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L^∞ -uniqueness of Schrödinger operators restricted in an open domain*

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Abstract

Consider the Schrödinger operator $\mathcal{A} = -\frac{\Delta}{2} + V$ acting on space $C_0^\infty(D)$, where D is an open domain in \mathbb{R}^d . The main purpose of this paper is to present the $L^\infty(D, dx)$ -uniqueness for Schrödinger operators which is equivalent to the $L^1(D, dx)$ -uniqueness of weak solutions of the heat diffusion equation associated to the operator \mathcal{A} .

Key Words: C_0 -semigroups; L^∞ -uniqueness of Schrödinger operators; L^1 -uniqueness of the heat diffusion equation.

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1 Preliminaries

Let D be an open domain in \mathbb{R}^d with its boundary ∂D . We denote by $C_0^\infty(D)$ the space of all infinitely differentiable real functions on D with compact support. Consider the *Schrödinger operator* $\mathcal{A} = -\frac{\Delta}{2} + V$ acting on space $C_0^\infty(D)$, where Δ is the Laplace operator and $V : \mathbb{R}^d \rightarrow \mathbb{R}$ is a Borel measurable potential.

The *essential self-adjointness* of Schrödinger operator in $L^2(\mathbb{R}^d, dx)$, equivalent to the unique solvability of Schrödinger equation in $L^2(\mathbb{R}^d, dx)$, has been studied by KATO [Ka'84], REED and SIMON [RS'75], SIMON [Si'82] and others because of its importance in Quantum Mechanics. In the case where V is bounded, it is not difficult to prove that $(\mathcal{A}, C_0^\infty(\mathbb{R}^d))$ is essentially self-adjoint in $L^2(\mathbb{R}^d, dx)$. But in almost all interesting situations in quantum physics, the potential V is unbounded. In this situation we need to consider the *Kato class*, used first by SCHECHTER [Sch'71] and KATO [Ka'72]. A real valued measurable function V is said to be in the *Kato class* \mathcal{K}^d on \mathbb{R}^d if

$$\limsup_{\delta \searrow 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y| \leq \delta} |g(x-y)V(y)| dy = 0$$

where

$$g(x) = \begin{cases} \frac{1}{|x|^{d-2}} & , \text{ if } d \geq 3 \\ \ln \frac{1}{|x|} & , \text{ if } d = 2 \\ 1 & , \text{ if } d = 1. \end{cases}$$

If $V \in L^2_{loc}(\mathbb{R}^d, dx)$ is such that V^- belongs to the Kato class on \mathbb{R}^d , it is well known that the Schrödinger operator $(\mathcal{A}, C_0^\infty(\mathbb{R}^d))$ is essentially self-adjoint and the unique solution in L^2 of the heat equation is given by the famous *Feynmann-Kac semigroup*

$\{P_t^V\}_{t \geq 0}$

$$P_t^V f(x) := \mathbb{E}^x f(B_t) \exp \left(- \int_0^t V(B_s) ds \right)$$

where f is a nonnegative measurable function, $(B_t)_{t \geq 0}$ is the Brownian Motion in \mathbb{R}^d defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \mathbb{R}^d})$ with $\mathbb{P}_x(B_0 = x) = 1$ for any initial point $x \in \mathbb{R}^d$ and \mathbb{E}^x means the expectation with respect to \mathbb{P}_x .

In the case where D is a strict sub-domain, sharp results are known only when $d = 1$ or, in the multidimensional case, only in some special situations.

Consequently of an intuitive probabilistic interpretation of uniqueness, WU [Wu'98] introduced and studied the uniqueness of Schrödinger operators in $L^1(D, dx)$. One says that $(\mathcal{A}, C_0^\infty(D))$ is $L^1(D, dx)$ -unique if \mathcal{A} is closable and its closure is the generator of some C_0 -semigroup on $L^1(D, dx)$. This uniqueness notion was also studied in ARENDT [Ar'86], EBERLE [Eb'97], DJELLOUT [Dj'97], RÖCKNER [Rö'98], WU [Wu'98] and [Wu'99] and others in the Banach spaces setting.

