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# Representation Theorems for Backward Doubly Stochastic Differential Equations

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## Abstract

In this paper we study the class of backward doubly stochastic differential equation (BDSDE, for short) whose terminal value depends on the history of forward diffusion. We first establish a probabilistic representation for the spatial gradient of the stochastic viscosity solution to a quasilinear parabolic SPDE in the spirit of the Feynman-Kac formula, without using the derivatives of the coefficients of the corresponding BDSDE. Then such a representation leads to a closed-form representation of the martingale integrand of BDSDE, under only standard Lipschitz condition on the coefficients.

**Key Words:** Backward doubly SDEs, Stochastic Partial Differential equation, Adapted solution, anticipating stochastic calculus, stochastic viscosity solutions.

## 1 Introduction

Backward stochastic differential equations (BSDE, for short) were firstly been considered in its linear form by Bismut [1, 2] in the context of optimal stochastic control. However, nonlinear BSDEs and their theory have been introduced by Pardoux and Peng [16]. It has been enjoying a great interest in the last ten year because of its connection with applied fields. We can cite stochastic control and stochastic games (see [10], ) and mathematical finance (see [6], ). BSDE also provide a probabilistic interpretation for solutions to elliptic or parabolic nonlinear partial differential equation generalizing the classical Feynman-Kac formula [17, 18].

In an other work, to give a probabilistic representation to solutions of quasi-linear stochastic PDEs, Pardoux and Peng [19] introduced a new type of BSDE with presence of two stochastic integrals driven by two independent Brownian motions  $B$  and  $W$ . The so-called backward

doubly stochastic differential equations (BDSDE in short) is of the form

$$\begin{aligned} Y_s &= \xi + \int_s^T f(r, Y_r, Z_r) dr + \int_s^T g(r, Y_r, Z_r) \downarrow dB_r \\ &\quad - \int_s^T Z_r dW_r. \quad s \in [t, T], \end{aligned} \quad (1.1)$$

where  $\xi$  is a square integrable variable. Let us remark that the integral driven by  $\{B_r\}_{r \geq 0}$  is a backward Itô integral and the other one driven by  $\{W_r\}_{r \geq 0}$  is the standard forward Itô integral. Under the global Lipschitz condition on the coefficient, they prove that this BSDE have a unique solution  $(Y, Z)$ . However in the markovian framework we point out that our version is in fact a time reversal of BDSDE (1.1), due to the set-up of our problem. (see Buckdahn and Ma [3, 4] for more detail). For each  $(t, x) \in [0, T] \times \mathbb{R}^d$ , we consider the following

$$X_s^{x,t} = x + \int_s^t b(r, X_r^{x,t}) dr + \int_s^t \sigma(r, X_r^{x,t}) \downarrow dW_r \quad s \in [0, t], \quad (1.2)$$

$$\begin{aligned} Y_s^{t,x} &= l(X_0^{x,t}) + \int_0^s f(r, X_r^{x,t}, Y_r^{x,t}, Z_r^{x,t}) dr + \int_0^s g(r, X_r^{x,t}, Y_r^{x,t}) dB_r \\ &\quad - \int_0^s Z_r^{x,t} \downarrow dW_r, \quad s \in [0, t]. \end{aligned} \quad (1.3)$$

In this case adding more stringent regularity conditions on the coefficients ( $b, \sigma$  and  $f, g, l$  are respectively  $C^1$  and  $C^3$  in their spatial variable) Pardoux and Peng [19] provide that

$$Y_s^{x,t} = u(t, X_s^{x,t}) \text{ and } Z_s^{x,t} = \partial u_x(t, X_s^{x,t}) \sigma(t, X_s^{x,t}), \quad s \in [0, t] \quad (1.4)$$

where  $\{u(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$  the classical solution of the backward stochastic PDE in the form

$$\begin{aligned} du(t, x) &= [\mathcal{L}u(t, x) + f(t, x, u(t, x), (\nabla u \sigma)(t, x))] dt \\ &\quad + g(t, x, u(t, x)) dB_t, \quad (t, x) \in (0, T) \times \mathbb{R}^d, \end{aligned} \quad (1.5)$$

$$u(0, x) = l(x), \quad x \in \mathbb{R}^d.$$

with  $\mathcal{L}$  the infinitesimal operator generated by the solution of forward SDE in the form

$$\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^n \sum_{l=1}^k \sigma_{il} \sigma_{lj}(x) \partial_{x_i x_j}^2 + \sum_{j=1}^n b_j(x) \partial_{x_j}.$$

In order to weak the condition given in Pardoux and Peng [19], Buckdahn and Ma [3, 4] introduce notion of stochastic viscosity solution of SPDE (1.5). Assuming the coefficient of BDSDE  $f$  and  $l$  only Lipschitz, they get that

$$Y_t^{x,t} = u(t, x) = \mathbb{E} \left\{ l(X_0^{t,x}) + \int_0^t f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) ds / \mathcal{F}_t^B \right\}$$

is a stochastic viscosity solution of SPDE (1.5). On the hand let us remark that a other notion of stochastic viscosity solution of SPDE (1.5) have been involved by Lions and Souganidis [12]. However there is no link between the two kind of viscosity solution of SPDE.

In this paper we consider the one use by Buckdahn and Ma [3, 4] that in our mind is natural and coincide (if  $g \equiv 0$ ) with the well-know viscosity solution of PDE introduced by Crandall and al [5]. Moreover we have two goal. First we show that if the coefficients  $f$  and  $l$  are continuously differentiable,  $u$  the stochastic viscosity solution (in the sense of [3, 4]) of the SPDE (1.5) will have a continuous spatial gradient  $\partial_x u(t, x)$  and more precisely, the following probabilistic representation holds:

$$\partial_x u(t, x) = \mathbb{E} \left\{ l(X_0^{t,x}) N_0^t + \int_0^s f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) N_r^s dr / \mathcal{F}_t^B \right\} \quad (1.6)$$

where  $N^s$  is some process defined on  $[0, s]$ , depending only on the solution of the forward equation (1.2) and the solution of its variational equation. The importance of this formula is that with the help of equality (1.4) we can derive a similar representation for the martingale integrand  $Z$ , under only Lipschitz condition on  $f$  and  $l$ . This representation can be thought of as new type of nonlinear Feynman-Kac formula for the derivative of solution of SPDE which is new in the literature. Let us recall that this two representations have already be given by Ma and Zhang [13] in the case of probabilistic representation for solutions of PDE via BSDE. Consequently our approach is be inspired by the one use in Ma and Zhang. However, because of the presence of stochastic term in equation (1.5), some particularities hold. The viscosity solution  $u$  is a random field that respect the filtration generated by the Brownian motion  $B$ . So the derivative notion is take in the general sense (for fixed  $\omega_1$ ). Moreover since the family of  $\sigma$ -algebraic  $\mathcal{F}_t = \mathcal{F}_t^B \otimes \mathcal{F}_{t,T}^W$  is not a filtration, there is real difficulty to proved the continuity of the representation of the process  $Z$ . The rest of this paper is organized as follows. In section 2 we give all the necessary preliminaries. In section 3 we establish the new Feynman-Kac formula between coupled forward backward SDE (1.1), (1.2) and the SPDE (1.5), under only the  $C^1$ -assumption of the coefficients. The section 4 is devoted to give the main representation theorem assuming only the Lipschitz condition of the coefficients  $l$  and  $f$ . In section 5 we study the path regularity of the process  $Z$ .

