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Backward stochastic nonlinear Volterra integral equation with local Lipschitz drift

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Abstract

In this paper, we study backward stochastic nonlinear Volterra integral equations. Under local Lipschitz continuity condition on the drift, we prove existence and uniqueness result. We also established a stability property for this kind of equations.

MSC Subject Classification: 60H20; 60H05; 60Gxx

Key Words: Backward stochastic differential equation; Volterra integral equation; Adapted process.

1 Introduction

A linear version of backward stochastic differential equations (BSDE's in short) was first considered by Bismut ([7], [8]) in the context of optimal stochastic control. Nonlinear BSDE's have been independently introduced by Pardoux and Peng [22] and Duffie and Epstein [10]. These equations have been intensively investigated in the last years. The main reason for this great interest in these equations is their connections with many other fields of research such as: mathematical finance, (see El Karoui et *al.*[11]), stochastic control and stochastic games, (see Hamadène and Lepeltier [17]). These equations also provide probabilistic interpretation for solutions to both elliptic and parabolic nonlinear partial differential equations (see Pardoux and Peng [23], Peng [24]). Indeed, coupled with a forward SDE, such BSDE's give an extension of the celebrated Feynman-Kac formula to nonlinear case.

The classical condition on the drift for proving existence and uniqueness result is a global Lipschitz one. Many authors have attempted to relax this condition. For instance, several

works treat BSDE's with continuous or local Lipschitz drift (see Hamadène [15], [16], Lepeltier and San Martin [20], N'zi and Ouknine [21] and the references therein). In one dimensional case, the essential tool is the comparison-theorem technique. In multidimensional case, the improvements of the Lipschitz condition on the generator concern, generally the variable y only and the conditions considered are global. It seems that the first works treating multidimensional BSDE's with both local conditions on the drift and only square integrable terminal data are Bahlali [2], [3]. This author considered BSDE's with locally Lipschitz coefficients both in y and z . This study has been continued by Bahlali *et al.* [4], Aman and N'zi [1] and Essaky *et al* [14].

Recently, Backward stochastic nonlinear Volterra integral equations (BSNVIEs in short) have been studied by Lin [27] under global Lipschitz condition on the drift. His work is a continuation of a previous one of Hu and Peng [18] where backward semi-linear stochastic evolution equations with values in a complete separable Hilbert space have been considered. More precisely, Lin [27] gives an existence and uniqueness result for the following nonlinear BSDE of Volterra type:

$$Y(t) + \int_t^T f(t, s, Y(s), Z(t, s))ds + \int_t^T [g(t, s, Y(s)) + Z(t, s)] dW(s) = \xi. \quad (1.1)$$

On the other hand, ordinary stochastic Volterra integral equations have been investigated by Berger and Mizel [5, 6], Pardoux and Protter [25], Protter [26], Kolodh [19] and have found applications in mathematical finance (see [9], [13]).

In this paper, we are concerned with equation (1.1) and our aim is to weaken the global Lipschitz condition on the drift to a local one. The paper is organized as follows. In Section 2, we give essential notions on backward stochastic nonlinear Volterra equations and Section 3 deals with the main result. Finally, Section 4 is devoted to a stability result.

2 Assumptions and Formulation of the problem

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions and $\{W(t), t \in [0, T]\}$ the d -dimensional standard Brownian motion defined on it. Define

$\mathcal{D} = \{(t, s) \in \mathbb{R}_+^2; 0 \leq t \leq s \leq T\}$ and denote by \mathcal{P} the σ -algebra of $\mathcal{F}_{t \vee s}$ -progressively measurable subsets of $\Omega \times \mathcal{D}$.

Let $M^2(t, T; \mathbb{R}^k)$ (resp. $M^2(\mathcal{D}; \mathbb{R}^{k \times d})$) be the set of \mathbb{R}^k -valued (resp. $\mathbb{R}^{k \times d}$ -valued), $\mathcal{F}_{t \vee s}$ -progressively measurable processes which are square integrable with respect to $\mathbb{P} \otimes \lambda \otimes \lambda$ (here λ denotes Lebesgue measure over $[0, T]$). For $X \in \mathbb{R}^k$, $|X|$ will denote its Euclidean norm. An element $Y \in \mathbb{R}^{k \times d}$ will be considered as a $k \times d$ matrix; its Euclidean norm is given by $|Y| = \sqrt{Tr(YY^*)}$ and $\langle Y, Z \rangle = Tr(YZ^*)$.

Moreover, we are given the following objects and assumptions:

- **(A1)** $f : \Omega \times \mathcal{D} \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \longrightarrow \mathbb{R}^k$ is a $\mathcal{P} \otimes \mathcal{B}_k \otimes \mathcal{B}_{k \times d} / \mathcal{B}_k$ measurable functions satisfying

$$\left\{ \begin{array}{l} (i) f(., ., 0, 0) \in M^2(\mathcal{D}; \mathbb{R}^k), \\ (ii) \text{ there exist two constants } K > 0 \text{ (} K \text{ sufficiently large) and } 0 \leq \alpha < 1 \text{ such that} \\ |f(t, s, y, z)| \leq K(1 + |y| + |z|)^\alpha, \mathbb{P} - a.s., a.e (t, s) \in \mathcal{D}, \\ (iii) \text{ for every } N \in \mathbb{N}, \text{ there exists a constant } L_N > 0 \text{ such that,} \\ |f(t, s, y, z) - f(t, s, y', z')| \leq L_N|y - y'| + K|z - z'|, \forall |y| \leq N, |y'| \leq N, \\ z \in \mathbb{R}^{k \times d}, z' \in \mathbb{R}^{k \times d}, \text{ where } K \text{ is the constant in (A1-ii).} \end{array} \right.$$
- **(A2)** $g : \Omega \times \mathcal{D} \times \mathbb{R}^k \longrightarrow \mathbb{R}^{k \times d}$ is a $\mathcal{P} \otimes \mathcal{B}_k / \mathcal{B}_{k \times d}$ -measurable function which satisfies

$$\left\{ \begin{array}{l} (i) g(., ., 0) \in M^2(\mathcal{D}; \mathbb{R}^{k \times d}) \\ (ii) |g(t, s, y) - g(t, s, y')| \leq K|y - y'|, \forall y, y' \in \mathbb{R}^k, \\ \text{where } K \text{ is the constant in (A1-ii).} \end{array} \right.$$
- **(A3)** ξ is a square integrable k -dimensional \mathcal{F}_T -measurable random vector.

Remark 2.1 Note that **(A1)** and **(A2)** imply $f(., ., Y(\cdot), Z(., \cdot)) \in M^2(\mathcal{D}; \mathbb{R}^k)$ and $g(., ., Y(\cdot)) \in M^2(\mathcal{D}; \mathbb{R}^{k \times d})$, whenever $Y \in M^2(t, T; \mathbb{R}^k)$, $Z \in M^2(\mathcal{D}; \mathbb{R}^{k \times d})$.

Definition 2.2 A solution to BSDE of Volterra type with data (ξ, f, g) is a pair of $\mathcal{F}_{t \vee s}$ -adapted processes $\{(Y(s), Z(t, s)); (t, s) \in \mathcal{D}\}$ with values in $M^2(t, T; \mathbb{R}^k) \times M^2(\mathcal{D}; \mathbb{R}^{k \times d})$ which solves (1.1).

3 Existence and uniqueness

Before stating the main result, let us give some preliminaries.

Lemma 3.1 Let f denotes a process satisfying assumption **(A1)**. Then, there exists a sequence of processes $(f_n)_{n \geq 1}$ such that for every $n \geq 1$, f_n is $\mathcal{P} \otimes \mathcal{B}_k \otimes \mathcal{B}_{k \times d} / \mathcal{B}_k$ -measurable, Lipschitzian, satisfies **(A1i)**, **(A1ii)** and $\rho_N(f_n - f) \rightarrow 0$ as $n \rightarrow +\infty$ for every fixed N , where

$$\rho_N(f) = \mathbb{E} \left(\int_{\mathcal{D}} \sup_{|y| \leq N, z \in \mathbb{R}^{k \times d}} |f(t, s, y, z)|^2 d\lambda(t, s) \right)^{\frac{1}{2}}.$$

Proof. Let ψ_n be a sequence of smooth functions with support in the ball $B(0, n+1)$ such that $\psi_n = 1$ in the ball $B(0, n)$ and $\sup \psi_n = 1$. One can show easily that the sequence $(f_n)_{n \geq 1}$ of truncated functions defined by $f_n = f\psi_n$, satisfies all the properties quoted above. ■

Let $(f_n)_{n \geq 1}$ associated with f by Lemma 3.1. Thanks to Lin [27], for every $n \geq 1$, there exists a unique couple of processes $\{(Y_n(s), Z_n(t, s)) : (t, s) \in \mathcal{D}\}$ element of $M^2(t, T; \mathbb{R}^k) \times M^2(\mathcal{D}; \mathbb{R}^{k \times d})$ solution to the BSDE of Volterra type with data (ξ, f_n, g) . We build the unique solution to equation (1.1) by studying convergence of the sequence $\{(Y_n(s), Z_n(t, s)) : (t, s) \in \mathcal{D}\}$.

