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# Prediction of long memory processes on same-realisation

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## Abstract

Let observations come from an infinite autoregressive long memory process. For predicting the future values, we use the least-squares predictor where the forecast coefficients are estimated on the same realisation. The aim of this paper is to express the rate of convergence of this predictor as the number of observations goes to infinity. We generalise the results of Ing and Wei (2003) for short memory processes. We first have to prove sharp moment bound for the inverse empirical covariance matrix. We then obtain an asymptotic expression of the mean-squared prediction error of this predictor. The second order term of this expression is the sum of two terms: the first corresponds to the complexity of the predictor and the second to goodness of fit of the theoretical predictor. Finally we prove a central limit theorem for this predictor.

## 1 Introduction

Consider  $(X_n)_{n \in \mathbb{Z}}$  a stationary process with zero mean and finite variance. We wish to predict  $X_{n+1}$  from the observed past  $(X_1, \dots, X_n)$  using linear predictor i.e. a linear combination of the observed data. Firstly we define the coefficients of the optimal predictor in  $L^2$  sense assuming that the covariance function is known. Then we need to estimate its coefficients. The second step is often realised under the following restrictive hypothesis: we predict another future independent series which has exactly the same probabilistic structure as the observed one, which is used to compute the forecast coefficients (see for example Bhansali (1978), Lewis and Reinsel (1985) or Godet (2007b)). This assumption makes the mathematical analysis easier since the prediction problem can be reduced to an estimation problem by conditioning. But the practitioner rarely have two independent observations: one to estimate the model, one to predict. Then he must estimate the forecast coefficients on the same realisation as the forecast one. In the following we concentrate on this case called same-realisation prediction.

The performance of the predictor depends on two parameters: the dimension of the subspace on which we project and the number of available data to estimate the forecast coefficients. To reduce prediction error, it is reasonable to increase the height of the space, onto which we project, as more

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and more observations become available. But when the complexity of the assumed model increases, the estimation of the model become more difficult and can affect the mean-squared error.

When the spectral density exists, is bounded and bounded away from 0 (the situation generally corresponds to the short memory case) Ing and Wei (2003) and Kunitomo and Yamamoto (1985) have studied the mean-squared prediction error for same-realisation prediction. The second order mean squared error for same realisation prediction can be approximate by the sum of two terms: one due to the goodness of fit and one due to the model complexity. It is interesting to remark that the approximation of the mean-squared error for same or independent realisation prediction is the same. The performance of least-squares predictor of long memory time series is still left unanswered. And in this case, the asymptotic equivalence between second order mean-squared error in same and independent realisation should not be taken for granted since the autocovariance function decays more slowly as in short memory case.

The paper is organised as follows. In Section 2 and 3 we generalise the results of Ing and Wei (2003) to find an asymptotic expression of the mean-squared error for the long memory time series. The second order term of the mean-squared error is approximated by the same function as in the short memory case but under more restrictive conditions on the number of available observations and the model complexity. In the last section, we prove a central limit theorem. More precisely, we prove the convergence in distribution of the normalised difference between our predictor and the Wiener-Kolmogorov predictor, which is the least square predictor knowing all the past. The normalisation is different as in short memory case since it is given by the goodness of fit of the model.

**Definition of the Predictor** Let  $(X_t)_{t \in \mathbb{Z}}$  a stationary process with zero mean and finite variance. We assume here that the autocovariance function of the process  $\sigma$  is known. Our goal is to predict  $X_{n+1}$ , using the last  $k$  observed data. The optimal linear predictor is defined as the projection mapping onto the closed span of the subset  $\{X_n, \dots, X_{n-k+1}\}$  of the Hilbert space  $L^2(\Omega, \mathcal{F}, \mathbb{P})$  with inner product  $\langle X, Y \rangle = \mathbb{E}(XY)$ . It is the least-squares predictor knowing  $(X_{n-k+1}, \dots, X_n)$ . We denote by  $\tilde{X}_{n+1}(k)$  this predictor and by  $-a_{j,k}$  the theoretical forecast coefficients i.e.:

$$\tilde{X}_{n+1}(k) = \sum_{j=1}^k (-a_{j,k}) X_{n+1-j}. \quad (1)$$

They are given by (see Brockwell and Davis (1988) Section 5.1):

$$\begin{pmatrix} a_{1,k} \\ \vdots \\ a_{k,k} \end{pmatrix} = -\Sigma_k^{-1} \begin{pmatrix} \sigma(1) \\ \vdots \\ \sigma(k) \end{pmatrix} \quad (2)$$

where  $\Sigma_k$  is the covariance matrix of the vector  $(X_1, \dots, X_k)$ .

**Estimation of the Forecast Coefficients** When the autocovariance function  $\sigma$  of the process  $(X_t)_{t \in \mathbb{Z}}$  is unknown, we need to plug-in an estimate of the forecast coefficients in (1). The estimate is constructed from the last  $n$  observations  $(X_n, \dots, X_1)$  and our predictor is the projection of the last  $k$  observations ( $k \leq n$ ). The matrix covariance is estimated by:

$$\hat{\Sigma}_n(k) := \frac{1}{n - K_n + 1} \sum_{j=K_n}^n \mathbf{X}_j(k) \mathbf{X}_j'(k) \quad (3)$$

where  $X'$  denotes the transpose of the vector  $X$  and

$$\mathbf{X}'_j(k) := (X_j, \dots, X_{j-k+1}). \quad (4)$$

The forecast coefficients  $a_{i,k}$  are estimated on  $(X_1, \dots, X_n)$  by:

$$\hat{\mathbf{a}}(k) = (-\hat{a}_{1,k}, \dots, -\hat{a}_{k,k}) = \hat{\Sigma}_n^{-1}(k) \frac{1}{n - K_n + 1} \sum_{j=K_n}^{n-1} \mathbf{X}'_j(k) X_{j+1}.$$

The resulting one-step predictor is:

$$\hat{X}_{n+1}(k) = \mathbf{X}'_n(k) \hat{\mathbf{a}}(k) \quad (5)$$

In this paper we use  $C$  to denote a generic positive constant independent of the sample size  $n$  but  $C$  may depend of the distributional properties of the process  $(X_t)_{t \in \mathbb{Z}}$ . It also may have different values in different equations.

The following assumptions on the process  $(X_t)_{t \in \mathbb{Z}}$  are essential to our analysis:

- H.1 The Gaussian stationary process  $(X_t)_{t \in \mathbb{Z}}$  admits an infinite moving average representation and an infinite autoregressive representation as follows:

$$\varepsilon_t = \sum_{j=0}^{+\infty} a_j X_{t-j} \text{ and } X_t = \sum_{j=0}^{+\infty} b_j \varepsilon_{t-j}$$

with for any  $j \geq 1$  and for any  $\delta > 0$ ,  $|a_j| \leq C j^{-d-1+\delta}$  and  $|b_j| \leq C j^{d-1+\delta}$ . These assumptions on the coefficients simultaneously allow long memory and short memory processes;

- H.2 The covariance  $\sigma(k) \sim L(k)k^{2d-1}$  as  $k$  goes to infinity and where  $L$  is a slowly varying function (i.e. for every  $\alpha > 0$ ,  $x^\alpha L(x)$  is ultimately increasing and  $x^{-\alpha} L(x)$  is ultimately decreasing), and  $d$  is in  $[0, 1/2]$ . Under this assumption the autocovariances are not absolutely summable and thus the process is long memory process;

- H.3 The spectral density of the process  $(X_t)_{t \in \mathbb{Z}}$  exists and has a positive lower bound.

## 2 Moment bounds

In this section, we establish moment bounds for the inverse sample covariance matrix and apply these results to obtain the rate of convergence of  $\hat{\Sigma}_n(k)$  to  $\Sigma_n(k)$ .

Throughout the paper,  $\lambda_{\min}(Y)$  and  $\lambda_{\max}(Y)$  are respectively the smallest and the largest eigenvalue of the matrix  $Y$ . We equip the set of matrices with the norm  $\|Y\|^2 = \lambda_{\max}(Y'Y)$  (see for example Dahlhaus (1989)). For a symmetric matrix, this norm is equal to the spectral radius and for a vector  $(X_1, \dots, X_n)$ , it is equal to  $\sqrt{\sum_{i=1}^n X_i^2}$ . This norm is a matrix norm having the propertie: for any matrices  $A$  and  $B$ :

$$\|AB\| \leq \|A\| \|B\|. \quad (6)$$

**Lemma 2.1.** *Let  $(K_n)_{n \in \mathbb{N}}$  an increasing sequence of positive integers satisfying  $K_n = o(\sqrt{n})$ . Assume (H.1). Then, for any  $q > 0$  and for any  $\theta > 0$ ,*

$$\mathbb{E} \left[ \lambda_{\min}^{-q} \left( \widehat{\Sigma}_n(K_n) \right) \right] = O \left( K_n^{(2+\theta)q} \right)$$

*Proof.* The sketch of the proof is the same as this of Lemma 1 of Ing and Wei (2003). The arguments are the following:

1. the series  $\sum_{j=1}^{+\infty} a_j$  converges absolutely;
2. the cumulative distribution function of the random variable  $\varepsilon_t$  is a Lipschitz function and we may choose a Lipschitz constant independent of  $t$ . For any integer  $t$  and for any reals  $x$  and  $y$ , there exists  $C$  independent of  $t$  such that:

$$|\mathbb{P}(\varepsilon_t < x) - \mathbb{P}(\varepsilon_t < y)| \leq C|x - y|.$$

In our context these two conditions are satisfied. The sequence  $(a_j)_{j \in \mathbb{N}}$  is summable under assumption H.1. Since we have assumed that the process  $(X_t)_{t \in \mathbb{Z}}$  is Gaussian,  $(\varepsilon_t)_{t \in \mathbb{Z}}$  is a sequence of independent and identically distributed Gaussian random variables. The distribution function of the process  $\varepsilon_t$  is independent of  $t$  and is a Lipschitz function.  $\square$

Lemma 2.1 guarantees that  $\widehat{\Sigma}_n^{-1}(K_n)$  almost surely exists for all large  $n$  since for all large  $n$ , the minimum eigenvalue of  $\widehat{\Sigma}_n(K_n)$  is positive. Hence we obtain an upper bound for the maximum eigenvalue of  $\widehat{\Sigma}_n^{-1}(K_n)$ . But this upper bound is not bounded as  $K_n \rightarrow +\infty$ . This bound is a foundation for further improvement which is given in the following theorem.

