



**HAL**  
open science

# Relaxation Approximation of some Initial-Boundary Value Problem for p-Systems

Gilles Carbou, Bernard Hanouzet

► **To cite this version:**

Gilles Carbou, Bernard Hanouzet. Relaxation Approximation of some Initial-Boundary Value Problem for p-Systems. Communications in Mathematical Sciences, 2007, 5 (1), pp.187-203. hal-00196163

**HAL Id: hal-00196163**

**<https://hal.science/hal-00196163>**

Submitted on 12 Dec 2007

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# RELAXATION APPROXIMATION OF SOME INITIAL-BOUNDARY VALUE PROBLEM FOR P-SYSTEMS \*

GILLES CARBOU † AND BERNARD HANOUZET ‡

**Abstract.** We consider the Suliciu model which is a relaxation approximation of the  $p$ -system. In the case of the Dirichlet boundary condition we prove that the local smooth solution of the  $p$ -system is the zero limit of the Suliciu model solutions.

**Key words.** Zero relaxation limit,  $p$ -system, Suliciu model, boundary conditions.

**subject classifications.** 35L50, 35Q72, 35B25.

## 1. Introduction

We study a relaxation approximation of the following  $p$ -system

$$\begin{cases} \partial_t u_1 - \partial_x u_2 = 0, \\ \partial_t u_2 - \partial_x p(u_1) = 0. \end{cases} \quad (1.1)$$

For the viscoelastic case, Suliciu introduces in [19] the following approximation

$$\begin{cases} \partial_t u_1 - \partial_x u_2 = 0, \\ \partial_t u_2 - \partial_x v = 0, \\ \partial_t v - \mu \partial_x u_2 = \frac{1}{\varepsilon} (p(u_1) - v), \end{cases} \quad (1.2)$$

where  $\varepsilon$  and  $\mu$  are positive.

The aim of this paper is to prove convergence results for the initial-boundary value problem when the relaxation coefficient  $\varepsilon$  tends to zero.

Under the classical assumption

$$\forall \xi \in \mathbb{R}, p'(\xi) > 0, \quad (1.3)$$

the  $p$ -system is strictly hyperbolic with eigenvalues

$$\lambda_1(u_1) = -\sqrt{p'(u_1)} < \lambda_2(u_1) = \sqrt{p'(u_1)}. \quad (1.4)$$

The semi-linear approximation system (1.2) is strictly hyperbolic with 3 constant eigenvalues

$$\mu_1 = -\sqrt{\mu} < \mu_2 = 0 < \mu_3 = \sqrt{\mu}. \quad (1.5)$$

In all the paper we assume that  $\mu$  is chosen great enough so that the subcharacteristic-type condition holds

$$\mu > p'(u_1) \quad (1.6)$$

---

\*

† Université Bordeaux 1, Institut de Mathématiques de Bordeaux, 351 cours de la Libération, 33405 Talence cedex, France, (carbou@math.u-bordeaux1.fr).

‡ Université Bordeaux 1, Institut de Mathématiques de Bordeaux, 351 cours de la Libération, 33405 Talence cedex, France, (hanouzet@math.u-bordeaux1.fr).

for all the values of  $u_1$  under consideration.

Formally, when  $\varepsilon$  tends to zero, the behaviour of the solution  $w^\varepsilon = (u^\varepsilon, v^\varepsilon) = ((u_1^\varepsilon, u_2^\varepsilon), v^\varepsilon)$  for the relaxation system (1.2) is the following:  $p(u_1^\varepsilon) - v^\varepsilon$  tends to zero, so that  $u^\varepsilon$  tends to a solution  $u = (u_1, u_2)$  of the p-system (1.1).

