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# $L^\infty$ -uniqueness of Schrödinger operators restricted in an open domain\*

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## Abstract

Consider the Schrödinger operator  $\mathcal{A} = -\frac{\Delta}{2} + V$  acting on space  $C_0^\infty(D)$ , where  $D$  is an open domain in  $\mathbb{R}^d$ . The main purpose of this paper is to present the  $L^\infty(D, dx)$ -uniqueness for Schrödinger operators which is equivalent to the  $L^1(D, dx)$ -uniqueness of weak solutions of the heat diffusion equation associated to the operator  $\mathcal{A}$ .

**Key Words:**  $C_0$ -semigroups;  $L^\infty$ -uniqueness of Schrödinger operators;  $L^1$ -uniqueness of the heat diffusion equation.

**2000 AMS Subject Classification** Primary: 47D03, 47F05 Secondary: 60J60

**Running head:**  $L^\infty$ -uniqueness of Schrödinger operators.

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# 1 Preliminaries

Let  $D$  be an open domain in  $\mathbb{R}^d$  with its boundary  $\partial D$ . We denote by  $C_0^\infty(D)$  the space of all infinitely differentiable real function on  $D$  with compact support. Consider the Schrödinger operator  $\mathcal{A} = -\frac{\Delta}{2} + V$  acting on space  $C_0^\infty(D)$ , where  $\Delta$  is the Laplace operator and  $V : \mathbb{R}^d \rightarrow \mathbb{R}$  is a Borel measurable potential. It is well known that the Laplacian  $\Delta$  generate a free Brownian Motion  $(B_t)_{t \geq 0}$  in  $\mathbb{R}^d$  defined on some filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \mathbb{R}^d})$  with  $\mathbb{P}_x(B_0 = x) = 1$  for any initial point  $x \in \mathbb{R}^d$ , and  $\mathbb{E}^x$  means the expectation with respect to  $\mathbb{P}_x$ . The essential self-adjointness of Schrödinger operator in  $L^2(D, dx)$ , equivalent to the unique solvability of Schrödinger equation in  $L^2(D, dx)$ , has been studied by KATO [Ka'84], REED and SIMON [RS'75], SIMON [Si'82] and others because of its importance in Quantum Mechanics. Very general results are known when  $D$  is the whole space  $\mathbb{R}^d$ . In the case where  $D$  is a strict sub-domain, sharp results are known only when  $d = 1$  or, in the multidimensional case, only in some special situations. Consequently of an intuitive probabilistic interpretation of uniqueness, WU [Wu'98] introduced and studied the uniqueness of Schrödinger operators in  $L^1(D, dx)$  and, later, in  $L^p(\mathbb{R}^d, dx)$ , for  $p \in (1, \infty)$  (see [Wu'01]). In fact, to have  $L^1(D, dx)$ -uniqueness, intuitively the repulsive potential  $V^+$  should grow rapidly to infinity near  $\partial D$  so that a particle with starting point inside  $D$  can not reach the boundary  $\partial D$  (see [Wu'98, Theorem 1.1, p. 279]).

The main purpose of this paper is to study the  $L^\infty(D, dx)$ -uniqueness of the Schrödinger operator  $(\mathcal{A}, C_0^\infty(D))$ . First we must remark that the natural topology for studying  $C_0$ -semigroups on  $L^\infty(D, dx)$  is *the topology of uniform convergence on compact subsets of  $L^1(D, dx)$* , denoted by  $\mathcal{C}(L^\infty, L^1)$  (see [WZ'02, p. 701]). More precisely, for an

arbitrary point  $y_0 \in L^\infty(D, dx)$ , a basis of neighborhoods with respect to  $\mathcal{C}(L^\infty, L^1)$  is given by

$$N(y_0; K, \varepsilon) := \left\{ y \in L^\infty \left| \sup_{x \in K} |\langle x, y \rangle - \langle x, y_0 \rangle| < \varepsilon \right. \right\}$$

where  $K$  runs over all compact subsets of  $L^1(D, dx)$  and  $\varepsilon > 0$ . If  $\{T(t)\}_{t \geq 0}$  is a  $C_0$ -semigroup on  $L^1(D, dx)$  with generator  $\mathcal{L}$ , by [WZ'02, Corollary 1, p. 701] it follows that  $\{T^*(t)\}_{t \geq 0}$  is a  $C_0$ -semigroup on  $(L^\infty, \mathcal{C}(L^\infty, L^1))$  with generator  $\mathcal{L}^*$ .

