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# On Linear-Quadratic Stackelberg Games with Time Preference Rates

Marc Jungers

## Abstract

This note deals with linear-quadratic Stackelberg differential games including time preference rates with an open-loop information structure. The properties of the characteristic matrix associated with the necessary conditions for a Stackelberg strategy are pointed out. It is shown that such a matrix exhibits a special symmetry property of its eigenvalues. Sufficient conditions to guarantee a predefined degree of stability are given based on the distribution of the eigenvalues in the complex plane.

## Index Terms

Game Theory, Stackelberg Strategy, Riccati Equation, Time Preference Rate,  $\alpha$ -Stability, Hamiltonian Matrix.

## I. INTRODUCTION

The Stackelberg strategies are an elegant concept for dealing with hierarchical differential games [1], [2]. In the framework of an open-loop information structure [3], the necessary conditions are well known and could be obtained explicitly within the context of linear-quadratic problems [1], [2]. Nevertheless it seems that explicit solution, coping with differential games with criteria including time preference rates does not exist.

It was proved in [4] that the linear-quadratic optimal control problem with a single criterion including a constant time preference rate  $\alpha$  could be restated as a standard one, with a shift of the eigenvalues of the drift matrix by  $\alpha$ . The reformulation uses a change of variable, which is closely connected with asymptotic stability of degree  $\alpha$ .

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Besides, a criterion with a time preference rate is quite frequent especially in economic applications of game theory (see [5], [6], [7, p. 77] for more details) and are recognized as the discount rate associated with the cost functionals. In order to emphasize the fact that each player has its own objective, the time preference rates are not necessarily identical [8].

When there is no time preference rate, the necessary conditions for obtaining an open-loop Stackelberg equilibrium, are characterized by a Hamiltonian matrix, (see [9]–[11] for an overview). This leads to a symmetry of the eigenvalues with respect to the origin of the complex plane. However, for the general case where time preference rates are different and not null, this property does not hold. The main contribution of this note is to consider such general case. Two points are examined. First, the eigenvalues distribution of the characteristic matrix associated with an open-loop Stackelberg strategy applied on the differential game is studied. Second, it is shown that a predefined degree of stability could be imposed to the controlled system.

The note is organized as follows. In section II, the Stackelberg strategy with an open-loop information structure is recalled and the associated necessary conditions are derived. The cases of finite and infinite time horizon are considered. The characteristic matrix and the corresponding coupled Riccati equations are presented. A non trivial symmetry for the eigenvalues is described in section III. Sufficient conditions for a strict  $\alpha$ –stability are provided in the same section, followed by an interpretation in terms of game theory. An example illustrates the main result. Some concluding remarks make up section IV.

## II. STACKELBERG STRATEGY

### A. Problem statement

Consider a two-players linear-quadratic differential game, on a finite time horizon, defined by

$$\dot{x}(t) = Ax(t) + B_1 u_1(t) + B_2 u_2(t), \quad x(t_0) = x_0, \quad (1)$$

where  $x \in \mathbb{R}^n$ ,  $u_i \in \mathcal{U}_{ad,i} \subset \mathbb{R}^{r_i}$  ( $i \in \{1, 2\}$  and  $n, r_i \in \mathbb{N}$ ,  $\mathcal{U}_{ad,i}$  is the admissible set of the controls  $u_i$ ) and with the cost functionals  $J_i$  ( $i \in \{1, 2\}$ ) including a time preference rate  $\alpha_i$

$$J_i = \frac{1}{2} x_f^T e^{2\alpha_i t_f} K_{if} x_f + \frac{1}{2} \int_{t_0}^{t_f} e^{2\alpha_i t} (x^T Q_i x + u_1^T R_{i1} u_1 + u_2^T R_{i2} u_2) d\tau, \quad (2)$$

where  $x_f = x(t_f)$ . All weighting matrices are constant and symmetric with  $Q_i = C_i^T C_i \geq 0$ ,  $K_{if} \geq 0$ ,  $R_{ij} \geq 0$  ( $i \neq j$ ) and  $R_{ii} > 0$ . The matrices  $C_i$  are of full rank  $C_i \in \mathbb{R}^{m_i \times n}$ .