Lemma 3.2 *Assume (A1) – (A3). Then there exists a constant $C > 0$, depending only on T , K and ξ such that for every $n \geq 1$*

$$\mathbb{E} \int_t^T |Y_n(s)|^2 ds + \mathbb{E} \int_t^T ds \int_s^T |Z_n(s, u)|^2 du \leq C, \quad \forall t \in [T - \eta, T]$$

where $\eta < \frac{1}{24K^2}$.

Proof. Since $\{(Y_n(s), Z_n(t, s)) : (t, s) \in \mathcal{D}\}$ is the unique solution to the BSDE of Volterra type with data (ξ, f_n, g) , we have

$$Y_n(t) + \int_t^T f_n(t, s, Y_n(s), Z(t, s)) ds + \int_t^T [g(t, s, Y_n(s)) + Z_n(t, s)] dW(s) = \xi. \quad (3.1)$$

Let $\mathcal{D}_\eta = \{(t, s) : T - \eta \leq t \leq s \leq T\}$ where η will be precised later. In view of Lemma 2.1 of [27], for every $(t, s) \in \mathcal{D}_\eta$, we have

$$\begin{aligned} & \mathbb{E}|Y_n(s)|^2 + \mathbb{E} \int_s^T |Z_n(s, u)|^2 du \\ &= \mathbb{E}|\xi|^2 - 2\mathbb{E} \int_s^T \langle f_n(s, u, Y_n(u), Z_n(s, u)), Y_n(u) \rangle du \\ & \quad - 2\mathbb{E} \int_s^T \langle f_n(s, u, Y_n(u), Z_n(s, u)), A_n(s, u) \rangle du \\ & \quad - 2\mathbb{E} \int_s^T \langle g(s, u, Y_n(u)), Z_n(s, u) \rangle du - \mathbb{E} \int_s^T |g(s, u, Y_n(u))|^2 du \\ &\leq \mathbb{E}|\xi|^2 + 2\mathbb{E} \int_s^T |\langle f_n(s, u, Y_n(u), Z_n(s, u)), Y_n(u) \rangle| du \\ & \quad + 2\mathbb{E} \int_s^T |\langle f_n(s, u, Y_n(u), Z_n(s, u)), A_n(s, u) \rangle| du \\ & \quad + 2\mathbb{E} \int_s^T |\langle g(s, u, Y_n(u)), Z_n(s, u) \rangle| du, \end{aligned} \quad (3.2)$$

where

$$A_n(s, u) = \int_u^T (f_n(u, v, Y_n(v), Z_n(u, v)) - f_n(s, v, Y_n(v), Z_n(s, v))) dv.$$

In view of assumptions **(A1)**, **(A2)** on f_n and g , Young inequality $2ab \leq \beta a^2 + \frac{b^2}{\beta}$ for every $\beta > 0$, we derive the following inequalities. Indeed,

$$\begin{aligned}
& 2|\langle f_n(s, u, Y_n(u), Z_n(s, u)), Y_n(u) \rangle| \\
& \leq 2|f_n(s, u, Y_n(u), Z_n(s, u))||Y_n(u)| \\
& \leq \frac{1}{\beta_1}|f_n(s, u, Y_n(u), Z_n(s, u))|^2 + \beta_1|Y_n(u)|^2 \\
& \leq \frac{3K^2}{\beta_1}(1 + |Y_n(u)|^2 + |Z_n(s, u)|^2) + \beta_1|Y_n(u)|^2 \\
& \leq (\beta_1 + \frac{3K^2}{\beta_1})|Y_n(u)|^2 + \frac{3K^2}{\beta_1}|Z_n(s, u)|^2 + \frac{3K^2}{\beta_1}.
\end{aligned} \tag{3.3}$$

Since

$$\begin{aligned}
& 2|\langle f_n(s, u, Y_n(u), Z_n(s, u)), A_n(s, u) \rangle| \\
& \leq \frac{1}{\beta_2}|f_n(s, u, Y_n(u), Z_n(s, u))|^2 + \beta_2|A_n(s, u)|^2 \\
& \leq \frac{3K^2}{\beta_2}(1 + |Y_n(u)|^2 + |Z_n(s, u)|^2) + \beta_2|A_n(s, u)|^2,
\end{aligned}$$

and

$$\begin{aligned}
\beta_2|A_n(s, u)|^2 & \leq 2\beta_2(T - u) \int_u^T |f_n(u, v, Y_n(v), Z_n(u, v))|^2 dv \\
& \quad + 2\beta_2(T - u) \int_u^T |f_n(u, v, Y_n(v), Z_n(s, v))|^2 dv \\
& \leq 6\beta_2(T - u)K^2 \int_u^T (1 + |Y_n(v)|^2 + |Z_n(u, v)|^2) dv \\
& \quad + 6\beta_2(T - u)K^2 \int_u^T (1 + |Y_n(v)|^2 + |Z_n(s, v)|^2) dv,
\end{aligned}$$

we have

$$\begin{aligned}
& 2|\langle f_n(s, u, Y_n(u), Z_n(s, u)), A_n(s, u) \rangle| \\
& \leq \frac{3K^2}{\beta_2}(1 + |Y_n(u)|^2 + |Z_n(s, u)|^2) \\
& \quad + 12\beta_2(T - u)^2K^2 + 12\beta_2(T - u)K^2 \int_u^T |Y_n(v)|^2 dv \\
& \quad + 6\beta_2(T - u)K^2 \int_u^T (|Z_n(u, v)|^2 + |Z_n(s, v)|^2) dv.
\end{aligned} \tag{3.4}$$

Now,

$$\begin{aligned}
2\langle g(s, u, Y_n(u)), Z_n(s, u) \rangle &\leq 2\beta_3 K^2 |Y_n(u)|^2 + \frac{1}{\beta_3} |Z_n(s, u)|^2 \\
&\quad + 2\beta_3 |g(s, u, 0)|^2.
\end{aligned} \tag{3.5}$$

Combining (3.2)-(3.5), we get

$$\begin{aligned}
&\mathbb{E}|Y_n(s)|^2 + \mathbb{E} \int_s^T |Z_n(s, u)|^2 du \\
\leq &\mathbb{E}|\xi|^2 + (\beta_1 + \frac{3K^2}{\beta_1} + \frac{3K^2}{\beta_2} + 2\beta_3 K^2) \mathbb{E} \int_s^T |Y_n(u)|^2 du \\
&+ 12\beta_2 K^2 \mathbb{E} \int_s^T (T-u) du \int_u^T |Y_n(v)|^2 dv \\
&+ 6\beta_2 K^2 \mathbb{E} \int_s^T (T-u) du \int_u^T (|Z_n(u, v)|^2 + |Z_n(s, v)|^2) dv \\
&+ (\frac{3K^2}{\beta_1} + \frac{3K^2}{\beta_2} + \frac{1}{\beta_3}) \mathbb{E} \int_s^T |Z_n(s, u)|^2 du \\
&+ 12\beta_2 K^2 \mathbb{E} \int_s^T (T-u)^2 du + \left(\frac{3K^2}{\beta_1} + \frac{3K^2}{\beta_2} \right) (T-s) + 2\beta_3 \mathbb{E} \int_s^T |g(s, u, 0)|^2 du.
\end{aligned} \tag{3.6}$$

Moreover, it is not difficult to show that for every process $\{h(s) : s \in [0, T]\}$, we have

$$\mathbb{E} \int_s^T (T-u) du \int_u^T |h(v)|^2 dv \leq \frac{1}{2} (T-s)^2 \mathbb{E} \int_s^T |h(u)|^2 du. \tag{3.7}$$

