When do stochastic max-plus linear systems have a cycle time ?

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Cycle time of stochastic max-plus linear systems. Version of December 16, 2007

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Abstract

We analyze the asymptotic behavior of the sequence of random variables $(x(n,x_0))_{n\in\mathbb{N}}$ defined by $x(0,x_0)=x_0$ and $x(n+1,x_0)=A(n)x(n,x_0)$, where $(A(n))_{n\in\mathbb{N}}$ is a stationary and ergodic sequence of random matrices with entries in the semiring $(\mathbb{R}\cup\{-\infty\},\max,+)$.

Such sequences model a large class of discrete event systems, among which timed event graphs, 1-bounded Petri nets, some queuing networks, train or computer networks.

We give a necessary condition for $\left(\frac{1}{n}x(n,x_0)\right)_{n\in\mathbb{N}}$ to converge almost-surely, which proves to be sufficient when the $A(n)$ are i.i.d.

Moreover, we construct a new example, in which $(A(n))_{n\in\mathbb{N}}$ is strongly mixing, that condition is satisfied, but $\left(\frac{1}{n}x(n,x_0)\right)_{n\in\mathbb{N}}$ do not converge almost-surely.

1 Introduction

1.1 Model

We analyze the asymptotic behavior of the sequence of random variables $(x(n,x_0))_{n\in\mathbb{N}}$ defined by:

$$\left\{\begin{array}{lcl} x(0,x_0) & = & x_0 \\ x(n+1,x_0) & = & A(n)x(n,x_0) \end{array}\right. , \tag{1}$$

where $(A(n))_{n\in\mathbb{N}}$ is a stationary and ergodic sequence of random matrices with entries in the semiring $\mathbb{R}_{\max}=\mathbb{R}\cup\{-\infty\}$ whose addition is the max and whose multiplication is $+$. Such sequences are called stochastic recurrent sequences driven by $(A(n))_{n\in\mathbb{N}}$.

We also define the product of matrices $A(k,n):=A(k)\cdots A(n)$ such that $x(n,x_0)=A(n-1,0)x_0$ and, if the sequence has indices in $\mathbb{Z}$, which is possible up to a change of probability space, $y(n,x_0):=A(-1,-n)x_0$.

On the coefficients, Relation (1) reads

$$x_i(n+1,x_0)=\max_j\left(A_{ij}(n)+x_j(n,x_0)\right),$$

and the product of matrices is defined by

$$A(n-1,0)_{ij}=\max_{i_0=j,i_n=i}\sum_{l=0}^{n-1}A_{i_{l+1}i_l}(l). \tag{2}$$

In most cases, we assume that $A(n)$ has at least one finite entry on each row, which is a necessary and sufficient condition for $x(n, x_0)$ to be finite. (Otherwise, some coefficients can be $-\infty$.)

Such sequences model a large class of discrete event dynamical systems. This class includes some models of operations research like timed event graphs (F. Baccelli [1]), 1-bounded Petri nets (S. Gaubert and J. Mairesse [11]) and some queuing networks (J. Mairesse [16], B. Heidergott [13]) as well as many concrete applications. Let us cite job-shops models (G. Cohen et al.[8]), train networks (H. Braker [7], A. de Kort and B. Heidergott [10]), computer networks (F. Baccelli and D. Hong [3]) or a statistical mechanics model (R. Griffiths [12]). For more details about modelling, see the books by F. Baccelli and al. [2] and by B. Heidergott and al. [14].

1.2 Law of large numbers

The sequences satisfying Equation (1) have been studied in many papers. If a matrix A has at least one finite entry on each row, then $x \mapsto Ax$ is non-expanding for the L^∞ norm. Therefore, we can assume $x_0 = 0$, and we do it from now on.

We say that they satisfy the strong law of large numbers (SLLN) if $\left(\frac{1}{n}x(n,0)\right)_{n\in\mathbb{N}}$ converge almost surely. The limit in the law of large numbers is called the cycle time of the system.

Some sufficient conditions for the existence of this cycle time were given by J.E. Cohen [9], F. Baccelli and Z. Liu [4, 1], D. Hong [15] and more recently by T. Bousch and J. Mairesse [6], the author [17] or B. Heidergott et al. [14].

T. Bousch and J. Mairesse proved (Cf. [6]) that, if $A(0)0$ is integrable, then the sequence $\left(\frac{1}{n}y(n,0)\right)_{n\in\mathbb{N}}$ converges almost-surely and in mean and that $\left(\frac{1}{n}x(n,0)\right)_{n\in\mathbb{N}}$ converges almost-surely if and only if the limit of $\left(\frac{1}{n}y(n,0)\right)_{n\in\mathbb{N}}$ is deterministic.

The other results can be seen as sufficient conditions for this to happen. Some results only assumed ergodicity of $(A(n))_{n\in\mathbb{N}}$, some others independence. But, even in the i.i.d. case, it was not clear when the sequence converge.

Based on the part of the result in [1] that do not rely on the additional hypotheses of that article, we give a necessary condition for the limit of $\left(\frac{1}{n}y(n,x_0)\right)_{n\in\mathbb{N}}$ to be deterministic. This condition involves a graph defined by the support of law of $A(1)$. Moreover, whenever it exists, the limit of $\left(\frac{1}{n}x(n,0)\right)_{n\in\mathbb{N}}$ is given by Formula (1), which was proved in [1] under several additional assumptions but unknown when the convergence was proved by other means, like in [6]. Those results are gathered in Theorem 2.3.

Conversely, we show (Theorem 2.4) that the necessary condition is sufficient when the $A(n)$ are i.i.d. As a first step, we extend (Theorem 3.11) a result of [15]. Then, we perform an induction, thanks to Proposition 3.5 and Lemma 3.10.

In Section 2, we state our results and give examples to show that the hypotheses are necessary. In Section 3, we successively prove the theorems 2.3 and 2.4 .

2 Results

2.1 Theorems

In this section we attach a graph to our sequence of random matrices, in order to define the necessary condition and to split the problem for the inductive proof of the converse theorem.

Before defining the graph, we need the following, which directly follows from Kingman's theorem and can be traced back to J.E. Cohen [9]:

Theorem-Definition 2.1 (Maximal Lyapunov exponent).
If $(A(n))_{n\in\mathbb{N}}$ *is an ergodic sequence of random matrices with entries in* $\mathbb{R}_{\max}$ *such that the positive part of* $\max_{ij} A_{ij}(0)$ *is integrable, then the sequences* $\left(\frac{1}{n}\max_i x_i(n,0)\right)_{n\in\mathbb{N}}$ *and* $\left(\frac{1}{n}\max_i y_i(n,0)\right)_{n\in\mathbb{N}}$ *converge almost-surely to the same constant* $\gamma \in \mathbb{R}_{\max}$*, which is called maximal Lyapunov exponent of* $(A(n))_{n\in\mathbb{N}}$.