$\mathcal{A}$  is said to be a *pre-generator* on  $(L^\infty, \mathcal{C}(L^\infty, L^1))$  if there exists some  $C_0$ -semigroup on  $(L^\infty, \mathcal{C}(L^\infty, L^1))$  such that its generator  $\mathcal{L}$  extends  $\mathcal{A}$ . We say that  $\mathcal{A}$  is an *essential generator* (or  *$L^\infty(D, dx)$ -unique*) on  $(L^\infty, \mathcal{C}(L^\infty, L^1))$  if  $\mathcal{A}$  is closable and its closure  $\overline{\mathcal{A}}$  with respect to topology  $\mathcal{C}(L^\infty, L^1)$  is the generator of some  $C_0$ -semigroup on  $(L^\infty, \mathcal{C}(L^\infty, L^1))$ . This uniqueness notion was studied in ARENDT [Ar'86], EBERLE [Eb'97], DJELLOUT [Dj'97], RÖCKNER [Rö'98], WU [Wu'98] and [Wu'99] and others in the Banach spaces setting and WU and ZHANG [WZ'06] in the case of locally convex spaces.

## 2 $L^\infty(D, dx)$ -uniqueness of Schrödinger operators

The main purpose of this section is to study the  $L^\infty(D, dx)$ -uniqueness of the Schrödinger operator  $(\mathcal{A}, C_0^\infty(D))$ . The next result is well known (see [WZ'02, theorem 5, p. 704]):

**Theorem 2.1.** *Let  $V \geq 0$ . Then  $(\mathcal{A}, C_0^\infty(D))$  is  $L^\infty(D, dx)$ -unique with respect to the topology  $\mathcal{C}(L^\infty, L^1)$ .*

In the case where  $V$  is bounded, it is not difficult to prove that  $(\mathcal{A}, C_0^\infty(D))$  is  $L^\infty(D, dx)$ -unique. But in almost all interesting situations in quantum physics, the

potential  $V$  is unbounded. In this case, we need to consider the potential of *Kato class*, used first time by SCHECHTER [Sch'71] and KATO [Ka'72].

**Definition 2.2.** *A real valued measurable function  $V$  is said to be in the Kato class  $\mathcal{K}^d$  on  $\mathbb{R}^d$  if*

$$\limsup_{\delta \searrow 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y| \leq \delta} |g(x-y)V(y)| dy = 0$$

where

$$g(x) = \begin{cases} \frac{1}{|x|^{d-2}} & , \text{ if } d \geq 3 \\ \ln \frac{1}{|x|} & , \text{ if } d = 2 \\ 1 & , \text{ if } d = 1. \end{cases}$$

For any  $p \in [1, \infty]$  we define

$$\left\| P_t^{D,V} \right\|_p := \sup_{\substack{f \geq 0 \\ \|f\|_\infty \leq 1}} \left\| P_t^{D,V} f \right\|_p .$$

The following properties of the Feynman-Kac semigroup are very useful (see ALBEVERIO et MA [AM'91, Theorem 4.1, p. 343]).

**Lemma 2.3.** *The following properties are equivalent:*

- (i)  $\left\| P_t^{D,V} \right\|_2 < \infty$ ;
- (ii)  $\left\{ P_t^{D,V} \right\}_{t \geq 0}$  is a  $C_0$ -semigroup on  $L^2(D, dx)$ ;
- (iii) the quadratic form

$$\mathcal{E}^V(f, g) := \int \nabla f \cdot \nabla g dx + \int V f g dx \quad , \quad \forall f, g \in C_0^\infty(D)$$

associated with  $(\mathcal{A}, C_0^\infty(D))$  is lower bounded, i.e.,

$$\lambda(D, V) := \inf \{ \mathcal{E}^V(f, f) \mid \|f\|_2 \leq 1 \} > -\infty .$$

In that case, we have

$$\left\| P_t^{D,V} \right\|_2 = e^{-\lambda(D,V)t} \quad , \quad \forall t \geq 0.$$

The proof of the lemma below is close to that of [Wu'98, Lemma 2.3, p. 288].