So, by integrating (3.6) from t to T , we have

$$\begin{aligned}
&\mathbb{E} \int_t^T |Y_n(s)|^2 ds + \int_t^T ds \mathbb{E} \int_s^T |Z_n(s, u)|^2 du \\
\leq &T \mathbb{E}|\xi|^2 + (\beta_1 + \frac{3K^2}{\beta_1} + \frac{3K^2}{\beta_2} + 2\beta_3 K^2 + 6\beta_2 K^2 \eta^2) \int_t^T ds \mathbb{E} \int_s^T |Y_n(u)|^2 du \\
&+ (\frac{3K^2}{\beta_1} + \frac{3K^2}{\beta_2} + \frac{1}{\beta_3} + 3\beta_2 K^2 \eta^2) \int_t^T ds \mathbb{E} \int_s^T |Z_n(s, u)|^2 du \\
&+ 6\beta_2 K^2 T \int_t^T ds \int_s^T du \mathbb{E} \int_u^T |Z_n(u, v)|^2 dv + \beta_2 K^2 T^4 + \left(\frac{3K^2}{\beta_1} + \frac{3K^2}{\beta_2} \right) T^2 \\
&+ 2\beta_3 \mathbb{E} \int_t^T ds \int_s^T |g(s, u, 0)|^2 du.
\end{aligned}$$

Let us put

$$U_n(t) = \mathbb{E} \int_t^T |Y_n(s)|^2 ds \quad \text{and} \quad V_n(t) = \mathbb{E} \int_t^T |Z_n(t, s)|^2 ds.$$

By choosing $\beta_1 = \beta_2 = 24K^2$, $\beta_3 = 8$ and $\eta < \frac{1}{24K^2}$, we deduce that there exist K_1 and K_2 depending only on ξ, T and K such that

$$U_n(t) + \frac{1}{2} \int_t^T V_n(s) ds \leq K_1 \int_t^T U_n(s) ds + K_2 \left(1 + \int_t^T ds \int_s^T V_n(u) du \right). \quad (3.8)$$

From now on let $C = C(K, T, \xi)$ be a constant depending only on K, T, ξ which may vary from line to line. By virtue of (3.8), we have

$$-\frac{d}{dt} \left(e^{K_1 t} \tilde{U}_n(t) \right) + \frac{1}{2} e^{K_1 t} \tilde{V}_n(t) \leq C \left(1 + \int_t^T e^{K_1 s} \tilde{V}_n(s) ds \right) \quad (3.9)$$

where

$$\tilde{U}_n(t) = \int_t^T U_n(s) ds \text{ and } \tilde{V}_n(t) = \int_t^T V_n(s) ds.$$

Integrating (3.9) from t to T , we obtain

$$\tilde{U}_n(t) e^{K_1 t} + \frac{1}{2} \int_t^T e^{K_1 s} \tilde{V}_n(s) ds \leq C \left(1 + \int_t^T ds \int_s^T e^{K_1 r} \tilde{V}_n(r) dr \right).$$

So, in view of Gronwall inequality we get that for every $n \geq 1$, $t \in [T - \eta, T]$

$$\int_t^T \tilde{V}_n(s) ds \leq C \text{ and } \tilde{U}_n(t) \leq C. \quad (3.10)$$

Putting (3.10) in (3.8), it follows again from Gronwall inequality that there exists a constant $C = C(\xi, T, K)$ such that for every $n \geq 1$, $t \in [T - \eta, T]$

$$\mathbb{E} \int_t^T |Y_n(s)|^2 ds + \mathbb{E} \int_t^T ds \int_s^T |Z_n(s, u)|^2 du \leq C.$$

■

Theorem 3.3 *Assume (A1) – (A3). If*

$$\lim_{N \rightarrow +\infty} \frac{1}{(2L_N + 2L_N^2)N^{2(1-\alpha)}} \exp[(2L_N + 2L_N^2)T] = 0 \quad (\mathbf{A})$$

then there is a unique process $\{(Y(s), Z(t, s)) : (t, s) \in \mathcal{D}\}$ with values in $M^2(t, T; \mathbb{R}^k) \times M^2(\mathcal{D}; \mathbb{R}^{k \times d})$ solution of equation (1.1).

Before the proof of Theorem 3.3, let us make the following

Remark 3.4 *The condition (A) is fulfilled if there exists $L \geq 0$, such that $(2L_N + 2L_N^2)T \leq L + (1 - \alpha) \log N$.*

Proof of Theorem 3.3.

• **Uniqueness**

Let $\{(Y(s), Z(t, s)) : (t, s) \in \mathcal{D}\}$ $\{(Y'(s), Z'(t, s)) : (t, s) \in \mathcal{D}\}$ be two solutions of equation (1.1).

Define $\Delta Y(t) = Y(s) - Y'(s)$, $\Delta Z(t, s) = Z(t, s) - Z'(t, s)$, $\Delta f(t, s) = f(t, s, Y(s), Z(t, s)) - f(t, s, Y'(s), Z'(t, s))$, $\Delta g(t, s) = g(t, s, Y(s)) - g(t, s, Y'(s))$. For every $N \geq 1$, we set

$$A^N = \{(\omega, t, s) \in \Omega \times \mathcal{D}_\eta, |Y(s)| + |Z(t, s)| + |Y'(s)| + |Z'(t, s)| \geq N\}$$

and $\bar{A}^N = (\Omega \times \mathcal{D}_\eta) \setminus A^N$.

In the sequel C is a positive constant depending only on K, T , and ξ which may vary from line to line.

We have

$$\Delta Y(s) + \int_s^T \Delta f(s, u) du + \int_s^T [\Delta g(s, u) + \Delta Z(s, u)] dW_u = 0.$$

Therefore, Lemma 2.1. in [27] yields

$$\begin{aligned} & \mathbb{E}|\Delta Y(s)|^2 + \mathbb{E} \int_s^T |\Delta Z(s, u)|^2 du \\ &= -2\mathbb{E} \int_s^T \langle \Delta f(s, u), \Delta Y(u) \rangle du - 2\mathbb{E} \int_s^T \langle \Delta f(s, u), A(s, u) \rangle du \\ & \quad - 2\mathbb{E} \int_s^T \langle \Delta g(s, u), \Delta Z(s, u) \rangle du - \mathbb{E} \int_s^T |\Delta g(s, u)|^2 du \\ &\leq 2\mathbb{E} \int_s^T |\Delta f(s, u)| |\Delta Y(u)| (\mathbf{1}_{A^N}(s, u) + \mathbf{1}_{\bar{A}^N}(s, u)) du \\ & \quad + 2\mathbb{E} \int_s^T |\Delta f(s, u)| |A(s, u)| (\mathbf{1}_{A^N}(s, u) + \mathbf{1}_{\bar{A}^N}(s, u)) du \\ & \quad + 2\mathbb{E} \int_s^T |\Delta g(s, u)| |\Delta Z(s, u)| du \\ &= J_1 + J_2 + J_3 + J_4 + J_5, \end{aligned} \tag{3.11}$$

where

$$A(s, u) = \int_u^T (\Delta f(u, v) - \Delta f(s, v)) dv.$$

In view of assumptions **(A1)**-**(A3)**, Hölder inequality and Young inequality, we derive the following inequalities.

$$\begin{aligned}
J_1 &= 2\mathbb{E} \int_s^T |\Delta f(s, u)| |\Delta Y(u)| \mathbf{1}_{A^N}(s, u) du \\
&\leq \mathbb{E} \int_s^T |\Delta Y(u)|^2 du + \mathbb{E} \int_s^T |\Delta f(s, u)|^2 \mathbf{1}_{A^N}(s, u) du \\
&\leq \mathbb{E} \int_s^T |\Delta Y(u)|^2 \mathbf{1} du + 4K^2 \mathbb{E} \int_s^T (1 + |Y(u)| + |Z(s, u)| + |Y'(u)| + |Z'(s, u)|)^{2\alpha} \mathbf{1}_{A^N}(s, u) du.
\end{aligned}$$