## 2 Preliminaries

Let  $T > 0$  a fixed final time. Throughout this paper we consider two complete probability spaces:  $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$  on which is defined a  $d$ -dimensional Brownian motion  $\{B_t, 0 \leq t \leq T\}$  and  $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$  on which we define a other  $d$ -Brownian motion  $\{W_t, 0 \leq t \leq T\}$  independent to the first one. Let  $\mathbf{F}^B = \{\mathcal{F}_t^B\}_{t \geq 0}$  denote the natural filtration generated by  $B$ , augmented by  $\mathbb{P}_1$ -null set of  $\mathcal{F}_1$ ; and let  $\mathcal{F}^B = \mathcal{F}_\infty^B$ . On the hand we consider the following family of  $\sigma$ -fields:

$$\mathcal{F}_{t,T}^W = \sigma\{W_s - W_T, t \leq s \leq T\} \vee \mathcal{N}_2,$$

where  $\mathcal{N}_2$  denotes all the  $\mathbb{P}_2$ -null set in  $\mathcal{F}_2$ . Denote  $F_T^W = \{\mathcal{F}_{t,T}^W\}_{0 \leq t \leq T}$ . Next we consider the product space  $(\Omega, \mathcal{F}, \mathbb{P})$  where

$$\Omega = \Omega_1 \times \Omega_2, \quad \mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2 \quad \text{and} \quad \mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2;$$

and define  $\mathcal{F}_t = \mathcal{F}_t^B \otimes \mathcal{F}_{t,T}^W$  for  $0 \leq t \leq T$ . We should note that  $\mathbf{F} = \{\mathcal{F}_t, t \in [0, T]\}$  is neither increasing nor decreasing. Therefore it is not a filtration; but in other words they are complete. Further, we assume that random variables  $\xi(\omega_1), \omega_1 \in \Omega_1$  and  $\zeta(\omega_2), \omega_2 \in \Omega_2$  are considered as random variables on  $\Omega$  via the following identification:

$$\xi(\omega_1, \omega_2) = \xi(\omega_1); \quad \zeta(\omega_1, \omega_2) = \zeta(\omega_2).$$

Let us give the space given that will be frequently used in the sequel;  $\mathcal{X}$  denoting the a generic Banach space with it's norm  $\|\cdot\|_{\mathcal{X}}$  and  $\mathbf{E}$  the generic Euclidean space (or  $\mathbf{E}_i, i = 1, 2, \dots$ )

- for  $t \in [0, T], L^0([t, T]; \mathcal{X})$  is the space of all measurable functions  $\varphi : [t, T] \mapsto \mathcal{X}$ ;
- for  $0 \leq t \leq T, C([t, T]; \mathcal{X})$  is the space of all continuous functions  $\varphi : [t, T] \mapsto \mathcal{X}$ ; further, for any  $p > 0$  we denote  $|\varphi|_{t,T}^{*,p} = \sup_{t \leq s \leq T} \|\varphi(s)\|_{\mathcal{X}}^p$  when the context is clear;
- for integers  $k$  and  $n, C^{k,n}([0, T] \times \mathbf{E}; \mathbf{E}_1)$  is the space of all  $\mathbf{E}_1$ -valued functions  $\varphi(t, e), (t, e) \in [0, T] \times \mathbf{E}$ , such that they are  $k$ -times continuously differentiable in  $t$  and  $n$ -times continuously differentiable in  $e$ ;
- $C_b^1([0, T] \times \mathbf{E}; \mathbf{E}_1)$  is the space of those  $\varphi \in C^1([0, T] \times \mathbf{E}; \mathbf{E}_1)$  such that all the partial derivatives are uniformly bounded;
- $W^{1,\infty}(\mathbf{E}, \mathbf{E}_1)$  is the space of all measurable functions  $\psi : \mathbf{E} \mapsto \mathbf{E}_1$ , such that for some constant  $K > 0$  it holds that  $|\psi(x) - \psi(y)|_{\mathbf{E}_1} \leq K|x - y|_{\mathbf{E}}, \forall x, y \in \mathbf{E}$ ;
- for any sub- $\sigma$ -field  $\mathcal{G} \subseteq \mathcal{F}_T$  and  $0 \leq p < \infty, L^p(\mathcal{G}; \mathbf{E})$  denote all  $\mathbf{E}$ -valued  $\mathcal{G}$ -measurable random variable  $\xi$  such that  $\mathbb{E}|\xi|^p < \infty$ . Moreover,  $\xi \in L^\infty(\mathcal{G}; \mathbf{E})$  means it is  $\mathcal{G}$ -measurable and bounded;
- for  $0 \leq p < \infty, L^p(\mathbf{F}, [0, T]; \mathcal{X})$  is the space of all  $\mathcal{X}$ -valued,  $\mathbf{F}$ -adapted processes  $\xi$  satisfying  $\mathbb{E} \left( \int_0^T \|\xi_t\|_{\mathcal{X}}^p dt \right) < \infty$ ; and also,  $\xi \in L^\infty(\mathbf{F}, [0, T]; \mathbb{R}^d)$  means that the process  $\xi$  is uniformly essentially bounded in  $(t, \omega)$ ;
- $C(\mathbf{F}, [0, T] \times \mathbf{E}; \mathbf{E}_1)$  is the space of  $\mathbf{E}_1$ -valued, continuous random field  $\varphi : \Omega \times [0, T] \times \mathbf{E}$ , such that for fixed  $e \in \mathbf{E}, \varphi(\cdot, \cdot, e)$  is an  $\mathbf{F}$ -adapted process.

Now we consider coefficients  $(b, \sigma)$  of the FSDE and  $(f, g, l)$  of BDSDE with the following properties:

(A1) The functions  $\sigma \in C_b^{0,1}([0, T] \times \mathbb{R}^d; \mathbb{R}^{d \times d})$ ,  $b \in C_b^{0,1}([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$ ; and all the partial derivatives of  $b$  and  $\sigma$  (with respect to  $x$ ) are uniformly bounded by a common constant  $K > 0$ . Further, there exists constants  $c > 0$ , such that

$$\xi^T \sigma(t, x) \sigma(t, x)^T \xi \geq c |\xi|^2, \forall x, \xi \in \mathbb{R}^d, t \in [0, T]. \quad (2.1)$$

The function  $f \in C(\mathcal{F}^B, [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d; \mathbb{R}) \cap W^{1,\infty}([0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d; \mathbb{R})$ . Furthermore, we denote the Lipschitz constants of  $f$  by a constant  $K > 0$  as in A1; and we assume that

$$\sup_{0 \leq t \leq T} \{|b(t, 0)| + |\sigma(t, 0)| + |f(t, 0, 0, 0)|\} \leq K. \quad (2.2)$$

(A3) The function  $g \in C_b^{0,2,3}([0, T] \times \mathbb{R}^d \times \mathbb{R}; \mathbb{R}^d)$

(A4) The function  $l: \mathbb{R}^d \mapsto \mathbb{R}$  is continuous, such that for some constants  $K, p > 0$ ,

$$l(x) \leq K(1 + |x|^p), x \in \mathbb{R}^d. \quad (2.3)$$

We will give the following results that are either standard or slight variations of the well-know results of FSDE (see [9]) and BDSDE (see [19])

**Lemma 2.1** *Suppose that  $b \in C(\mathbf{F}, [0, T] \times \mathbb{R}^d; \mathbb{R}^d) \cap L^0(\mathbf{F}, [0, T]; W^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d))$ ,  $\sigma \in C(\mathbf{F}, [0, T] \times \mathbb{R}^d; \mathbb{R}^{d \times d}) \cap L^0(\mathbf{F}, [0, T]; W^{1,\infty}(\mathbb{R}^d; \mathbb{R}^{d \times d}))$ , with a common Lipschitz constant  $K > 0$ . Suppose also that  $b(t, 0) \in L^2(\mathbf{F}, [0, T]; \mathbb{R}^d)$  and  $\sigma(t, 0) \in L^2(\mathbf{F}, [0, T]; \mathbb{R}^{d \times d})$ . Let  $X$  be the unique solution of the following forward SDE*

$$X_s = x + \int_s^t b(r, X_r) dr + \int_s^t \sigma(r, X_r) dW_r. \quad (2.4)$$

Then for any  $p \geq 2$ , there exists a constant  $C > 0$  depending only on  $p, T$  and  $K$ , such that

$$E(|X|_{0,T}^{*,p}) \leq C \left\{ |x|^p + \mathbb{E} \int_0^T [ |b(s, 0)|^p + |\sigma(s, 0)|^p ] ds \right\} \quad (2.5)$$

**Lemma 2.2** *Assume  $f \in C(\mathbf{F}, [0, T] \times \mathbb{R} \times \mathbb{R}^d; \mathbb{R}) \cap L^0(\mathbf{F}, [0, T]; W^{1,\infty}(\mathbb{R} \times \mathbb{R}^d; \mathbb{R}))$ , with a uniform Lipschitz constant  $K > 0$ , such that  $f(s, 0, 0) \in L^2(\mathbf{F}, [0, T])$  and  $g \in C(\mathbf{F}, [0, T] \times \mathbb{R} \times \mathbb{R}^d; \mathbb{R}^d) \cap L^0(\mathbf{F}, [0, T]; W^{1,\infty}(\mathbb{R} \times \mathbb{R}^d; \mathbb{R}^d))$  with a common uniform Lipschitz constant  $K > 0$  with respect the first variable and  $0 < \alpha < 1$  a Lipschitz constant with respect the second one such that  $g(s, 0, 0) \in L^2(\mathbf{F}, [0, T])$ . For any  $\xi \in L^2(\mathcal{F}_0; \mathbb{R})$ , let  $(Y, Z)$  be the adapted solution to the BDSDE:*

$$Y_s = \xi + \int_0^s f(r, Y_r, Z_r) dr + \int_0^s g(r, Y_r) dB_r - \int_0^s Z_r \downarrow dW_r. \quad (2.6)$$