Stackelberg strategy with an open-loop information structure is applied for the differential game (1)-(2). Player 2 is assumed to be the leader while player 1 is the follower. The hierarchy in the game comes from the fact that the leader knows the rational reaction of the follower and reveals first his/her strategy. The follower does not know the rational reaction of the leader and must optimize his/her criterion  $J_1$  for a given control  $u_2^*(t)$  of the leader. Define the rational reaction set of the follower  $\mathcal{R}_1(u)$

$$\{\tilde{u}_1 \mid J_1(\tilde{u}_1, u) \leq J_1(u_1, u), \forall u_1 \in \mathcal{U}_{ad,1}\}. \quad (3)$$

For a differential game with an open-loop information structure [12], i.e. the players are committed to follow a predetermined strategy or no state measurements are available, the leader (player 2) is seeking a  $J_2$ -minimizing strategy  $u_2^*(t)$ , as a function of time only, that he expresses before the game starts knowing the follower's rational reaction. The follower (player 1) will then minimize his cost functional  $J_1$  with the strategy  $u_1^*(t)$ , a function of time only. Mathematically, the definition of a Stackelberg equilibrium  $(u_1^*, u_2^*)$  is

$$\left\{ \begin{array}{l} u_1^* \in \mathcal{R}_1(u_2^*) \\ \text{and} \\ \max_{u_1 \in \mathcal{R}_1(u_2^*)} J_2(u_1, u_2^*) \leq \max_{u_1 \in \mathcal{R}_1(u_2)} J_2(u_1, u_2), \quad \forall u_2 \in \mathcal{U}_{ad,2}. \end{array} \right. \quad (4)$$

The necessary conditions for a Stackelberg strategy with an open-loop information structure were derived in [1], [2] (see also [5], chapter 5 for application in economy). Let  $\psi_1$  be the costate vector of the follower associated with the dynamic constraint (1) and  $\psi_2$  and  $\gamma$ , the costate vectors of the leader, associated respectively with the dynamic constraint (1) and the rational reaction set of the follower  $\mathcal{R}_1(\cdot)$ . The open-loop Stackelberg controls are

$$\left\{ \begin{array}{l} u_1^*(t) = -R_{11}^{-1} B_1^T \psi_1(t), \\ u_2^*(t) = -R_{22}^{-1} B_2^T \psi_2(t), \end{array} \right. \quad (5)$$

with

$$\begin{pmatrix} \dot{x} \\ \dot{\psi}_1 \\ \dot{\psi}_2 \\ \dot{\gamma} \end{pmatrix} = M_{\alpha_1 \alpha_2} \begin{pmatrix} x \\ \psi_1 \\ \psi_2 \\ \gamma \end{pmatrix}, \quad (6)$$

and the transversality conditions are

$$\begin{cases} x(t_0) &= x_0, \\ \psi_1(t_f) &= K_{1f}x_f, \\ \psi_2(t_f) &= K_{2f}x_f - K_{1f}\gamma(t_f), \\ \gamma(t_0) &= 0, \end{cases} \quad (7)$$

where

$$M_{\alpha_1\alpha_2} = \begin{bmatrix} A & -S_1 & -S_2 & 0_n \\ -Q_1 & -A^T - 2\alpha_1 I_n & 0_n & 0_n \\ -Q_2 & 0_n & -A^T - 2\alpha_2 I_n & Q_1 \\ 0_n & -S_{21} & S_1 & A - 2(\alpha_2 - \alpha_1)I_n \end{bmatrix}. \quad (8)$$

The notation  $S_{ij} = B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j^T$  and  $S_i = S_{ii}$  are used. The matrix  $M_{\alpha_1\alpha_2}$  defined by (6) and (8) characterizes the two point boundary value problem, in association with conditions (7). Note that for  $\alpha_1 = \alpha_2 = 0$ , one can recognize the Hamiltonian structure of  $M_{\alpha_1\alpha_2} = M_{0,0}$  pointed out in [9].