By virtue of Hölder inequality and Chebychev inequality, we deduce that

$$J_1 \leq \mathbb{E} \int_s^T |\Delta Y(u)|^2 du + \frac{C}{N^{2(1-\alpha)}}. \quad (3.12)$$

$$\begin{aligned}
J_2 &= 2\mathbb{E} \int_s^T |\Delta f(s, u)| |\Delta Y(u)| \mathbf{1}_{\bar{A}^N}(s, u) du \\
&\leq 2\mathbb{E} \int_s^T (L_N |\Delta Y(u)|^2 + K |\Delta Z(s, u)|) |\Delta Y(u)| \mathbf{1}_{\bar{A}^N}(s, u) du \\
&\leq (2L_N + \beta_1) \int_s^T |\Delta Y(u)|^2 du + \frac{K^2}{\beta_1} \mathbb{E} \int_s^T |\Delta Z(s, u)|^2 du.
\end{aligned} \quad (3.13)$$

$$\begin{aligned}
J_3 &= 2\mathbb{E} \int_s^T |\Delta f(s, u)| |A(s, u)| \mathbf{1}_{A^N}(s, u) du \\
&\leq \mathbb{E} \int_s^T |\Delta f(s, u)|^2 \mathbf{1}_{A^N}(s, u) du + \mathbb{E} \int_s^T |A(s, u)|^2 du \\
&= I_1 + I_2.
\end{aligned}$$

We have

$$\begin{aligned}
I_1 &\leq 4K^2 \mathbb{E} \int_s^T (1 + |Y(u)| + |Z(s, u)| + |Y'(u)| + |Z'(s, u)|)^{2\alpha} \mathbf{1}_{A^N}(s, u) du \\
&\leq \frac{C}{N^{2(1-\alpha)}}
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= \mathbb{E} \int_s^T |A(s, u)|^2 du \\
&\leq 2\mathbb{E} \int_s^T (T - u) du \int_u^T |\Delta f(u, v)|^2 dv \\
&\quad + 2\mathbb{E} \int_s^T (T - u) du \int_u^T |\Delta f(s, v)|^2 dv.
\end{aligned}$$

Let $\eta < \frac{1}{24K^2}$, in view of (3.7), for $(s, u) \in \mathcal{D}_\eta$, we have

$$\begin{aligned}
I_2 &\leq \eta^2 \mathbb{E} \int_s^T |\Delta f(s, u)|^2 (\mathbf{1}_{A^N}(s, u) + \mathbf{1}_{\bar{A}^N}(s, u)) du \\
&\quad + 2\mathbb{E} \int_s^T (T-u) du \int_u^T |\Delta f(u, v)|^2 (\mathbf{1}_{A^N}(u, v) + \mathbf{1}_{\bar{A}^N}(u, v)) dv \\
&\leq 4\eta^2 K^2 \mathbb{E} \int_s^T (1 + |Y(u)| + |Z(s, u)| + |Y'(u)| + |Z'(s, u)|)^{2\alpha} \mathbf{1}_{A^N}(s, u) du \\
&\quad + 4\eta^2 L_N^2 \mathbb{E} \int_s^T |\Delta Y(u)|^2 du + 4\eta^2 K^2 \mathbb{E} \int_s^T |\Delta Z(s, u)|^2 du \\
&\quad + 8K^2 \mathbb{E} \int_s^T (T-u) du \int_u^T (1 + |Y(v)| + |Z(u, v)| + |Y'(v)| + |Z'(u, v)|)^{2\alpha} \mathbf{1}_{A^N}(u, v) dv \\
&\quad + 4K^2 T \mathbb{E} \int_s^T du \int_u^T |\Delta Z(u, v)|^2 dv. \tag{3.14}
\end{aligned}$$

By virtue of Hölder inequality and Chebychev inequality, we deduce that

$$\begin{aligned}
J_3 &\leq 4L_N^2 \eta^2 \mathbb{E} \int_s^T |\Delta Y(u)|^2 du + 4K^2 \eta^2 \mathbb{E} \int_s^T |\Delta Z(s, u)|^2 du \\
&\quad + \frac{(1 + \eta^2) C}{N^{2(1-\alpha)}} + 4K^2 T \mathbb{E} \int_s^T du \int_u^T |\Delta Z(u, v)|^2 dv \tag{3.15}
\end{aligned}$$

$$\begin{aligned}
J_4 &= 2\mathbb{E} \int_s^T |\Delta f(s, u)| |A(s, u)| \mathbf{1}_{\bar{A}^N}(s, u) du \\
&\leq 2\mathbb{E} \int_s^T (L_N |\Delta Y(u)| + K |\Delta Z(s, u)|) |A(s, u)| \mathbf{1}_{\bar{A}^N}(s, u) du \\
&\leq L_N^2 \mathbb{E} \int_s^T |\Delta Y(u)|^2 du + (\beta_2 + 1) \mathbb{E} \int_s^T |A(s, u)|^2 du + \frac{K^2}{\beta_2} \mathbb{E} \int_s^T |\Delta Z(s, u)|^2 du.
\end{aligned}$$

Therefore, by (3.14) we have

$$\begin{aligned}
J_4 &\leq [4(\beta_2 + 1)\eta^2 + 1] L_N^2 \mathbb{E} \int_s^T |\Delta Y(u)|^2 du \\
&\quad + \left[4(\beta_2 + 1)\eta^2 + \frac{1}{\beta_2} \right] K^2 \mathbb{E} \int_s^T |\Delta Z(s, u)|^2 du \\
&\quad + 4(\beta_2 + 1) K^2 T \mathbb{E} \int_s^T \left(\int_u^T |\Delta Z(u, v)|^2 dv \right) du \\
&\quad + (\beta_2 + 1) \eta^2 \frac{C}{N^{2(1-\alpha)}}. \tag{3.16}
\end{aligned}$$

$$\begin{aligned}
J_5 &= 2\mathbb{E} \int_s^T |\Delta g(s, u)| |\Delta Z(s, u)| du \\
&\leq \beta_3 \mathbb{E} \int_s^T |\Delta g(s, u)|^2 du + \frac{1}{\beta_3} \mathbb{E} \int_s^T |\Delta Z(s, u)|^2 du \\
&\leq \beta_3 \mathbb{E} \int_s^T |\Delta Y(u)|^2 du + \frac{K^2}{\beta_3} \mathbb{E} \int_s^T |\Delta Z(s, u)|^2 du.
\end{aligned} \tag{3.17}$$

By combining (3.11)-(3.17) and integrating from t to T , we obtain

$$\begin{aligned}
&\mathbb{E} \int_t^T |\Delta Y(s)|^2 ds + \int_t^T ds \mathbb{E} \int_s^T |\Delta Z(s, u)|^2 du \\
&\leq (1 + 2L_N + \beta_1 + [4(\beta_2 + 2)\eta^2 + 1]L_N^2 + \beta_3) \mathbb{E} \int_t^T ds \int_s^T |\Delta Y(u)|^2 du \\
&\quad + \left(\frac{1}{\beta_1} + \frac{1}{\beta_2} + \frac{1}{\beta_3} + 4(\beta_2 + 2)\eta^2\right) K^2 \mathbb{E} \int_t^T ds \int_s^T |\Delta Z(s, u)|^2 du \\
&\quad + [2 + (\beta_2 + 2)\eta^2] \frac{C}{N^{2(1-\alpha)}} + 4(\beta_2 + 2)K^2 T \mathbb{E} \int_t^T ds \int_s^T du \int_u^T |\Delta Z(u, v)|^2 dv.
\end{aligned}$$

Let us choose $\beta_1 = \beta_2 = \beta_3 = 8K^2$ and put

$$U(t) = \mathbb{E} \int_t^T |\Delta Y(s)|^2 ds \text{ and } V(t) = \mathbb{E} \int_t^T |\Delta Z(t, s)|^2 ds.$$

Then we have

$$U(t) + \frac{1}{2} \int_t^T V(s) ds \leq K_1 \int_t^T U(s) ds + \frac{C}{N^{2(1-\alpha)}} + K_2 \int_t^T ds \int_s^T V(u) du \tag{3.18}$$

where $K_1 = 1 + 16K^2 + 2L_N + 2L_N^2$, $K_2 = 4(8K^2 + 2)K^2 T$. It follows that

$$-\frac{d}{dt}(e^{K_1 t} \tilde{U}(t)) + \frac{1}{2} e^{K_1 t} \tilde{V}(t) \leq K_2 \int_t^T e^{K_1 s} \tilde{V}(s) ds + \frac{C}{N^{2(1-\alpha)}} e^{K_1 t} \tag{3.19}$$

where

$$\tilde{U}(t) = \int_t^T U(s) ds \quad \text{and} \quad \tilde{V}(t) = \int_t^T V(s) ds.$$

Integrating (3.19) from t to T , we get

$$e^{K_1 t} \tilde{U}(t) + \frac{1}{2} \int_t^T e^{K_1 s} \tilde{V}(s) ds \leq K_2 \int_t^T ds \int_s^T e^{K_1 r} \tilde{V}(u) du + \frac{C e^{K_1 T}}{K_1 N^{2(1-\alpha)}}. \tag{3.20}$$

Therefore, Gronwall inequality implies that for $t \in [T - \eta, T]$

$$\int_t^T e^{K_1 s} \tilde{V}(s) ds \leq \frac{C}{(2L_N + 2L_N^2)N^{2(1-\alpha)}} \exp[(2L_N + 2L_N^2)T].$$

Passing to the limit on N , we deduce that for each $t \in [T - \eta, T]$ we have $\tilde{V}(t) = 0$ and $\tilde{U}(t) = 0$. Therefore, $Y(s) = Y'(s)$ and $Z(t, s) = Z'(t, s)$ for a.e. $(t, s) \in [T - \eta, T] \times [t, T]$.