2 $L^\infty(D, dx)$ -uniqueness of Schrödinger operators

Our purpose is to study the $L^\infty(D, dx)$ -uniqueness of the Schrödinger operator $(\mathcal{A}, C_0^\infty(D))$ in the case where D is a strict sub-domain on \mathbb{R}^d . But how we can define the uniqueness in $L^\infty(D, dx)$? One can prove rather easily that *the killed Feynmann-Kac* semigroup

$\{P_t^{D,V}\}_{t \geq 0}$

$$P_t^{D,V} f(x) := \mathbb{E}^x 1_{[t < \tau_D]} f(B_t) \exp \left(- \int_0^t V(B_s) ds \right)$$

where $\tau_D := \inf\{t > 0 : B_t \notin D\}$ is the first exiting time of D , is a semigroup of bounded operators on $L^p(D, dx)$ for any $1 \leq p \leq \infty$, which is strongly continuous for

$1 \leq p < \infty$, but never strongly continuous in $(L^\infty(D, dx), \|\cdot\|_\infty)$. Moreover, a well known result of LOTZ [Lo'86, Theorem 3.6, p. 57] says that the generator of any strongly continuous semigroup on $(L^\infty(D, dx), \|\cdot\|_\infty)$ must be bounded.

To obtain a correct definition of $L^\infty(D, dx)$ -uniqueness, we should introduce a weaker topology of $L^\infty(D, dx)$ such that $\{P_t^{D,V}\}_{t \geq 0}$ becomes a strongly continuous semigroup with respect to this new topology. Remark that the natural topology for studying C_0 -semigroups on $L^\infty(D, dx)$ used first by WU and ZHANG [WZ'06] is *the topology of uniform convergence on compact subsets of $L^1(D, dx)$* , denoted by $\mathcal{C}(L^\infty, L^1)$. More precisely, if we denote

$$\langle f, g \rangle := \int_D f(x)g(x)dx$$

for all $f \in L^1(D, dx)$ and $g \in L^\infty(D, dx)$, then for an arbitrary point $g_0 \in L^\infty(D, dx)$, a basis of neighborhoods with respect to $\mathcal{C}(L^\infty, L^1)$ is given by

$$N(g_0; K, \varepsilon) := \left\{ g \in L^\infty(D, dx) : \sup_{f \in K} |\langle f, g \rangle - \langle f, g_0 \rangle| < \varepsilon \right\}$$

where K runs over all compact subsets of $L^1(D, dx)$ and $\varepsilon > 0$.

Remark that $(L^\infty(D, dx), \mathcal{C}(L^\infty, L^1))$ is a locally convex space and if $\{T(t)\}_{t \geq 0}$ is a C_0 -semigroup on $L^1(D, dx)$ with generator \mathcal{L} , by [WZ'06, Theorem 1.4, p. 564] it follows that $\{T^*(t)\}_{t \geq 0}$ is a C_0 -semigroup on $(L^\infty(D, dx), \mathcal{C}(L^\infty, L^1))$ with generator \mathcal{L}^* .

Now we can introduce the uniqueness notion in $L^\infty(D, dx)$. Let \mathbf{A} be a linear operator on $L^\infty(D, dx)$ with domain \mathcal{D} which is assumed to be dense in $L^\infty(D, dx)$ with respect to the topology $\mathcal{C}(L^\infty, L^1)$.

Definition 2.1. *The operator \mathbf{A} is said to be a pre-generator on $L^\infty(D, dx)$ if there exists some C_0 -semigroup on $(L^\infty(D, dx), \mathcal{C}(L^\infty, L^1))$ such that its generator \mathcal{L} extends*

A. We say that \mathbf{A} is $L^\infty(D, dx)$ -unique if \mathbf{A} is closable and its closure with respect to the topology $\mathcal{C}(L^\infty, L^1)$ is the generator of some C_0 -semigroup on $(L^\infty(D, dx), \mathcal{C}(L^\infty, L^1))$.