**Theorem 1.** *Assume that the process  $(X_t)_{t \in \mathbb{Z}}$  verifies the hypothesis H.1-H.3*

- *if  $d \in ]0, 1/4[$  and if there exists  $\delta > 0$  such that  $K_n^{2+\delta} = O(n)$  then for all  $q > 0$  :*

$$\mathbb{E} \|\widehat{\Sigma}_n^{-1}(K_n)\|^q = O(1) \tag{7}$$

and

$$\mathbb{E} \|\widehat{\Sigma}_n^{-1}(K_n) - \Sigma^{-1}(K_n)\|^{q/2} \leq C \left( \frac{K_n^2}{n - K_n + 1} \right)^{q/4} \tag{8}$$

for sufficiently large  $n$  ;

- *if  $d \in ]1/4, 1/2[$  and if there exists  $\delta > 0$  such that  $K_n^{2+\delta} = O(n^{2-4d})$  then for all  $q > 0$ :*

$$\mathbb{E} \|\widehat{\Sigma}_n^{-1}(K_n)\|^q = O(1) \tag{9}$$

and

$$\mathbb{E} \|\widehat{\Sigma}_n^{-1}(K_n) - \Sigma^{-1}(K_n)\|^{q/2} \leq C \left( \frac{K_n^2}{(n - K_n + 1)^{2-4d}} \right)^{q/4} \tag{10}$$

for sufficiently large  $n$ .

We now consider the following lemma, which we will need in the proof of Theorem 1.

**Lemma 2.2.** *If the process  $(X_t)_{t \in \mathbb{Z}}$  verifies (H.2) and*

- if  $d \in ]0, 1/4[$ , then for all  $q > 0$ ,

$$\mathbb{E} \|\widehat{\Sigma}_n(K_n) - \Sigma(K_n)\|^q \leq C \left( \frac{K_n^2}{n - K_n + 1} \right)^{\frac{q}{2}}; \quad (11)$$

- if  $d \in ]1/4, 1/2[$ , then for all  $q > 0$ ,

$$\mathbb{E} \|\widehat{\Sigma}_n(K_n) - \Sigma(K_n)\|^q \leq C \left( \frac{K_n^2 L^2 (n - K_n + 1)}{(n - K_n + 1)^{2-4d}} \right)^{\frac{q}{2}}. \quad (12)$$

*Proof.* We only prove the inequalities (11) and (12) for  $q > 2$  because this and Jensen's inequality can easily yield the result for  $q > 0$ . We consider the matrix norm  $\|\cdot\|_E$  (see Ciarlet (1982)) by: for all matrix  $Y = (y_{i,j})_{1 \leq i,j \leq K_n}$

$$\|Y\|_E = \sqrt{\sum_{i=1}^{K_n} \sum_{j=1}^{K_n} y_{i,j}^2}.$$

Moreover since the matrix  $\widehat{\Sigma}_n(K_n) - \Sigma(K_n)$  is symmetric, we have

$$\|\widehat{\Sigma}_n(K_n) - \Sigma(K_n)\| \leq \|\widehat{\Sigma}_n(K_n) - \Sigma(K_n)\|_E$$

We obtain:

$$\begin{aligned} \|\widehat{\Sigma}_n(K_n) - \Sigma(K_n)\|^q &\leq \|\widehat{\Sigma}_n(K_n) - \Sigma(K_n)\|_E^q \\ &\leq \left( \sum_{i=1}^{K_n} \sum_{j=1}^{K_n} (\hat{\sigma}_{i,j} - \sigma(i-j))^2 \right)^{q/2} \end{aligned} \quad (13)$$

where  $\hat{\sigma}_{i,j}$  and  $\sigma(i-j)$  denote respectively the  $(i, j)$  entries of the matrices  $\widehat{\Sigma}_n(K_n)$  and  $\Sigma(K_n)$ . Applying Jensen's inequality to (13) because  $q/2 > 1$ , we have:

$$\|\widehat{\Sigma}_n(K_n) - \Sigma(K_n)\|^q \leq \frac{K_n^q}{K_n^2} \sum_{i=1}^{K_n} \sum_{j=1}^{K_n} |\hat{\sigma}_{i,j} - \sigma(i-j)|^q.$$

It follows that:

$$\mathbb{E} \|\widehat{\Sigma}_n(K_n) - \Sigma(K_n)\|^q \leq K_n^{q-2} \sum_{i=1}^{K_n} \sum_{j=1}^{K_n} \mathbb{E} |\hat{\sigma}_{i,j} - \sigma(i-j)|^q \quad (14)$$

Now we derive the limiting distribution of  $\hat{\sigma}_{i,j} - \sigma(i-j)$  to find an asymptotic expression of  $\mathbb{E} |\hat{\sigma}_{i,j} - \sigma(i-j)|^q$ . We shall work with the definition of the empirical covariances. By (3), we have:

$$\hat{\sigma}_{i,j} = \frac{1}{n - K_n + 1} \sum_{l=K_n}^n X_{l+i-1} X_{l+j-1} \stackrel{\mathcal{L}}{=} \frac{1}{n - K_n + 1} \sum_{l=1}^{n-K_n+1} X_l X_{l+(j-i)}, \quad (15)$$

where the second equality is ensured by the strict stationarity of the process. Without loss of generality, we assume  $j \geq i$ . The right term of (15) can be write as a quadratic form:

$$\frac{1}{n - K_n + 1} \sum_{l=1}^{n-K_n+1} X_l X_{l+(j-i)} = \frac{1}{n - K_n + 1} \mathbf{X}'_1 (n - K_n + 1 + j - i) B \mathbf{X}_1 (n - K_n + 1 + j - i) \quad (16)$$

where  $\mathbf{X}_1(n - K_n + 1 + j - i)$  is defined in (4) the entries of the matrix  $B$  verify

$$b_{s,t} = \begin{cases} 1/2 & \text{if } |s - t| = j - i \\ 0 & \text{otherwise.} \end{cases}$$

$B$  is a Toeplitz matrix with symbol  $g(x) = \cos((j - i)x)$  i.e.  $b_{s,t} = \int_{-\pi}^{\pi} g(x)e^{i(t-s)x} dx$ .

Undo Assumption H.2 with  $d \in ]0, 1/4[$ , the spectral density verifies in a neighbourhood of 0:

$$f(x) = O\left(x^{-1/2}\right)$$

(see Zygmund (1968) Chap. 5 Theorem 2.6). It is shown by applying Theorem 2 of Fox and Taqqu (1987) to relation (16) that we obtain the following convergence in distribution:

$$\frac{(n - K_n + 1)(\hat{\sigma}_{i,j} - \sigma(i - j))}{\sqrt{n - K_n + 1 + j - i}} \xrightarrow[n \rightarrow +\infty]{} \mathcal{N}\left(0, 16\pi^3 \int_{-\pi}^{\pi} f^2(\lambda)(2\pi)^{-2} \cos^2((i - j)\lambda) d\lambda\right)$$

This convergence in distribution is proved by the convergence of all the cross-cumulants and then the convergence of all the raw moments. If  $q$  is even, we have an asymptotic equivalent as  $n \rightarrow +\infty$  of:

$$\mathbb{E}|\hat{\sigma}_{i,j} - \sigma(i - j)|^q \underset{n \rightarrow +\infty}{\sim} \left(\frac{\sqrt{n - K_n + 1 + j - i}}{n - K_n + 1}\right)^q \mathbb{E}[|Y|^q], \quad (17)$$

where  $Y$  is a Gaussian random variable with mean 0 and variance

$$\sigma_Y^2 := 4\pi \int_{-\pi}^{\pi} f^2(\lambda) \cos^2((i - j)\lambda) d\lambda.$$

The  $q$ th-order absolute moment has the form:

$$E[|Y|^q] = \frac{q!}{2^{q/2}(q/2)!} \sigma_Y^q \quad (18)$$

Moreover notice that for all  $(i, j)$ :

$$\sigma_Y \leq \sqrt{4\pi \int_{-\pi}^{\pi} f^2(\lambda) d\lambda} := M. \quad (19)$$

Thus (17), (18) and (19) imply for sufficiently large  $n$ :

$$\begin{aligned} \mathbb{E}|\hat{\sigma}_{i,j} - \sigma(i - j)|^q &\leq \left(\frac{\sqrt{n - K_n + 1 + j - i}}{n - K_n + 1}\right)^q \frac{q!}{(q/2)!} M^q \\ &\leq C \left(\frac{1}{\sqrt{n - K_n + 1}}\right)^q \end{aligned}$$

with  $C$  constant independent of  $q$ . The result follows from the previous inequality and the inequality (14). By Jensen's inequality and the concavity of the function  $x \mapsto x^r$  on  $\mathbb{R}^+$  if  $r < 1$ , the result can be obtained for any  $q > 0$ .