When $t \in [T - 2\eta, T - \eta]$, we have

$$\Delta Y(s) + \int_s^{T-\eta} \Delta f(s, u) du + \int_s^{T-\eta} [\Delta g(s, u) + \Delta Z(s, u)] dW_u = 0.$$

Use above same procedure and we can deduce that for a.e. $(t, s) \in [T - 2\eta, T - \eta] \times [t, T]$, $Y(s) = Y'(s)$ and $Z(t, s) = Z'(t, s)$ a.s. hence, we can prove uniqueness of (1.1).

• **Existence**

For every $n, m \in \mathbb{N}^*$ and $(t, s) \in \mathcal{D}_\eta$, let us set

$$\begin{aligned} A_{m,n}^N &= \{(\omega, t, s) \in \Omega \times \mathcal{D}_\eta, |Y_n(s)| + |Z_n(t, s)| + |Y_m(s)| + |Z_m(t, s)| \geq N\} \\ \text{and } \bar{A}_{m,n}^N &= (\Omega \times \mathcal{D}_\eta) \setminus A_{m,n}^N. \end{aligned}$$

We have

$$\begin{aligned} & \mathbb{E}|Y_n(t) - Y_m(t)|^2 + \mathbb{E} \int_t^T |Z_n(t, s) - Z_m(t, s)|^2 ds \\ &= 2\mathbb{E} \int_t^T \langle f_n(t, s, Y_n(s), Z_n(t, s)) - f_m(t, s, Y_m(s), Z_m(t, s)), Y_n(s) - Y_m(s) \rangle ds \\ & \quad - 2\mathbb{E} \int_t^T \langle f_n(t, s, Y_n(s), Z_n(t, s)) - f_m(t, s, Y_m(s), Z_m(t, s)), B_{m,n}(t, s) \rangle ds \\ & \quad - 2\mathbb{E} \int_t^T \langle g(t, s, Y_n(s)) - g(t, s, Y_m(s)), Z_n(t, s) - Z_m(t, s) \rangle ds \\ & \quad - \mathbb{E} \int_s^T |g(t, s, Y_n(s)) - g(t, s, Y_m(s))|^2 ds \\ &\leq 2\mathbb{E} \int_t^T |f_n(t, s, Y_n(s), Z_n(t, s)) - f_m(t, s, Y_m(s), Z_m(t, s))| |Y_n(s) - Y_m(s)| \\ & \quad \times (\mathbf{1}_{A_{m,n}^N}(t, s) + \mathbf{1}_{\bar{A}_{m,n}^N}(t, s)) ds \\ & \quad + 2\mathbb{E} \int_s^T |f_n(t, s, Y_n(s), Z_n(t, s)) - f_m(t, s, Y_m(s), Z_m(t, s))| |B_{m,n}(t, s)| \\ & \quad \times (\mathbf{1}_{A_{m,n}^N}(t, s) + \mathbf{1}_{\bar{A}_{m,n}^N}(t, s)) ds \\ & \quad + 2\mathbb{E} \int_s^T |g(t, s, Y_n(s)) - g(t, s, Y_m(s))| |Z_n(t, s) - Z_m(t, s)| ds \\ &= J_1 + J_2 + J_3 + J_4 + J_5 \end{aligned} \tag{3.21}$$

where

$$\begin{aligned}
B_{m,n}(t, s) &= \int_s^T (f_n(s, u, Y_n(u), Z_n(s, u)) - f_m(s, u, Y_m(u), Z_m(s, u))) du \\
&\quad - \int_s^T (f_n(t, u, Y_n(u), Z_n(t, u)) - f_m(t, u, Y_m(u), Z_m(t, u))) du.
\end{aligned}$$

In view of Lemma 3.2 and using the same calculations in its proof, we have

$$\begin{aligned}
J_1 &= 2\mathbb{E} \int_t^T |f_n(t, s, Y_n(s), Z_n(t, s)) - f_m(t, s, Y_m(s), Z_m(t, s))| \\
&\quad \times |Y_n(s) - Y_m(s)| \mathbf{1}_{A_{m,n}^N}(t, s) ds \\
&\leq \mathbb{E} \int_t^T |Y_n(s) - Y_m(s)|^2 ds + \frac{C}{N^{2(1-\alpha)}}. \tag{3.22}
\end{aligned}$$

$$\begin{aligned}
J_2 &= 2\mathbb{E} \int_s^T |f_n(t, s, Y_n(s), Z_n(t, s)) - f_m(t, s, Y_m(s), Z_m(t, s))| \\
&\quad \times |Y_n(s) - Y_m(s)| \mathbf{1}_{A_{m,n}^N}(t, s) ds \\
&\leq 2\mathbb{E} \int_t^T |f_n(t, s, Y_n(s), Z_n(t, s)) - f(t, s, Y_n(s), Z_n(t, s))| \\
&\quad \times |Y_n(s) - Y_m(s)| \mathbf{1}_{A_{m,n}^N}(t, s) ds \\
&\quad + 2\mathbb{E} \int_t^T |f(t, s, Y_n(s), Z_n(t, s)) - f(t, s, Y_m(s), Z_m(t, s))| \\
&\quad \times |Y_n(s) - Y_m(s)| \mathbf{1}_{A_{m,n}^N}(t, s) ds \\
&\quad + 2\mathbb{E} \int_t^T |f_m(t, s, Y_m(s), Z_m(t, s)) - f(t, s, Y_m(s), Z_m(t, s))| \\
&\quad \times |Y_n(s) - Y_m(s)| \mathbf{1}_{A_{m,n}^N}(t, s) ds \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

Since

$$\begin{aligned}
I_1 &\leq \mathbb{E} \int_t^T |Y_n(s) - Y_m(s)| ds \\
&\quad + \mathbb{E} \int_t^T |(f_n - f)(t, s, Y_n(s), Z_n(t, s))|^2 \mathbf{1}_{A_{m,n}^N}(t, s) ds, \\
I_2 &\leq (2L_N + \beta_1) \mathbb{E} \int_t^T |Y_n(s) - Y_m(s)| ds + \frac{K^2}{\beta_1} \mathbb{E} \int_t^T |Z_n(t, s) - Z_m(t, s)|^2 ds,
\end{aligned}$$

and

$$\begin{aligned}
I_3 &\leq \mathbb{E} \int_t^T |Y_n(s) - Y_m(s)| ds \\
&\quad + \mathbb{E} \int_t^T |(f_m - f)(t, s, Y_m(s), Z_m(t, s))|^2 \mathbf{1}_{\bar{A}_{m,n}^N}(t, s) ds.
\end{aligned}$$

We have

$$\begin{aligned}
J_2 &\leq (2L_N + \beta_1 + 2) \mathbb{E} \int_t^T |Y_n(s) - Y_m(s)| ds \\
&\quad + \mathbb{E} \int_t^T |(f_n - f)(t, s, Y_n(s), Z_n(t, s))|^2 \mathbf{1}_{\bar{A}_{m,n}^N}(t, s) ds \\
&\quad + \mathbb{E} \int_t^T |(f_m - f)(t, s, Y_m(s), Z_m(t, s))|^2 \mathbf{1}_{\bar{A}_{m,n}^N}(t, s) ds \\
&\quad + \frac{K^2}{\beta_1} \mathbb{E} \int_s^T |Z_n(t, s) - Z_m(t, s)|^2 du, \tag{3.23}
\end{aligned}$$

$$\begin{aligned}
J_3 &= 2 \mathbb{E} \int_t^T |f_n(t, s, Y_n(s), Z_n(t, s)) - f_m(t, s, Y_m(s), Z_m(t, s))| |B_{m,n}(t, s)| \mathbf{1}_{A_{m,n}^N}(t, s) ds \\
&\leq \mathbb{E} \int_t^T |f_n(t, s, Y_n(s), Z_n(t, s)) - f_m(t, s, Y_m(s), Z_m(t, s))|^2 \mathbf{1}_{A_{m,n}^N}(t, s) ds \\
&\quad + \mathbb{E} \int_t^T |B_{m,n}(t, s)|^2 ds \\
&= I_4 + I_5. \tag{3.24}
\end{aligned}$$