Then there exists a constant  $C > 0$  depending only on  $T$  and the Lipschitz constants  $K, \alpha$  of  $f$  and  $g$ , such that

$$\mathbb{E} \int_0^T |Z_s|^2 ds \leq C \mathbb{E} \left\{ |\xi|^2 + \int_0^T [|f(s, 0, 0)|^2 + |g(s, 0, 0)|^2] ds \right\}. \quad (2.7)$$

Moreover, for all  $p \geq 2$ , there exists a constant  $C_p > 0$ , such that

$$\mathbb{E}(|Y_{0,T}^{*,p}|) \leq C_p \mathbb{E} \left\{ |\xi|^p + \int_0^T [|f(s, 0, 0)|^p + |g(s, 0, 0)|^p] ds \right\} \quad (2.8)$$

To continue we give the basic facts of the malliavin calculus, especially those related to BDSDE (see Pardoux-Peng [19]). To begin with, let  $\mathcal{S}$  be the space of all random variables of the form

$$\xi = F \left( \int_0^T \varphi_1 dW_t, \dots, \int_0^T \varphi_n dW_s; \int_0^T \psi_1 dB_s, \dots, \int_0^T \psi_p dB_s \right)$$

where  $F \in C_b^\infty(\mathbb{R}^{n+p})$ ,  $\varphi_1, \dots, \varphi_n \in L^2([0, T], \mathbb{R}^d)$  and  $\psi_1, \dots, \psi_n \in L^2([0, T], \mathbb{R}^d)$ . We call a mapping  $D : \mathcal{S} \mapsto L([0, T] \times \Omega)$  the malliavin derivative operator with respect to  $W$  if for each  $\xi \in \mathcal{S}$  and  $t \in [0, T]$ ,

$$D_t \xi = \sum_{i=1}^n \frac{\partial F}{\partial x_i} \left( \int_0^T \varphi_1 dW_t, \dots, \int_0^T \varphi_n dW_s; \int_0^T \psi_1 dB_s, \dots, \int_0^T \psi_p dB_s \right) \varphi_i(t).$$

Next, we define a norm on  $\mathcal{S}$  by

$$\|\xi\|_{1,2}^2 = \mathbb{E}|\xi|^2 + \mathbb{E} \int_0^T |D_r \xi|^2 dr, \forall \xi \in \mathcal{S},$$

and denote  $\mathbb{D}^{1,2}$  to be the completion of  $\mathcal{S}$  in  $L^2(\Omega)$  under  $\|\cdot\|_{1,2}$ . We show (see, Nualart [14]) that  $D$  is a densely defined, closed linear operator from  $\mathbb{D}^{1,2}$  to  $L^2(\Omega \times [0, T]; \mathbb{R})$  with domain  $\mathbb{D}^{1,2}$ .

Now we shall apply the previous anticipative calculus to the following coupled forward backward SDEs

$$\begin{cases} X_s^{x,t} = x + \int_s^t b(r, X_r^{x,t}) dr + \int_s^t \sigma(r, X_r^{x,t}) dW_r; \\ Y_s^{t,x} = l(X_0^{t,x}) + \int_0^s f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr + \int_0^s g(r, X_r^{t,x}, Y_r^{t,x}) dB_r \\ \quad - \int_0^s Z_r^{t,x} \downarrow dW_r, \quad 0 \leq s \leq t. \end{cases} \quad (2.9)$$

Here the superscription  $^{t,x}$  indicates the dependence of the solution on the initial date  $(t, x)$ , and it will be omitted when the context is clear. Let us consider following variational equation

of 2.9 that is very important in this paper.

$$\left\{ \begin{array}{l} \nabla_i X_s = e_i + \int_s^t \partial_x b(r, X_r) \nabla_i X_r dr + \sum_{j=1}^d \int_s^t \partial_x \sigma^j(r, X_r) \nabla_i X_r \downarrow dW_r^j, \\ \nabla_i Y_s = \partial_x l(X_0) \nabla_i X_0 + \int_0^s [\partial_x f(r, \Theta(r)) \nabla_i X_r + \partial_y f(r, \Theta(r)) \nabla_i Y_r \\ \quad + \langle \partial_z f(r, \Theta(r)), \nabla_i Z_r \rangle] dr + \int_0^s [\partial_x g(r, \Theta(r)) \nabla_i X_r + \partial_y g(r, \Theta(r)) \nabla_i Y_r \\ \quad + \langle \partial_z g(r, \Theta(r)), \nabla_i Z_r \rangle] dB_r - \int_0^s \nabla_i Z_r \downarrow dW_r, \end{array} \right. \quad (2.10)$$

where  $e_i = (0, \dots, \overset{i}{1}, \dots, 0)^T$  is the  $i$ -th coordinate vector of  $\mathbb{R}^d$ ;  $\sigma^j(\cdot)$  is the  $j$ -th column of the matrix  $\sigma(\cdot)$  and  $\Theta(r) = (X_r, Y_r, Z_r)$ .

We remark that  $(\nabla X, \nabla Y, \nabla Z) \in L^2(\mathbf{F}; C([0, T]; \mathbb{R}^{d \times d})) \times C([0, T]; \mathbb{R}^d) \times L^2([0, T]; \mathbb{R}^{d \times d})$ ; the  $d \times d$ -matrix-valued process  $\nabla X$  satisfies a linear SDE and  $\nabla X_t = I$  so that  $\nabla[X_s]^{-1}$  exists for  $s \in [t, T]$ ,  $\mathbb{P} - a.s.$

The following lemma concerns the Malliavin-derivatives of the solution  $(X, Y, Z)$  to (2.9). The proof can be found in Pardoux-Peng [19].

**Lemma 2.3** *Assume that (A1) holds; and suppose that  $f \in C_b^{0,1}([0, T] \times \mathbb{R}^{2d+1})$ ,  $g \in C_b^{0,2,3}([0, T] \times \mathbb{R}^{d+1}; \mathbb{R}^d)$ . Then  $(X, Y, Z) \in L^2([0, T]; \mathbb{D}^{1,2}(\mathbb{R}^{2d+1}))$ , and there exists a version of  $(D_s X_r, D_s Y_r, D_s Z_r)$  that satisfies*

$$\left\{ \begin{array}{l} D_s X_r = \nabla X_r (\nabla X_s)^{-1} \sigma(s, X_s) \mathbf{1}_{\{s \leq r\}}, \\ D_s Y_r = \nabla Y_r (\nabla X_s)^{-1} \sigma(s, X_s) \mathbf{1}_{\{s \leq r\}}, \\ D_s Z_r = \nabla Z_r (\nabla X_s)^{-1} \sigma(s, X_s) \mathbf{1}_{\{s \leq r\}}, \end{array} \quad 0 \leq s, r \leq t; \right. \quad (2.11)$$

To end this section let us give the important properties of the anticipative integrals (see [14]).

**Lemma 2.4** *Suppose that  $F \in \mathbb{D}^{1,2}$ . Then*

(i) *(Integration by parts formula): for any  $u \in \text{Dom}(\delta)$  such that  $Fu \in L^2([0, T] \times \Omega; \mathbb{R}^d)$ , one has  $Fu \in \text{Dom}(\delta)$ , and it holds that*

$$\int_0^T \langle Fu_t, dW_t \rangle = \delta(Fu) = F \int_0^T \langle u_t, dW_t \rangle - \int_0^T D_t F u_t dt;$$

(ii) *(Clark-Hausman-Ocone formula):*

$$F = \mathbb{E}(F) + \int_0^T \mathbb{E}\{D_t F / \mathcal{F}_t\} dW_t.$$



### 3 Relations to Stochastic PDE Revisited

In this section we prove the relation (1.4) between the forward backward doubly SDE (2.8) and the quasi-linear SPDE (1.5), under the condition that the coefficients are only continuously differentiable. Indeed, since Buckdahn and Ma [3, 4] provide that, if  $f$  and  $l$  are only Lipschitz continuous, the quantity  $u(t, x) = Y_t^{t,x}$  is a stochastic viscosity solution of the quasi-linear SPDE (1.5), the second relation in (1.4) becomes questionable. Our objective is to fill this gap in the literature and to extend the results of Ma and Zhang [13] given in the case of the probabilistic interpretation of PDEs via the BSDEs.