For any  $d \in ]1/4, 1/2[$ , we apply the proposition of Rosenblatt (1979) which gives the following convergence in distribution:

$$\frac{(n - K_n + 1)(\hat{\sigma}_{i,j} - \sigma(i - j))}{L(n - K_n + 1)(n - K_n + 1 + j - i)^{2d}} \xrightarrow[n \rightarrow +\infty]{} \mathcal{R}(1)$$

where  $\mathcal{R}$  is a Rosenblatt process. This convergence in distribution is obtained by proving the convergence of the cumulants of any order and then the convergence of the raw moments of any order. This limit does not depend on the difference  $(j - i)$ . By a similar procedure as previously, we show for any integer  $q$  even:

$$\mathbb{E}|\hat{\sigma}_{i,j} - \sigma(i - j)|^q \leq C \left( \frac{L(n - K_n + 1)}{L(n - K_n + 1)(n - K_n + 1)^{1-2d}} \right)^q$$

where  $C$  does not depend on  $(j - i)$ . This inequality and (14) yield the desired result.  $\square$

With this intermediate lemma, we can prove Theorem 1.

*Proof of Theorem 1.* Since  $\|\cdot\|$  is a matrix norm (see (6)),

$$\|\widehat{\Sigma}_n^{-1}(K_n) - \Sigma^{-1}(K_n)\|^q \leq \|\widehat{\Sigma}_n^{-1}(K_n)\|^q \|\widehat{\Sigma}_n(K_n) - \Sigma(K_n)\|^q \|\Sigma^{-1}(K_n)\|^q.$$

Furthermore on assumption H.3, the spectral density of the process  $(X_t)_{t \in \mathbb{Z}}$  has a positive lower bound. Thus by Grenander and Szegö (1958), there exists a constant  $C$  such that for all  $n \geq 0$ :

$$\|\Sigma^{-1}(K_n)\|^q \leq C. \quad (20)$$

Using Hölder's inequality with  $1/p' + 1/q' = 1$ , we obtain:

$$\mathbb{E}\|\widehat{\Sigma}_n^{-1}(K_n) - \Sigma^{-1}(K_n)\|^q \leq C \left( \mathbb{E}\|\widehat{\Sigma}_n^{-1}(K_n)\|^{qq'} \right)^{1/q'} \left( \mathbb{E}\|\widehat{\Sigma}_n(K_n) - \Sigma(K_n)\|^{qp'} \right)^{1/p'}.$$

By Lemma 2.1, we have for all  $\theta > 0$  and for large  $n$ :

$$\left( \mathbb{E}\|\widehat{\Sigma}_n^{-1}(K_n)\|^{qq'} \right)^{1/q'} \leq C(K_n^{2+\theta})^q$$

then

$$\mathbb{E}\|\widehat{\Sigma}_n^{-1}(K_n) - \Sigma^{-1}(K_n)\|^q \leq C(K_n^{2+\theta})^q \left( \mathbb{E}\|\widehat{\Sigma}_n(K_n) - \Sigma(K_n)\|^{qp'} \right)^{1/p'}. \quad (21)$$

We now apply Lemma 2.2. In order to not separate each case for  $d$ , we define  $g(n)$  by:

$$g(n) = \begin{cases} \frac{K_n^2}{n - K_n + 1} & \text{if } d \in ]0, 1/4[ \\ \frac{K_n^2 L^2 (n - K_n + 1)}{(n - K_n + 1)^{2-4d}} & \text{if } d \in ]1/4, 1/2[ \end{cases}$$

For large  $n$ , we then obtain:

$$\left( \mathbb{E}\|\widehat{\Sigma}_n(K_n) - \Sigma(K_n)\|^{qp'} \right)^{1/p'} \leq C(g(n))^{q/2} \quad (22)$$

Noting inequality (21) and the bound (22), we have that there exists  $\theta > 0$  such that for sufficiently large  $n$ :

$$\mathbb{E}\|\widehat{\Sigma}_n^{-1}(K_n) - \Sigma^{-1}(K_n)\|^q \leq C(K_n^{4+\theta} g(n))^{q/2}. \quad (23)$$



By inequalities (23) and (20), we have:

$$\mathbb{E}\|\widehat{\Sigma}_n^{-1}(K_n)\|^q \leq C \left( 1 + \left( K_n^{4+\theta} g(n) \right)^{q/2} \right).$$

This inequality is not sufficient to obtain (7) and (9) since under the assumptions of Theorem 1,  $(K_n^{4+\theta} g(n))^{q/2}$  is not necessarily bounded. We then have to improve this intermediate inequality. The Cauchy-Schwarz inequality gives:

$$\mathbb{E}\|\widehat{\Sigma}_n^{-1}(K_n) - \Sigma^{-1}(K_n)\|^{q/2} \leq C \left( \mathbb{E}\|\widehat{\Sigma}_n^{-1}(K_n)\|^q \right)^{1/2} \left( \mathbb{E}\|\widehat{\Sigma}_n(K_n) - \Sigma(K_n)\|^q \right)^{1/2}. \quad (24)$$

And there exists  $C > 0$  independent  $q$  such that:

$$\mathbb{E}\|\widehat{\Sigma}_n^{-1}(K_n)\|^{q/2} \leq C \left( \mathbb{E}\|\Sigma^{-1}(K_n)\|^{q/2} + \mathbb{E}\|\widehat{\Sigma}_n^{-1}(K_n) - \Sigma^{-1}(K_n)\|^{q/2} \right).$$

Inequalities (24), (20) and Lemma 2.2 imply that:

$$\mathbb{E}\|\widehat{\Sigma}_n^{-1}(K_n)\|^{q/2} \leq C \left( 1 + \left( K_n^{4+\theta} g(n)^2 \right)^{q/4} \right).$$

Repeating this argument  $s - 1$  times, one has for large  $n$ :

$$\mathbb{E}\|\widehat{\Sigma}_n^{-1}(K_n)\|^{q2^{-s}} \leq C \left( 1 + \left( K_n^{4+\theta} g(n)^{(1+s)} \right)^{q/2^{-(s+1)}} \right). \quad (25)$$

By assumption there exists  $\delta > 0$  such that  $g(n)K_n^\delta$  converges to 0 as  $n$  tends to infinity, therefore there exists  $s$ , such that  $\mathbb{E}\|\widehat{\Sigma}_n^{-1}(K_n)\|^{q2^{-s}}$  is bounded. Since  $q$  in (25) is arbitrary, (7) and (9) are proved. Inequality (8) and (10) follow from (24) and inequalities obtained in Lemma 2.2.  $\square$

Using this sharp upper bound for  $\mathbb{E}\|\widehat{\Sigma}_n^{-1}(K_n)\|^q$  for any  $q > 0$ , we can establish an asymptotic expression for the mean-squared prediction error of the least-squares predictor.

### 3 The mean-squared prediction error of the least-squares predictor

In this section, our goal is to give an asymptotic expression of the mean-square prediction error of the predictor defined on (5). First we decompose the forecast error in:

$$X_{n+1} - \widehat{X}_{n+1}(k) = \varepsilon_{n+1} + f(k) + \mathcal{S}_n(k) \quad (26)$$

where  $\varepsilon_{n+1}$  is the innovation white noise at time  $n + 1$  which can not be forecast,  $\mathcal{S}_n(k)$  is the error due to the projection onto the closed span of the subset  $X_n, \dots, X_{n-k+1}$ , and  $f(k)$  is the error due to the estimation of the forecast coefficients. More precisely if we set  $a_{i,k} = 0$  for  $i > k$ , we have

$$\mathcal{S}_j(k) = \sum_{i=1}^{+\infty} (a_i - a_{i,k}) X_{j+1-i} \quad (27)$$

and with

$$\varepsilon_{j+1,k} = X_{j+1} - \sum_{l=1}^k a_{l,k} X_{j+1-l},$$

we have

$$f(k) = \mathbf{X}'_n(k) \widehat{\Sigma}_n^{-1}(k) \frac{1}{n - K_n + 1} \sum_{j=K_n}^{n-1} \mathbf{X}_j(k) \varepsilon_{j+1,k}$$

where  $\mathbf{X}'_n(k)$  is defined in (4). It can also be seen that  $\varepsilon_{j+1,k}$  is equal to the forecast error of  $X_{j+1}$  due to the projection onto  $(X_j, \dots, X_{j-k+1})$ :

$$\varepsilon_{j+1,k} = X_{j+1} - P_{[X_{j-k}, \dots, X_j]}(X_{j+1})$$

In view of (26), we obtain the decomposition of the mean-squared prediction errors as the sum of the variance of the white noise  $\sigma_\varepsilon^2$  and the error due to the prediction method  $\mathbb{E}(f(k) + \mathcal{S}_n(k))^2$ :

$$\mathbb{E} \left( X_{n+1} - \widehat{X}_{n+1}(k) \right)^2 = \sigma_\varepsilon^2 + \mathbb{E} (f(k) + \mathcal{S}_n(k))^2.$$

**Theorem 2.** *Undo assumptions H.1-H.3, if there exists  $\delta > 0$  such that the sequence  $(K_n)_{n \in \mathbb{N}}$  verifies :*

$$K_n^4 = o(n^{1-2d-\delta}), \quad (28)$$

then

$$\lim_{n \rightarrow +\infty} \max_{1 \leq k \leq K_n} \left| \frac{\mathbb{E} \left( X_{n+1} - \widehat{X}_{n+1}(k) \right)^2 - \sigma_\varepsilon^2}{L_n(k)} - 1 \right| = 0$$

where

$$L_n(k) = \mathbb{E} (\mathcal{S}_n(k))^2 + \frac{k}{n - K_n + 1} \sigma_\varepsilon^2 \quad (29)$$

where  $\mathcal{S}_n(k)$  is defined in (27).

**Remark** In our prediction approach, we fit a misspecified AR( $k$ ) model to the long-memory time series  $(X_n)_{n \in \mathbb{Z}}$ . Then  $L_n(k)$  can be viewed as the model's quality of prediction, which is the sum of the model complexity  $\frac{k}{n - K_n + 1} \sigma_\varepsilon^2$  and the goodness of fit  $\mathbb{E} (\mathcal{S}_n(k))^2$ .