The main result of this paper is

Theorem 2.2. *Let $V \in L^\infty_{loc}(D, dx)$ such that $V^- \in \mathcal{K}^d$. Then the Schrödinger operator $(\mathcal{A}, C_0^\infty(D))$ is $(L^\infty(D, dx), \mathcal{C}(L^\infty, L^1))$ -unique.*

Proof. First, we must remark that the existence assumption of pre-generator in [WZ'06, Theorem 2.1, p. 570] is satisfied. Indeed, if consider the killed Feynman-Kac semigroup $\{P_t^{D,V}\}_{t \geq 0}$ on $L^\infty(D, dx)$ and for any $p \in [1, \infty]$ we define

$$\left\| P_t^{D,V} \right\|_p := \sup_{\substack{f \geq 0 \\ \|f\|_p \leq 1}} \left\| P_t^{D,V} f \right\|_p,$$

next lemma show that \mathcal{A} is a pre-generator on $(L^\infty(D, dx), \mathcal{C}(L^\infty, L^1))$, i.e. \mathcal{A} is contained in the generator $\mathcal{L}_{(\infty)}^{D,V}$ of the killed Feynmann-Kac semigroup $\{P_t^{D,V}\}_{t \geq 0}$.

Lemma 2.3. *Let $V \in L^\infty_{loc}(D, dx)$ such that $V^- \in \mathcal{K}^d$ and let $\{P_t^{D,V}\}_{t \geq 0}$ be the killed Feynman-Kac semigroup on $L^\infty(D, dx)$. If $\left\| P_t^{D,V} \right\|_\infty$ is bounded over the compact intervals, then $\{P_t^{D,V}\}_{t \geq 0}$ is a C_0 -semigroup on $(L^\infty(D, dx), \mathcal{C}(L^\infty, L^1))$ and its generator $\mathcal{L}_{(\infty)}^{D,V}$ is an extension of $(\mathcal{A}, C_0^\infty(D))$.*

Proof. The proof is close to that of [Wu'98, Lemma 2.3, p. 288]. Let $\{P_t^{D,V}\}_{t \geq 0}$ be the killed Feynman-Kac semigroup on $L^\infty(D, dx)$. Remark that

$$\left| P_t^{D,V} f(x) \right| \leq P_t^{D,V} |f|(x) \leq P_t^{D,-V^-} |f|(x) \leq P_t^{-V^-} |f|(x)$$

from where we deduce that

$$\sup_{0 \leq t \leq 1} \left\| P_t^{D,V} \right\|_\infty \leq \sup_{0 \leq t \leq 1} \left\| P_t^{-V^-} \right\|_\infty < \infty$$

since $\left\|P_t^{-V^-}\right\|_\infty$ is uniformly bounded by the assumption that $V^- \in \mathcal{K}^d$ (see [AS'82]). Since $\left\|P_t^{D,V}\right\|_1 = \left\|P_t^{D,V}\right\|_\infty$ is bounded for t in compact intervals of $[0, \infty)$, using [Wu'01, Lemma 2.3, p. 59] it follows that $\left\{P_t^{D,V}\right\}_{t \geq 0}$ is a C_0 -semigroup on $L^1(D, dx)$. By [WZ'06, Theorem 1.4, p. 564] we find that $\left\{P_t^{D,V}\right\}_{t \geq 0}$ is a C_0 -semigroup on $L^\infty(D, dx)$ with respect to the topology $\mathcal{C}(L^\infty, L^1)$. We have only to show that its generator $\mathcal{L}_{(\infty)}^{D,V}$ is an extension of $(\mathcal{A}, C_0^\infty(D))$.

Step 1: the case $V \geq 0$. For $n \in \mathbb{N}$ we consider $V_n := V \wedge n$. By a theorem of bounded perturbation (see [Da'80, Theorem 3.1, p. 68]) it follows that

$$\mathcal{A}_n = -\frac{\Delta}{2} + V_n$$

is the generator of a C_0 -semigroup $\left\{P_t^{D,V_n}\right\}_{t \geq 0}$ on $(L^\infty(D, dx), \mathcal{C}(L^\infty, L^1))$. So for any $f \in C_0^\infty(D)$ we have

$$P_t^{D,V_n} f - f = \int_0^t P_s^{D,V_n} \mathcal{A}_n f \, ds \quad , \quad \forall t \geq 0.$$

