^\gamma \frac{|\sigma|}{h_\sigma} (\rho_K^{(m)} - \rho_L^{(m)})^2 \end{aligned}$$

Reordering the summations, we get:

$$T_8^{(m)} = \sum_{K \in \mathcal{M}} |K| \sum_{\sigma \in \mathcal{E}_K} |\mathbf{u}_\sigma^{(m)}|^2$$

and thus, the term $T_8^{(m)}$ is controlled by $\|\mathbf{u}^{(m)}\|_{L^2(\Omega)}$, while, by the *a priori* estimate (4.1), $T_9^{(m)}$ tends to zero for $\gamma < 2$.

We now turn to the terms $T_4^{(m)}$ and $T_5^{(m)}$. Since the sequence $(\rho^{(m)})_{m \in \mathbb{N}}$ is bounded in $L^2(\Omega)$, the term $T_4^{(m)}$ tends to zero as soon as $\alpha > 0$. Let us denote by ψ_K the mean value of ψ over K . With this notation, $T_5^{(m)}$ reads:

$$T_5^{(m)} = \sum_{K \in \mathcal{M}^{(m)}} \psi_K \left(\sum_{\sigma=K|L} (h_K + h_L)^\beta \frac{|\sigma|}{h_\sigma} \left(\rho_K^{(m)} + \rho_L^{(m)} \right) \left(\rho_K^{(m)} - \rho_L^{(m)} \right) \right)$$

Reordering the sum, we get:

$$T_5^{(m)} = \sum_{K \in \mathcal{M}^{(m)}} \rho_K^{(m)} \left(\sum_{\sigma=K|L} (h_K + h_L)^\beta \frac{|\sigma|}{h_\sigma} (\rho_K^{(m)} + \rho_L^{(m)}) (\psi_K - \psi_L) \right)$$

By regularity of ψ , $|\psi_K - \psi_L| \leq c_\psi (h_K + h_L)$ and thus:

$$|T_5^{(m)}| \leq \sum_{K \in \mathcal{M}^{(m)}} \rho_K^{(m)} \left(\sum_{\sigma=K|L} (h_K + h_L)^{\beta+1} \frac{|\sigma|}{h_\sigma} (\rho_K^{(m)} + \rho_L^{(m)}) \right)$$

which, once again since the sequence $(\rho^{(m)})_{m \in \mathbb{N}}$ is bounded in $L^2(\Omega)$, tends to zero by regularity of the mesh for $\beta > 1$. The proof is thus complete. \square

6. DISCUSSION

To our knowledge, the convergence analysis performed in this paper seems to be the first result of this kind for the compressible Stokes problem (and, of course, more widely, for compressible Navier-Stokes equations). Besides the convergence of the scheme, it also provides an existence result for solutions of the continuous problem.

A puzzling fact is that this theory relies on two arguments which are usually non satisfied in practical applications. Firstly, the stabilisation term $T_{\text{stab},2}$ is crucial in our proof to ensure the convergence of the discretization of the mass convection flux $\text{div}(\rho \mathbf{u})$ and, to our knowledge, has never been introduced elsewhere. Secondly, the control of the pressure in $L^2(\Omega)$ relies on the stability of the discrete gradient (*i.e.* the satisfaction of the so-called discrete *inf-sup* condition), which is not verified by colocated discretizations; note that this argument is not needed for the stability of the scheme (see the proof of *a priori* estimates here and [9, 6] for stability studies of schemes for the Navier-Stokes equations). Assessing the effective relevance of these requirements for the discretization should deserve more work in the future.

An easy extension of this work consists in replacing in the first equation of the problem $-\Delta \mathbf{u}$ by $-\mu \Delta \mathbf{u} - \mu/3 \nabla(\text{div} \mathbf{u})$ with $\mu > 0$ (*i.e.* the usual form of the divergence of the shear stress tensor in a constant viscosity compressible flow). Another less straightforward issue is the extension to more general state equation (for instance, $p = \rho^\gamma$ with $\gamma > 1$); it will be the topic of a further paper. Finally, let us note that the fact that the pressure is approximated by piecewise constants appears crucial in both stability and convergence proofs: extending this study to higher degree finite element discretizations thus certainly represents a difficult task.