These necessary conditions are also sufficient if the criteria are convex for a finite time horizon [1], [2]. In the following subsections the determination of the open-loop Stackelberg controls is examined for both finite and infinite time horizon cases.

### B. Criteria on a finite horizon

The linearity of the differential equation (6) and of the transversality conditions (7) with respect to  $x(t)$  and  $x(t_0)$  allows to research Stackelberg controls  $u_1^*(t)$  and  $u_2^*(t)$  as following. Let us find  $K_1(t)$ ,  $K_2(t)$  and  $P(t)$  verifying

$$\begin{cases} \psi_1(t) &= K_1(t)x(t), \\ \psi_2(t) &= K_2(t)x(t), \\ \gamma(t) &= P(t)x(t). \end{cases} \quad (9)$$

From (9) and (8) we obtain the Riccati differential equations to be verified by  $K_1(t)$ ,  $K_2(t)$  and  $P(t)$

$$\begin{cases} \dot{K}_1 &= -(A^T + 2\alpha_1 I_n) K_1 - K_1 A - Q_1 + K_1 S_1 K_1 + K_1 S_2 K_2, \\ \dot{K}_2 &= -(A^T + 2\alpha_2 I_n) K_2 - K_2 A - Q_2 + Q_1 P + K_2 S_1 K_1 + K_2 S_2 K_2, \\ \dot{P} &= (A - 2(\alpha_2 - \alpha_1) I_n) P - P A + P S_1 K_1 + P S_2 K_2 - S_{21} K_1 + S_1 K_2, \end{cases} \quad (10)$$

with

$$\begin{cases} K_1(t_f) = K_{1f}, \\ K_2(t_f) = K_{2f} - K_{1f}P(t_f), \\ P(t_0) = 0. \end{cases} \quad (11)$$

In this case, the Stackelberg controls can be formulated as

$$\begin{cases} u_1^*(t) = -R_{11}^{-1}B_1^TK_1(t)\xi(t, t_0)x_0, \\ u_2^*(t) = -R_{22}^{-1}B_2^TK_2(t)\xi(t, t_0)x_0, \end{cases} \quad (12)$$

where

$$\begin{cases} \dot{\xi}(t, t_0) = (A - S_1K_1(t) - S_2K_2(t))\xi(t, t_0), \\ \xi(t, t) = I_n. \end{cases} \quad (13)$$

These controls are not functions of the measurement of the state  $x(t)$  but of the pre-determined state  $\xi(t, t_0)x_0$ , function of time.

The Coupled Riccati Equations (11) are associated with the matrix  $M_{\alpha_1\alpha_2}$ . In fact, decomposing  $M_{\alpha_1\alpha_2}$  into submatrices as follows

$$M_{\alpha_1\alpha_2} = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix} = \left[ \begin{array}{c|ccc} A & -S_1 & -S_2 & 0_n \\ \hline -Q_1 & -A^T - 2\alpha_1 I_n & 0_n & 0_n \\ -Q_2 & 0_n & -A^T - 2\alpha_2 I_n & Q_1 \\ 0_n & -S_{21} & S_1 & A - 2(\alpha_2 - \alpha_1)I_n \end{array} \right], \quad (14)$$

we could rewrite the equations (10) as a single non-symmetric Riccati equation

$$\begin{pmatrix} \dot{K}_1 \\ \dot{K}_2 \\ \dot{P} \end{pmatrix} = M_4 \begin{pmatrix} K_1 \\ K_2 \\ P \end{pmatrix} - \begin{pmatrix} K_1 \\ K_2 \\ P \end{pmatrix} M_1 + M_3 - \begin{pmatrix} K_1 \\ K_2 \\ P \end{pmatrix} M_2 \begin{pmatrix} K_1 \\ K_2 \\ P \end{pmatrix}. \quad (15)$$

Such an equation could be solved using the representation formula of solutions of Riccati equations [10].

### C. Criteria on an infinite horizon

In this subsection, the particular case of infinite horizon  $t_f \rightarrow +\infty$  is considered. Since only stabilizing controls are considered, the terminal costs  $\frac{1}{2}x_f^T e^{2\alpha_i t_f} K_{if} x_f$  in criteria (2) and the transversality conditions (7) become irrelevant.