*Proof.* By (26), we have:

$$\left| \frac{\mathbb{E} \left( X_{n+1} - \widehat{X}_{n+1}(k) \right)^2 - \sigma_\varepsilon^2}{L_n(k)} - 1 \right| = \left| \frac{\mathbb{E} (f(k) + \mathcal{S}_n(k))^2}{L_n(k)} - 1 \right|.$$

Our proof is divided into three steps:

1. we approximate  $\mathbb{E}(f(k))^2$  by  $\mathbb{E}(f_1(k))^2$  which is easier to estimate;
2. we show that the asymptotic equivalent of  $\mathbb{E}(f_1(k))^2$  is  $\frac{k}{n - K_n + 1} \sigma_\varepsilon^2$ ;
3. we prove that the cross-product term  $\mathbb{E}(f(k)\mathcal{S}_n(k))$  is negligible with respect to  $L_n(k)$ .

**First step** We introduce:

$$f_1(k) := \mathbf{X}_n^{*'}(k)\Sigma^{-1}(k)\frac{1}{n-K_n+1}\sum_{j=K_n}^{n-\sqrt{n}-1}\mathbf{X}_j(k)\varepsilon_{j+1},$$

with

$$\mathbf{X}_n^{*'}(k) = \left( \sum_{j=0}^{\sqrt{n}/2-K_n} b_j \varepsilon_{n-j}, \dots, \sum_{j=0}^{\sqrt{n}/2-K_n} b_j \varepsilon_{n-k+1-j} \right).$$

**Lemma 3.1.** *If the assumptions of Theorem 2 hold, then*

$$\lim_{n \rightarrow +\infty} \max_{1 \leq k \leq K_n} \mathbb{E} \left( \sqrt{\frac{1}{L_n(k)}} (f(k) - f_1(k)) \right)^2 = 0. \quad (30)$$

*Proof.* See the appendix. □

**Second step** Prove that:

$$\lim_{n \rightarrow +\infty} \max_{1 \leq k \leq K_n} \left| \mathbb{E} \left( \frac{n-K_n+1}{k\sigma_\varepsilon^2} f_1^2(k) \right) - 1 \right| = 0. \quad (31)$$

First observe that:

$$\begin{aligned} \mathbb{E} \left( \frac{n-K_n+1}{k\sigma_\varepsilon^2} f_1^2(k) \right) &= \frac{n-K_n+1}{k\sigma_\varepsilon^2} \mathbb{E} \left( \mathbf{X}_n^{*'}(k)\Sigma^{-1}(k)\frac{1}{n-K_n+1}\sum_{j=K_n}^{n-\sqrt{n}-1}\mathbf{X}_j(k)\varepsilon_{j+1} \right)^2 \\ &= \frac{n-K_n+1}{k\sigma_\varepsilon^2} \mathbb{E} \left[ \text{trace} \left( \mathbf{X}_n^{*'}(k)\Sigma^{-1}(k)\frac{1}{n-K_n+1}\sum_{j=K_n}^{n-\sqrt{n}-1}\mathbf{X}_j(k)\varepsilon_{j+1} \right)^2 \right] \end{aligned}$$

Since the vector  $\mathbf{X}_n^{*'}$  and  $\sum_{j=K_n}^{n-\sqrt{n}-1}\mathbf{X}_j(k)\varepsilon_{j+1}$  are uncorrelated:

$$\begin{aligned} \mathbb{E} \left( \frac{n-K_n+1}{k\sigma_\varepsilon^2} f_1^2(k) \right) &= \frac{n-K_n+1}{k\sigma_\varepsilon^2(n-K_n+1)^2} \text{trace} (\Sigma^{-1}(k)(n-K_n+1-\sqrt{n})\Sigma(k)\Sigma^{-1}(k)\Sigma^*(k)) \\ &= \text{trace} (\Sigma^{-1}(k)\Sigma^*(k)k^{-1}) (n-K_n+1-\sqrt{n})(n-K_n+1)^{-1}, \end{aligned}$$

where  $\Sigma^*(k)$  is the covariance matrix of the vector  $\mathbf{X}_n^*(k)$ . We remark that:

$$(n-K_n+1-\sqrt{n})(n-K_n+1)^{-1} \rightarrow 1 \text{ as } n \rightarrow +\infty.$$

Then we only have to study  $\text{trace} (\Sigma^{-1}(k)\Sigma^*(k)k^{-1}) - 1$ . Since the matrix  $\Sigma^{-1}(k)(\Sigma^*(k) - \Sigma(k))$  is symmetric we can use for it the following well known relation:  $|\text{trace}(Y)| \leq k\|Y\|$  for all symmetric matrix  $Y$ . We obtain:

$$\begin{aligned} \max_{1 \leq k \leq K_n} |\text{trace} (\Sigma^{-1}(k)\Sigma^*(k)k^{-1}) - 1| &= \max_{1 \leq k \leq K_n} |\text{trace} (\Sigma^{-1}(k)(\Sigma^*(k) - \Sigma(k))k^{-1})| \\ &\leq \max_{1 \leq k \leq K_n} \|\Sigma^{-1}(k)(\Sigma^*(k) - \Sigma(k))\| \end{aligned}$$

Using property (6) of  $\|\cdot\|$ :

$$\max_{1 \leq k \leq K_n} \|\Sigma^{-1}(k)(\Sigma^*(k) - \Sigma(k))\| \leq \max_{1 \leq k \leq K_n} \|\Sigma^{-1}(k)\| \max_{1 \leq k \leq K_n} \|\Sigma^*(k) - \Sigma(k)\|.$$

$\Sigma(k) - \Sigma^*(k)$  is also a symmetric matrix as difference between two symmetric matrices, and its spectral norm is bounded by every other matrix norm. We use the subordinate norm defined for all matrix  $Y = (y_{i,j})_{1 \leq i,j \leq k}$  by:

$$\|Y\|_1 = \max_j \sum_{i=1}^k |y_{i,j}|.$$

$$\begin{aligned} \max_{1 \leq k \leq K_n} \|\Sigma^{-1}(k)\| \max_{1 \leq k \leq K_n} \|\Sigma^*(k) - \Sigma(k)\| &\leq \max_{1 \leq k \leq K_n} \|\Sigma^{-1}(k)\| \max_{1 \leq k \leq K_n} \|\Sigma(k) - \Sigma^*(k)\|_1 \\ &\leq \max_{1 \leq k \leq K_n} \|\Sigma^{-1}(k)\| \max_{1 \leq k \leq K_n} k \max_{0 \leq j \leq k-1} \sum_{l=\sqrt{n}/2-K_n+1}^{+\infty} |b_l b_{l+j}| \\ &= O\left(\frac{K_n}{(\sqrt{n})^{1-2d-\delta}}\right) \end{aligned}$$

for all  $\delta > 0$ . And

$$\max_{1 \leq k \leq K_n} \|\Sigma^{-1}(k)\| \max_{1 \leq k \leq K_n} \|\Sigma^*(k) - \Sigma(k)\| = o(1)$$

follows from condition (28).

**Third step** We consider the cross-product term and show that it is negligible. The proof is similar to Theorem 3 (step 4) of Ing and Wei (2003). Ing and Wei (2003) have proved that:

$$|\mathbb{E}(f(k)\mathcal{S}_n(k)L_n^{-1}(k))| = |\mathbb{E}((f(k) - f_1(k))\mathcal{S}_n(k)L_n^{-1}(k))|.$$

By the Cauchy-Schwarz inequality:

$$\begin{aligned} &\max_{1 \leq k \leq K_n} |\mathbb{E}((f(k) - f_1(k))\mathcal{S}_n(k)L_n^{-1}(k))| \\ &\leq \left[ \max_{1 \leq k \leq K_n} \mathbb{E}((f(k) - f_1(k))^2 L_n^{-1}(k)) \max_{1 \leq k \leq K_n} \mathbb{E}(\mathcal{S}_n^2(k)L_n^{-1}(k)) \right]^{1/2}. \end{aligned}$$

By (30), we obtain:

$$\max_{1 \leq k \leq K_n} \mathbb{E}((f(k) - f_1(k))^2 L_n^{-1}(k)) = o(1)$$

and by the definition (29) of  $L_n(k)$ , we have

$$\max_{1 \leq k \leq K_n} \mathbb{E}(\mathcal{S}_n^2(k)L_n^{-1}(k)) = O(1).$$

Finally we have

$$\max_{1 \leq k \leq K_n} |\mathbb{E}((f(k) - f_1(k))\mathcal{S}_n(k)L_n^{-1}(k))| = o(1).$$

□

In this theorem, we have obtain an asymptotic expression of the mean squared prediction error for  $\widehat{X}_{n+1}(K_n)$ , which holds uniformly for all  $1 \leq k \leq K_n$ . The second order term have the same expression as in the short memory case (Ing and Wei (2003)). It is the sum of two terms: the first term  $(k/n)\sigma_\varepsilon^2$  proportional to the order of the model is a measure of the complexity of the predictor, the second term  $\mathcal{S}_n^2(k)$  corresponds to the goodness of fit of the model. This second term has not the same asymptotic behaviour in short and long memory case : for short memory time series it decays exponentially fast as a function of  $k$ , for long memory time series it decays geometrically. In the following section ,we will use the proof of this theorem to obtain a central limit theorem for our predictor.

## 4 Central limit theorem

We search a normalisation to obtain a convergence in distribution between our predictor  $\widehat{X}_{n+1}(K_n)$  and the Wiener-Kolmogorov predictor (see Bhansali (1978) for short memory time series in univariate case and Lewis and Reinsel (1985) in multivariate case). The Wiener-Kolmogorov predictor  $\widetilde{X}_{n+1} = -\sum_{j=1}^{+\infty} a_j X_{n+1-j}$  is the linear least-squares predictor. The Wiener-Kolmogorov predictor is also equal to  $\widehat{X}_{n+1}(K_n) + f(K_n) + \mathcal{S}_n(K_n)$ .