**Theorem 3.1** *Assume (A1), (A3) and suppose that  $f \in C_b^{0,1}([0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d)$  and  $l \in C_b^1(\mathbb{R}^d)$ . Let  $(X^{t,x}, Y^{t,x}, Z^{t,x})$  be the adapted solution to the FBDSDE (2.9), and define  $u(t, x) = Y_t^{t,x}$  the stochastic viscosity of SPDE (1.5). Then,*

(i)  $\partial_x u(t, x)$  exists for all  $(t, x) \in [0, T] \times \mathbb{R}^d$ ; and for each  $(t, x)$  and  $i=1, \dots, d$ , the following representation holds:

$$\begin{aligned} \partial_{x_i} u(t, x) &= \mathbb{E} \left\{ \partial_x l(X_0^{t,x}) \nabla_i X_0 + \int_0^t [\partial_x f(r, \Theta^{t,x}(r)) \nabla_i X_r + \partial_y f(r, \Theta^{t,x}(r)) \nabla_i Y_r \right. \\ &\quad \left. + \partial_z f(r, \Theta^{t,x}(r)) \nabla_i Z_r] dr / \mathcal{F}_t^B \right\} \end{aligned} \quad (3.1)$$

where  $\Theta^{t,x}(r) = (X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})$  and  $(\nabla X, \nabla Y, \nabla Z)$  are respectively the solution to equation (2.9) and its variational equation (2.10);

(ii)  $\partial_x u(t, x)$  is continuous on  $[0, T] \times \mathbb{R}^d$ ;

(iii)  $Z_s^{t,x} = \partial_x u(s, X_s^{t,x}) \sigma(s, X_s^{t,x}), \forall s \in [0, t], \mathbb{P}$ - a.s.

**Proof.** For the simple presentation we take  $d = 1$ . the higher dimensional case can be treated in the same way without substantial difficulty. We use the simpler notations  $l_x, (f_x, f_y, f_z), (g_x, g_y, g_z)$  for the partial derivatives of  $l, f$  and  $g$ .

The proof is inspired by the approach of Ma and Zhang [13] (see Theorem 3.1). Nevertheless, there exists important difference due in the fact that the solution of the SPDE is not deterministic but a random field; more precisely  $u(t, x)$  is a conditional expectation with respect the filtration  $\mathbf{F}^B$ .

(i) Let  $(t, x) \in [0, T] \times \mathbb{R}$  be fixed. For  $h \neq 0$ , we define:

$$\nabla X_s^h = \frac{X_s^{t,x+h} - X_s^{t,x}}{h}; \nabla Y_s^h = \frac{Y_s^{t,x+h} - Y_s^{t,x}}{h}; \nabla Z_s^h = \frac{Z_s^{t,x+h} - Z_s^{t,x}}{h} \quad s \in [0, t].$$

By the roughly computation of BDSDEs ( see Ma and Zhang [13] for BSDE case) we show that

$$\mathbb{E}\{|\Delta Y^h|_{0,T}^{*,2} = \mathbb{E}\{|\nabla Y^h - \nabla Y|_{0,T}^{*,2}\} \rightarrow 0 \text{ as } h \rightarrow 0. \quad (3.2)$$

Moreover since,  $u(t, x) = Y_t^{x,t}$  and  $u(t, x + h) = Y_t^{x+h,t}$ , we have  $\nabla Y_t^h = \frac{1}{h}[u(t, x) - u(t, x + h)]$ . Then, in view of (3.2) we obtain

$$\begin{aligned}\partial_x u(t, x) &= \lim_{h \rightarrow 0} \frac{1}{h}[u(t, x) - u(t, x + h)] = \lim_{h \rightarrow 0} \nabla Y_t^h \\ &= \nabla Y_t.\end{aligned}\tag{3.3}$$

On the other hand  $Y_s^{x,t}$ ,  $Y_s^{x+h,t}$ ,  $\nabla Y_s^h$  and  $\Delta Y_s^h$  are all  $\mathcal{F}_s^B \otimes \mathcal{F}_{s,t}^W$ -measurable. In particular  $Y_t^{x,t}$ ,  $Y_t^{x+h,t}$ ,  $\nabla Y_t^h$  and  $\Delta Y_t^h$  are all  $\mathcal{F}_t^B \otimes \mathcal{F}_{t,t}^W$ -measurable so is  $\partial_x u(t, x)$ . But since  $W$  is a Brownian motion on  $(\Omega, \mathcal{F}, \mathbb{P})$  applying the Blumenthal 0 – 1 law (see, e.g, [11]) we conclude that  $\partial_x u$  is independent of (or a constant with respect to)  $\omega_2 \in \Omega_2$ . Therefore, we can identify the random field  $\partial_x u$  as one that is defined on  $\Omega_1 \times [0, T] \times \mathbb{R}^d$ , and is  $\mathcal{F}_t^B$ -measurable for each  $t \in [0, T]$ . Finally, taking the conditional expectation on the both sides of (2.10) at  $s = t$  we claim that the representation (3.1) hold. Indeed by the proof of Theorem 3.1 in [22], we have

$$\begin{aligned}\mathbb{E} \int_0^s [\partial_x g(r, \Theta^{t,x}(r)) \nabla X_r + \partial_y g(r, \Theta^{t,x}(r)) \nabla Y_r + \partial_z g(r, \Theta^{t,x}(r)) \nabla Z_r] dB_r &= 0 \\ \mathbb{E} \int_0^s \nabla Z_r \downarrow dW_r &= 0\end{aligned}$$

which finish the prove of (i)

We now prove (ii). Let us remark that  $\partial_x u$  is not deterministic as in the paper of Ma and Zhang [13] but a random field. Since this difference, the prove of (ii) is a slight modification. It is subdivided in two step. Firstly we get continuity *w.r.t* time variable. Indeed, for fixed  $x \in \mathbb{R}$ , the process with parameter  $x$   $\{A_t(x), 0 \leq t \leq T\}$  define by

$$\begin{aligned}A_t(x) &= \partial_x l(X_0^{t,x}) \nabla X_0 + \int_0^t [\partial_x f(r, \Theta^{t,x}(r)) \nabla X_r + \partial_y f(r, \Theta^{t,x}(r)) \nabla Y_r \\ &\quad + \partial_z f(r, \Theta^{t,x}(r)) \nabla Z_r] dr,\end{aligned}\tag{3.4}$$

is uniformly integrable. Therefore, applying Theorem VI – 47 and Remark VI – 50 (f) of Dellacherie-Meyer [7] we see that  $\partial_x u(t, x)$  is the  $\mathcal{F}_t^B$ -optional projection of  $A_t(x)$ , for all  $t \in [0, T]$ . Thus it has càdlàg paths. To show that those paths are actually continuous, we note that the filtration  $\mathcal{F}_t^B$  is Brownian, whence quasi-left-continuous. Thus every  $\mathcal{F}_t^B$ -stopping time  $\tau > t$  is accessible. That is, there exists a sequence of  $\mathcal{F}_t^B$ -stopping times  $\{\tau_k, k \geq 0\}$  such that  $\tau_k < \tau, \forall k, \mathbb{P}$ - a.s., and that  $\tau_k \uparrow \tau$ , as  $k \rightarrow \infty$ . Note that

$$\begin{aligned}\partial_x u(x, \tau_k) - \partial_x u(x, \tau) &= \mathbb{E} \{A_{\tau_k}(x) / \mathcal{F}_{\tau_k}^B\} - \mathbb{E} \{A_\tau(x) / \mathcal{F}_\tau^B\} \\ &= \mathbb{E} \{A_{\tau_k}(x) - A_\tau(x) / \mathcal{F}_{\tau_k}^B\} \\ &\quad + (\mathbb{E} \{A_\tau(x) / \mathcal{F}_{\tau_k}^B\} - \mathbb{E} \{A_\tau(x) / \mathcal{F}_\tau^B\}).\end{aligned}\tag{3.5}$$