With criteria on infinite horizon, it is possible to look for only constant solutions of (15), by replacing the left hand side of (15) by a extended zero matrix. Some necessary conditions of existence of solution of this equation are developed in [13].

If  $K_1$ ,  $K_2$  and  $P$  are constant solutions of the Coupled Riccati Equations (10), the  $n$  eigenvalues of  $(A - S_1K_1 - S_2K_2)$  are included in the  $4n$  of  $M_{\alpha_1\alpha_2}$ . A method to find all solutions  $K_1$ ,  $K_2$  and  $P$  of the coupled Riccati equations is to study the invariant subspaces of the matrix

$M_{\alpha_1\alpha_2}$ , with the form  $\begin{pmatrix} I_n \\ K_1 \\ K_2 \\ P \end{pmatrix}$  (see [10], [13] for more details). Therefore the distribution of the eigenvalues of  $M_{\alpha_1\alpha_2}$  has to be studied.

### III. EIGENVALUES SYMMETRY AND CONDITIONS FOR STRICT $\alpha_2$ -STABILITY

In this section, firstly a property of symmetry of the matrix  $M_{\alpha_1\alpha_2}$  is emphasized.

*Theorem 1 (Symmetry):* The matrix  $(M_{\alpha_1\alpha_2} + \alpha_2 I_{4n})$  is Hamiltonian.

*Proof:* Let

$$\tilde{A}_{\alpha_1,\alpha_2} = \begin{bmatrix} A + \alpha_2 I_n & -S_1 \\ -Q_1 & -A^T + (\alpha_2 - 2\alpha_1)I_n \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_2 \\ 0_{n \times r_2} \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} C_2 & 0_{m_2 \times n} \\ 0_{r_1 \times n} & R_{11}^{-1} B_1^T \end{bmatrix}.$$

The matrix  $M_{\alpha_1\alpha_2}$  verifies

$$M_{\alpha_1\alpha_2} + \alpha_2 I_{4n} = \begin{bmatrix} \tilde{A}_{\alpha_1,\alpha_2} & -\tilde{B}R_{22}^{-1}\tilde{B} \\ -\tilde{C}^T \begin{bmatrix} I_{m_2} & 0_{m_2 \times r_1} \\ 0_{r_1 \times m_2} & R_{21} \end{bmatrix} \tilde{C} & -\tilde{A}_{\alpha_1,\alpha_2}^T \end{bmatrix}. \quad (16)$$

By noting  $\mathcal{J} = \begin{bmatrix} 0_{2n} & I_{2n} \\ -I_{2n} & 0_{2n} \end{bmatrix}$ , (note that  $\mathcal{J}^T = -\mathcal{J} = \mathcal{J}^{-1}$ ), it is easy to verify that

$$\mathcal{J}^T (M_{\alpha_1\alpha_2} + \alpha_2 I_{4n}) \mathcal{J} = -(M_{\alpha_1\alpha_2} + \alpha_2 I_{4n})^T$$

which proves that  $(M_{\alpha_1\alpha_2} + \alpha_2 I_{4n})$  is an Hamiltonian matrix. ■

This Hamiltonian symmetry property depends only on  $\alpha_2$ , the time preference rate of the leader. The hierarchical structure of the game allows the leader to impose his time preference rate to the system (1). The follower's time preference rate does not influence the symmetry.

It may be seen as a further degree of freedom in the control design. To obtain a  $\alpha$ -degree of stability, one should only impose  $\alpha_2 \leq \alpha$ .