We denote this constant by $\gamma\left((A(n))_{n\in\mathbb{N}}\right)$*, or* $\gamma(A)$.

Remark 2.1. The constant $\gamma(A)$ is well-defined even if $(A(n))_{n\in\mathbb{N}}$ has a row without finite entry.
The variable $\max_i x_i(n,0)$ is equal to $\max_{ij} A(n-1,0)_{ij}$.

Let us define the graph attached to our sequence of random matrices as well as some subgraphs. We also set the notations for the rest of the text.

Definition 2.2 (Graph of a random matrix). For every $x \in \mathbb{R}_{\max}^{[1,\cdots,d]}$ and every subset $I \subset [1,\cdots,d]$, we define the subvector $x^I := (x_i)_{i\in I}$.

Let $(A(n))_{n\in\mathbb{N}}$ be a stationary sequence of random matrices with values in $\mathbb{R}_{\max}^{d\times d}$.

i) The graph of $(A(n))_{n\in\mathbb{N}}$, denoted by $\mathcal{G}(A)$, is the directed graph whose nodes are the integers between 1 and d and whose arcs are the pairs (i,j) such that $\mathbb{P}(A_{ij}(0) \neq -\infty) > 0$.

ii) To each strongly connected components (s.c.c) c of $\mathcal{G}(A)$, we attach the submatrices $A^{(c)}(n) := (A_{ij}(n))_{i,j\in c}$ and the exponent $\gamma^{(c)} := \gamma(A^{(c)})$.

Nodes which are not in a circuit are assumed to be alone in their s.c.c. Those s.c.c are called trivial and they satisfy $A^{(c)} = -\infty a.s.$ and therefore $\gamma^{(c)} = -\infty$.

iii) A s.c.c $\tilde{c}$ is reachable from a s.c.c c (or from a node i) if $c = \tilde{c}$ (or $i \in c$) or if there exists a path on $\mathcal{G}(A)$ from a node in c (resp. from i) to a node in $\tilde{c}$. In this case, we write $c \to \tilde{c}$. (resp. $i \to \tilde{c}$).

Remark 2.2 *(Paths on* $\mathcal{G}(A)$*).* 1. Equation (2) can be read as '$A(-1,-n)_{ij}$ is the maximum of the weights of paths from i to j with length n on $\mathcal{G}(A)$, the weight of the k^{th} arc being given by $A(-k)$'. Thus, $y_i(n,0)$ is the maximum of the weights of paths on $\mathcal{G}(A)$ with initial node i and length n and $\gamma(A)$ is a kind of maximal average weight of infinite paths.

2. Previous author used such a graph, in the fixed support case, that is when $P(A_{ij}(0) = -\infty)) \in \{0, 1\}$. In that case, the (random) weight where almost surely finite. Here, we can have weight $-\infty$, with probability strictly less than one.

3. Some previous author used the isomorphic graph with weight A_{ji} on arc (i, j). This is natural to multiply on the left an compute $x(n, 0)$. Since we mainly work with $y(n, 0)$ and thus multiply on the right, our definition is more convenient.

With those definitions, we can state the announced necessary condition for $(x(n, X_0))_{n\in\mathbb{N}}$ to satisfy a strong law of large numbers:

Theorem 2.3. *Let $(A(n))_{n\in\mathbb{N}}$ be a stationary and ergodic sequence of random matrices with values in $\mathbb{R}_{\max}^{d\times d}$ and almost-surely at least one finite entry on each row, such that the positive part of $\max_{ij} A_{ij}(0)$ is integrable.*

If the limit of $\left(\frac{1}{n}y(n, 0)\right)_{n\in\mathbb{N}}$ is deterministic, then it is given by:

$$\forall i \in [1, d], \lim_n \frac{1}{n} y_i(n, 0) = \max_{i\to c} \gamma^{(c)} \text{ a.s.}, \tag{3}$$

That being the case, for every s.c.c c of $\mathcal{G}(A)$, the submatrix $A^{\{c\}}$ of $A(0)$ whose indices are in

$$\left\{ i \in [1, d] \,\middle|\, \exists \tilde{c}, c \to i \to \tilde{c}, \gamma^{(\tilde{c})} = \max_{c\to\hat{c}} \gamma^{(\overline{c})} \right\}$$

almost-surely has at least one finite entry on each row.

If $\left(\frac{1}{n}x(n, 0)\right)_{n\in\mathbb{N}}$ converges almost-surely, then its limit is deterministic and is equal to that of $\left(\frac{1}{n}y(n, 0)\right)_{n\in\mathbb{N}}$, that is we have:

$$\forall i \in [1, d], \lim_n \frac{1}{n} x_i(n, 0) = \max_{i\to c} \gamma^{(c)} \text{ a.s.}, \tag{4}$$

To make the submatrices $A^{\{c\}}$ more concrete, we give on Fig. 1 an example of a graph $\mathcal{G}(A)$ with the exponent $\gamma^{(k)}$ attached to each s.c.c and compute the submatrix $A^{\{c_2\}}$.

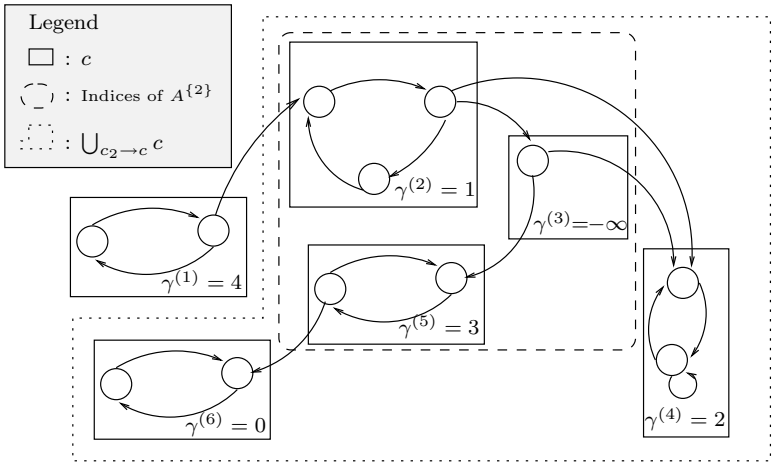
The necessary and sufficient condition in the i.i.d. case reads

Theorem 2.4 (Independent case). *If $(A(n))_{n\in\mathbb{N}}$ is a sequence of i.i.d. random matrices with values in $\mathbb{R}_{\max}^{d\times d}$ and almost-surely at least one finite entry on each row, such that $\max_{A_{ij}(0)\neq-\infty} |A_{ij}(0)|$ is integrable, then the sequence $\left(\frac{1}{n}x(n, 0)\right)$ converges almost-surely if and only if for every s.c.c c, the submatrix $A^{\{c\}}$ of $A(0)$ defined in Theorem 2.3 almost-surely has at least one finite entry on each row. That being the case the limit is given by Equation (4).*

Remark 2.3. We also prove that, when $A(0)0 \in \mathbb{L}^1$, the limit of $\left(\frac{1}{n}y(n, 0)\right)$ is deterministic if and only if the matrices $A^{\{c\}}$ almost-surely have at least one finite entry on each row.