**Lemma 2.4.** *Let  $V \in L_{loc}^\infty(D, dx)$  and let  $\{P_t^{D,V}\}_{t \geq 0}$  be the killed Feynman-Kac semigroup on  $L^\infty(D, dx)$ . If  $\|P_t^{D,V}\|_\infty$  is bounded over the compact intervals, then  $\{P_t^{D,V}\}_{t \geq 0}$  is a  $C_0$ -semigroup on  $(L^\infty, \mathcal{C}(L^\infty, L^1))$  and its generator  $\mathcal{L}_{(\infty)}^{D,V}$  is an extension of  $(\mathcal{A}, C_0^\infty(D))$ .*

**Proof.** Let  $\{P_t^{D,V}\}_{t \geq 0}$  be the Feynman-Kac semigroup on  $L^\infty(\mathbb{R}^d, dx)$ . Since  $\|P_t^{D,V}\|_1 = \|P_t^{D,V}\|_\infty$  is bounded for  $t$  in compact intervals of  $[0, \infty)$ , using [Wu'01, Lemma 2.3, p. 59] it follows that  $\{P_t^{D,V}\}_{t \geq 0}$  is a  $C_0$ -semigroup on  $L^1(\mathbb{R}^d, dx)$ . By [WZ'06, Theorem 1.4] we find that  $\{P_t^{D,V}\}_{t \geq 0}$  is a  $C_0$ -semigroup on  $L^\infty(\mathbb{R}^d, dx)$  with respect to the topology  $\mathcal{C}(L^\infty, L^1)$ . We have only to show that its generator  $\mathcal{L}_{(\infty)}^{D,V}$  is an extension of  $(\mathcal{A}, C_0^\infty(D))$ .

Let  $V \geq 0$ . For  $n \in \mathbb{N}$  we consider  $V_n := V \wedge n$ . By a theorem of bounded perturbation (see [Da'80, Theorem 3.1, p. 68]) it follows that

$$\mathcal{A}_n = -\frac{\Delta}{2} + V_n$$

is the generator of a  $C_0$ -semigroup  $\{P_t^{D,V_n}\}_{t \geq 0}$  on  $(L^\infty, \mathcal{C}(L^\infty, L^1))$ . So for any  $f \in C_0^\infty(D)$  we have

$$P_t^{D,V_n} f - f = \int_0^t P_s^{D,V_n} \mathcal{A}_n f \, ds \quad , \quad \forall t \geq 0.$$

Letting  $n \rightarrow \infty$ , we have pointwisely on  $D$ :

$$P_t^{D,V_n} f \rightarrow P_t^{D,V} f$$

and

$$P_t^{D,V_n} \mathcal{A}_n f \rightarrow P_t^{D,V} \mathcal{A} f \quad .$$

Moreover, for any  $x \in D$  we have:

$$\left| P_t^{D,V_n} f(x) \right| \leq P_t^{D,V} |f|(x)$$

and

$$\left| P_t^{D, V^n} \mathcal{A}_n f(x) \right| \leq P_t^{D, V} \left( \left| \frac{\Delta}{2} \right| + |Vf| \right) (x) \quad .$$

Hence by the dominated convergence we derive that

$$P_t^{D, V} f - f = \int_0^t P_s^{D, V} \mathcal{A} f ds \quad , \quad \forall t \geq 0.$$

It follows that  $f$  is in the domain of the generator  $\mathcal{L}_{(\infty)}^{D, V}$  of  $C_0$ -semigroup  $\left\{ P_t^{D, V} \right\}_{t \geq 0}$ . In the general case, let  $V^n = V \vee (-n)$ , for  $n \in \mathbb{N}$ . Considering the case  $V \geq 0$ , we have

$$P_t^{D, V^n} f - f = \int_0^t P_s^{D, V^n} \mathcal{A}^n f ds \quad , \quad t \geq 0.$$