Letting  $k \rightarrow \infty$  we see that  $\mathbb{E} \{A_\tau(x) / \mathcal{F}_{\tau_k}^B\} - \mathbb{E} \{A_\tau(x) / \mathcal{F}_\tau^B\} \rightarrow 0$ , thanks to the quasi-left continuity of  $\{\mathcal{F}_t^B\}_{0 \leq t \leq T}$ . Moreover, we claim that  $\mathbb{E} \{A_{\tau_k}(x) - A_\tau(x) / \mathcal{F}_{\tau_k}^B\} \rightarrow 0$ . Indeed in

view of (3.4), standard computation and conditions of  $l$ ,  $f$  and  $g$  it follows easily that

$$\mathbb{E}|A_{\tau_k}(x) - A_\tau(x)|^2 \leq C|\tau_k - \tau|^2. \quad (3.6)$$

Thus  $\partial_x u(\tau-, x) = \partial_x u(\tau, x)$ ,  $\mathbb{P}$ -a.s. Since  $\partial_x u(\cdot, x)$  is càdlàg and  $\tau$  is arbitrary, we conclude that  $\partial_x u(\cdot, x)$  is in fact continuous on  $[0, T]$ , almost surely which finish the first step. In the second step we study continuity w.r.t spacial variable. We fixe  $t \in [0, T]$  and consider the family random variables  $\{\partial_x u(t, x), x \in \mathbb{R}\}$ . By using the same argument as previous we get

$$\mathbb{E}|\partial_x u(t, x_1) - \partial_x u(t, x_2)|^2 \leq C|x_1 - x_2|^2. \quad (3.7)$$

Thus the mapping  $x \mapsto \partial_x u(t, x)$  is continuous. Finally by the two step we conclude that  $\partial_x u(\cdot, \cdot)$  is continuous on  $[0, T] \times \mathbb{R}^d$  and finished the prove of (ii).

It remains to prove (iii). For a continuous function  $\varphi$ , let us consider  $\{\varphi^\varepsilon\}_{\varepsilon>0}$  a family of  $C^{0,\infty}$  functions that converges to  $\varphi$  uniformly. Since  $b$ ,  $\sigma$ ,  $l$ ,  $f$  are all uniformly Lipschitz continuous, we may assume that the first order partial derivatives of  $b^\varepsilon$ ,  $\sigma^\varepsilon$ ,  $l^\varepsilon$ ,  $f^\varepsilon$  are all uniformly bounded, by the corresponding Lipschitz constants of  $b, \sigma, l, f$  uniformly in  $\varepsilon > 0$ . Now we consider the family of FBDSDEs parameterized by  $\varepsilon > 0$ :

$$\begin{cases} X_s^{t,x} = x + \int_s^t b^\varepsilon(r, X_r^{t,x})dr + \int_s^t \sigma^\varepsilon(r, X_r^{t,x}) \downarrow dW_r; \\ Y_s^{t,x} = l^\varepsilon(X_0^{t,x}) + \int_0^s f^\varepsilon(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr + \int_0^s g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dB_r - \int_0^s Z_r^{t,x} \downarrow dW_r \end{cases} \quad (3.8)$$

and denote it solution by  $(X^{t,x}(\varepsilon), Y^{t,x}(\varepsilon), Z^{t,x}(\varepsilon))$ . We define  $u^\varepsilon(t, x) = Y_t^{t,x}(\varepsilon)$ . Applying Theorem 3.2 of [19] it follow that  $u^\varepsilon$  is the classical solution of stochastic PDE

$$\begin{aligned} du^\varepsilon(t, x) &= [\mathcal{L}^\varepsilon u(t, x) + f^\varepsilon(t, x, u^\varepsilon(t, x), (\nabla u^\varepsilon \sigma^\varepsilon)(t, x))] dt \\ &\quad + g(t, x, u^\varepsilon(t, x)) dB_t, \quad (t, x) \in (0, T) \times \mathbb{R}^d, \end{aligned} \quad (3.9)$$

$$u^\varepsilon(0, x) = l^\varepsilon(x), \quad x \in \mathbb{R}^d.$$

For any  $\{x^\varepsilon\} \subset \mathbb{R}^n$  such that  $x^\varepsilon \rightarrow x$  as  $\varepsilon \rightarrow 0$ , define

$(X^\varepsilon, Y^\varepsilon, Z^\varepsilon) = (X^{t,x^\varepsilon}(\varepsilon), Y^{t,x^\varepsilon}(\varepsilon), Z^{t,x^\varepsilon}(\varepsilon))$ . Then it is well know according the work of Pardoux and Peng [19] that

$$Y_s^\varepsilon = u^\varepsilon(s, X_s^\varepsilon); \quad Z_s^\varepsilon = \partial_x u^\varepsilon(s, X_s^\varepsilon) \sigma^\varepsilon(s, X_s^\varepsilon), \quad \forall s \in [0, t], \quad \mathbb{P} - a.s. \quad (3.10)$$

Now by Lemma 2.1 and Lemma 2.2, for all  $p \geq 2$  it hold that

$$\mathbb{E} \left\{ |X^\varepsilon - X|_{0,T}^{*,p} + |Y^\varepsilon - Y|_{0,T}^{*,p} + \int_0^T |Z_s^\varepsilon - Z_s|^2 ds \right\} \rightarrow 0 \quad (3.11)$$

as  $\varepsilon \rightarrow 0$ . Moreover let us recall  $(\nabla X^\varepsilon, \nabla Y^\varepsilon, \nabla Z^\varepsilon)$  the unique solution of the variational equation of (3.8). Using again Lemma 2.1 and Lemma 2.2 we get

$$\mathbb{E} \left\{ |\nabla X^\varepsilon - \nabla X|_{0,T}^{*,p} + |\nabla Y^\varepsilon - \nabla Y|_{0,T}^{*,p} + \int_0^T |\nabla Z_s^\varepsilon - \nabla Z_s|^2 ds \right\} \rightarrow 0, \quad (3.12)$$

as  $\varepsilon \rightarrow 0$ . Thus, using the dominated convergence theorem one derives that

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left\{ l_x^\varepsilon(X_0^\varepsilon) \nabla X_0^\varepsilon + \int_0^t [f_x^\varepsilon \nabla X_r^\varepsilon + f_y^\varepsilon \nabla Y_r^\varepsilon + f_z^\varepsilon \nabla Z_r^\varepsilon] dr \mathbf{1}_A \right\} \\ &= \mathbb{E} \left\{ l_x(X_0) \nabla X_0 + \int_0^t [f_x \nabla X_r + f_y \nabla Y_r + f_z \nabla Z_r] dr \mathbf{1}_A \right\} \end{aligned}$$

for any  $A \in \mathcal{F}_t^B$ . Then

$$\partial_x u^\varepsilon(t, x^\varepsilon) \rightarrow \partial_x u(t, x), \quad \text{as } \varepsilon \rightarrow 0 \quad \mathbb{P} - a.s. \quad \forall (t, x).$$

Consequently, possibly along a subsequence, we get

$$Z_s^\varepsilon = \lim_{\varepsilon \rightarrow 0} \partial u^\varepsilon(s, X_s^\varepsilon) \sigma^\varepsilon(s, X_s^\varepsilon) = \partial u(s, X_s) \sigma(s, X_s), \quad ds \times d\mathbb{P} - a.e.$$

Since for  $\mathbb{P} - a.e.$   $\omega$ ,  $\partial_x u(\cdot, \cdot)$  and  $X$  are both continuous, the above equalities actually holds for all  $s \in [0, t]$ ,  $\mathbb{P}$ -a.s., proving (iii) and end the proof.  $\blacksquare$

The following corollary is the direct consequence of the Theorem 3.1. The convention on the generic constant  $C > 0$  still true.