Recent papers are devoted to the zero relaxation limit in the case of the Cauchy problem. In [22] Wen-An Yong establishes a general framework to study the strong convergence for the smooth solutions. This convergence result is obtained describing the boundary layer which appears at  $t=0$ . We can apply Yong's tools for the Suliciu approximation

$$\begin{cases} \partial_t u_1^\varepsilon - \partial_x u_2^\varepsilon = 0, \\ \partial_t u_2^\varepsilon - \partial_x v^\varepsilon = 0, \\ \partial_t v^\varepsilon - \mu \partial_x u_2^\varepsilon = \frac{1}{\varepsilon} (p(u_1^\varepsilon) - v^\varepsilon), \end{cases} \quad (1.7)$$

for  $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$ , with the smooth initial data:

$$w^\varepsilon(0, x) = w_0(x), x \in \mathbb{R}. \quad (1.8)$$

We give more details about this question in the annex at the end of this paper.

Since the lifespan for a smooth solution  $u$  of the Cauchy problem for the p-system is generally finite (see [12]), the strong convergence of the solution  $u^\varepsilon$  to  $u$  can only be obtained locally in time. Nevertheless, under the assumption

$$\forall \xi \in \mathbb{R}, p'(\xi) \leq \Gamma < \mu, \quad (1.9)$$

if  $w_0$  is smooth, the solution for the semi-linear Cauchy problem (1.7)-(1.8) is global and smooth. In this case, the question is: what about the global convergence ?

Under further additional assumptions (in particular  $p'(\xi) \geq \gamma > 0$ ) the weak convergence to a global weak solution of the p-system is obtained by Tzavaras in [21] using the compactness methods of [17].

Other convergence results in some particular cases can be found in [8] and [10].

For other connected papers see also [13, 16, 20]...

In this paper we study the zero relaxation limit for the initial-boundary value problem. To our knowledge general convergence results are not available for hyperbolic relaxation systems in domains with boundary in the literature. A special well investigated problem is the semi-linear relaxation approximation to the boundary value problem for a scalar quasilinear equation, see [11, 15, 9, 14], and [5, 1] for related numerical considerations.

A first example of convergence result for a particular p-system (1.1) is obtained in [4]. In that paper the p-system is the one-dimensionnal Kerr model, so  $p$  is the inverse function of  $\xi \mapsto (1 + \xi^2)\xi$ . The relaxation approximation is given by the Kerr-Debye model which is the following quasilinear hyperbolic system

$$\begin{cases} \partial_t u_1^\varepsilon - \partial_x u_2^\varepsilon = 0, \\ \partial_t u_2^\varepsilon - \partial_x ((1 + v^\varepsilon)^{-1} u_1^\varepsilon) = 0, \\ \partial_t v^\varepsilon = \frac{1}{\varepsilon} ((1 + v^\varepsilon)^{-2} (u_1^\varepsilon)^2 - v^\varepsilon). \end{cases}$$

For these two models we consider the ingoing wave boundary condition. In the case of the smooth solutions we obtained a local strong convergence result. The main tool of the proof is the use of the entropic variables as proposed in [7]. In these variables, the system is symmetrized and the equilibrium manifold is linearized.

Here we study the zero relaxation limit for the Suliciu approximation

$$\begin{cases} \partial_t u_1^\varepsilon - \partial_x u_2^\varepsilon = 0, \\ \partial_t u_2^\varepsilon - \partial_x v^\varepsilon = 0, \\ \partial_t v^\varepsilon - \mu \partial_x u_2^\varepsilon = \frac{1}{\varepsilon} (p(u_1^\varepsilon) - v^\varepsilon), \end{cases} \quad (1.10)$$

for  $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^+$ , with the null initial data

$$w^\varepsilon(0, x) = 0, x \in \mathbb{R}^+, \quad (1.11)$$

and with the Dirichlet boundary condition

$$u_2^\varepsilon(t, 0) = \varphi(t), t \in \mathbb{R}^+. \quad (1.12)$$

For the null initial data to be in equilibrium we assume that  $p(0) = 0$ . We prove the strong convergence of  $u^\varepsilon$  to the smooth solution of the initial-boundary value problem for the p-system

$$\begin{cases} \partial_t u_1 - \partial_x u_2 = 0, \\ \partial_t u_2 - \partial_x p(u_1) = 0, \end{cases} \quad (1.13)$$

for  $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^+$ , with the initial-boundary conditions