The stronger integrability ensures the convergence of $\left(\frac{1}{n}x(n, 0)\right)$ to this limit, like in [6]. There, it appeared as the specialization of a general condition for

Figure 1: An example of computations on $\mathcal{G}(A)$



uniformly topical operators, whereas in this paper it ensures that $B0$ is integrable for every submatrix B of $A(0)$ with at least one finite entry on each row.

Actually, we prove that $\left(\frac{1}{n}x(n,0)\right)$ converges, provided that $\forall c, A^{\{c\}}0 \in \mathbb{L}^1$, (see Proposition 3.5). We chose to give a slightly stronger integrability condition, which is easier to check because it does not depend on $\mathcal{G}(A)$.

2.2 Examples

To end this section, bellow are three examples that show that the independence is necessary but not sufficient to ensure the strong law of large numbers and that the integrability condition is necessary.

Example 1 *(Independence is necessary).* Let A and B be defined by

$$A = \begin{pmatrix} 1 & -\infty \\ -\infty & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} -\infty & 0 \\ 0 & -\infty \end{pmatrix}.$$

For any positive numbers γ_1 and γ_2 such that $\gamma_1 + \gamma_2 < 1$, we set $\delta = \frac{1-\gamma_1-\gamma_2}{2}$. Let $(A(n), i_n)_{n\in\mathbb{N}}$ be a stationnary version of the irreducible Markov chain on $\{A, B\} \times \{1, 2\}$ with transition probabilities given by the following diagram :

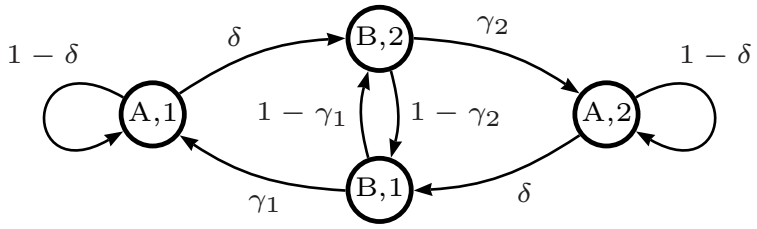
Then, $(A(n))_{n\in\mathbb{N}}$ is a strongly mixing sequence of matrices whose support is the full subshit $\{A, B\}^{\mathbb{N}}$, but we have

$$\mathbb{P}\left(\lim_n \frac{1}{n} y_1(n,0) = \gamma_1\right) = \gamma_1 + \delta \text{ and } \mathbb{P}\left(\lim_n \frac{1}{n} y_1(n,0) = \gamma_2\right) = \gamma_2 + \delta, \quad (5)$$

and thus, according to Theorem 2.3, $\left(\frac{1}{n}x(n,0)\right)_{n\in\mathbb{N}}$ do not converge.

To prove Equation (5), let us denote by τ the permutation between 1 and 2 and by $g(C,i)$ the only finite entry on the i^{th} row of C. It means that for any i, $g(A,i) = A_{ii}$ and $g(B,i) = B_{i\tau(i)}$. Since all arcs of the diagram arriving to a

Figure 2: Transition probabilities of $(A(n), i_n)_{n\in\mathbb{N}}$



node (A, i) are coming from a node (C, i), while those arriving at a node (B, i) are coming from a node $(C, \tau(i))$, we almost surely have

$$x_{i_n}(n+1,0)-x_{i_{n-1}}(n,0) = g(A(n), i_n)) \text{ and } x_{\tau(i_n)}(n+1,0)-x_{\tau(i_{n-1})}(n,0) = g(A(n), \tau(i_n))),$$

and thus

$$x_{i_{n-1}}(n,0) = \sum_{k=0}^{n-1} g(A(k), i_k) \quad \text{and} \quad x_{\tau(i_{n-1})}(n,0) = \sum_{k=0}^{n-1} g(A(k), \tau(i_k)),$$
$$y_{i_{-1}}(n,0) = \sum_{k=1}^{n} g(A(-k), i_{-k}) \quad \text{and} \quad y_{\tau(i_{-1})}(n,0) = \sum_{k=1}^{n} g(A(-k), \tau(i_{-k})).$$

It is easily checked, that the invariant distribution of the Markov chain is given by the following table :

x	$(A,1)$	$(B,2)$	$(A,2)$	$(B,1)$
$\mathbb{P}((A(n), i_n) = x)$	γ_1	δ	γ_2	δ

and that g is equal to 0 except in $(A, 1)$.

Therefore, we have

$$\begin{aligned}\lim_n \frac{1}{n} y_{i_{-1}}(n,0) &= \mathbb{E}\left(g(A(0), i_0)\right) = \mathbb{P}\left((A(0), i_0) = (A,1)\right) = \gamma_1 \\ \lim_n \frac{1}{n} y_{\tau(i_{-1})}(n,0) &= \mathbb{E}\left(g(A(0), \tau(i_0))\right) = \mathbb{P}\left((A(0), \tau(i_0)) = (A,2)\right) = \gamma_2\end{aligned}$$

and consequently

$$\lim_n \frac{1}{n} y(n,0) = (\gamma_{i_{-1}}, \gamma_{\tau(i_{-1})})' a.s.,$$

which implies Equation (5).

The next example shows that independence is not enough.