Notice that

$$\left| P_s^{D, V^n} \mathcal{A}^n f(x) \right| \leq P_s^{D, V} \left( \left| \frac{\Delta}{2} f \right| + |Vf| \right) (x)$$

which is uniformly bounded in  $L^\infty(D)$  over  $[0, t]$ . By Fubini's theorem we have

$$\int_0^t P_s^{D, V} \left( \left| \frac{\Delta}{2} f \right| + |Vf| \right) (x) ds < \infty$$

dx-a.e. on  $D$ . On the other hand, for any  $x \in D$  fixed such that

$$P_s^{D, V} \left( \left| \frac{\Delta}{2} f \right| + |Vf| \right) (x) < \infty$$

then by the Feynman-Kac formula and dominated convergence we find

$$P_s^{D, V^n} \left( -\frac{\Delta}{2} + V^n \right) f(x) \longrightarrow P_s^{D, V} \left( -\frac{\Delta}{2} + V \right) f(x) \quad .$$

Thus by dominated convergence we have dx-a.e. on  $D$ ,

$$\int_0^t P_s^{D, V^n} \left( -\frac{\Delta}{2} + V^n \right) f ds \longrightarrow \int_0^t P_s^{D, V} \left( -\frac{\Delta}{2} + V \right) f ds \quad , \quad \forall t \geq 0.$$

The same argument shows that

$$P_t^{D,V^n} f - f \rightarrow P_t^{D,V} f - f \quad .$$

By consequence

$$P_t^{D,V} f - f = \int_0^t P_s^{D,V} \left( -\frac{\Delta}{2} + V \right) f ds \quad , \quad \forall t \geq 0.$$

Hence  $f$  is in the domain of generator  $\mathcal{L}_{(\infty)}^{D,V}$  of semigroup  $\left\{ P_t^{D,V} \right\}_{t \geq 0}$ . So  $\mathcal{L}_{(\infty)}^{D,V}$  is an extension of the operator  $(\mathcal{A}, C_0^\infty(D))$ . ■

The main result of this section is

**Theorem 2.5.** *Let  $V \in L_{loc}^\infty(D, dx)$  such that  $V^- \in \mathcal{K}^d$ . Then  $(\mathcal{A}, C_0^\infty(D))$  is a  $L^\infty(D, dx)$ -essential generator with respect to the topology  $\mathcal{C}(L^\infty, L^1)$ .*

**Proof.** First, we must remark that the existence assumption of [WZ'06, Theorem 2.1] is satisfied. Indeed, if we consider the killed Feynman-Kac semigroup  $\left\{ P_t^{D,V} \right\}_{t \geq 0}$  on  $L^\infty(D, dx)$ , we have

$$\left| P_t^{D,V} f(x) \right| \leq P_t^{D,V} |f|(x) \leq P_t^{D,-V^-} |f|(x) \leq P_t^{-V^-} |f|(x)$$

hence we deduce that

$$\sup_{0 \leq t \leq 1} \left\| P_t^{D,V} \right\|_\infty \leq \sup_{0 \leq t \leq 1} \left\| P_t^{-V^-} \right\|_\infty < \infty$$

since  $\left\| P_t^{-V^-} \right\|_\infty$  is uniformly bounded by the assumption that  $V^- \in \mathcal{K}^d$  and by [AS'82]. Using Lemma 2.4 it follows that  $\left\{ P_t^{D,V} \right\}_{t \geq 0}$  is a  $C_0$ -semigroup on  $(L^\infty, \mathcal{C}(L^\infty, L^1))$  and its generator  $\mathcal{L}_{(\infty)}^{D,V}$  is an extension of  $(\mathcal{A}, C_0^\infty(D))$ .

By [WZ'06, Theorem 2.1], we deduce that the operator  $(\mathcal{A}, C_0^\infty(D))$  is  $L^\infty(D, dx)$ -unique if and only if for some  $\lambda > \omega$ , the range  $(\lambda I - \mathcal{A})(C_0^\infty(D))$  is dense in  $(L^\infty, \mathcal{C}(L^\infty, L^1))$ . It is enough to show that for any  $h \in L^1(D, dx)$  which satisfies the equality

$$\langle h, (\lambda I - \mathcal{A})f \rangle = 0 \quad , \quad \forall f \in C_0^\infty(D)$$

for some  $\lambda > \omega$ , it follows  $h = 0$ .