$$u(0, x) = 0, x \in \mathbb{R}^+, \quad (1.14)$$

$$u_2(t, 0) = \varphi(t), t \in \mathbb{R}^+. \quad (1.15)$$

## 2. Main Results

Let us specify the assumptions on the source term  $\varphi$  in the boundary condition (1.12) or (1.15). In order to simplify we chose  $\varphi$  smooth enough on  $\mathbb{R}$  and such that  $\text{supp } \varphi \subset [0, b]$ , with  $b > 0$ . In this case the boundary conditions and the null initial data (1.11) and (1.14) match each other so both initial-boundary value problem (1.10)-(1.11)-(1.12) and (1.13)-(1.14)-(1.15) admit local smooth solutions.

First we consider the solutions for the second problem (1.13)-(1.14)-(1.15) and using the methods of [12] we establish that the lifespan  $T^*$  is generally finite with formation of shock waves.

**THEOREM 2.1.** *Assume the property (1.3). Let  $\varphi \in C^\infty(\mathbb{R})$  with  $\text{supp } \varphi \subset [0, b]$ ,  $b > 0$ ,  $\varphi \neq 0$ . Let  $g$  the function defined by*

$$g(\xi) = \int_0^\xi \sqrt{p'(s)} ds.$$

*We assume that*

$$p'' \text{ does not vanish on the interval } g^{-1}(-\varphi(\mathbb{R})). \quad (2.1)$$

Then the local smooth solution of (1.13)-(1.14)-(1.15) exhibits a shock wave at the time  $T^* < +\infty$  and we have

$$\|u\|_{L^\infty([0, T^*] \times \mathbb{R}^+)} \leq C \|\varphi\|_{L^\infty(\mathbb{R})}. \quad (2.2)$$

We now investigate the smooth solutions of the initial-boundary value problem (1.10)-(1.11)-(1.12) for a fixed  $\varepsilon > 0$ . The system is semi-linear strictly hyperbolic and the boundary  $\{x=0\}$  is characteristic. It is easy to prove that the local smooth solution  $w$  exists and, if the lifespan  $T_\varepsilon^*$  is finite, we have

$$\|w\|_{L^\infty([0, T_\varepsilon^*] \times \mathbb{R}^+)} = +\infty \quad (2.3)$$

(for general semi-linear hyperbolic systems, see [18]).

If we assume that  $p$  is globally lipschitz we establish that the smooth solutions are global.

**THEOREM 2.2.** *Assume the properties (1.3) and (1.9). Let  $\varphi \in H^3(\mathbb{R})$  with  $\text{supp } \varphi \subset \mathbb{R}^+$ . Then the solution of (1.10)-(1.11)-(1.12) is global and*

$$w \in \mathcal{C}^0(\mathbb{R}^+; H^1(\mathbb{R})), \partial_t w \in \mathcal{C}^0(\mathbb{R}^+; L^2(\mathbb{R})). \quad (2.4)$$

Finally, let us describe the convergence result.

**THEOREM 2.3.** *We suppose (1.3). Let  $\varphi \in H^3(\mathbb{R})$  with  $\text{supp } \varphi \subset \mathbb{R}^+$ . We consider a smooth solution  $u = (u_1^0, u_2^0)$  of (1.13)-(1.14)-(1.15) defined on  $[0, T^*]$ . We suppose that*

$$\mu > \sup_{(t,x) \in [0, T^*] \times \mathbb{R}^+} p'(u_1^0(t, x)). \quad (2.5)$$

*Let  $T < T^*$ . For  $\varepsilon$  small enough, the relaxation problem (1.10)-(1.11)-(1.12) admits a solution  $w^\varepsilon = (u^\varepsilon, v^\varepsilon)$  defined on  $[0, T]$  such that*

$$u^\varepsilon = u^0 + \varepsilon u_\varepsilon^1,$$

*and there exists a constant  $K$  such that*

$$\|u_\varepsilon^1\|_{L^\infty(0, T; H^1(\mathbb{R}^+))} + \|\partial_t u_\varepsilon^1\|_{L^\infty(0, T; L^2(\mathbb{R}^+))} \leq K. \quad (2.6)$$

In this result we can remark that no boundary layer appears in the time variable because the null initial data belongs to the equilibrium manifold  $\mathcal{V} = \{v = p(u_1)\}$ . For the space variable, we have the same boundary condition for both systems, so no space boundary layer appears.