APPENDIX A. PROOF OF LEMMA 3.2

We only sketch the idea of the proof in the two-dimensionnal case and with quasi-uniform meshes (*i.e.* the bound we prove here blows up when $\max_{K \in \mathcal{M}} (h/h_K)$ tends to infinity). The extension to the three-dimensional case is straightforward, and obtaining bounds only depending on the parameter θ defined by relation (2.1) would only necessitate additional cumbersome efforts of notation.

Let \mathcal{M} be a triangulation of a two-dimensional domain Ω , K a triangular cell of \mathcal{M} and σ an edge of K . Without loss of generality, we suppose that σ is the segment $(0, h_\sigma) \times 0$ and we denote by ξ_K the diameter of the largest ball included

in K and by h_K the diameter of K . We denote by z_σ the opposite vertex to σ ; the first coordinate of z_σ is necessarily lower than h_K while its second coordinate is necessarily greater than ξ_K (in the opposite case, no ball of diameter ξ_K would be included in K). It thus follows (see figure 1):

- (1) that the rectangular domain $\omega_\sigma = (h_\sigma/3, 2h_\sigma/3) \times (0, h_\sigma \xi_K / (12h_K))$ is included in K ,
- (2) that, if the similar construction is performed for another edge σ' of K to obtain $\omega_{\sigma'}$, ω_σ and $\omega_{\sigma'}$ do not intersect.

We denote by d_σ the quantity $d_\sigma = h_\sigma \xi_K / (12h_K)$. We thus have $d_\sigma \geq (\theta/12) h_\sigma$, where θ is the parameter defined by 2.1.

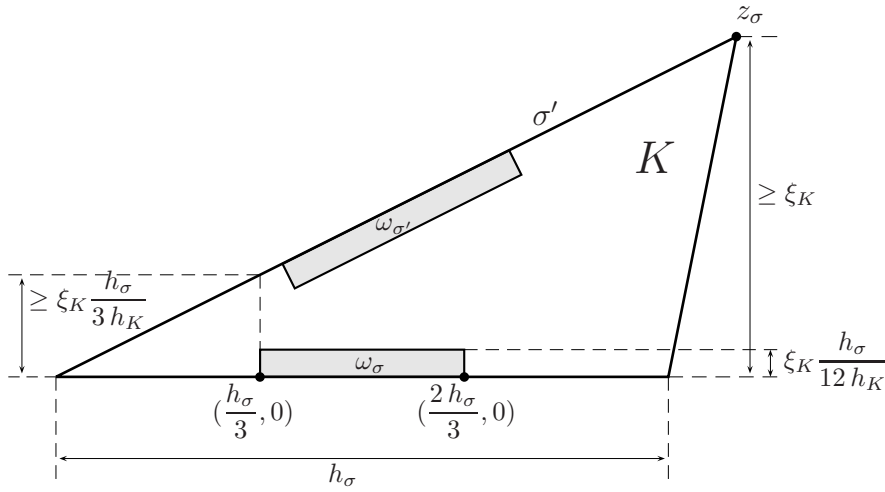


FIGURE 1. Notations for the control volume K

We now perform this construction for each edge σ of the mesh. If $\sigma \in \mathcal{E}_{\text{ext}}$, there is only one possible choice for K (the adjacent cell to σ); if $\sigma \in \mathcal{E}_{\text{int}}$, $\sigma = K|L$, we choose either K or L . Let \mathbf{x} and \mathbf{y} be two points of \mathbb{R}^d . Let $\mathbf{t}_{(\mathbf{x}, \mathbf{y})}$ be the vector given by:

$$\mathbf{t}_{(\mathbf{x}, \mathbf{y})} = \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|}$$

and $\mathbf{n}_{(\mathbf{x}, \mathbf{y})}$ a normal vector to $\mathbf{t}_{(\mathbf{x}, \mathbf{y})}$. We denote by $S_{(\mathbf{x}, \mathbf{y})}$ the rectangle defined by:

$$S_{(\mathbf{x}, \mathbf{y})} = \{\mathbf{x} + \alpha \mathbf{t}_{(\mathbf{x}, \mathbf{y})} + \beta \mathbf{n}_{(\mathbf{x}, \mathbf{y})}, \alpha \in (-h, |\mathbf{y} - \mathbf{x}| + h), \beta \in (-h, +h)\}$$