Example 2 *(Independence is not sufficient[5]).* Let $(A(n))_{n\in\mathbb{N}}$ be the sequence of i.i.d. random variables taking values

$$B = \begin{pmatrix} 0 & -\infty & -\infty \\ 0 & -\infty & -\infty \\ 0 & 1 & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & -\infty & -\infty \\ 0 & -\infty & 0 \\ 0 & 0 & -\infty \end{pmatrix}$$

with probabilities $p > 0$ and $1 - p > 0$. Let us compute the action of B and C on vectors of type $(0, x, y)'$, with $x, y \geq 0$:

$$B(0, x, y)' = (0, 0, \max(x, y) + 1)' \text{ and } C(0, x, y)' = (0, y, x)'.$$

Therefore $x_1(n, 0) = 0$ and $\max_i x_i(n+1, 0) = \#\{0 \leq k \leq n | A(k) = B\}$. In particular, if $A(n) = B$, then $x(n + 1, 0) = (0, 0, \#\{0 \leq k \leq n | A(k) = B\})'$, and if $A(n) = C$ and $A(n-1) = B$, then $x(n+1, 0) = (0, \#\{0 \leq k \leq n | A(k) = B\}, 0)'$. Since $\left(\frac{1}{n}\#\{0 \leq k \leq n | A(k) = B\}\right)_{n\in\mathbb{N}}$ converges almost-surely to p, we see:

$$\begin{array}{c} \lim_n \frac{1}{n} x_1(n, 0) = 0 \text{ a.s.} \\ \forall i \in \{2, 3\}, \liminf_n \frac{1}{n} x_i(n, 0) = 0 \text{ and } \limsup_n \frac{1}{n} x_i(n, 0) = p \text{ a.s.}. \end{array} \tag{6}$$

Therefore the sequence $\left(\frac{1}{n} x(n, 0)\right)_{n\in\mathbb{N}}$ almost-surely does not converge.

We notice that $\mathcal{G}(A)$ has two s.c.c $c_1 = \{1\}$ and $c_2 = \{2, 3\}$, with Lyapunov exponents $\gamma^{(c_1)} = 0$ and $\gamma^{(c_2)} = p$, and $2 \to 1$. Therefore we check that the first row of $A^{\{c_2\}}(n)$ has no finite entry with probability p.

The last example shows that Theorem 2.4 do not holds under the weaker integrability condition $A(0)0 \in \mathbb{L}^1$.

Example 3 *(Integrability).* Let $(X_n)_{n\in\mathbb{Z}}$ be a sequence of i.i.d. real variables satisfying $X_n \geq 1$ a.s. and $\mathbb{E}(X_n) = +\infty$. The sequence of matrices is defined by:

$$A(n) = \begin{pmatrix} -X_n & -X_n & 0 \\ -\infty & 0 & 0 \\ -\infty & -\infty & -1 \end{pmatrix}$$

A straightforward computation shows that $x(n, 0) = (\max(-X_n, -n), 0, -n)'$ and $y(n, 0) = (\max(-X_0, -n), 0, -n)'$. It follows from Borel-Cantelli lemma that $\lim_n \frac{1}{n} X_n = 0$ a.s. if and only if $\mathbb{E}(X_n) < \infty$. Hence $\left(\frac{1}{n} x(n, 0)\right)_{n\in\mathbb{N}}$ converges to $(0, 0, -1)'$ in probability but the convergence do not occur almost-surely.

Let us notice that the limit of $\left(\frac{1}{n} y(n, 0)\right)_{n\in\mathbb{N}}$ is given by Remark 2.3 : each s.c.c has exactly one node, $\gamma^{(1)} = -\mathbb{E}(X_n) = -\infty$, $\gamma^{(2)} = 0$ and $\gamma^{(3)} = -1$.

3 Proofs

3.1 Necessary conditions

3.1.1 Additional notations

To interpret the results in terms of paths on $\mathcal{G}(A)$, and prove them, we redefine the $A^{\{c\}}$ and some intermediate submatrices.

Definition 3.1. To each s.c.c c, we attach three sets of elements.

i) Those that only depend on c itself.

$$x^{(c)}(n,x_0) := A^{(c)}(n-1,0)(x_0)^c \text{ and } y^{(c)}(n,x_0) := A^{(c)}(-1,-n)(x_0)^c$$

ii) Those that depend on the graph downstream of c.

$$E_c := \{\tilde{c} | c \to \tilde{c}\},\ \gamma^{[c]} := \max_{\tilde{c}\in E_c} \gamma^{(\tilde{c})},$$

$$F_c := \bigcup_{\tilde{c}\in E_c} \tilde{c},\ A^{[c]}(n) := (A_{ij}(n))_{i,j\in F_c}$$

$$x^{[c]}(n,x_0) := A^{[c]}(n-1,0)(x_0)^{F_c} \text{ and } y^{[c]}(n,x_0) := A^{[c]}(-1,-n)(x_0)^{F_c}.$$

iii) Those that depend on the indices of $A.^{\{c\}}$.

$$G_c := \{\tilde{c} \in E_c | \exists \hat{c}, c \to \tilde{c} \to \hat{c}, \gamma^{(\hat{c})} = \gamma^{[c]}\},$$

$$H_c := \bigcup_{\tilde{c}\in G_c} \tilde{c}\ ,\ A^{\{c\}}(n) := (A_{ij}(n))_{i,j\in H_c}$$

$$x^{\{c\}}(n,x_0) := A^{\{c\}}(n-1,0)(x_0)^{H_c} \text{ and } y^{\{c\}}(n,x_0) := A^{\{c\}}(-1,-n)(x_0)^{H_c}.$$

iv) A s.c.c c is called dominating if $G_c = \{c\}$, that is if for every $\tilde{c} \in E_c \backslash \{c\}$, we have: $\gamma^{(c)} > \gamma^{(\tilde{c})}$.

As in Remark 2.2, we notice that the coefficients $y_i^{(c)}(n,0)$, $y_i^{[c]}(n,0)$ and $y_i^{\{c\}}(n,0)$ are the maximum of the weights of paths on the subgraph of $\mathcal{G}(A)$ with nodes in c, F_c and H_c respectively.

Consequently $\gamma^{(c)}$, $\gamma(A^{[c]})$ and $\gamma(A^{\{c\}})$ are the maximal average weight of infinite paths on c, F_c and G_c respectively. Since $\gamma^{[c]}$ is the maximum of the $\gamma^{(\tilde{c})}$ for s.c.c $\tilde{c}$ downstream of c, the interpretation suggests it might be equal to $\gamma(A^{[c]})$ and $\gamma(A^{\{c\}})$. Corollary 3.4, due to F. Baccelli [1] says it is.

Clearly, $\gamma(A^{[c]}) \geq \gamma(A^{\{c\}}) \geq \gamma(A^{[c]})$, but the maximum is actuallyy taken for finite paths, so that the converse inequalities are not obvious.

3.1.2 Formula for the limit

Let L be the limit of $\left(\frac{1}{n}y(n,0)\right)_{n\in\mathbb{N}}$, which exists according to [6] and is assumed to be deterministic.

By definition of $\mathcal{G}(A)$, if (i,j) is an arc of $\mathcal{G}(A)$, then, with positive probability, we have $A_{ij}(-1) \neq -\infty$ and

$$L_i = \lim_n \frac{1}{n} y_i(n,0) \geq \lim_n \frac{1}{n}(A_{ij}(-1) + y_j(n,0)\circ\theta^{-1}) = 0 + L_j \circ \theta^{-1} = L_j.$$

If $c \to \tilde{c}$, then for every $i \in c$ and $j \in \tilde{c}$, there exists a path on $\mathcal{G}(A)$ from i to j, therefore $L_i \geq L_j$. Since this holds for every $j \in F_c$, we have:

$$L_i = \max_{j\in F_c} L_j \tag{7}$$

To show that $\max_{j\in F_c} L_j = \gamma^{[c]}$, we have to study the Lyapunov exponents of sub-matrices.