By virtue of Hölder inequality, Chebychev inequality and lemma 3.2

$$I_4 \leq \frac{C}{N^{2(1-\alpha)}}. \tag{3.25}$$

On the other hand

$$\begin{aligned}
I_5 &\leq 2 \mathbb{E} \int_t^T (T - s) ds \int_s^T |f_n(s, u, Y_n(u), Z_n(s, u)) - f_m(s, u, Y_m(u), Z_m(s, u))|^2 du \\
&\quad + 2 \mathbb{E} \int_t^T (T - s) ds \int_s^T |f_n(t, u, Y_n(u), Z_n(t, u)) - f_m(t, u, Y_m(u), Z_m(t, u))|^2 du.
\end{aligned}$$

By using (3.7) we obtain

$$\begin{aligned}
I_5 &\leq \eta^2 \mathbb{E} \int_t^T |f_n(t, s, Y_n(s), Z_n(t, s)) - f_m(t, s, Y_m(s), Z_m(t, s))|^2 \\
&\quad \times \left(\mathbf{1}_{A_{m,n}^N}(t, s) + \mathbf{1}_{\bar{A}_{m,n}^N}(t, s) \right) ds \\
&\quad + 2\mathbb{E} \int_t^T (T-s) ds \times \\
&\quad \int_s^T |f_n(s, u, Y_n(u), Z_n(s, u)) - f_m(s, u, Y_m(u), Z_m(s, u))|^2 \\
&\quad \times \left(\mathbf{1}_{A_{m,n}^N}(s, u) + \mathbf{1}_{\bar{A}_{m,n}^N}(s, u) \right) du.
\end{aligned}$$

Therefore Hölder inequality, Chebychev inequality and Lemma 3.2 yield

$$\begin{aligned}
I_5 &\leq \eta^2 \frac{C}{N^{2(1-\alpha)}} + 3\eta^2 \mathbb{E} \int_t^T |(f_n - f)(t, s, Y_n(s), Z_n(t, s))|^2 \mathbf{1}_{\bar{A}_{m,n}^N}(t, s) du \\
&\quad + 3\eta^2 \mathbb{E} \int_t^T |(f_m - f)(t, s, Y_m(s), Z_m(t, s))|^2 \mathbf{1}_{\bar{A}_{m,n}^N}(t, s) ds \\
&\quad + 6\mathbb{E} \int_t^T (T-s) ds \int_s^T |(f_n - f)(s, u, Y_n(u), Z_n(s, u))|^2 \mathbf{1}_{\bar{A}_{m,n}^N}(s, u) du \\
&\quad + 6\mathbb{E} \int_t^T (T-s) ds \int_s^T |(f_m - f)(s, u, Y_m(u), Z_m(s, u))|^2 \mathbf{1}_{\bar{A}_{m,n}^N}(s, u) du \\
&\quad + 12\eta^2 L_N^2 \mathbb{E} \int_t^T |Y_n(s) - Y_m(s)|^2 ds + 6\eta^2 K^2 \mathbb{E} \int_t^T |Z_n(t, s) - Z_m(t, s)|^2 ds \\
&\quad + 12K^2 T \mathbb{E} \int_t^T ds \int_s^T |Z_n(s, u) - Z_m(s, u)|^2 du. \tag{3.26}
\end{aligned}$$

$$\begin{aligned}
J_4 &= 2\mathbb{E} \int_t^T |f_n(t, s, Y_n(s), Z_n(t, s)) - f_m(t, s, Y_m(s), Z_m(t, s))| |B_{m,n}(t, s)| \mathbf{1}_{\overline{A}_{m,n}^N}(t, s) ds \\
&\leq 2\mathbb{E} \int_t^T |(f_n - f)(t, s, Y_n(s), Z_n(t, s))| |B_{m,n}(t, s)| \mathbf{1}_{\overline{A}_{m,n}^N}(t, s) ds \\
&\quad + 2\mathbb{E} \int_t^T |f(t, s, Y_n(s), Z(t, s)) - f(t, s, Y_m(s), Z_m(t, s))| |B_{m,n}(t, s)| \mathbf{1}_{\overline{A}_{m,n}^N}(t, s) ds \\
&\quad + 2\mathbb{E} \int_t^T |(f_m - f)(t, s, Y_m(s), Z_m(t, s))| |B_{m,n}(t, s)| \mathbf{1}_{\overline{A}_{m,n}^N}(t, s) ds \\
&\leq \mathbb{E} \int_t^T |(f_n - f)(t, s, Y_n(s), Z_n(t, s))|^2 \mathbf{1}_{\overline{A}_{m,n}^N}(t, s) ds \\
&\quad + \mathbb{E} \int_t^T |(f_m - f)(t, s, Y_m(s), Z_m(t, s))|^2 \mathbf{1}_{\overline{A}_{m,n}^N}(t, s) ds \\
&\quad + L_N^2 \mathbb{E} \int_t^T |Y_n(s) - Y_m(s)|^2 ds + (\beta_2 + 3) \mathbb{E} \int_s^T |B_{m,n}(t, s)|^2 ds \\
&\quad + \frac{K^2}{\beta_2} \mathbb{E} \int_t^T |Z_n(t, s) - Z_m(t, s)|^2 ds
\end{aligned}$$

and by using (3.26) we have

$$\begin{aligned}
J_4 &\leq \eta^2 \frac{C}{N^{2(1-\alpha)}} + [3(\beta_2 + 3)\eta^2 + 1] \mathbb{E} \int_t^T |(f_n - f)(t, s, Y_n(s), Z_n(t, s))|^2 \mathbf{1}_{\overline{A}_{m,n}^N}(t, s) ds \\
&\quad + [3(\beta_2 + 3)\eta^2 + 1] \mathbb{E} \int_t^T |(f_m - f)(t, s, Y_m(s), Z_m(t, s))|^2 \mathbf{1}_{\overline{A}_{m,n}^N}(t, s) ds \\
&\quad + 6(\beta_2 + 3) \mathbb{E} \int_t^T (T - s) du \int_s^T |(f_n - f)(s, u, Y_n(u), Z_n(s, u))|^2 \mathbf{1}_{\overline{A}_{m,n}^N}(s, u) du \\
&\quad + 6(\beta_2 + 3) \mathbb{E} \int_t^T (T - s) du \int_s^T |(f_m - f)(s, u, Y_m(u), Z_m(s, u))|^2 \mathbf{1}_{\overline{A}_{m,n}^N}(s, u) du \\
&\quad + [12(\beta_2 + 3)\eta^2 + 1] L_N^2 \mathbb{E} \int_t^T |Y_n(s) - Y_m(s)|^2 ds \\
&\quad + \left[6(\beta_2 + 3)\eta^2 + \frac{1}{\beta_2} \right] K^2 \mathbb{E} \int_t^T |Z_n(t, s) - Z_m(t, s)|^2 ds \\
&\quad + 12(\beta_2 + 3) K^2 T \mathbb{E} \int_t^T ds \int_s^T |Z_n(s, u) - Z_m(s, u)|^2 du. \tag{3.27}
\end{aligned}$$

We have also

$$J_5 \leq \beta_3 \mathbb{E} \int_t^T |Y_n(s) - Y_m(s)|^2 ds + \frac{K^2}{\beta_3} \mathbb{E} \int_t^T |Z_n(t, s) - Z_m(t, s)|^2 ds. \tag{3.28}$$