Let  $h \in L^1(D, dx)$  be such that for some  $\lambda > \omega$  one have

$$\langle h, (\lambda I - \mathcal{A})f \rangle = 0 \quad , \quad \forall f \in C_0^\infty(D)$$

or

$$(\lambda I - \mathcal{A})h = 0 \quad \text{in the sense of distribution.}$$

Since  $V \in L_{loc}^\infty(D, dx)$ , by applying [AS'82, Theorem 1.5, p. 217] we can see that  $h$  is a continuous function. By the mean value theorem due to AIZENMAN and SIMON [AS'82, Corollary 3.9, p. 231], there exists some constant  $C > 0$  such as

$$|h(x)| \leq C \int_{|x-y| \leq 1} |h(y)| dy \quad , \quad \forall x \in D.$$

As  $V^- \in \mathcal{K}^d$ ,  $C$  may be chosen independently of  $x \in D$ . Since  $h \in L^1(D, dx)$ , it follows that  $h$  is bounded and, consequently,  $h \in L^2(D, dx)$ . Now by the  $L^2(D, dx)$ -uniqueness of  $(\mathcal{A}, C_0^\infty(D))$  and [WZ'06, Theorem 2.1],  $h$  belongs to the domain of the generator  $\mathcal{L}_{(2)}^{D,V}$  of  $\left\{ P_t^{D,V} \right\}_{t \geq 0}$  on  $L^2$  and

$$\mathcal{L}_{(2)}^{D,V} h = \left( -\frac{\Delta}{2} + V \right) h = -\lambda h \quad .$$

Hence

$$P_t^{D,V} h = e^{-\lambda t} h \quad , \quad \forall t \geq 0.$$

If we take  $\lambda \in (\omega, \lambda(D, V))$ , then the last equality is possible only for  $h = 0$ , because  $\|P_t^{D, V}\|_2 = e^{-\lambda(D, V)t}$ . ■

**Remarque 2.6.** By analogy with the uniqueness in  $L^1(D, dx)$ , the  $L^\infty(D, dx)$ -uniqueness of  $(\mathcal{A}, C_0^\infty(D))$  means that a particle starting from the boundary  $\partial D$  can not enter in  $D$ . Unfortunately, here we have a problem:  $L^\infty(D, dx)$ -uniqueness of  $\mathcal{A}$  is equivalent to the existence of a unique boundary condition for  $\mathcal{A}^*$ . It is well known that there are many boundary conditions (Dirichlet, Neumann, etc.). But in the case of  $L^1(D, dx)$ -uniqueness of  $\mathcal{A}$ , the effect of the boundary condition for  $\mathcal{A}^*$  is eliminated by some condition (see the condition  $(C_1)$  in [Wu'98, p. 277]) for potential. To find such condition in the case of  $L^\infty(D, dx)$ -uniqueness is very difficult. In this moment we can present here an interesting partial result from [WZ'06]:

**Proposition 2.7.** *Let  $D$  be a nonempty open domain of  $\mathbb{R}^d$ . If the Laplacian  $(\Delta, C_0^\infty(D))$  is  $L^\infty(D, dx)$ -unique, then  $D^C = \emptyset$  or  $D = \mathbb{R}^d$ .*

For the heat diffusion equation we can formulate the next result

**Corollary 2.8.** *If  $V \in L_{loc}^\infty(\mathbb{R}^d, dx)$  and  $V^- \in \mathcal{K}^d$ , then for every  $h \in L^1(\mathbb{R}^d, dx)$ , the heat diffusion equation*

$$\begin{cases} \partial_t u(t) = (\frac{\Delta}{2} - V) u(t) \\ u(0) = h \end{cases}$$

*has one  $L^1(\mathbb{R}^d, dx)$ -unique weak solution which is given by  $u(t) = P_t^V h$ .*

**Proof.** The assertion follows by [WZ'06, Theorem 2.1] and Theorem 2.5. ■

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