Letting $n \rightarrow \infty$, we have pointwisely on D :

$$P_t^{D,V_n} f \rightarrow P_t^{D,V} f$$

and

$$P_t^{D,V_n} \mathcal{A}_n f \rightarrow P_t^{D,V} \mathcal{A} f \quad .$$

Moreover, for any $x \in D$ we have:

$$\left|P_t^{D,V_n} f(x)\right| \leq P_t^{D,V} |f|(x)$$

and

$$\left|P_t^{D,V_n} \mathcal{A}_n f(x)\right| \leq P_t^{D,V} \left(\left|\frac{\Delta}{2}\right| + |Vf|\right)(x) \quad .$$

Hence by the dominated convergence we derive that

$$P_t^{D,V} f - f = \int_0^t P_s^{D,V} \mathcal{A} f ds \quad , \quad \forall t \geq 0.$$

It follows that f is in the domain of the generator $\mathcal{L}_{(\infty)}^{D,V}$ of C_0 -semigroup $\{P_t^{D,V}\}_{t \geq 0}$.

Step 2: the general case. Setting $V^n = V \vee (-n)$, for $n \in \mathbb{N}$, and denoting by

$$\mathcal{A}^n = -\frac{\Delta}{2} + V^n$$

the generator of the C_0 -semigroup $\{P_t^{D,V^n}\}_{t \geq 0}$ on $(L^\infty(D, dx), \mathcal{C}(L^\infty, L^1))$, we have by Step 1

$$P_t^{D,V^n} f - f = \int_0^t P_s^{D,V^n} \mathcal{A}^n f ds \quad , \quad t \geq 0.$$

Notice that

$$|P_s^{D,V^n} \mathcal{A}^n f(x)| \leq P_s^{D,V} \left(\left| \frac{\Delta}{2} f \right| + |V f| \right) (x)$$

which is uniformly bounded in $L^\infty(D, dx)$ over $[0, t]$. By Fubini's theorem we have

$$\int_0^t P_s^{D,V} \left(\left| \frac{\Delta}{2} f \right| + |V f| \right) (x) ds < \infty \quad \text{dx-a.e. on } D.$$

On the other hand, for any $x \in D$ fixed such that

$$P_s^{D,V} \left(\left| \frac{\Delta}{2} f \right| + |V f| \right) (x) < \infty$$

then by dominated convergence we find

$$P_s^{D,V^n} \left(-\frac{\Delta}{2} + V^n \right) f(x) \longrightarrow P_s^{D,V} \left(-\frac{\Delta}{2} + V \right) f(x) \quad .$$

Thus by dominated convergence we have dx-a.e. on D ,

$$\int_0^t P_s^{D,V^n} \left(-\frac{\Delta}{2} + V^n \right) f ds \longrightarrow \int_0^t P_s^{D,V} \left(-\frac{\Delta}{2} + V \right) f ds \quad , \quad \forall t \geq 0.$$

The same argument shows that

$$P_t^{D,V^n} f - f \rightarrow P_t^{D,V} f - f \quad .$$

By consequence

$$P_t^{D,V} f - f = \int_0^t P_s^{D,V} \left(-\frac{\Delta}{2} + V \right) f ds \quad , \quad \forall t \geq 0.$$

Hence f is in the domain of generator $\mathcal{L}_{(\infty)}^{D,V}$ of semigroup $\{P_t^{D,V}\}_{t \geq 0}$. So $\mathcal{L}_{(\infty)}^{D,V}$ is an extension of the operator $(\mathcal{A}, C_0^\infty(D))$ and the lemma is proved.

Next we prove the $L^\infty(D, dx)$ -uniqueness of \mathcal{A} . By [WZ'06, Theorem 2.1, p. 570], we deduce that the operator $(\mathcal{A}, C_0^\infty(D))$ is $L^\infty(D, dx)$ -unique if and only if for some λ , the range $(\lambda I - \mathcal{A})(C_0^\infty(D))$ is dense in $(L^\infty(D, dx), \mathcal{C}(L^\infty, L^1))$. It is enough to show that for any $h \in L^1(D, dx)$ which satisfies the equality

$$\langle h, (\lambda I + \mathcal{A})f \rangle = 0 \quad , \quad \forall f \in C_0^\infty(D)$$

it follows $h = 0$.