For this result,we need an another assumption:

H.4 The coefficients  $a_j$  verify:

$$a_j \underset{j \rightarrow +\infty}{\sim} L(j)j^{-d-1} \quad (32)$$

with  $L$  a slowly varying function.

**Theorem 3.** *Undo assumptions H.1-H.4 and if*

$$K_n^4 = O(n) \quad \text{and} \quad K_n^{1+2d} = o\left(n^{1-2d}\right), \quad (33)$$

then

$$\frac{1}{\sqrt{\mathbb{E}[\mathcal{S}_n^2(K_n)]}} \left( \widetilde{X}_{n+1} - \widehat{X}_{n+1}(K_n) \right) \xrightarrow[n \rightarrow +\infty]{} \mathcal{N}(0, 1).$$

*Proof.* The difference between our predictor  $\widehat{X}_{n+1}(K_n)$  and the Wiener-Kolmogorov predictor  $\widetilde{X}_{n+1}$  is equal to:

$$\widetilde{X}_{n+1} - \widehat{X}_{n+1}(K_n) = f(K_n) + \mathcal{S}_n(K_n). \quad (34)$$

Our proof is divided into two steps:

1. to prove that:

$$\frac{1}{\sqrt{\mathbb{E}[\mathcal{S}_n^2(K_n)]}} f(K_n) \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0$$

2. to prove that  $\mathcal{S}_n(K_n)$  is a Gaussian random variable with zero mean.

The result follows immediately from these two steps.

**First step** First we search a bound for  $1/\mathbb{E}[\mathcal{S}_n^2(K_n)]$ . For all integers  $l$  and  $K_n$ ,

$$\mathbb{E} \left( \sum_{i=1}^l (a_i - a_{i,K_n}) X_{n+1-i} \right)^2 \geq 2\pi \underline{f} \sum_{i=1}^l (a_i - a_{i,K_n})^2$$

because the spectral density  $f$  Toeplitz symbol of the covariance matrix is bounded below by a positive constant  $\underline{f}$  (see Grenander and Szegö (1958)). By taking the limit as  $l \rightarrow +\infty$ , we obtain:

$$\begin{aligned} E \left( \sum_{i=1}^{+\infty} (a_i - a_{i,K_n}) X_{j+1-i} \right)^2 &\geq 2\pi \underline{f} \sum_{i=1}^{+\infty} (a_i - a_{i,K_n})^2 \\ &\geq 2\pi \underline{f} \sum_{i=K_n+1}^{+\infty} a_i^2 \end{aligned}$$

since  $a_{i,K_n} = 0$  when  $i > K_n$ . Under assumption H.4, the remainder of the series is equivalent to:

$$\sum_{i=K_n+1}^{+\infty} a_i^2 \underset{n \rightarrow +\infty}{\sim} \frac{1}{1+2d} K_n^{-2d-1} L^2(K_n)$$

(see Proposition 1.5.10 of Bingham et al. (1987)). Then for any  $\delta > 0$ , there exists  $C > 0$  such that:

$$\frac{1}{\mathbb{E}[\mathcal{S}_n^2(K_n)]} \leq C K_n^{2d+1+\delta} \quad (35)$$

By introducing  $f_1$  defined in the proof of Theorem 2, we decompose the proof of the mean-squared convergence in two parts. We will first show that:

$$\frac{1}{\sqrt{\mathbb{E}[\mathcal{S}_n^2(K_n)]}} (f(K_n) - f_1(K_n)) \xrightarrow[n \rightarrow +\infty]{L^2} 0 \quad (36)$$

then

$$\frac{1}{\sqrt{\mathbb{E}[\mathcal{S}_n^2(K_n)]}} f_1(K_n) \xrightarrow[n \rightarrow +\infty]{L^2} 0. \quad (37)$$

More precisely we will prove the mean-squared convergence (36), using the previous decomposition used in proof of Lemma 3.1. Using (41) and (35) it follows that:

$$\mathbb{E} \left( \frac{1}{\sqrt{\mathbb{E}[\mathcal{S}_n^2(K_n)]}} \mathbf{X}_n^*(K_n) \Sigma^{-1}(K_n) \frac{1}{n - K_n + 1} \left[ \sum_{j=K_n}^{n-\sqrt{n}-1} \mathbf{X}_j(K_n) \varepsilon_{j+1} - \sum_{j=K_n}^{n-1} \mathbf{X}_j(K_n) \varepsilon_{j+1} \right] \right)^2 = O \left( \frac{K_n^{3+2d}}{n^{5/4}} \right).$$

Under assumption (33), it converges to 0.

Similarly using (45) and (35):

$$\mathbb{E} \left( \frac{1}{\sqrt{\mathbb{E}[\mathcal{S}_n^2(K_n)]}} \mathbf{X}_n^*(K_n) \left[ \Sigma^{-1}(K_n) - \widehat{\Sigma}_n^{-1}(K_n) \right] \sum_{j=K_n}^{n-1} \mathbf{X}_j(K_n) \varepsilon_{j+1} \right)^2 = O \left( \frac{K_n^{5+2d}}{(n - K_n + 1)^2} \right)$$

which converges to 0 under the assumption (33).

For the third term by (49) and (35) we obtain:

$$\mathbb{E} \left( \frac{1}{\sqrt{\mathbb{E}[\mathcal{S}_n^2(K_n)]}} \left[ \mathbf{X}_n^*(K_n) - \mathbf{X}'_n(K_n) \right] \widehat{\Sigma}_n^{-1}(K_n) \sum_{j=K_n}^{n-1} \mathbf{X}_j(K_n) \varepsilon_{j+1} \right)^2 = \mathcal{O} \left( \frac{K_n^{3+2d}}{(n - K_n + 1)^{\frac{3-2d}{2}}} \right)$$

which converges to 0 under the assumption (33).

Finally the estimation of the fourth term is directly given in (52):

$$\begin{aligned} & \mathbb{E} \left( \sqrt{\frac{1}{\mathbb{E}(\mathcal{S}_n(K_n))^2}} \mathbf{X}'_n(K_n) \widehat{\Sigma}_n^{-1}(K_n) \frac{1}{n - K_n + 1} \sum_{j=K_n}^{n-1} \mathbf{X}_j(K_n) [\varepsilon_{j+1, K_n} - \varepsilon_{j+1}] \right)^2 \\ &= \mathcal{O} \left( \frac{K_n^{1+2d+\delta}}{(n - K_n + 1)^{1-2d-\delta}} \right) \end{aligned}$$

which converges to 0 under the condition (33).

Now we will prove the mean-squared convergence expressed in (37).

By (31):

$$\mathbb{E} \left( \frac{n - K_n + 1}{K_n \sigma_\varepsilon^2} f_1^2(K_n) \right) \underset{n \rightarrow +\infty}{\sim} \frac{K_n \sigma_\varepsilon^2}{n - K_n + 1}.$$

Under the condition (33), the bound (35) implies:

$$\frac{1}{\mathbb{E}[\mathcal{S}_n^2(K_n)]} \frac{K_n \sigma_\varepsilon^2}{n - K_n + 1} \xrightarrow{n \rightarrow +\infty} 0.$$

Then we have:

$$\lim_{n \rightarrow +\infty} \frac{1}{\mathbb{E}[\mathcal{S}_n^2(K_n)]} \mathbb{E}(f_1^2(K_n)) = 0.$$

**Second step** Since  $(X_t)_{t \in \mathbb{Z}}$  is a Gaussian process with mean 0,  $(\sum_{i=1}^l (a_i - a_{i, K_n}) X_{t+1-i})_{t \in \mathbb{Z}}$  is a Gaussian random variable with mean 0 for any integer  $l$ . But  $(\sum_{i=1}^l (a_i - a_{i, K_n}) X_{t+1-i})_{t \in \mathbb{Z}}$  converges in mean-squared sense and thus in distribution to  $\mathcal{S}_n(K_n)$  as  $l$  tends to infinity. Then  $\mathcal{S}_n(K_n)$  is Gaussian random variable with mean 0. □

**Remark 1** The normalisation in Theorem 3 is not an explicit function of  $K_n$ . Nevertheless we have a good idea of its rate of convergence to 0. We have shown in (35) that:

$$\exists C, CK_n^{-2d-1} \leq \mathbb{E}[\mathcal{S}_n^2(K_n)]$$

under assumption H.4. In Godet (2007a), an upper bound for the rate of convergence is proved [see Theorem 3.3.1] assuming H.1-H.2

$$\exists C, \mathbb{E}[\mathcal{S}_n^2(K_n)] \leq CK_n^{-1}.$$

In the case of fractionally integrated noise, i.e.  $(X_t)_{t \in \mathbb{Z}}$  is the stationary solution of the difference equation:

$$(1 - B)^d X_t = \varepsilon_t$$

where  $(\varepsilon_t)_{t \in \mathbb{Z}}$  is a white noise with mean 0 and constant finite variance  $\sigma_\varepsilon^2$  and  $B$  is the backward-shift operator. The rate of convergence is equal to:

$$\exists C, \mathbb{E}[\mathcal{S}_n^2(K_n)] \sim CK_n^{-1}.$$

**Remark 2** In central limit theorems for short memory processes in univariate case Bhansali (1978) or in multivariate case Lewis and Reinsel (1985), the normalisation is an explicit function of  $K_n$  and  $n$ . It comes from the rate of convergence of the forecast coefficients to the autoregressive representation coefficients. In the long memory case the normalisation is given by the rate of convergence of the predictor knowing a finite past to the linear least square predictor knowing the infinite past. It can be explain. In the short memory case, the rate of convergence due to the projection decays exponentially fast and is negligible in view of the rate of convergence due to the estimation of the forecast coefficients. In the long memory case the rate of convergence due to the projection decays hyperbolically and is then the main term of the global error of prediction.