**Corollary 3.2** *Assume that the same conditions as in Theorem 3.1 hold, and let  $(X, Y, Z)$  be the solution of FBDSDE (2.9). Then, there exists a constant  $C > 0$  depending only on  $K, T$  and a positive  $L^p(\Omega, \mathcal{F}_t^B, \mathbb{P}_1)$ - process  $\Gamma$ , ( $p \geq 1$ ), such that*

$$|\partial_x u(t, x)| \leq C \Gamma_t, \quad \forall (t, x) \in [0, T] \times \mathbb{R}^d, \quad \mathbb{P} - a.s. \quad (3.13)$$

Consequently, one has

$$|Z_s| \leq C \Gamma_s (1 + |X_s|), \quad \forall s \in [0, t], \quad \mathbb{P} - a.s. \quad (3.14)$$

Furthermore,  $\forall p > 1$ , there exists a constant  $C_p > 0$ , depending on  $K, T$ , and  $p$  such that

$$\mathbb{E} \{ |X|_{0,T}^{*,p} + |Y|_{0,T}^{*,p} + |Z|_{0,T}^{*,p} \} \leq C_p (1 + |x|^p). \quad (3.15)$$

**Proof.** Let us assume that  $p \geq 2$ . The case for  $1 < p < 2$  then follows easily from the Hölder inequality. By the representation (3.1) and the condition on  $l$  and  $f$  it follows that

$$|\partial_x u(t, x)| \leq C \mathbb{E} \left( |\nabla X_0| + \int_0^t [|\nabla X_s| + |\nabla Y_s| + |\nabla Z_s|] ds / \mathcal{F}_t^B \right).$$

Let us denote  $\Gamma_t = \mathbb{E} \left( |\nabla X_0| + \int_0^t [|\nabla X_s| + |\nabla Y_s| + |\nabla Z_s|] ds / \mathcal{F}_t^B \right)$ . Then using Lemma 2.1 and Lemma 2.2 we obtain

$$\mathbb{E} |\Gamma_t|^p \leq \mathbb{E} \{ |\nabla X|_{0,T}^{*,p} + |\nabla Y|_{0,T}^{*,p} + \int_0^T |\nabla Z_r|^2 dr \} \leq C.$$

In view of (iii) of Theorem 3.1 it follows that

$$|Z_s| \leq C\Gamma_s(1 + |X_s|), \quad \forall s \in [0, T], \quad \mathbb{P} - a.s.$$

Applying again Lemma 2.1, Lemma 2.2 and hölder's inequality for  $p \geq 2$  we get (3.15). ■

The following last theorem in this section reduces the conditions on  $l$  and  $f$  to **(A2)** and **(A4)**. Let us remark that this assumptions is much weaker than that one of Theorem 3.1 in which the functions  $l$  and  $f$  are assumed to be continuously differentiable in all spatial variables with uniformly bounded partial derivatives.

**Theorem 3.3** *Assume **(A1)** – **(A4)**, and let  $(X, Y, Z)$  be the solution to the FBDSDE (2.9). Then for all  $p > 0$ , there exists a constant  $C_p > 0$  such that*

$$\mathbb{E} \left\{ |X|_{0,T}^{*,p} + |Y|_{0,T}^{*,p} + \text{ess sup}_{0 \leq s \leq T} |Z_s|^p \right\} \leq C_p(1 + |x|^p). \quad (3.16)$$

**Proof.** In the light of the corollary 3.1, we need only consider  $p \geq 2$ . By Lemma 2.1 and Lemma 2.2 it follows that for any  $p > 0$  there exists  $C_p > 0$  such that

$$\mathbb{E} \{ |X|_{0,T}^{*,p} + |Y|_{0,T}^{*,p} \} \leq C_p(1 + |x|^p). \quad (3.17)$$

Next using the same argument as in the proof of part (iii) of Theorem 3.1 we consider two sequences of smooth functions  $\{f^\varepsilon\}_\varepsilon$  and  $\{l^\varepsilon\}_\varepsilon$  such that

$$\lim_{\varepsilon \rightarrow 0} \left\{ \sup_{(t,x,y,z)} |f^\varepsilon(t, x, y, z) - f(t, x, y, z)| + \sup_x |l^\varepsilon(x) - l(x)| \right\} = 0$$

and that the first order derivatives of  $f^\varepsilon$ 's and  $l^\varepsilon$ 's in  $(x, y, z)$  are uniformly bounded, uniformly in  $t$  and  $\varepsilon$ . Let  $(X^\varepsilon, Y^\varepsilon, Z^\varepsilon)$  be the unique solution of the corresponding FBDSDEs. Thus apply Corollary 3.1 it follows that for any  $p \geq 2$  there exists a constant  $C_p > 0$ , independent of  $\varepsilon$ , such that

$$\mathbb{E} (|Z^\varepsilon|_{0,T}^{*,p}) \leq C_p(1 + |x|^p). \quad (3.18)$$

Furthermore, we know that  $\mathbb{E} \int_0^T |Z_s^\varepsilon - Z_s|^2 ds \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Thus, possibly along a sequence say  $(\varepsilon_n)_{n \geq 1}$  we have  $\lim_{n \rightarrow \infty} Z^{\varepsilon_n} = Z$   $ds \times d\mathbb{P}$  - a.s. Applying Fatou's lemma to (3.18) we obtain

$$\mathbb{E} \left\{ \text{ess sup}_{0 \leq s \leq T} |Z_s|^p \right\} \leq C_p(1 + |x|^p)$$

which end the proof. ■

## 4 The Representation Theorem

Now let us give the second main result of the paper. The following theorem can be regarded as an extension of the nonlinear Feynman-Kac formula obtained by Pardoux-Peng [19]. It gives a probabilistic representation of the gradient w.r.t the space variable of the viscosity solution, whenever it exists, to a quasi-linear parabolic stochastic PDE. The main feature of our representation lies in the fact that it does not depend on the partial derivatives of the functions  $f$  and  $l$  as in (3.1). Such representation give the argument for us to study the path regularity of the process  $Z$  in the BDSDE with non-smooth coefficients. To Begin let denote the following processes that will play an important role in the sequel.

$$M_r^s = \int_r^s [\sigma^{-1}(\tau, X_\tau) \nabla X_\tau]^T \downarrow dW_\tau$$

and

$$N_r^s = \frac{1}{s-r} (M_r^s)^T [\nabla X_r]^{-1}, \quad 0 \leq r < s \leq t.$$

Let us recall that

$$\begin{aligned} \mathbb{E}|M_r^s|^{2p} &\leq C_p \mathbb{E} \left( \int_r^s |\sigma^{-1}(\tau, X_\tau) \nabla X_\tau|^2 d\tau \right)^p \\ &\leq C_p (s-r)^p \mathbb{E} (|\nabla X_\tau|_{s,r}^{*,2p}) \leq C_p (s-r)^p, \end{aligned} \quad (4.1)$$

where  $C_p > 0$  is a generic constant.

Before we give our main result let us introduce this two notations  $\mathcal{F}^t = \{\mathcal{F}_s^B \otimes \mathcal{F}_{s,t}^W, 0 \leq s \leq t\}$ ,  $\mathcal{G}^t = \{\mathcal{F}_s^B \otimes \mathcal{F}_{0,t}^W, 0 \leq s \leq t\}$  and give the following lemma that is slight modification version of Lemma 4.1 in Ma and Zhang [13].

**Lemma 4.1** *For any fixed  $t \in [0, T]$  and any  $H \in L^\infty(\mathbf{F}^t, [0, T]; \mathbb{R})$  we have*

- (i)  $E|\int_0^s \frac{1}{s-r} H_s M_s^r dr| < +\infty$
- (ii) for  $\mathbb{P}$ .a.e.,  $\omega \in \Omega$ , the mapping  $s \mapsto \int_0^s \frac{1}{s-r} H_r(\omega) M_s^r(\omega) dr(\omega)$  is Hölder- $([p_0-2]/[p_0(p_0+2)])$  continuous on  $[0, t]$
- (iii) for  $\mathbb{P}$ .a.e.,  $\omega \in \Omega$ , the mapping  $s \mapsto \mathbb{E}\{\int_0^s \frac{1}{s-r} H_r M_s^r dr / \mathcal{G}_s^t\}(\omega)$  is continuous on  $[0, t]$

**Proof.** First, for any  $0 \leq \tau < t$  we denote

$$A_\tau^s = \begin{cases} \int_\tau^s \frac{1}{s-r} H_r M_r^s dr, & 0 \leq \tau < s \\ 0, & \text{if } s = \tau. \end{cases} \quad (4.2)$$

To simplify notation, when  $\tau = 0$  we denote  $A_0^s = A_s$ . Further, let  $q_0 > 0$  be such that  $\frac{1}{p_0} + \frac{1}{q_0} = 1$ , and define  $\beta = \frac{p_0}{2+p_0}$  and  $\alpha = 1 - \beta$ . It is readily seen that  $\beta > \frac{1}{q_0}$  and  $\alpha < \frac{1}{2}$ . Consider the random variable

$$M^* = \sup_{0 \leq t_1 < t_2 \leq t} \frac{|M_{t_1}^{t_2}|}{(t_2 - t_1)^\alpha}. \quad (4.3)$$

According (4.2) and Theorem 2.1 of Revuz-Yor [21], we have  $\mathbb{E}[M^*]^2 < +\infty$ .

By Hölder's inequality one has

$$\begin{aligned} \mathbb{E}|A_\tau^s| &\leq C(s - \tau)^{(1/q_0) - \beta} \|H\|_{p_0} M^* \\ &\leq C \|H\|_{L^{p_0}([0, T] \times \Omega)} \|M^*\|_{L^2(\Omega)} < \infty. \end{aligned} \quad (4.4)$$

Setting  $\tau = 0$  in (4.4) we proved (i).