So by virtue of (3.21) -(3.28) we deduce that

$$\begin{aligned}
& \mathbb{E}|Y_n(t) - Y_m(t)|^2 + \mathbb{E} \int_t^T |Z_n(t, s) - Z_m(t, s)|^2 ds \\
\leq & (3 + \beta_1 + \beta_3 + 2L_N + [1 + 12(\beta_2 + 4)\eta^2])L_N^2 \mathbb{E} \int_t^T |Y_n(s) - Y_m(s)|^2 ds \\
& + \left(\frac{1}{\beta_1} + \frac{1}{\beta_2} + \frac{1}{\beta_3} + 6(\beta_2 + 4)\eta^2\right)K^2 \mathbb{E} \int_t^T |Z_n(t, s) - Z_m(t, s)|^2 ds \\
& + (2 + 3(\beta_2 + 4)\eta^2) \mathbb{E} \int_t^T |(f_n - f)(t, s, Y_n(s), Z_n(t, s))|^2 \mathbf{1}_{\overline{A}_{m,n}^N}(t, s) ds \\
& + (2 + 3(\beta_2 + 4)\eta^2) \mathbb{E} \int_t^T |(f_m - f)(t, s, Y_m(s), Z_m(t, s))|^2 \mathbf{1}_{\overline{A}_{m,n}^N}(t, s) ds \\
& + 6(\beta_2 + 4) \mathbb{E} \int_t^T (T - s) du \int_s^T |(f_n - f)(s, u, Y_n(u), Z_n(s, u))|^2 \mathbf{1}_{\overline{A}_{m,n}^N}(s, u) du \\
& + 6(\beta_2 + 4) \mathbb{E} \int_t^T (T - s) du \int_s^T |(f_m - f)(s, u, Y_m(u), Z_m(s, u))|^2 \mathbf{1}_{\overline{A}_{m,n}^N}(s, u) du \\
& + 12(\beta_2 + 4)K^2 T \mathbb{E} \int_t^T ds \int_s^T |Z_n(s, u) - Z_m(s, u)|^2 du \\
& + \frac{C}{N^{2(1-\alpha)}}(1 + \eta^2).
\end{aligned}$$

Let us choose $\beta_1 = \beta_2 = \beta_3 = 8K^2$ and define

$$U_{m,n}(t) = \mathbb{E} \int_t^T |Y_n(s) - Y_m(s)|^2 ds, \quad V_{m,n}(t) = \mathbb{E} \int_t^T |Z_n(t, s) - Z_m(t, s)|^2 ds.$$

Then we have

$$\begin{aligned}
& -\frac{d}{dt}(e^{K_1 t} U_{m,n}(t)) + \frac{1}{2} e^{K_1 t} V_{m,n}(t) \\
\leq & 4 \mathbb{E} e^{K_1 t} \int_t^T |(f_n - f)(t, s, Y_n(s), Z_n(t, s))|^2 \mathbf{1}_{\overline{A}_{m,n}^N}(t, s) ds \\
& + 4 \mathbb{E} e^{K_1 t} \int_t^T |(f_m - f)(t, s, Y_m(s), Z_m(t, s))|^2 \mathbf{1}_{\overline{A}_{m,n}^N}(t, s) ds \\
& + 6(8K^2 + 4) \mathbb{E} e^{K_1 t} \int_t^T (T - s) ds \int_s^T |(f_n - f)(s, u, Y_n(u), Z_n(s, u))|^2 \mathbf{1}_{\overline{A}_{m,n}^N}(s, u) du \\
& + 6(8K^2 + 4) \mathbb{E} e^{K_1 t} \int_t^T (T - s) ds \int_s^T |(f_m - f)(s, u, Y_m(u), Z_m(s, u))|^2 \mathbf{1}_{\overline{A}_{m,n}^N}(s, u) du \\
& + 12(8K^2 + 4)K^2 T \mathbb{E} e^{K_1 t} \int_t^T V_{m,n}(s) ds + \frac{C}{N^{2(1-\alpha)}} e^{K_1 t}, \tag{3.29}
\end{aligned}$$

where $K_1 = 3 + 16K^2 + 2L_N + 2L_N^2$ and C are constants depending only on K, T, ξ . Integrating (3.29) from t to T , we obtain

$$\begin{aligned}
& e^{K_1 t} U_{m,n}(t) + \frac{1}{2} \int_t^T e^{K_1 s} V_{m,n}(s) ds \\
& \leq (4 + 3(8K^2 + 4)\eta^2) [\rho_N^2(f_n - f) + \rho_N^2(f_m - f)] e^{K_1 T} \\
& \quad + 12(8K^2 + 4)K^2 T \int_t^T ds \int_s^T e^{K_1 u} V_{m,n}(u) du \\
& \quad + \frac{C}{K_1 N^{2(1-\alpha)}} e^{K_1 T}. \tag{3.30}
\end{aligned}$$

By virtue of Gronwall inequality, we deduce that

$$\int_t^T e^{K_1 s} V_{m,n}(s) ds \leq \left(K_2 [\rho_N^2(f_n - f) + \rho_N^2(f_m - f)] e^{K_1 T} + \frac{C}{K_1 N^{2(1-\alpha)}} e^{K_1 T} \right) \exp(K_3 T),$$

where K_2 and K_3 are constants depending only on K, T, ξ .

With condition **(A)**, passing to the limit successively on N, n and m we have

$$\int_t^T V_{m,n}(s) ds \longrightarrow 0 \tag{3.31}$$

Substituting (3.31) to (3.30) and passing to the limit on N, n and m we have

$$U_{m,n}(t) \longrightarrow 0.$$

The previous result prove that $(Y_n, Z_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in the Banach space $M^2([t, T]; \mathbb{R}^k) \times M^2([T - \eta, T] \times [t, T]; \mathbb{R}^{k \times d})$.

We put

$$Y(s) = \lim_n Y_n(s) \quad \text{and} \quad Z(t, s) = \lim_n Z_n(t, s).$$

On the other hand, if we denote

$$\begin{aligned}
A_n^N &= \{(\omega, t, s) \in \Omega \times \mathcal{D}_\eta, 1 + |Y(s)| + |Z(t, s)| + |Y_n(s)| + |Z_n(t, s)| > N\} \\
\text{and } \bar{A}_n^N &= \Omega \setminus A_n^N
\end{aligned}$$

then

$$\begin{aligned}
& \int_t^T ds \mathbb{E} \int_s^T |f_n(s, u, Y_n(u), Z_n(s, u)) - f(s, u, Y(u), Z(s, u))|^2 du \\
\leq & \int_t^T ds \mathbb{E} \int_s^T |f_n(s, u, Y_n(u), Z_n(s, u)) - f(s, u, Y(u), Z(s, u))|^2 \mathbf{1}_{A_n^N}(s, u) du \\
& + 2 \int_t^T ds \mathbb{E} \int_s^T |(f_n - f)(s, u, Y_n(u), Z_n(s, u))|^2 \mathbf{1}_{\bar{A}_n^N}(s, u) du \\
& + 2 \int_t^T ds \mathbb{E} \int_s^T |f(s, u, Y_n(u), Z_n(s, u)) - f(s, u, Y(u), Z(s, u))|^2 \mathbf{1}_{\bar{A}_n^N}(s, u) du \\
\leq & \frac{C}{N^{2(1-\alpha)}} + 2\rho_N^2 (f - f_n) + 4L_N^2 \mathbb{E} \int_t^T |Y_n(s) - Y_m(s)|^2 ds \\
& + 4K^2 \mathbb{E} \int_s^T ds \int_u^T |Z_n(s, u) - Z_m(s, u)|^2 du.
\end{aligned}$$

Passing to the limit successively on N and n , in virtue of the previous result we obtain

$$\int_t^T ds \mathbb{E} \int_s^T |f_n(s, u, Y_n(u), Z_n(s, u)) - f(s, u, Y(u), Z(s, u))|^2 du \longrightarrow 0 \quad \forall t \in [T - \eta, T].$$

Then taking the limit in (3.1), it follows that (Y, Z) solves equation (1.1) for $(t, s) \in [T - \eta, T] \times [t, T]$.