## 5 Appendix

### 5.1 Preliminary lemmas

In the following lemmas we prove subsidiary asymptotic results, which we need in proof of Theorem 2.

**Lemma 5.1.** *Assume H.2. If  $q \geq 1$ , then:*

$$\forall \delta > 0, \mathbb{E} \left\| \frac{1}{\sqrt{n - K_n + 1}} \sum_{j=K_n}^{n-1} \mathbf{X}_j(k) (\varepsilon_{j+1,k} - \varepsilon_{j+1}) \right\|^q \leq C \left( k(n - K_n + 1)^{2d+\delta} \mathbb{E} (\mathcal{S}_n(k))^2 \right)^{q/2} \quad (38)$$

*Proof.* We have

$$\frac{1}{\sqrt{n - K_n + 1}} \sum_{j=K_n}^{n-1} \mathbf{X}_j(k) (\varepsilon_{j+1,k} - \varepsilon_{j+1}) = \frac{1}{\sqrt{n - K_n + 1}} \sum_{j=K_n}^{n-1} \mathbf{X}_j(k) \sum_{i=1}^{+\infty} (a_i - a_{i,k}) X_{j+1-i}.$$

Without loss of generality, we assume that  $q > 2$  since the result for  $q > 1$  can be obtained from the result for  $q > 2$  and Jensen's inequality. Observe that:

$$\begin{aligned} & \left\| \frac{1}{\sqrt{n - K_n + 1}} \sum_{j=K_n}^{n-1} \mathbf{X}_j(k) \sum_{i=1}^{+\infty} (a_i - a_{i,k}) X_{j+1-i} \right\|^q \\ &= (n - K_n + 1)^{-q/2} \left( \sum_{l=0}^{k-1} \left( \sum_{j=K_n}^{n-1} X_{j-l} \sum_{i=1}^{+\infty} (a_i - a_{i,k}) X_{j+1-i} \right)^2 \right)^{q/2} \end{aligned}$$

Since the function  $x \mapsto x^{q/2}$  is convex on  $\mathbb{R}^+$  if  $q > 2$ , we obtain by Jensen's inequality:

$$\left( \sum_{l=0}^{k-1} \left( \sum_{j=K_n}^{n-1} X_{j-l} \sum_{i=1}^{+\infty} (a_i - a_{i,k}) X_{j+1-i} \right)^2 \right)^{q/2} \leq k^{-1} \sum_{l=0}^{k-1} k^{q/2} \left| \sum_{j=K_n}^{n-1} X_{j-l} \sum_{i=1}^{+\infty} (a_i - a_{i,k}) X_{j+1-i} \right|^q.$$



Consequently

$$\begin{aligned} & \mathbb{E} \left\| \frac{1}{\sqrt{n - K_n + 1}} \sum_{j=K_n}^{n-1} \mathbf{X}_j(k) (\varepsilon_{j+1,k} - \varepsilon_{j+1}) \right\|^q \\ & \leq k^{q/2-1} \sum_{l=0}^{k-1} \mathbb{E} \left( (n - K_n + 1)^{-q/2} \left| \sum_{j=K_n}^{n-1} X_{j-l} \sum_{i=1}^{+\infty} (a_i - a_{i,k}) X_{j+1-i} \right|^q \right). \end{aligned} \quad (39)$$

Furthermore

$$\begin{aligned} \mathbb{E} \left( X_{j-l} \sum_{i=1}^{+\infty} (a_i - a_{i,k}) X_{j+1-i} \right) &= \mathbb{E} \left( X_{j-l} \left[ \varepsilon_{j+1} - X_{j+1} - \sum_{i=1}^k a_{i,k} X_{j+1-i} \right] \right) \\ &= -\sigma_{l+1} - \sum_{i=1}^k a_{i,k} \sigma_{l+1-i} \\ &= 0 \end{aligned} \quad (40)$$

for any  $l \in \llbracket 1, k \rrbracket$  by definition of  $(a_{i,k})_{1 \leq i \leq k}$ . By applying Theorem 1 of Ing and Wei (2003), we obtain a bound for  $\mathbb{E} \left( (n - K_n + 1)^{-q/2} \left| \sum_{j=K_n}^{n-1} X_{j-l} \sum_{i=1}^{+\infty} (a_i - a_{i,k}) X_{j+1-i} \right|^q \right)$  since the mean defined in (40) is equal to 0. We have:

$$\begin{aligned} & \mathbb{E} \left( (n - K_n + 1)^{-q/2} \left| \sum_{j=K_n}^{n-1} X_{j-l} \sum_{i=1}^{+\infty} (a_i - a_{i,k}) X_{j+1-i} \right|^q \right) \\ & \leq C \left( \frac{1}{n - K_n + 1} \sum_{s=K_n}^{n-1} \sum_{t=K_n}^{n-1} \sigma(s-t) \sigma^*(s-t) \right)^{q/2}, \end{aligned}$$

where  $\sigma^*(\cdot)$  is the autocovariance function of the process  $(\sum_{i=1}^{+\infty} (a_i - a_{i,k}) X_{t+1-i})_{t \in \mathbb{Z}}$  i.e.

$$\sigma^*(s-t) = \mathbb{E} \left[ \left( \sum_{i=1}^{+\infty} (a_i - a_{i,k}) X_{s+1-i} \right) \left( \sum_{i=1}^{+\infty} (a_i - a_{i,k}) X_{t+1-i} \right) \right].$$

We use as bound for  $\sigma^*(s-t)$  the variance  $\sigma^*(0)$ :

$$\begin{aligned} & \mathbb{E} \left( (n - K_n + 1)^{-q/2} \left| \sum_{j=K_n}^{n-1} X_{j-l} \sum_{i=1}^{+\infty} (a_i - a_{i,k}) X_{j+1-i} \right|^q \right) \\ & \leq C \left( \frac{1}{n - K_n + 1} \sigma^*(0) \sum_{s=K_n}^{n-1} \sum_{t=K_n}^{n-1} |\sigma(s-t)| \right)^{q/2} \\ & \leq C \left( \frac{1}{n - K_n + 1} \sigma^*(0) \sum_{s=1}^{n-K_n+1} \sum_{t=1}^{n-K_n+1} |\sigma(s-t)| \right)^{q/2}. \end{aligned}$$

Under assumption H.2, we have for any  $\delta > 0$

$$\sum_{s=1}^{n-K_n+1} \sum_{t=1}^{n-K_n+1} |\sigma(s-t)| \leq C(n - K_n + 1)^{2d+\delta+1}.$$

Then we have shown

$$\forall \delta > 0, \mathbb{E} \left( (n - K_n + 1)^{-q/2} \left| \sum_{j=K_n}^{n-1} X_{j-l} \sum_{i=1}^{+\infty} (a_i - a_{i,k}) X_{j+1-i} \right|^q \right) \leq C \left( (n - K_n + 1)^{2d+\delta} \sigma^*(0) \right)^{q/2}.$$

Notice that:

$$\sigma^*(0) = \mathbb{E} (\mathcal{S}_n(k))^2.$$

And this remark allows us to conclude.  $\square$

**Lemma 5.2.** *Assume that the assumptions of Theorem 2 hold. If  $q > 1$ , then*

$$\mathbb{E} \left\| \frac{1}{\sqrt{n - K_n + 1}} \sum_{j=K_n}^{n-1} \mathbf{X}_j(k) \varepsilon_{j+1} \right\|^q \leq C k^{q/2}.$$

*Proof.* The arguments are similar to those used for verifying Lemma 5.1. Without loss of generality we assume that  $q > 2$ , since this result and Jensen's inequality allow to conclude for  $q > 1$ . Reasoning as for (39), we have by convexity:

$$\mathbb{E} \left\| \frac{1}{\sqrt{n - K_n + 1}} \sum_{j=K_n}^{n-1} \mathbf{X}_j(k) \varepsilon_{j+1} \right\|^q \leq k^{q/2} k^{-1} \sum_{l=0}^{k-1} \mathbb{E} \left[ (n - K_n + 1)^{-q/2} \left| \sum_{j=K_n}^{n-1} X_{j-l} \varepsilon_{j+1} \right|^q \right].$$

Applying again Theorem 1 of Ing and Wei (2003):

$$\mathbb{E} \left[ (n - K_n + 1)^{-q/2} \left| \sum_{j=K_n}^{n-1} X_{j-l} \varepsilon_{j+1} \right|^q \right] \leq C \left( \frac{1}{n - K_n + 1} \sum_{s=K_n}^{n-1} \sum_{t=K_n}^{n-1} \sigma(s-t) \sigma_\varepsilon(s-t) \right)^{q/2}$$

where  $\sigma_\varepsilon(\cdot)$  is the autocovariance function of the process  $(\varepsilon_t)_{t \in \mathbb{Z}}$  i.e.

$$\sigma_\varepsilon(s-t) = \mathbb{E}(\varepsilon_t \varepsilon_s) = \begin{cases} 0 & \text{if } s \neq t \\ 1 & \text{otherwise} \end{cases}.$$

We obtain that:

$$\begin{aligned} \mathbb{E} \left[ (n - K_n + 1)^{-q/2} \left| \sum_{j=K_n}^{n-1} X_{j-l} \varepsilon_{j+1} \right|^q \right] &\leq C \left( \frac{1}{n - K_n + 1} (n - K_n + 1) \sigma(0) \right)^{q/2} \\ &= O(1) \end{aligned}$$

That concludes the proof.  $\square$

## 5.2 Proof of Lemma 3.1

We decompose the difference between  $f_1(k)$  and  $f(k)$  into 4 parts, which we estimate separately.