The following proposition states some easy consequences of Definition 3.1 which will be useful in the sequel.

Proposition 3.2. *The notations are those of Definition 3.1 .*

i) for every s.c.c. c, $x^{[c]}(n, x_0) = x^{F_c}(n, x_0)$.

ii) for every s.c.c. m, and every $i \in c$, we have:

$$x_i(n,0) = x_i^{[c]}(n,0) \geq x_i^{\{c\}}(n,0) \geq x_i^{(c)}(n,0).$$

$$y_i(n,0) = y_i^{[c]}(n,0) \geq y_i^{\{c\}}(n,0) \geq y_i^{(c)}(n,0). \tag{8}$$

iii) Relation $\rightarrow$ is a partial order, for both the nodes and the s.c.c. .

iv) If $A(0)$ almost-surely has at least one finite entry on each row, then for every s.c.c. c, $A^{[c]}(0)$ almost-surely has at least one finite entry on each row.

v) for every $\tilde{c} \in E_c$, we have $\gamma^{(\tilde{c})} \leq \gamma^{[\tilde{c}]} \leq \gamma^{[c]}$ and $G_c = \{\tilde{c} \in E_c | \gamma^{[\tilde{c}]} = \gamma^{[c]}\}$.

The next result is about Lyapunov exponents. It is already in [1, 4] and its proof does not uses the additional hypotheses of those articles. For a point by point checking, see [17].

Theorem 3.3 ([1, 4, 2]). *If $(A(n))_{n\in\mathbb{N}}$ is a stationary and ergodic sequence of random matrices with values in $\mathbb{R}_{\max}^{d\times d}$ such that the positive part of $\max_{i,j} A_{ij}$ is integrable, then $\gamma(A) = \max_{\tilde{c}} \gamma^{(\tilde{c})}$.*

Applying this theorem to sequences $\left(A^{[c]}(n)\right)_{n\in\mathbb{N}}$ and $\left(A^{\{c\}}(n)\right)_{n\in\mathbb{N}}$, we get the following corollary.

Corollary 3.4. *For every s.c.c. c, we have*

$$\gamma(A^{\{c\}}) = \gamma(A^{[c]}) = \gamma^{[c]}.$$

It follows from proposition 3.2 and the definition of Lyapunov exponents that for every s.c.c c of $\mathcal{G}(A)$,

$$\max_{i\in F_c} L_i = \lim_n \frac{1}{n} \max_{i\in F_c} y_i(n,0) = \gamma(A^{[c]}).$$

Combining this with Equation (7) and Corollary 3.4, we deduce that the limit of $\left(\frac{1}{n}y(n,0)\right)_{n\in\mathbb{N}}$ is given by Equation (3).

3.1.3 $A^{\{c\}}(0)$ has at least one finite entry on each row

We still have to show that for every s.c.c c, $A^{\{c\}}(0)$ almost-surely has at least one finite entry on each row. Let us assume it does not. It means that there exists a s.c.c. c and an $i \in c$ such that the set

$$\{\forall j \in H_c, A_{ij}(-1) = -\infty\}$$

has positive probability. On this set, we have:

$$y_i(n,0) \leq \max_{j \in F_c \setminus H_c} A_{ij}(-1) + \max_{j \in F_c \setminus H_c} y_j(n-1,0) \circ \theta^{-1}.$$

Dividing by n and letting n to $+\infty$, we have $L_i \leq \max_{j \in F_c \setminus H_c} L_j$. Replacing L according to Equation (3) we get $\gamma^{[c]} \leq \max_{k \in E_c \setminus G_c} \gamma^{[k]}$. This last inequality contradicts proposition 3.2 v). Therefore, $A^{\{c\}}(0)$ almost-surely has at least one finite entry on each row.

3.1.4 The limit is constant

Let us assume that $\left(\frac{1}{n}x(n,0)\right)_{n\in\mathbb{N}}$ converges almost-surely to a limit L'. Up to a change of probability space, we can assume that $A(n) = A \circ \theta^n$, where A is a random variable and $(\Omega, \theta, \mathbb{P})$ is an invertible ergodic measurable dynamical system.

It follows from [6] that $\left(\frac{1}{n}y(n,0)\right)_{n\in\mathbb{N}}$ converges almost-surely, thus we have

$$\frac{1}{n}y(n,0) - \frac{1}{n+1}y(n+1,0) \xrightarrow{\mathbb{P}} 0.$$

We compound each term of this relation by θ^{n+1} and, since $x(n,0) = y(n,0) \circ \theta^n$, it proves that:

$$\frac{1}{n}x(n,0) \circ \theta - \frac{1}{n+1}x(n+1,0) \xrightarrow{\mathbb{P}} 0.$$

When n tends to $+\infty$, it becomes $L' \circ \theta - L' = 0$. Since θ is ergodic, this implies that L' is constant.

Since $\frac{1}{n}y(n,0) = \frac{1}{n}x(n,0) \circ \theta^n$, L' and L have the same law. Since L' is constant, $L = L'$ almost-surely, therefore L is also the limit of $\left(\frac{1}{n}x(n,0)\right)_{n\in\mathbb{N}}$. This proves formula (4) and concludes the proof of Theorem 2.3 .

3.2 Sufficient conditions

3.2.1 Right products

In this section, we prove the following proposition, which is a converse to Theorem 2.3.

Proposition 3.5. *Let $(A(n))_{n\in\mathbb{N}}$ be an ergodic sequence of random matrices with values in $\mathbb{R}_{\max}^{d\times d}$ such that the positive part of $\max_{ij} A_{ij}(0)$ is integrable and that satisfy the three following hypotheses:*

1. *for every s.c.c c of $\mathcal{G}(A)$, $A^{\{c\}}(0)$ almost-surely has at least one finite entry on each row.*
2. *for every dominating s.c.c c of $\mathcal{G}(A)$, $\lim_n \frac{1}{n}y^{(c)}(n,0) = \gamma^{(c)}\mathbf{1}$ a.s..*

3. *for every subsets I and J of $[1, \cdots, d]$, such that random matrices $\tilde{A}(n) = (A_{ij}(n))_{i,j\in I\cup J}$ almost-surely have at least one finite entry on each row and split along I and J following the equation*

$$\tilde{A}(n) =: \begin{pmatrix} B(n) & D(n) \\ -\infty & C(n) \end{pmatrix}, \tag{9}$$

such that $\mathcal{G}(B)$ is strongly connected and $D(n)$ is not almost-surely $(-\infty)^{I\times J}$, we have:

$$\mathbb{P}\left(\{\exists i \in I, \forall n \in \mathbb{N}, (B(-1)\cdots B(-n)D(-n-1)0)_i = -\infty\}\right) = 0. \tag{10}$$

Then the limit of $\left(\frac{1}{n}y(n,0)\right)_{n\in\mathbb{N}}$ is given by Equation (3).