To prove (ii) let  $\tau = 0$  and observe that, for  $0 < s_1 < s_2 \leq t$ ,

$$\begin{aligned} A_{s_1} - A_{s_2} &= \int_{s_1}^{s_2} \frac{1}{s_2 - r} H_r M_r^{s_2} dr - \int_0^{s_1} \frac{1}{s_2 - r} H_r M_r^{s_2} dr \\ &\quad + \int_0^{s_1} \left( \frac{1}{s_1 - r} - \frac{1}{s_2 - r} \right) H_r M_r^{s_1} dr = \Gamma_1 + \Gamma_2 + \Gamma_3, \end{aligned} \quad (4.5)$$

where  $\Gamma_i$ 's are defined in an obvious way. Comparing to (4.2) and in view of (4.4),

$$\Gamma_1 \leq C(s_2 - s_1)^{(1/q_0) - \beta} \|H\|_{p_0} M^*. \quad (4.6)$$

Further, by definition (4.3) we see that

$$|\Gamma_2| \leq C(s_2 - s_1)^{(1/q_0) - \beta} \|H\|_{p_0} M^*. \quad (4.7)$$

Finally it not difficult to prove

$$|\Gamma_3| \leq C(s_2 - s_1)^{(1/q_0) - \beta} \|H\|_{p_0} M^*. \quad (4.8)$$

Combining (4.6) – (4.8) we obtain that

$$|A_{s_2} - A_{s_1}| \leq C(s_2 - s_1)^{(1/q_0) - \beta} \|H\|_{p_0} M^*. \quad (4.9)$$

We should note that by (4.4) with  $\tau = 0$  we see that (4.9) holds true even when  $s_1 = 0$ . This, together with the fact  $\frac{1}{q_0} - \beta = \frac{p_0 - 2}{p_0(p_0 + 2)}$ , proves (ii).

The proof of (iii) is the same as this one of Ma and Zhang [13] (see Lemma 4.1 (iii)). ■

**Theorem 4.2** *Assume that the assumptions (A1) - (A4) hold, and let  $(X, Y, Z)$  be the adapted solution to FBDSDE (2.9). Then*

(i) the following identity holds  $\mathbb{P}$ - almost surely:

$$Z_s = \mathbb{E} \left\{ l(X_0)N_0^s + \int_0^s f(r, X_r, Y_r, Z_r)N_r^s dr / \mathcal{F}_s^t \right\} \sigma(s, X_s). \quad (4.10)$$

(ii) There exist a version of  $Z$  such that for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ , the mapping  $s \mapsto Z_s(\omega)$  is continuous;

(iii) If in addition the functions  $f$  and  $l$  satisfy assumptions of Theorem 3.1, then for all  $(t, x) \in [0, T] \times \mathbb{R}^d$  it holds that

$$\partial_x u(t, x) = \mathbb{E} \left\{ l(X_0)N_0^t + \int_0^t f(s, X_s, Y_s, Z_s)N_s^t dr / \mathcal{F}_t^B \right\} \quad (4.11)$$

**Proof.** Again we shall consider only the case  $d = 1$ . We assume first that  $l \in C_b^1(\mathbb{R})$  and  $f \in C_b^{0,1}([0, T] \times \mathbb{R}^3)$ . Using the nonlinear Feynman-Kac formula of Pardoux and Peng [19] we obtain that for  $0 \leq s \leq t$ ,

$$u(s, X_s) = Y_s = \mathbb{E} \left\{ l(X_0) + \int_0^s f(r, X_r, Y_r, Z_r) dr / \mathcal{F}_s^t \right\}. \quad (4.12)$$

According the step (i) in the proof of Theorem 4.1 in Ma and Zhang [13] we get

$$\partial_x u(s, X_s) = \mathbb{E} \left\{ l(X_0)N_0^s + \int_0^s f(r, \Theta(r))N_r^s dr / \mathcal{F}_s^t \right\}.$$

In particular, setting  $s = t$  we obtain (4.11), this proves (iii).

We now consider the general case. First we fix  $s \in [t, T]$ . For  $\varphi = l, f$ , let  $\varphi^\varepsilon \in C^\infty, \varepsilon > 0$ , be the mollifiers of  $\varphi$ , and let  $(Y^\varepsilon, Z^\varepsilon)$  be the solution of the BDSDE in (2.9) with coefficients  $(l^\varepsilon, f^\varepsilon, g)$ . Then for each  $\varepsilon > 0$ , as the previous we get

$$Z_s^\varepsilon = \mathbb{E} \left\{ l^\varepsilon(X_0)N_0^s + \int_0^s f^\varepsilon(r, X, Y^\varepsilon, Z^\varepsilon)N_r^s dr / \mathcal{F}_s^t \right\} \sigma(s, X_s). \quad (4.13)$$

Passing to limit as  $\varepsilon$  goes to zero in (4.13), we get (4.10)  $\mathbb{P}$ - a.s., for each fixed  $s \in [0, t]$ .

However in order to prove part (i) we still need to show that (4.10) actually holds for all  $s \in [0, t]$ ,  $\mathbb{P}$ - a.s. But this will follow from part (ii), i.e., the process  $Z$  has the a continuous version. Thus it remain to prove only (ii) For instance let us remark that according some adaptation version of Pardoux and Peng work (see the proof of proposition 1.2) one can prove that

$$Z_s = \mathbb{E} \left\{ l(X_0)N_0^s + \int_0^s f(r, X, Y, Z)N_r^s dr / \mathcal{G}_s^t \right\} \sigma(s, X_s).$$



Hence it follows by Lemma 4.1 that mapping

$$s \mapsto \mathbb{E} \left\{ \int_s^T f(r, X, Y, Z) N_r^s dr / \mathcal{G}_s^t \right\}$$

is a.s. continuous on  $[0, t]$ . Next, since  $l(\cdot)$  is uniformly Lipschitz, in view of proposition 1.23 of Nualart [14], there exists  $\xi \in L^2(\Omega)$ , such applying integration by part formula (Lemma 2.5) again we have

$$\begin{aligned} \mathbb{E} \{ l(X_0) N_0^s / \mathcal{G}_s^t \} &= \frac{1}{s} \mathbb{E} \left\{ l(X_0) \int_0^s \sigma^{-1}(r, X_r) \nabla X_r dW_r / \mathcal{G}_s^t \right\} (\nabla X_s)^{-1} \\ &= \frac{1}{s} \mathbb{E} \left\{ \int_0^s [D_r l(X_T)] \sigma^{-1}(r, X_r) \nabla X_r dr / \mathcal{F}_s^t \right\} (\nabla X_s)^{-1} \\ &= \mathbb{E} \{ \xi \nabla X_0 / \mathcal{G}_s^t \} (\nabla X_s)^{-1}. \end{aligned}$$

Thus using again Lemma 4.1 the mapping  $s \mapsto \mathbb{E} \{ l(X_0) N_0^s / \mathcal{G}_s^t \}$  is also continuous on  $[0, t]$ . Consequently, the right side of (4.10) is a.s. continuous on  $[0, t]$ , and hence (4.10) holds for all  $s \in [0, t]$ ,  $P$ - a.s., proving (ii), whence the theorem. ■

**Remark 4.3** *A direct consequence of Theorem 4.1 that might be useful in application is the following improvement of Theorem 3.3: assume that (A1) and (A2) hold, then for all  $p > 0$ , there exists a constant  $C_p > 0$  depending only on  $T, K$  and  $p$  such that*

$$\mathbb{E} \{ |X|_{t,T}^{*,p} + |Y|_{t,T}^{*,p} + |Z|_{t,T}^{*,p} \} \leq C_p (1 + |x|^p) \quad (4.14)$$

Indeed, since by Theorem 4.1,  $Z$  has a continuous version, thus (3.16) becomes (4.14)

## 5 Discrete function case

Let us recall that we have proved in Theorem 4.1 that the process  $Z$  in the solution to the FBDSDE (2.1) has continuous paths, under the condition that the coefficients  $f$  and  $l$  are only uniformly Lipschitz continuous. While such a result is already an improvement of that of Pardoux and Peng [19], it still within the paradigm of the standard FBDSDE in the literature, to wit, the terminal condition of the BDSDE is of the form  $l(X_T)$  (see also [19]). In this section we consider the class of BDSDEs whose terminal conditions are path dependent. More precisely, we assume that the terminal condition of the BDSDE is the form  $\xi = l(X_{t_0}, X_{t_1}, \dots, X_{t_n})$ , where  $0 = t_0 < t_1 < \dots < t_n = t$  is any partition of  $[0, t]$ . We shall prove a new representation theorem for the process  $Z$ , and will extend the path regularity result to such a case.