It is well known that for a Hamiltonian matrix if  $\lambda$  is an eigenvalue,  $-\lambda$  is also an eigenvalue. Here, matrix  $(M_{\alpha_1\alpha_2} + \alpha_2 I_{4n})$  is Hamiltonian and real. Therefore, the matrix  $M_{\alpha_1\alpha_2}$  admits the point  $(-\alpha_2; 0)$  as a center of symmetry in the complex plane for its eigenvalues distribution. However it is possible that eigenvalues of matrix  $M_{\alpha_1\alpha_2}$  has a real part equal to  $-\alpha_2$ . To guarantee an exponential convergence rate of degree greater than  $\alpha_2$  for the system (13), one should prove that at least  $n$  eigenvalues of  $M_{\alpha_1\alpha_2}$  are located to the left of the vertical symmetry axis  $-\alpha_2 + j\mathbb{R}$ .

In the following, the sufficient conditions for strict  $\alpha_2$ -stability, associated with this Hamiltonian symmetry is studied. By noting  $\lambda_k$  ( $k \in \{1, \dots, 4n\}$ ) the  $4n$  eigenvalues of  $M_{\alpha_1\alpha_2}$ , a permutation allows to have

$$\operatorname{Re}(\lambda_1) \leq \dots \leq \operatorname{Re}(\lambda_{2n}) \leq -\alpha_2 \leq \operatorname{Re}(\lambda_{2n+1}) \leq \dots \leq \operatorname{Re}(\lambda_{4n}). \quad (17)$$

In the case  $\alpha_1 = \alpha_2 = 0$ , the relation

$$\operatorname{Re}(\lambda_{2n}) < 0 < \operatorname{Re}(\lambda_{2n+1}), \quad (18)$$

is obtained if the pairs  $(C_1, A)$  and  $(C_2, A)$  are detectable and  $(A, B_1)$  is stabilizable [10]. The presence of eigenvalues on the vertical symmetry axis means that a predefined stability of degree  $\alpha_2$  cannot be guaranteed. Sufficient conditions to ensure that

$$\operatorname{Re}(\lambda_{2n}) < -\alpha_2 < \operatorname{Re}(\lambda_{2n+1}). \quad (19)$$

are derived hereafter.

The proof of the main theorem requires two lemmas.

*Lemma 1:* Assuming that  $R > 0$  and  $W > 0$ , the Hamiltonian matrix  $\begin{bmatrix} P & -DR^{-1}D \\ -E^TWE & -P^T \end{bmatrix}$  does not admit imaginary eigenvalue if

- the pair  $(E, P)$  does not admit imaginary unobservable mode.
- the pair  $(P, D)$  does not admit imaginary uncontrollable mode.

This lemma is well known in LQ problem. A sketch of the proof can be found for example in [10, p. 59].



*Lemma 2 (Sylvester's Inequality):* Let  $U_1$  and  $U_2$  be  $q \times w$  and  $w \times p$  matrices with coefficients in the same field. Then by noting  $\rho(U_1)$  the rank of the matrix  $U_1$ , one have

$$\rho(U_1) + \rho(U_2) - w \leq \rho(U_1 U_2) \leq \min(\rho(U_1), \rho(U_2)). \quad (20)$$

A proof of Sylvester's Inequality can be found in [14].

*Theorem 2 (General strict  $\alpha_2$ -stability):* For each couple  $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ , the relation (19) is verified under the sufficient conditions

- i) the pair  $(C_1, A + (2\alpha_1 - \alpha_2)I_n)$  does not admit imaginary unobservable mode,
- ii) the pair  $(C_2, A + \alpha_2 I_n)$  does not admit imaginary unobservable mode,
- iii) the pair  $(A + (2\alpha_1 - \alpha_2)I_n, B_1)$  does not admit imaginary uncontrollable mode,
- iv) and furthermore, if  $\alpha_1 \neq \alpha_2$ , it is assumed that  $m_1 \leq r_2$  and that the triplet  $(A + \alpha_2 I_n, B_2, C_1)$  does not admit imaginary transition zero.