If Hypothesis 1. is strengthened by demanding that $A^{\{c\}}(0)0$ is integrable, then the sequence $\left(\frac{1}{n}x(n,0)\right)_{n\in\mathbb{N}}$ converges almost-surely and its limit is given by Equation (4).

Hypothesis 1. is necessary according to Theorem 2.3, Hypothesis 2 ensures the basis of the inductive proof, while Hypothesis 3 ensures the inductive step.

Remark 3.1 *(Non independent case).* Proposition 3.5 does not assume the independence of the $A(n)$. Actually, it also imply that $\left(\frac{1}{n}x(n,0)\right)_{n\in\mathbb{N}}$ almost surely if the $A(n)$ have fixed support (that is $\mathbb{P}(A_{ij}(n) = -\infty) \in \{0,1\}$) and the powers of the shift are ergodic, which is an improvement of [1]. It also allows to prove the convergence when the diagonal entries of the $A(n)$ are almost surely finite, under weaker integrability conditions than in [6]. (see [18] or [17] for details)

Remark 3.2 *(Paths on $\mathcal{G}(A)$, continued).* Let us interpret the three hypotheses with the paths on $\mathcal{G}(A)$.

1. The hypothesis on $A^{\{c\}}(0)$ means that, whatever the initial condition $i \in c$, there is always an infinite path beginning in i and not leaving H_c.

2. The hypothesis on dominating s.c.c means that, whatever the initial condition i in dominating s.c.c c, there is always a path beginning in i with average weight $\gamma^{(c)}$. The proof of Theorem 3.3 (see [1] or [17]) can be adapted to show that it is a necessary condition.

3. We will use the last hypothesis with $\tilde{A}(n) = A^{\{c\}}(n)$, $B(n) = A^{(c)}(n)$. It means there is a path from $i \in c$, to $H_c\backslash c$. Once we know that the limit of $\left(\frac{1}{n}y(n,0)\right)_{n\in\mathbb{N}}$ is given by Equation (3) this hypothesis is obviously necessary when $\gamma^{(c)} < \gamma^{[c]}$.

It follows from propositions 3.2 and 3.4 and the definition of Lyapunov exponents that we have, for every s.c.c c of $\mathcal{G}(A)$,

$$\limsup_n \frac{1}{n} y^c(n,0) \leq \gamma^{[c]}\mathbf{1} \text{ a.s.}. \tag{11}$$

Therefore, it is sufficient to show that $\liminf_n \frac{1}{n} y^c(n,0) \geq \gamma^{[c]}\mathbf{1}$ a.s.. Because of proposition 3.2 *i*), it is stronger to show that

$$\lim_n \frac{1}{n} y^{\{c\}}(n,0) = \gamma^{[c]}\mathbf{1}. \tag{12}$$

We prove Equation (12) by induction on the size of G_c. The initialization of the induction is exactly Hypothesis 2. of Proposition 3.5.

Let us assume that Equation (12) is satisfied by every m such that the size of G_c is less than N, and let m be such that the size of G_c is $N+1$. Let us take $I = c$ and $J = H_c \backslash c$. If c is not trivial, it is the situation of Hypothesis 3. with $\tilde{A} = A^{\{c\}}$, which almost-surely has at least one finite entry on each row thanks to Hypothesis 1.. Therefore, Equation (10) is satisfied. If c is trivial, $\mathcal{G}(B)$ is not strongly connected, but Equation (10) is still satisfied because $D(-1)0 = (\tilde{A}(-1)0)^I \in \mathbb{R}^I$.

Moreover, J is the union of the $\tilde{c}$ such that $\tilde{c} \in G_c \backslash \{m\}$, thus the induction hypothesis implies that:

$$\forall j \in J, j \in \tilde{c} \Rightarrow \lim_n \frac{1}{n} \left(C(-1,-n)0\right)_j = \lim_n \frac{1}{n} y_j^{\{\tilde{c}\}}(n,0) = \gamma^{[\tilde{c}]} \text{ a.s..}$$

Because of Corollary 3.4 *ii*), $\gamma^{[\tilde{c}]} = \gamma^{[c]}$, therefore the right side of the last equation is $\gamma^{[c]}$ and we have:

$$\lim_n \frac{1}{n} (y^{\{c\}})^J(n,0) = \lim_n \frac{1}{n} C(-1,-n)0 = \gamma^{[c]}\mathbf{1} \text{ a.s..} \tag{13}$$

Equation (10) ensures that, for every $i \in I$, there exists almost-surely a $T \in \mathbb{N}$ and a $j \in J$ such that $(B(-1,-T)D(-T-1))_{ij} \neq -\infty$. Since we have $\lim_n \frac{1}{n} \left(C(-T,-n)0\right)_j = \gamma^{[c]}$ a.s., it implies that:

$$\begin{aligned} &\liminf_n \frac{1}{n} y_i^{\{c\}}(n,0) \\ &\quad \geq \lim_n \frac{1}{n} \left(B(-1,-T)D(-T-1)\right)_{ij} + \lim_n \frac{1}{n} \left(C(-T,-n)0\right)_j = \gamma^{[c]} \text{ a.s.} \end{aligned}$$

Because of upper bound (11) and inequality (8), it implies that

$$\lim_n \frac{1}{n} (y^{\{c\}})^I(n,0) = \gamma^{[c]}\mathbf{1} \text{ a.s..,}$$

which, because of Equation (13), proves Equation (12). This concludes the induction and the proof of Proposition 3.5.

3.2.2 Left products

As recalled in the introductions, T. Bousch an J. Mairesse proved that $\left(\frac{1}{n} x(n,0)\right)_{n\in\mathbb{N}}$ converges almost-surely as soon as the limit of $\left(\frac{1}{n} y(n,0)\right)_{n\in\mathbb{N}}$ is deterministic. Therefore, the hypotheses of Proposition 3.5 should imply the SLLN for $(x(n,0))_{n\in\mathbb{N}}$. But the theorem in [6] assumes a reinforced integrability assumption, that is not necessary in our situation. We will prove the following in this section :

Proposition 3.6. *Let $(A(n))_{n\in\mathbb{N}}$ be an ergodic sequence of random matrices with values in $\mathbb{R}_{\max}^{d\times d}$ such that the positive part of $\max_{ij} A_{ij}(0)$ is integrable and that satisfy the three hypotheses of Proposition 3.5.*

If Hypothesis 1. is strengthened by demanding that $A^{\{c\}}(0)0$ is integrable, then the sequence $\left(\frac{1}{n}x(n,0)\right)_{n\in\mathbb{N}}$ converges almost-surely and its limit is given by Equation (4).