Let $h \in L^1(D, dx)$ be such that for some λ one have

$$\langle h, (\lambda I + \mathcal{A})f \rangle = 0 \quad , \quad \forall f \in C_0^\infty(D)$$

or

$$(\lambda I + \mathcal{A})h = 0 \quad \text{in the sense of distribution.}$$

Since $V \in L_{loc}^\infty(D, dx)$, by applying [AS'82, Theorem 1.5, p. 217] we can see that h is a continuous function. By the mean value theorem due to AIZENMAN and SIMON [AS'82, Corollary 3.9, p. 231], there exists some constant $C > 0$ such as

$$|h(x)| \leq C \int_{|x-y| \leq 1} |h(y)| dy \quad , \quad \forall x \in D.$$

As $V^- \in \mathcal{K}^d$, C may be chosen independently of $x \in D$. Since $h \in L^1(D, dx)$, it follows that h is bounded and, consequently, $h \in L^2(D, dx)$. Now by the $L^2(D, dx)$ -uniqueness of $(\mathcal{A}, C_0^\infty(D))$ and [WZ'06, Theorem 2.1, p. 570], h belongs to the domain of the generator $\mathcal{L}_{(2)}^{D,V}$ of $\{P_t^{D,V}\}_{t \geq 0}$ on L^2 and

$$\mathcal{L}_{(2)}^{D,V} h = \left(-\frac{\Delta}{2} + V \right) h = -\lambda h \quad .$$

Hence

$$P_t^{D,V} h = e^{-\lambda t} h \quad , \quad \forall t \geq 0.$$

Let

$$\lambda(D, V) := \inf_{f \in C_0^\infty(D)} \left\{ \frac{1}{2} \int_D |\nabla f|^2 dx + V f^2 dx : \|f\|_2 \leq 1 \right\}.$$

be the lowest energy of the Schrödinger operator. If we take $\lambda < \lambda(D, V)$, then the last equality is possible only for $h = 0$, because $\|P_t^{D,V}\|_2 = e^{-\lambda(D,V)t}$ (see ALBEVERIO and MA [AM'91, Theorem 4.1, p. 343]).■

Remarque 2.4. Intuitively, to have $L^1(D, dx)$ -uniqueness, the repulsive potential V^+ should grow rapidly to infinity near ∂D , this means

$$(C_1) \quad \mathbb{P}_x \left(\int_0^{\tau_D} V^+(B_s) ds + \tau_D = \infty \right) = 1 \quad \text{for a.e. } x \in D$$

so that a particle with starting point inside D can not reach the boundary ∂D (see [Wu'98, Theorem 1.1, p. 279]).

By analogy with the uniqueness in $L^1(D, dx)$, the $L^\infty(D, dx)$ -uniqueness of $(\mathcal{A}, C_0^\infty(D))$ means that a particle starting from the boundary ∂D can not enter in D . Unfortunately, here we have a problem: $L^\infty(D, dx)$ -uniqueness of \mathcal{A} is equivalent to the existence of a unique boundary condition for \mathcal{A}^* . It is well known that there are many boundary conditions (Dirichlet, Neumann, etc.). Remark that in the case of $L^1(D, dx)$ -uniqueness

of \mathcal{A} , the effect of the boundary condition for \mathcal{A}^* is eliminated by the condition (C_1) for potential. To find such condition in the case of $L^\infty(D, dx)$ -uniqueness is very difficult. In this moment we can present here an interesting result from [WZ'06]:

Proposition 2.5. *Let D be a nonempty open domain of \mathbb{R}^d . If the Laplacian $(\Delta, C_0^\infty(D))$ is $L^\infty(D, dx)$ -unique, then $D^C = \emptyset$ or $D = \mathbb{R}^d$.*

For the heat diffusion equation we can formulate the next result

Corollary 2.6. *If $V \in L_{loc}^\infty(\mathbb{R}^d, dx)$ and $V^- \in \mathcal{K}^d$, then for every $h \in L^1(\mathbb{R}^d, dx)$, the heat diffusion equation*

$$\begin{cases} \partial_t u(t, x) = \left(-\frac{\Delta}{2} + V\right) u(t, x) \\ u(0, x) = h(x) \end{cases}$$

has one $L^1(\mathbb{R}^d, dx)$ -unique weak solution which is given by $u(t, x) = P_t^V h(x)$.

Proof. The assertion follows by [WZ'06, Theorem 2.1, p. 570] and Theorem 2.2.

■

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