*Proof:* In order to avoid imaginary eigenvalues of the Hamiltonian matrix  $(M_{\alpha_1 \alpha_2} + \alpha_2 I_{4n})$ , the lemma 1 and the Hautus approach are used. The matrix  $(M_{\alpha_1 \alpha_2} + \alpha_2 I_{4n})$  defined by (16) does not admit imaginary eigenvalues if for all  $\omega \in \mathbb{R}$ ,

- $\rho \left( \left[ \begin{array}{c|c} \tilde{A}_{\alpha_1, \alpha_2} - j\omega I_{2n} & \tilde{B} \\ \hline \tilde{C} & \end{array} \right] \right) = 2n,$
- $\rho \left( \left[ \begin{array}{c} \tilde{A}_{\alpha_1, \alpha_2} - j\omega I_{2n} \\ \hline \tilde{C} \end{array} \right] \right) = 2n.$

The above matrices are rearranged to show Hautus matrices.

$$\begin{aligned} \rho \left( \left[ \begin{array}{c} \tilde{A}_{\alpha_1, \alpha_2} - j\omega I_n \\ \hline \tilde{C} \end{array} \right] \right) &= \rho \left( \left[ \begin{array}{cc} A + \alpha_2 I_n - j\omega I_n & -B_1 R_{11}^{-1} B_1^T \\ -Q_1 & -A^T + (\alpha_2 - 2\alpha_1 - j\omega)I_n \\ C_2 & 0_n \\ 0_{r_1 \times n} & R_{11}^{-1} B_1^T \end{array} \right] \right) \\ &= \rho \left( \left[ \begin{array}{c|c} A + \alpha_2 I_n - j\omega I_n & 0_n \\ \hline C_2 & 0_n \\ \hline -Q_1 & -A^T + (\alpha_2 - 2\alpha_1 - j\omega)I_n \\ 0_{r_1 \times n} & R_{11}^{-1} B_1^T \end{array} \right] \right). \end{aligned}$$

The conditions ii) and iii) in theorem 2 lead to  $\rho \left( \left[ \begin{array}{c} \tilde{A}_{\alpha_1, \alpha_2} - j\omega I_n \\ \hline \tilde{C} \end{array} \right] \right) = 2n$ . Now the condition on the pair  $(\tilde{A}_{\alpha_1, \alpha_2} - j\omega I_{2n}, \tilde{B})$  is studied.

$$\left[ \begin{array}{c} \tilde{A}_{\alpha_1, \alpha_2} - j\omega I_{2n} \\ \hline \tilde{B} \end{array} \right] = \left[ \begin{array}{cc|c} A + (\alpha_2 - j\omega)I_n & -S_1 & B_2 \\ -Q_1 & -A^T + (\alpha_2 - 2\alpha_1 - j\omega)I_n & 0_{n \times r_2} \end{array} \right] = U_1 U_2,$$

with

$$U_1 = \left[ \begin{array}{c|cc} I_n & 0_{n \times m_1} & 0_n \\ \hline 0_n & -C_1^T & -A^T + (\alpha_2 - 2\alpha_1 - j\omega)I_n \end{array} \right] \in \mathbb{R}^{2n \times (2n+m_1)}, \quad (21)$$

and

$$U_2 = \left[ \begin{array}{cc|c} A + (\alpha_2 - j\omega)I_n & B_2 & -S_1 \\ C_1 & 0_{m_1 \times r_2} & 0_{m_1 \times n} \\ \hline 0_n & 0_{n \times r_2} & I_n \end{array} \right] \in \mathbb{R}^{(2n+m_1) \times (2n+r_2)}. \quad (22)$$

Condition i) in theorem 2 gives  $\rho(U_1) = 2n$ , and Condition iv) gives  $\rho(U_2) = 2n + m_1$ . The Sylvester's Inequality in lemma 2 allows to verify that  $\rho(U_1 U_2) = 2n$ .  $\blacksquare$

*Remark 1:* It is noteworthy that for the case  $\alpha_1 = \alpha_2 = \alpha$ , the condition about the imaginary transition zero is redundant. Matrix  $\tilde{A}_{\alpha, \alpha}$  is Hamiltonian. From lemma 1,  $(C_1, A + \alpha I_n)$  has no imaginary unobservable mode and  $(A + \alpha I_n)$  has no imaginary uncontrollable mode which is enough to guarantee that  $\tilde{A}_{\alpha, \alpha} - j\omega I_{2n}$  is invertible for every  $\omega \in \mathbb{R}$  and that

$$\rho \left( \left[ \begin{array}{c} \tilde{A}_{\alpha, \alpha} - j\omega I_{2n} \\ \hline \tilde{B} \end{array} \right] \right) = 2n,$$

without consideration on the triplet  $(A + \alpha I_n, B_2, C_1)$ .