To deduce the results on $x(n,0)$ from those on $y(n,0)$, we introduce the following theorem-definition, which is a special case of the main theorem of [19] and directly follows from Kingman's theorem:

Theorem-Definition 3.7 ([19]). *If $(A(n))_{n\in\mathbb{Z}}$ is a stationary and ergodic sequence of random matrices with values in $\mathbb{R}_{\max}^{d\times d}$ and almost-surely at least one finite entry on each row such that $A(0)0$ is integrable, then there are two real numbers $\gamma(A)$ and $\gamma_b(A)$ such that*

$$\lim_n \frac{1}{n}\max_i x_i(n,0) = \frac{1}{n}\max_i y_i(n,0) = \gamma(A) \text{ a.s.}$$

$$\lim_n \frac{1}{n}\min_i x_i(n,0) = \frac{1}{n}\min_i y_i(n,0) = \gamma_b(A) \text{ a.s.}$$

It implies the following corollary, which makes the link between the results on $(y(n,0))_{n\in\mathbb{N}}$ and those on $(x(n,0))_{n\in\mathbb{N}}$ when all $\gamma^{[c]}$ are equal, that is when $\gamma(A) = \gamma_b(A)$.

Corollary 3.8. *If $(A(n))_{n\in\mathbb{Z}}$ is a stationary and ergodic sequence of random matrices with values in $\mathbb{R}_{\max}^{d\times d}$ and almost-surely at least one finite entry on each row such that $A(0)0$ is integrable then*

$$\lim_n \frac{1}{n}x(n,0) = \gamma(A)\mathbf{1} \text{ if and only if } \lim_n \frac{1}{n}y(n,0) = \gamma(A)\mathbf{1}.$$

Let us go back to the proof of the general result on $(x(n,0))_{n\in\mathbb{N}}$. Because of propositions 3.2 and 3.4 and the definition of Lyapunov exponents, we already have, for every s.c.c c of $\mathcal{G}(A)$,

$$\limsup_n \frac{1}{n}x^c(n,0) \leq \gamma^{[c]}\mathbf{1} \text{ a.s.}.$$

Therefore it is sufficient to show that $\liminf_n \frac{1}{n}x^c(n,0) \geq \gamma^{[c]}\mathbf{1}$ a.s.. and even that

$$\lim_n \frac{1}{n}x^{\{c\}}(n,0) = \gamma^{[c]}\mathbf{1}.$$

Because of corollary 3.8, it is equivalent to $\lim_n \frac{1}{n}y^{\{c\}}(n,0) = \gamma^{[c]}\mathbf{1}$. Since all s.c.c of $\mathcal{G}(A^{\{c\}})$ are s.c.c of $\mathcal{G}(A)$ and have the same Lyapunov exponent $\gamma^{[c]}$, it follows from the result on the $y(n,0)$ applied to $A^{\{c\}}$.

3.3 Independent case

In this section, we prove Theorem 2.4.

Because of Theorem 2.3, it is sufficient to show that, if, for every s.c.c c, $A^{\{c\}}$ almost-surely has at least one finite entry on each row, then the sequence $\left(\frac{1}{n}x(n,0)\right)$ converges almost-surely. To do this, we will prove that, in this situation, the hypotheses of Proposition 3.6 are satisfied. Hypothesis 1. is exactly Hypothesis 1. of Theorem 2.4 and Hypotheses 2. and 3. respectively follow from the next lemma and theorem.

Definition 3.9. For every matrix $A \in \mathbb{R}_{\max}^{d\times d}$, the pattern matrix $\widehat{A}$ is defined by $\widehat{A}_{ij} = -\infty$ if $A_{ij} = -\infty$ and $A_{ij} = 0$ otherwise.

For every matrix $A, B \in \mathbb{R}_{\max}^{d\times d}$, we have $\widehat{AB} = \widehat{A}\widehat{B}$.

Lemma 3.10. *Let $(A(n))_{n\in\mathbb{N}}$ be a stationary sequence of random matrices with values in $\mathbb{R}_{\max}^{d\times d}$ and almost-surely at least one finite entry on each row. Let us assume that there exists a partition (I,J) of $[1,\cdots,d]$ such that $A = \tilde{A}$ satisfy Equation (9), with $\mathcal{G}(B)$ strongly connected. For every $i \in I$, let us define*

$$\mathcal{A}_i := \{\forall n \in \mathbb{N}, (B(1,n)D(n+1)0)_i = -\infty\}.$$

1. *If $\omega \in \mathcal{A}_i$, then we have $\forall n \in \mathbb{N}, \exists i_n \in I\, (B(1,n))_{ii_n} \neq -\infty$.*
2. *If the set $\mathcal{E} = \left\{M \in \{0,-\infty\}^{d\times d} \,\middle|\, \mathbb{P}\left(\widehat{A}(1,n) = M\right) > 0\right\}$ is a semigroup, and if $\mathbb{P}\left(D = (-\infty)^{I\times J}\right) < 1$, then for every $i \in I$, we have $\mathbb{P}(\mathcal{A}_i) = 0$.*

Proof.

1. For every $\omega \in \mathcal{A}_i$, we prove our result by induction on n.

   Since the $A(n)$ almost-surely have at least one finite entry on each row, there exists an $i_1 \in [1,\cdots,d]$, such that $A_{ii_1}(1) \neq -\infty$. Since $(D(1)0)_i = -\infty$, every entry on row i of $D(1)$ is $-\infty$, that is $A_{ij}(1) = -\infty$ for every $j \in J$, therefore $i_1 \in I$ and $B_{ii_1}(1) = A_{ii_1}(1) \neq -\infty$.

   Let us assume that the sequence is defined up to rank n. Since $A(n+1)$ almost-surely has at least one finite entry on each row, there exists an $i_{n+1} \in [1,\cdots,d]$, such that $A_{i_n i_{n+1}}(n+1) \neq -\infty$.

   Since $\omega \in \mathcal{A}_i$, we have:

   $$-\infty = (B(1,n)D(n+1)0)_i \geq (B(1,n))_{ii_n} + (D(n+1)0)_{i_n},$$

   therefore $(D(n+1)0)_{i_n} = -\infty$.