**Change the index of the sum of vectors** In this part we want to prove the mean-squared convergence of:

$$\begin{aligned} & \sqrt{\frac{n-K_n+1}{k}} \mathbf{X}_n^{*'}(k) \Sigma^{-1}(k) \frac{1}{n-K_n+1} \left[ \sum_{j=K_n}^{n-\sqrt{n}-1} \mathbf{X}_j(k) \varepsilon_{j+1} - \sum_{j=K_n}^{n-1} \mathbf{X}_j(k) \varepsilon_{j+1} \right] \\ &= \sqrt{\frac{1}{k}} \mathbf{X}_n^{*'} \Sigma^{-1}(k) \frac{1}{\sqrt{n-K_n+1}} \sum_{j=n-\sqrt{n}-1}^{n-1} \mathbf{X}_j(k) \varepsilon_{j+1}. \end{aligned} \quad (41)$$

The Hölder's inequality applying twice with  $1/p + 1/q = 1$  and  $1/p' + 1/q' = 1$  gives:

$$\begin{aligned} & \mathbb{E} \left( \sqrt{\frac{1}{k}} \mathbf{X}_n^{*'} \Sigma^{-1}(k) \frac{1}{\sqrt{n-K_n+1}} \sum_{j=n-\sqrt{n}-1}^{n-1} \mathbf{X}_j(k) \varepsilon_{j+1} \right)^2 \\ & \leq \left( \mathbb{E} \left\| \frac{1}{\sqrt{k}} \mathbf{X}_n^{*'} \Sigma^{-1}(k) \right\|^{2q} \right)^{1/q} \left( \mathbb{E} \left\| \frac{1}{\sqrt{n-K_n+1}} \sum_{j=n-\sqrt{n}-1}^{n-1} \mathbf{X}_j(k) \varepsilon_{j+1} \right\|^{2p} \right)^{1/p} \\ & \leq \left( \mathbb{E} \left\| \frac{1}{\sqrt{k}} \mathbf{X}_n^{*'} \right\|^{2q'q} \right)^{1/(q'q)} \left( \mathbb{E} \|\Sigma^{-1}(k)\|^{2p'q} \right)^{1/(p'q)} \left( \mathbb{E} \left\| \frac{1}{\sqrt{n-K_n+1}} \sum_{j=n-\sqrt{n}-1}^{n-1} \mathbf{X}_j(k) \varepsilon_{j+1} \right\|^{2p} \right)^{1/p}. \end{aligned}$$

Under assumption H.3, Theorem 1 establish that for all  $p'$  and  $q'$ :

$$\left( \mathbb{E} \|\Sigma^{-1}(k)\|^{2p'q} \right)^{1/(p'q)} = O(1). \quad (42)$$

Furthermore by Lemma 2 of Wei (1987) and the convexity of the function  $x \mapsto x^{2q'q}$  on  $\mathbb{R}^+$

$$\begin{aligned} \left( \mathbb{E} \left\| \frac{1}{\sqrt{k}} \mathbf{X}_n^{*'} \right\|^{2q'q} \right)^{1/(q'q)} & \leq C \left( \sum_{j=0}^{\sqrt{n}/2-K_n} b_j^2 \right)^{1/2} \\ & = O(1) \end{aligned} \quad (43)$$

because the sequence  $(b_j^2)_{j \in \mathbb{N}}$  is summable.

Finally by Lemma 5.2:

$$\begin{aligned} & \left( \mathbb{E} \left\| \frac{1}{\sqrt{n-K_n+1}} \sum_{j=n-\sqrt{n}-1}^{n-1} \mathbf{X}_j(k) \varepsilon_{j+1} \right\|^{2p} \right)^{1/p} \\ & \leq \frac{(n+1)^{1/4}}{\sqrt{n-K_n}} \left( \mathbb{E} \left\| \frac{1}{(n+1)^{1/4}} \sum_{j=n-\sqrt{n}-1}^{n-1} \mathbf{X}_j(k) \varepsilon_{j+1} \right\|^{2p} \right)^{1/p} \\ & = O\left(\frac{k}{n^{1/4}}\right) \end{aligned} \quad (44)$$

By inequalities (42), (43) and (44):

$$\mathbb{E} \left( \sqrt{\frac{1}{k}} \mathbf{X}_n^{*'} \Sigma^{-1}(k) \frac{1}{\sqrt{n - K_n + 1}} \sum_{j=n-\sqrt{n}-1}^{n-1} \mathbf{X}_j(k) \varepsilon_{j+1} \right)^2 = O \left( \frac{k}{n^{1/4}} \right)$$

which converges to 0 as  $n$  tends to infinity under assumption (28).

**Replace  $\Sigma^{-1}(k)$  by  $\widehat{\Sigma}_n^{-1}(k)$**  Prove that:

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left( \sqrt{\frac{n - K_n + 1}{k}} \mathbf{X}_n^{*'}(k) \left[ \Sigma^{-1}(k) - \widehat{\Sigma}_n^{-1}(k) \right] \sum_{j=K_n}^{n-1} \mathbf{X}_j(k) \varepsilon_{j+1} \right)^2 = 0. \quad (45)$$

Applying twice Hölder's inequality, we have:

$$\begin{aligned} & \mathbb{E} \left( \sqrt{\frac{n - K_n + 1}{k}} \mathbf{X}_n^{*'}(k) \left[ \Sigma^{-1}(k) - \widehat{\Sigma}_n^{-1}(k) \right] \sum_{j=K_n}^{n-1} \mathbf{X}_j(k) \varepsilon_{j+1} \right)^2 \\ & \leq \left( \mathbb{E} \left\| \frac{1}{\sqrt{k}} \mathbf{X}_n^{*'}(k) \right\|^{2q'} \right)^{1/(q'q)} \left( \mathbb{E} \left\| \Sigma^{-1}(k) - \widehat{\Sigma}_n^{-1}(k) \right\|^{2p'q} \right)^{1/(p'q)} \\ & \quad \left( \mathbb{E} \left\| \frac{1}{\sqrt{n - K_n + 1}} \sum_{j=K_n}^{n-1} \mathbf{X}_j(k) \varepsilon_{j+1} \right\|^{2p} \right)^{1/p}. \end{aligned}$$

Applying Lemma 5.2 we show that:

$$\left( \mathbb{E} \left\| \frac{1}{\sqrt{n - K_n + 1}} \sum_{j=K_n}^{n-1} \mathbf{X}_j(k) \varepsilon_{j+1} \right\|^{2p} \right)^{1/p} = O(K_n). \quad (46)$$

Now we derive the mean-squared convergence to 0 for each case of  $d$ .

For  $d \in ]0, 1/4[$ , we apply Theorem 1 and we have:

$$\left( \mathbb{E} \left\| \Sigma^{-1}(k) - \widehat{\Sigma}_n^{-1}(k) \right\|^{2p'q} \right)^{1/(p'q)} = O \left( \frac{K_n^2}{n - K_n + 1} \right). \quad (47)$$

And follows from the three inequalities (43), (46) and (47) that:

$$\mathbb{E} \left( \sqrt{\frac{n - K_n + 1}{k}} \mathbf{X}_n^{*'}(k) \left[ \Sigma^{-1}(k) - \widehat{\Sigma}_n^{-1}(k) \right] \sum_{j=K_n}^{n-1} \mathbf{X}_j(k) \varepsilon_{j+1} \right)^2 = O \left( \frac{K_n^3}{n - K_n + 1} \right)$$

which converges to 0 if condition (28) holds.

If  $d \in ]1/4, 1/2[$ , we obtain by Theorem 1 that

$$\left( \mathbb{E} \left\| \Sigma^{-1}(k) - \widehat{\Sigma}_n^{-1}(k) \right\|^{2p'q} \right)^{1/(p'q)} = O \left( \frac{K_n^2}{(n - K_n + 1)^{2-4d}} \right). \quad (48)$$

The inequalities (43),(46) and (48) allow us to conclude that:

$$\mathbb{E} \left( \sqrt{\frac{n - K_n + 1}{k}} \mathbf{X}_n^{*'}(k) \left[ \Sigma^{-1}(k) - \widehat{\Sigma}_n^{-1}(k) \right] \sum_{j=K_n}^{n-1} \mathbf{X}_j(k) \varepsilon_{j+1} \right)^2 = \mathcal{O} \left( \frac{K_n^3}{(n - K_n + 1)^{2-4d}} \right)$$

which converges to 0 under assumption (28).