From the above proof, we know that when $(t, s) \in [T - \eta, T] \times [t, T]$, there exists unique $Y(T - \eta)$. Now, for $(t, s) \in [T - 2\eta, T - \eta] \times [t, T - \eta]$, we consider the equation

$$Y_n(t) + \int_t^{T-\eta} f_n(t, s, Y_n(s), Z(t, s)) ds + \int_t^{T-\eta} [g(t, s, Y_n(s)) + Z_n(t, s)] dW(s) = Y(T - \eta).$$

With the same argument as above, one can prove that $(Y_n, Z_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in the Banach space $M^2([T - 2\eta, T - \eta]; \mathbb{R}^k) \times M^2([T - 2\eta, T - \eta] \times [t, T - \eta]; \mathbb{R}^{k \times d})$. One can prove that its limit is the unique solution of BSDE of Volterra type with data (ξ, f, g) for $(t, s) \in [T - 2\eta, T - \eta] \times [t, T - \eta]$. Thus, we can prove the existence by continuing this procedure. ■

4 Stability result for BSNVIE with local Lipschitz drift

In this section, we prove a stability result for backward stochastic nonlinear Volterra integral equation assuming local Lipschitz drift. Let $(\xi_n, f_n, g_n)_{n \in \mathbb{N}^*}$ be sequences of processes that satisfy assumptions of Theorem 3.3. Let $(Y_n(s), Z_n(t, s)) \in \mathbb{H}$ be the solution to the BSDE of Volterra type with data (ξ_n, f_n, g_n) . Moreover we consider the following assumption:

- **(A4)** for each $N \in \mathbb{N}^* \setminus \{1\}$
 - (i) $\rho_N(f_n - f_0) \longrightarrow 0$ as $n \longrightarrow +\infty$
 - (ii) $\pi(g_n - g_0) \longrightarrow 0$ as $n \longrightarrow +\infty$
 - (iii) $\mathbb{E}|\xi_n - \xi_0|^2 \longrightarrow 0$ as $n \longrightarrow +\infty$

where $\pi(g_n - g_0) = \mathbb{E} \left(\int_{\mathcal{D}} \sup_y |g_n(t, s, y) - g_0(t, s, y)|^2 ds dt \right)^{1/2}$

Theorem 4.1 *Assume (A1) – (A4) and (A). Then*

$$(Y_n, Z_n) \longrightarrow (Y_0, Z_0) \text{ in } M(t, T, \mathbb{R}^k) \times M(\mathcal{D}, \mathbb{R}^{k \times d}) \text{ as } n \longrightarrow +\infty.$$

Proof. For each $(t, s) \in [T - \eta, T] \times [t, T]$ it follows from Lemma 2.1 of [27]

$$\begin{aligned} & \mathbb{E}|Y_n(t) - Y_0(t)|^2 ds + \mathbb{E} \int_t^T |Z_n(t, s) - Z_0(t, s)|^2 ds \\ = & \mathbb{E}|\xi_n - \xi_0|^2 + 2\mathbb{E} \int_t^T \langle f_n(t, s, Y_n(s), Z_n(t, s)) - f_0(t, s, Y_0(s), Z_0(t, s)), Y_0(s) - Y_0(s) \rangle ds \\ & - 2\mathbb{E} \int_t^T \langle f_n(t, s, Y_n(u), Z_n(t, s)) - f_0(t, s, Y_0(s), Z_0(t, s)), I_{n,0}(t, s) \rangle ds \\ & - 2\mathbb{E} \int_t^T \langle g_n(t, s, Y_n(s)) - g_0(t, s, Y_0(s)), Z_n(t, s) - Z_0(t, s) \rangle ds \\ & - \mathbb{E} \int_t^T |g_n(t, s, Y_n(s)) - g_0(t, s, Y_0(s))|^2 du \\ \leq & \mathbb{E}|\xi_n - \xi_0|^2 + 2\mathbb{E} \int_t^T |\langle f_n(t, s, Y_n(s), Z_n(t, s)) - f_0(t, s, Y_0(s), Z_0(t, s)), Y_n(s) - Y_0(s) \rangle| ds \\ & + 2\mathbb{E} \int_t^T |\langle f_n(t, s, Y_n(s), Z_n(t, s)) - f_0(t, s, Y_0(s), Z_0(t, s)), I_{n,0}(s) \rangle| ds \\ & + 2\mathbb{E} \int_t^T |\langle g_n(t, s, Y_n(s)) - g_0(t, s, Y_0(s)), Z_n(t, s) - Z_0(t, s) \rangle| ds \end{aligned}$$

where

$$\begin{aligned} I_{n,0}(t, s) &= \int_s^T (f_n(s, u, Y_n(u), Z_n(s, u)) - f_0(s, u, Y_0(u), Z_0(s, u))) du \\ &\quad - \int_s^T (f_n(t, u, Y_n(u), Z_n(t, u)) - f_0(t, u, Y_0(u), Z_0(t, u))) du. \end{aligned}$$

For each $N > 1$, let us consider L_N the Lipschitz constant of f in the ball $B(0, N)$ of \mathbb{R}^k and denote

$$D_{n,0}^N = \{(\omega, t, s) \in \Omega \times \mathcal{D}_\eta, |Y_n(s)| + |Z_n(t, s)| + |Y_0(s)| + |Z_0(t, s)| \geq N\}$$

$$\text{and } \bar{D}_{n,0}^N = (\Omega \times \mathcal{D}_\eta) \setminus D_{n,0}^N.$$

The same procedure as in the proof of existence of Theorem 3.3 yields

$$\begin{aligned} & |Y_n(t) - Y_0(t)|^2 + \mathbb{E} \int_t^T |Z_n(t, s) - Z_0(t, s)|^2 ds \\ \leq & \mathbb{E} |\xi_n - \xi_0|^2 + (2 + \beta_1 + \beta_3 + 2L_N + (1 + 8(\beta_2 + 3))\eta^2)L_N^2 \mathbb{E} \int_t^T |Y_n(s) - Y_0(s)|^2 ds \\ & + \left[\left(\frac{1}{\beta_1} + \frac{1}{\beta_2} + \frac{1}{\beta_3} + 4(\beta_2 + 3)\eta^2 \right) K^2 + \frac{1}{\beta_4} \right] \mathbb{E} \int_t^T \int_s^T |Z_n(t, s) - Z_0(t, s)|^2 ds \\ & + (2 + 2(\beta_2 + 3)\eta^2) \mathbb{E} \int_t^T |(f_n - f_0)(t, s, Y_n(s), Z_n(t, s))|^2 \mathbf{1}_{\bar{D}_{n,0}^N}(t, s) ds \\ & + 4(\beta_2 + 3) \mathbb{E} \int_t^T (T - s) du \int_s^T |(f_n - f_0)(s, u, Y_n(u), Z_n(s, u))|^2 \mathbf{1}_{\bar{D}_{n,0}^N}(s, u) du \\ & + \beta_4 \mathbb{E} \int_t^T |(g_n - g_0)(t, s, Y_n(s))|^2 ds \\ & + 8(\beta_2 + 3)K^2T \mathbb{E} \int_t^T ds \int_s^T |Z_n(s, u) - Z_0(s, u)|^2 du \\ & + (\eta^2 + 1) \frac{C}{N^{2(1-\alpha)}}. \end{aligned}$$

Let us choose $\beta_1 = \beta_2 = \beta_3 = 12K^2$, $\beta_4 = 8$ and $\eta < \frac{1}{24K^2}$. If we define

$$U_{n,0}(t) = \mathbb{E} \int_t^T |Y_n(s) - Y_0(s)|^2 ds, V_{n,0}(t) = \mathbb{E} \int_t^T |Z_n(t, s) - Z_0(t, s)|^2 ds,$$

then we obtain

$$\begin{aligned} & -\frac{d}{dt}(e^{K_1 t} U_{n,0}(t)) + \frac{1}{2} e^{K_1 t} V_{n,0}(t) \\ \leq & \mathbb{E}(e^{K_1 t} |\xi_n - \xi_0|^2) + 4 \mathbb{E} e^{K_1 t} \int_t^T |(f_n - f_0)(t, s, Y_n(s), Z_n(t, s))|^2 \mathbf{1}_{\bar{D}_{n,0}^N}(t, s) ds \\ & + 4(12K^2 + 3) \mathbb{E} e^{K_1 t} \int_t^T (T - s) ds \int_s^T |(f_n - f_0)(s, u, Y_n(u), Z_n(s, u))|^2 \mathbf{1}_{\bar{D}_{n,0}^N}(s, u) du \\ & + 8 \mathbb{E} e^{K_1 t} \int_t^T |(g_n - g_0)(t, s, Y_n(s))|^2 ds \\ & + 8(12K^2 + 3)K^2T \mathbb{E} e^{K_1 t} \int_t^T V_{n,0}(s) ds \\ & + e^{K_1 t} \frac{C}{N^{2(1-\alpha)}}. \end{aligned}$$

where $K_1 = 2 + 24K^2 + 2L_N + 2L_N^2$, K_2, K_3 and C are constants depending only on K, T , and ξ_0 . The rest of the proof is identical to that of existence part of Theorem 3.3. ■

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