   It means that every entry on row i_n of $D(n+1)$ is $-\infty$, that is $A_{i_n j}(n+1) = -\infty$ for every $j \in J$, therefore $i_{n+1} \in I$ and

   $$B_{i_n i_{n+1}}(n+1) = A_{i_n i_{n+1}}(n+1) \neq -\infty.$$

   Finally, we have:

   $$(B(1,n+1))_{ii_{n+1}} \geq (B(1,n))_{ii_n} + B_{i_n i_{n+1}}(n+1) \neq -\infty.$$

2. As a first step, we want to construct a matrix $M \in \mathcal{E}$ such that

$$\forall i \in I, \exists j \in J, M_{ij} = 0.$$

Since $\mathbb{P}\left(D = (-\infty)^{I\times J}\right) < 1$, there are $\alpha \in I$, $\beta \in J$ and $M^0 \in \mathcal{E}$ with $M^0_{\alpha\beta} = 0$. For any $i \in I$, since $\mathcal{G}(B)$ is strongly connected, there is $M \in \mathcal{E}$ such that $M \in \mathcal{E}$ and $M_{i\alpha} = 0$. Therefore $M^i = MM^0$ is in $\mathcal{E}$ and satisfies $M^i_{i\beta} = 0$.

Now let us assume $I = \{\alpha_1, \cdots, \alpha_c\}$ and define by induction the finite sequence of matrices P^k.

- $P^1 = M^{\alpha_1}$
- If there exists $j \in J$ such that $P^k_{\alpha_{k+1}j} = 0$, then $P^{k+1} = P^k$. Else, since the matrices have at least one finite entry on each row, there is an $i \in I$, such that $P^k_{\alpha_k i}$, and $P^{k+1} = P^k M^i$.

It is easily checked that such P^k satisfy,

$$\forall l \leq k, \exists j \in J, P^k_{\alpha_l j} = 0.$$

Therefore, we set $M = P^m$ and denote by p the smallest integer such that $\mathbb{P}\left(\widehat{A}(1,p) = M\right) > 0$

Now, it follows from the definition of $\mathcal{E}$ and the ergodicity of $(A(n))_{n\in\mathbb{N}}$ that there is almost surely an $N \in N$, such that $\widehat{A}(N+1, N+p) = M$.

On $\mathcal{A}_i$, that would define a random $j_N \in J$ such that $M_{i_N j_N} = 0$, where i_N is defined according to the first point of the lemma. Then, we would have

$$(A(1, N+p))_{ij_N} \geq (A(1,N))_{ii_N} + (A(N+1, N+p))_{i_N j_N} > -\infty$$

But $\mathcal{A}_i$ is defined as the event on which there is never a path from i to J, so that we should have $\forall n \in \mathbb{N}, \forall j \in J, A(1,n))_{ij} = -\infty$.

Finally, $\mathcal{A}_i$ is included in the negligible set $\left\{\forall n \in \mathbb{N}, \widehat{A}(n+1, n+p) \neq M\right\}$.

□

Theorem 3.11. *If $(A(n))_{n\in\mathbb{N}}$ is a sequence of i.i.d. random matrices with values in $\mathbb{R}^{d\times d}_{\max}$ such that the positive part of $\max_{ij} A_{ij}(0)$ is integrable, $A(0)$ almost-surely has at least one finite entry on each row and $\mathcal{G}(A)$ is strongly connected, then we have*

$$\forall i \in [1,d], \lim_n \frac{1}{n} y_i(n,0) = \gamma(A)$$

.

This theorem is stated by D. Hong in the unpublished [15], but the proof is rather difficult to understand and it is unclear if it holds when $A(1)$ takes infinitely many values. Building on [6], we now give a short proof of this result.

Proof. According to [6], $\left(\frac{1}{n}y(n,0)\right)_{n\in\mathbb{N}}$ converges a.s. We have to show that its limit is deterministic.

The sequence $R(n) := \widehat{A}(-1,-n)$ is a Markov chain whose states space is $\left\{M \in \{0,-\infty\}^{d\times d} \,|M0 = 0\right\}$ and whose transitions are defined by:

$$\mathbb{P}\left(R(n+1) = F | R(n) = E\right) = \mathbb{P}\left(\widehat{EA(1)} = F\right).$$

For every $i,j \in I$, we have $R_{ij}(n) = 0$ if and only if $(A(-1,-n))_{ij} \neq -\infty$.

Let i be any integer in $\{1,\cdots,d\}$ and E be a recurrent state of $(R(n))_{n\in\mathbb{N}}$. There exists a $j \in [1,\cdots,d]$ such that $E_{ij} = 0$. Since $\mathcal{G}(A)$ is strongly connected, there exists a $p \in \mathbb{N}$, such that $(B(-1,-p))_{ji} \neq -\infty$ with positive probability. Let G be such that $\mathbb{P}\left((B(-1,-p))_{ji} \neq -\infty, \widehat{B}(-1,-p) = G\right) > 0$. Now, $F = EG$ is a state of the chain, reachable from state E and such that $F_{ii} = 0$. Since E is recurrent, so is F and E and F belong to the same recurrence class.

Let $\mathcal{E}$ be a set with exactly one matrix F in each recurrence class, such that $F_{ii} = 0$. Let S_n be the n^{th} time $(R(m))_{m\in\mathbb{N}}$ is in $\mathcal{E}$.

Since the Markov chain has finitely many states and $\mathcal{E}$ intersects every recurrence class, S_n is almost-surely finite, and even integrable. Moreover, the $S_{n+1} - S_n$ are i.i.d. (we set $S_0 = 0$) and so are the $A(-S_n - 1, -S_{n+1})$. Since $P(S_1 > k)$ decreases exponentially fast, $A(-1,-S_1)0$ is integrable and thus the sequence $\left(\frac{1}{n}y(S_n,0)\right)_{n\in\mathbb{N}}$ converges a.s. Let us denote its limit by l.

Let us denote by $\mathcal{F}_0$ the σ-algebra generated by the random matrices $A(-S_n - 1, -S_{n+1})$. Then l is $\mathcal{F}_0$ measurable, and the independence of the $A(-S_n - 1, -S_{n+1})$ means that $(\Omega, \mathcal{F}_0, \mathbb{P}, \theta^{S_1})$ is an ergodic measurable dynamical system. Because of the choice of S_1, we have $l_i \geq l_i \circ \theta^{S_1}$, so that l_i is deterministic.

Now, let us notice that the limit of $\frac{1}{n}y_i(n,0)$ is that of $\frac{1}{S_n}y_i(S_n,0)$, that is $\frac{l_i}{\mathbb{E}(S_1)}$, which is deterministic.

This means that $\lim \frac{1}{n}y_i(n,0)$ is deterministic for any i, and, according to Theorem 2.3, it implies that it is equal to $\gamma(A)$. □

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