*Remark 2:* Condition iv) can be interpreted in a game-theoretic sense. In fact, this triplet corresponds to non-symmetric coupling between players. The action of the leader (matrix  $B_2$ ) on the system (matrix  $A$ ) can be viewed by the follower (matrix  $C_1$ ), if there is no imaginary transition zero. Note that if  $m_1 > r_2$  holds, there is too follower's outputs with respect to the number of leader's inputs. Then the leader is not able to impose arbitrary all the outputs of the follower. The leader is not anymore omnipotent. That's why  $m_1 \leq r_2$  is required.

*Remark 3:* It is worth noting that in all cases for  $\alpha_1$  and  $\alpha_2$ , no conditions on the pair  $(A, B_2)$  are assumed. The controllability or only the stabilizability of the pair  $(A, B_2)$  of the leader does not influence the main result. This emphasizes the privileged position of the leader.

In order to illustrate the main result, the following example is taken.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -3 & -1 \\ 2 & 1 & -5 \end{bmatrix}; \quad B_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}; \quad B_2 = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$$

and

$$C_1 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}; \quad C_2 = \begin{bmatrix} 2 & -1 & 1 \end{bmatrix}.$$

The pairs  $(C_1, A)$  and  $(C_2, A)$  are observable, and the pair  $(A, B_1)$  is controllable. The different conditions (except the condition about the imaginary zero transmission) of theorem 2 are verified for all  $\alpha_1$  and  $\alpha_2$ . The distribution of eigenvalues is computed for the couple  $(\alpha_1, \alpha_2) = (1, 2)$  to illustrate the main result of symmetry. This couple verifies that the triplet  $(A + \alpha_2 I_n, B_2, C_1)$  does not admit imaginary eigenvalue. To sum up all assumptions of theorem 2 are verified. On Fig. 1, the eigenvalues of  $M_{\alpha_1 \alpha_2}$  are represented. It is obvious that the  $\alpha_2$ -translated imaginary axis is an axis of symmetry for the distribution of eigenvalues. Moreover there is no eigenvalue on this axis of symmetry, as claimed in theorem 2.

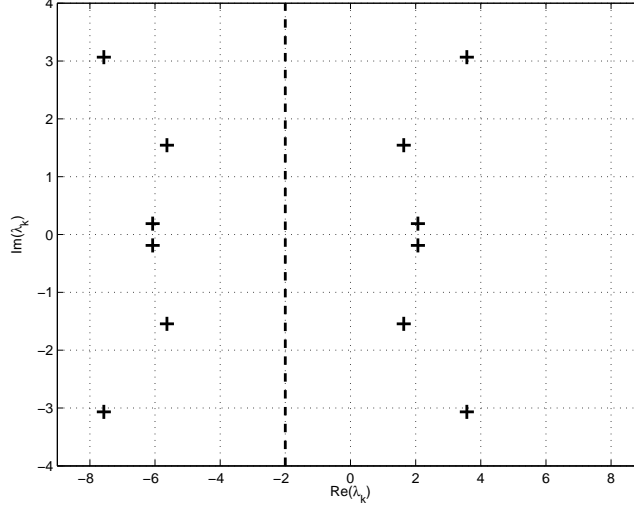


Fig. 1. Distribution of eigenvalues for  $(\alpha_1, \alpha_2) = (1, 2)$ .

#### IV. CONCLUSION

In this paper, the linear-quadratic games including time preference rates in the framework of Stackelberg strategy with open-loop information structure is studied. The general case of

different time preference rates is considered. It is shown that the time preference rate of the leader imposes the symmetry of the eigenvalues of the characteristic matrix associated with the necessary conditions. Also, sufficient conditions to guarantee a strict  $\alpha$ -stability for this type of differential games are derived.

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