**Theorem 5.4** *Assume that (A1) – (A3) hold; and in (A3),  $l : R^{d(n+1)} \rightarrow \mathbb{R}$ . Let  $\pi : 0 = t_0 < t_1 < \dots < t_n = t$  be a given partition of  $[0, t]$ , and let  $(X, Y, Z)$  be the unique adapted solution*

to the following FBDSDE:

$$\begin{aligned} X_s &= x + \int_s^t b(r, X_r)dr + \int_s^t \sigma(r, X_r)dW_r, \\ Y_s &= l(X_{t_0}, X_{t_1}, \dots, X_{t_n}) + \int_0^s f(r, X_r, Y_r, Z_r)dr \\ &\quad + \int_0^s g(r, X_r, Y_r, Z_r)dB_r - \int_0^s Z_r dW_r, \quad s \in [0, t]. \end{aligned} \quad (5.15)$$

Then on each interval  $(t_{i-1}, t_i), i = 1, \dots, n$ , the following identity holds:

$$\begin{aligned} Z_s &= \mathbb{E} \left\{ l(X_{t_0}, X_{t_1}, \dots, X_{t_n}) N_{t_{i-1}}^s \right. \\ &\quad \left. + \int_0^s f(r, X_r, Y_r, Z_r) N_{r \vee t_{i-1}}^s dr | \mathcal{F}_s^t \right\} \sigma(s, X_s). \quad s \in (t_{i-1}, t_i) \end{aligned} \quad (5.16)$$

Further, there exists a version of process  $Z$  that enjoys the following properties:

- (i) the mapping  $s \mapsto Z_s$  is a.s. continuous on each interval  $(t_{i-1}, t_i), i = 1, \dots, n$ ;
- (ii) limits  $Z_{t_i^-} = \lim_{s \uparrow t_i} Z_s$  and  $Z_{t_i^+} = \lim_{s \downarrow t_i} Z_s$  exist;
- (iii)  $\forall p > 0$ , there exists a constant  $C_p > 0$  depending only on  $T, K$  and  $p$  such that

$$\mathbb{E} |\Delta Z_{t_i}|^p \leq C_p (1 + |x|^p) \leq \infty. \quad (5.17)$$

Consequently, the process  $Z$  has both càdlàg and càglàd version with discontinuities  $t_0, \dots, t_n$  and jump sizes satisfying (5.17)

**Proof.** Recall again the argument of the proof of Theorem 4.1, we shall treat only the case when  $d = 1$ . and also assume first that  $f, l \in C_b^1$ . We remark also that our demarche is nearly identic as the one of Ma and Zhang. For this reason will give the important line. The reader can see detail in [13]. We firstly proved equality (5.16); For instance let us fix arbitrary index  $i$ . By virtue of the Malliavin operator, Theorem 2.4 and the uniqueness of the adapted solution to BDSDE, we obtain

$$\begin{aligned} Z_s &= \sum_{j \geq i} \partial_j l D_s X_{t_j} + \int_0^s [f_x(r) D_s X_r + f_y(r) D_s Y_r + f_z(r) D_s Z_r] dr \\ &\quad + \int_0^s [g_x(r) D_s X_r + g_y(r) D_s Y_r + g_z(r) D_s Z_r] dB_r - \int_0^s D_s Z_r dW_r \\ &= \left\{ \sum_{j \geq i} \partial_j l \nabla X_{t_j} + \int_0^s [f_x(r) \nabla_s X_r + f_y(r) \nabla^i Y_r + f_z(r) \nabla^i Z_r] dr \right. \\ &\quad \left. + \int_0^s [g_x(r) \nabla X_r + g_y(r) \nabla^i Y_r + g_z(r) \nabla^i Z_r] dB_r - \int_0^s \nabla^i Z_r dW_r \right\} (\nabla X_s)^{-1} \sigma(s, X_s) \\ &= \nabla^i Y_s (\nabla X_s)^{-1} \sigma(s, X_s), \quad t_{i-1} < s < t_i. \end{aligned} \quad (5.18)$$

Taking the conditional expectation  $\mathbb{E}\{.\mid\mathcal{F}_s^t\}$  on two sides of (5.18) we then get

$$Z_s = \mathbb{E} \left\{ \sum_{j \geq i} \partial_j l \nabla X_{t_j} + \int_0^s [f_x(r) \nabla_s X_r + f_y(r) \nabla_s^i Y_r + f_z(r) \nabla^i Z_r] dr \mid \mathcal{F}_s^t \right\} (\nabla X_s)^{-1} \sigma(s, X_s). \quad (5.19)$$

The rest of the proof is similar in the BSDE case. Now we will prove (i) – (iii) to end the proof of this Theorem. For instance let us note that (i) is obvious, in light of Theorem 4.2 and thanks to representation (5.16). Property (ii) is a small variation of Lemma 4.1, with 0 here replaced by  $t_{i-1}$ , for each  $i$ . We shall only check (iii). For this way, we define  $\Delta Z_{t_i} = Z_{t_{i+}} - Z_{t_{i-}}$ . From (5.18) it not difficult to check that

$$Z_{t_{i-}} = \nabla^i Y_{t_i} [\nabla X_{t_i}]^{-1} \sigma(t_i, X_{t_i}) \quad Z_{t_{i+}} = \nabla^{i+1} Y_{t_i} [\nabla X_{t_i}]^{-1} \sigma(t_i, X_{t_i})$$

Denoting  $\alpha_s^i = -(\nabla^{i+1} Y_s - \nabla^i Y_s)$ ,  $i = 1, \dots, n$ , we have

$$\Delta Z_{t_i} = (\nabla^{i+1} Y_s - \nabla^i Y_s) \sigma(t_i, X_{t_i}) = -\alpha_{t_i}^i \sigma(t_i, X_{t_i}). \quad (5.20)$$

Further, since  $(\nabla^i Y, \nabla^i Z)$  denotes the adapted solution of the following BDSDE

$$\begin{aligned} \nabla^i Y_\tau &= \sum_{j \geq i} \partial_j l \nabla X_{t_j} + \int_0^\tau [f_x(r) \nabla X_r + f_y(r) \nabla^i Y_r + f_z(r) \nabla Z_r] dr \\ &+ \int_0^\tau [g_x(r) \nabla X_r + g_y(r) \nabla^i Y_r + g_z(r) \nabla^i Z_r] dB_r - \int_0^\tau \nabla^i Z_r dW_r, \quad \tau \in [t_{i-1}, t], \end{aligned}$$

if we denote  $\beta_{t_i}^i = -(\nabla^{i+1} Z_s - \nabla^i Z_s)$ , then we have

$$\begin{aligned} \alpha_s^i &= \partial_i l \nabla_{t_i} + \int_0^s [f_y(r) \alpha_r^i + f_z(r) \beta_r^i] dr + \int_0^s [g_y(r) \alpha_r^i + g_z(r) \beta_r^i] dB_r \\ &- \int_0^s \beta_r^i dW_r, \quad s \in [0, t]. \end{aligned} \quad (5.21)$$

So  $(\alpha^i, \beta^i)$  is the adapted solution to the linear BDSDE (5.21). It follows by Lemma 2.2 that  $\forall p > 0$  there exists a  $C_p > 0$  such that  $\mathbb{E}\{|\alpha_{t_i}^i|^p\} \leq C_p$ . On the other hand the same estimate holds for  $\sigma(s, X_s)$  because of assumption **(A1)** and Theorem 3.3; for  $[\nabla X]^{-1}$  since it is solution of a appropriated SDE. It readily seen that (5.17) follows from (5.20) which prove (iii).

Finally, we note that when  $f$  and  $l$  are only Lipschitz, (5.16) still holds, modulo a standard approximation the same as that in Theorem 4.2. Thus properties (i) and (ii) are obvious. To prove (iii) we should observe that the standard approximation yield that  $\Delta Z_{t_i}^\varepsilon \rightarrow \Delta Z_{t_i}$  a.s. So if (5.17) holds for  $\Delta Z_{t_i}^\varepsilon$ , then letting  $\varepsilon \rightarrow 0$ , (5.17) remains true for  $\Delta Z_{t_i}$ , according the Fatou's lemma; that end the proof. ■

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