**Replace  $\mathbf{X}_n^{*'}(k)$  by  $\mathbf{X}'_n(k)$**  Prove that:

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left( \sqrt{\frac{n - K_n + 1}{k}} \left[ \mathbf{X}_n^{*'}(k) - \mathbf{X}'_n(k) \right] \widehat{\Sigma}_n^{-1}(k) \sum_{j=K_n}^{n-1} \mathbf{X}_j(k) \varepsilon_{j+1} \right)^2 = 0. \quad (49)$$

Using twice Holder's inequality, we have:

$$\begin{aligned} & \mathbb{E} \left( \sqrt{\frac{n - K_n + 1}{k}} \left[ \mathbf{X}_n^{*'}(k) - \mathbf{X}'_n(k) \right] \widehat{\Sigma}_n^{-1}(k) \sum_{j=K_n}^{n-1} \mathbf{X}_j(k) \varepsilon_{j+1} \right)^2 \\ & \leq \left( \mathbb{E} \left\| \frac{1}{\sqrt{k}} \left[ \mathbf{X}_n^{*'}(k) - \mathbf{X}'_n(k) \right] \right\|^{2q'} \right)^{1/(q'q)} \left( \mathbb{E} \left\| \widehat{\Sigma}_n^{-1}(k) \right\|^{2p'q} \right)^{1/(p'q)} \\ & \quad \left( \mathbb{E} \left\| \frac{1}{\sqrt{n - K_n + 1}} \sum_{j=K_n}^{n-1} \mathbf{X}_j(k) \varepsilon_{j+1} \right\|^{2p} \right)^{1/p}. \end{aligned}$$

In view of Lemma 2 of Wei (1987) and the convexity of the function  $x \mapsto x^{qq'}$ , we have:

$$\begin{aligned} & \max_{1 \leq k \leq K_n} \left( \mathbb{E} \left\| \frac{1}{\sqrt{k}} \left[ \mathbf{X}_n^{*'}(k) - \mathbf{X}'_n(k) \right] \right\|^{2q'} \right)^{1/(q'q)} \\ & = \max_{1 \leq k \leq K_n} \left( \mathbb{E} \left( \frac{1}{k} \sum_{l=0}^{k-1} \left( \sum_{j=\sqrt{n}/2 - K_n + 1}^{+\infty} b_j \varepsilon_{n-j-l} \right)^2 \right)^{q'q} \right)^{1/(q'q)} \\ & \leq C \left( \sum_{j=\sqrt{n}/2 - K_n + 1}^{+\infty} b_j^2 \right) \\ & \leq C n^{\frac{2d-1}{2}}. \end{aligned} \quad (50)$$

Then by Theorem 1:

$$\left( \mathbb{E} \left\| \widehat{\Sigma}_n^{-1}(k) \right\|^{2p'q} \right)^{1/(p'q)} = \mathcal{O}(1). \quad (51)$$

By inequalities (46), (50) and (51), we obtain:

$$\mathbb{E} \left( \sqrt{\frac{n - K_n + 1}{k}} \left[ \mathbf{X}_n^{*'}(k) - \mathbf{X}'_n(k) \right] \widehat{\Sigma}_n^{-1}(k) \sum_{j=K_n}^{n-1} \mathbf{X}_j(k) \varepsilon_{j+1} \right)^2 = \mathcal{O} \left( K_n n^{\frac{2d-1}{2}} \right)$$

which converges to 0 as  $n$  tends to infinity if condition (28) holds.

**Replace  $\varepsilon_{j,k}$  by  $\varepsilon_j$**  We want to prove that:

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left( \sqrt{\frac{1}{\mathbb{E}(\mathcal{S}_n(k))^2}} \mathbf{X}'_n(k) \widehat{\Sigma}_n^{-1}(k) \frac{1}{n - K_n + 1} \sum_{j=K_n}^{n-1} \mathbf{X}_j(k) [\varepsilon_{j+1,k} - \varepsilon_{j+1}] \right)^2 = 0. \quad (52)$$

Using twice Hölder's inequality, we have:

$$\begin{aligned} & \mathbb{E} \left( \sqrt{\frac{1}{\mathbb{E}(\mathcal{S}_n(k))^2}} \mathbf{X}'_n(k) \widehat{\Sigma}_n^{-1}(k) \frac{1}{n - K_n + 1} \sum_{j=K_n}^{n-1} \mathbf{X}_j(k) [\varepsilon_{j+1,k} - \varepsilon_{j+1}] \right)^2 \\ & \leq \frac{1}{(n - K_n + 1) \mathbb{E}(\mathcal{S}_n(k))^2} \left( \mathbb{E} \left\| \frac{1}{\sqrt{n - K_n + 1}} \sum_{j=K_n}^{n-1} \mathbf{X}_j(k) [\varepsilon_{j+1,k} - \varepsilon_{j+1}] \right\|^{2q'} \right)^{1/(q'q)} \\ & \quad \left( \mathbb{E} \left\| \widehat{\Sigma}_n^{-1}(k) \right\|^{2p'q} \right)^{1/(p'q)} \left( \mathbb{E} \left\| \mathbf{X}'_n(k) \right\|^{2p} \right)^{1/p}. \end{aligned}$$

Applying Lemma 5.1, we prove that:

$$\forall \delta > 0, \left( \mathbb{E} \left\| \frac{1}{\sqrt{n - K_n + 1}} \sum_{j=K_n}^{n-1} \mathbf{X}_j(k) [\varepsilon_{j+1,k} - \varepsilon_{j+1}] \right\|^{2q'} \right)^{1/(q'q)} = O \left( k(n - K_n + 1)^{2d+\delta} \mathbb{E}(\mathcal{S}_n(k))^2 \right). \quad (53)$$

Finally we choose  $p = 2$  and we have:

$$\begin{aligned} \left( \mathbb{E} \left\| \mathbf{X}'_n(k) \right\|^4 \right)^{1/2} &= \sqrt{\mathbb{E} \left( \sum_{j=1}^k X_j^2 \right)^2} \\ &= \sqrt{\left( \sum_{j=1}^k \sigma(j) \right)^2 + 2 \sum_{j=1}^k \sum_{l=1}^k \sigma(j-l)^2} \end{aligned}$$

since the process  $(X_t)_{t \in \mathbb{Z}}$  is Gaussian. Using the assumption on the covariances H.2, we verify that:

$$\begin{aligned} \forall \delta > 0, \left( \mathbb{E} \left\| \mathbf{X}'_n(k) \right\|^4 \right)^{1/2} &\leq C \sqrt{k^{4d+\delta}} \\ &\leq C \sqrt{k} \end{aligned} \quad (54)$$

if  $d \in ]0, 1/4[$  with  $C$  constant. With these three inequalities(51), (53) and (54), we conclude that:

$$\begin{aligned} \forall \delta > 0, & \quad \mathbb{E} \left( \sqrt{\frac{1}{\mathbb{E}(\mathcal{S}_n(k))^2}} \mathbf{X}'_n(k) \widehat{\Sigma}_n^{-1}(k) \frac{1}{n - K_n + 1} \sum_{j=K_n}^{n-1} \mathbf{X}_j(k) (\varepsilon_{j+1,k} - \varepsilon_{j+1}) \right)^2 \\ & \leq C \frac{1}{(n - K_n + 1) \mathbb{E}(\mathcal{S}_n(k))^2} \left( \sqrt{k} k(n - K_n + 1)^{2d+\delta} \mathbb{E}(\mathcal{S}_n(k))^2 \right) \\ & \leq C \frac{K_n^{3/2}}{(n - K_n + 1)^{1-2d-\delta}}. \end{aligned}$$

which converges to 0 under condition (28).

On the other hand if  $d \in ]1/4, 1/2[$ , inequality (54) becomes:

$$\forall \delta > 0, \left( \mathbb{E} \|\mathbf{X}'_n(k)\|^4 \right)^{1/2} \leq Ck^{4d+\delta} \quad (55)$$

Using inequalities (51), (53) and (55), we have for all  $\delta > 0$

$$\mathbb{E} \left( \sqrt{\frac{1}{\mathbb{E}(\mathcal{S}_n(k))^2}} \mathbf{X}'_n(k) \widehat{\Sigma}_n^{-1}(k) \frac{1}{n - K_n + 1} \sum_{j=K_n}^{n-1} \mathbf{X}_j(k) [\varepsilon_{j+1,k} - \varepsilon_{j+1}] \right)^2 \leq O \left( \frac{K_n^{1+2d+\delta}}{(n - K_n + 1)^{1-2d-\delta}} \right)$$

which converges to 0 under condition (28).

We have proved in each case for  $d$  that:

$$\lim_{n \rightarrow +\infty} \max_{1 \leq k \leq K_n} \mathbb{E} \left( \sqrt{\frac{1}{L_n(k)}} (f(k) - f_1(k)) \right)^2 = 0.$$

## References

- Bhansali, R. (1978). Linear prediction by autoregressive model fitting in the time domain. *Ann. Stat.*, 6:224–231.
- Bingham, N., Goldie, C., and Teugels, J. L. (1987). *Regular variation*. Encyclopedia of Mathematics and its applications, Vol. 27. Cambridge etc.: Cambridge University Press.
- Brockwell, P. and Davis, R. (1988). Simple consistent estimation of the coefficients of a linear filter. *Stochastic Processes and their Applications*.
- Ciarlet, P. (1982). *Introduction à l'analyse numérique matricielle et à l'optimisation*. Collection Mathématiques Appliquées pour la Maîtrise. Paris.
- Dahlhaus, R. (1989). Efficient parameter estimation for self-similar processes. *The Annals of Statistics*.
- Fox, R. and Taqqu, M. S. (1987). Central limit theorems for quadratic forms in random variables having long-range dependence. *Probab. Theory Relat. Fields*, 74:213–240.
- Godet, F. (2007a). Linear prediction of long-memory processes: Asymptotic results on mean-squared errors. <http://arxiv.org/abs/0705.1927v1>.
- Godet, F. (2007b). Linear Prediction of Long-Range Dependent Time Series. <http://arxiv.org/abs/math/0702485>.
- Grenander, U. and Szegő, G. (1958). *Toeplitz forms and their applications*. (California Monographs in Mathematical Sciences). Berkeley and Los Angeles: University of California Press VIII .
- Ing, C.-K. and Wei, C.-Z. (2003). On same-realization prediction in an infinite-order autoregressive process. *J. Multivariate Anal.*, 85(1):130–155.

- Kunitomo, N. and Yamamoto, T. (1985). Properties of predictors in misspecified autoregressive time series models. *J. Amer. Statist. Assoc.*, 80(392):941–950.
- Lewis, R. and Reinsel, G. (1985). Prediction of multivariate time series by autoregressive model fitting. *Journal of multivariate analysis*.
- Rosenblatt, M. (1979). Some limit theorems for partial sums of quadratic forms in stationary Gaussian variables. *Z. Wahrscheinlichkeitstheor. Verw. Geb.*, 49:125–132.
- Wei, C.-Z. (1987). Adaptive prediction by least-squares predictors in stochastic regression models with applications to time series. *Ann. Stat.*, 15(4):1667–1682.
- Zygmund, A. (1968). *Trigonometric series*. Cambridge University Press.