For each edge intersected by the segment $[\mathbf{x}, \mathbf{y}]$ (*i.e.* for each edge σ such that $\mathcal{X}_\sigma(\mathbf{x}, \mathbf{y}) = 1$), the rectangle ω_σ is included in $S_{(\mathbf{x}, \mathbf{y})}$; thus, since these domains are disjoint:

$$\sum_{\sigma \in \mathcal{E}} \mathcal{X}_\sigma(\mathbf{x}, \mathbf{y}) |\omega_\sigma| \leq |S_{(\mathbf{x}, \mathbf{y})}|$$

and thus:

$$\sum_{\sigma \in \mathcal{E}} \mathcal{X}_\sigma(\mathbf{x}, \mathbf{y}) \frac{1}{3} d_\sigma h_\sigma \leq 2h(|\mathbf{y} - \mathbf{x}| + 2h)$$

which concludes the proof, since we have allow the right hand-side to depend on $\max_{K \in \mathcal{M}}(h/h_K)$ and on θ .

REFERENCES

- [1] F. Brezzi and J. Pitkäranta. On the stabilization of finite element approximations of the Stokes equations. In W. Hackbusch, editor, *Efficient Solution of Elliptic Systems*, volume 10 of *Notes Num.Fluid Mech.*, pages 11–19. Vieweg, 1984.
- [2] P. G. Ciarlet. Handbook of numerical analysis volume II : Finite elements methods – Basic error estimates for elliptic problems. In P. Ciarlet and J.L. Lions, editors, *Handbook of Numerical Analysis, Volume II*, pages 17–351. North Holland, 1991.
- [3] M. Crouzeix and P.-A. Raviart. Conforming and nonconforming finite element methods for solving the stationary Stokes equations I. *Revue Française d'Automatique, Informatique et Recherche Opérationnelle (R.A.I.R.O.)*, R-3:33–75, 1973.
- [4] A. Ern and J.-L. Guermond. Theory and practice of finite elements. Number 159 in Applied Mathematical Sciences. Springer, New York, 2004.
- [5] R. Eymard, T Gallouët, and R. Herbin. Finite volume methods. In P. Ciarlet and J.L. Lions, editors, *Handbook of Numerical Analysis, Volume VII*, pages 713–1020. North Holland, 2000.
- [6] R. Eymard and R. Herbin. Entropy estimate for the approximation of the compressible barotropic Navier-Stokes equations using a collocated finite volume scheme. *in preparation*, 2007.
- [7] R. Eymard, R. Herbin, and J.C. Latché. On a stabilized collocated finite volume scheme for the Stokes problem. *Mathematical Modelling and Numerical Analysis*, 40(3):501–528, 2006.
- [8] E. Feireisl. Dynamics of viscous compressible flows. volume 26 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, 2004.
- [9] T. Gallouët, L. Gastaldo, R. Herbin, and J.-C. Latché. An unconditionally stable pressure correction scheme for compressible barotropic Navier-Stokes equations. *to appear in Mathematical Modelling and Numerical Analysis*, 2008.
- [10] P.-L. Lions. Mathematical topics in fluid mechanics – volume 2 – compressible models. volume 10 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, 1998.
- [11] A. Novotný and I. Straškraba. Introduction to the mathematical theory of compressible flow. volume 27 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, 2004.
- [12] L.E. Payne and H.F. Weinberger. An optimal Poincaré-inequality for convex domains. *Archive for Rational Mechanics and Analysis*, 5:286–292, 1960.
- [13] R. Verfürth. Error estimates for some quasi-interpolation operators. *Mathematical Modelling and Numerical Analysis*, 33(4):695–713, 1999.

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