To prove Theorem 2.3 we don't use the method in [4]: as observed in [7], with the entropic variables, we lose the semi-linear character of the system (1.10). We prefer write the following expansion of  $w^\varepsilon$

$$w^\varepsilon = w^0 + \varepsilon w_\varepsilon^1 = ((u_1^0, u_2^0), p(u_1^0)) + \varepsilon w_\varepsilon^1$$

so that the rest term  $w_\varepsilon^1$  satisfies a semi-linear hyperbolic system. In order to estimate  $w_\varepsilon^1$ , we use the conservative-dissipative variables introduced in [2]. With these variables the system is symmetrized and its semi-linear character is preserved. Furthermore by this method we obtain a more precise result : for  $\varepsilon$  small enough the lifespan  $T_\varepsilon^*$  is greater than the lifespan  $T^*$  of the limit system solution and the convergence is proved on all compact subset of  $[0, T^*]$ .

### 3. Proof of Theorem 2.1

We use the methods proposed by Majda in [12] for the Cauchy problem. We denote by  $l$  and  $r$  the left and right Riemann invariants of the system (1.1):

$$\begin{cases} l = \frac{1}{2}(u_2 + g(u_1)), \\ r = \frac{1}{2}(u_2 - g(u_1)). \end{cases}$$

These variables define a diffeomorphism which inverse is given by

$$\begin{cases} u_1 = g^{-1}(l - r), \\ u_2 = l + r. \end{cases}$$

These invariants  $(l, r)$  satisfy the diagonal system

$$\begin{cases} \partial_t l - \nu(l - r)\partial_x l = 0, \\ \partial_t r + \nu(l - r)\partial_x r = 0, \\ l(0, x) = r(0, x) = 0, x > 0, \\ (l + r)(t, 0) = \varphi(t), t > 0, \end{cases} \quad (3.1)$$

where  $\nu(l - r) = \sqrt{p'(g^{-1}(l - r))}$ . The smooth solution of (3.1) is  $(0, r)$  where  $r$  is the solution of the scalar equation

$$\begin{cases} \partial_t r + \nu(-r)\partial_x r = 0, \\ r(0, x) = 0, x > 0, \\ r(t, 0) = \varphi(t), t > 0. \end{cases} \quad (3.2)$$

Under the assumptions (1.3) and (2.1) we will prove that the lifespan  $T^*$  of the solution of the problem (3.2) is finite and that this solution exhibits shock waves in  $T^*$ .

For solving (3.2) we can use the method of characteristics. The function  $r$  is constant on the characteristic curves which are the straight lines  $t = T + \frac{1}{\nu(-\varphi(T))}x$ ,  $T \in \mathbb{R}$ .

Denoting  $\alpha(s) = \frac{1}{\nu(-s)}$  we obtain then that

$$r(T, 0) = \varphi(T) = r(T + \alpha(\varphi(T))x, x).$$

Let us introduce the mapping

$$(T, X) \mapsto \Phi(T, X) = (t, x) = (T + \alpha(\varphi(T))X, X).$$

This map is a diffeomorphism for  $X < \bar{X}$  with

$$\bar{X} = \left[ \max_{T \in [0, b]} -\frac{d}{dT} \alpha(\varphi(T)) \right]^{-1}.$$

Under assumption (2.1) we have  $0 < \bar{X} < +\infty$  and we have

$$\|r\|_{L^\infty(\mathbb{R}^+ \times [0, \bar{X}])} \leq \|\varphi\|_{L^\infty(\mathbb{R})}.$$

The characteristic curves through  $(0,0)$  and  $(b,0)$  cut the straight line  $\{x = \bar{X}\}$  at times

$$T_1 = \sqrt{p'(0)^{-1}} \bar{X} \text{ and } T_2 = b + \sqrt{p'(0)^{-1}} \bar{X} \text{ so } T^* \in [T_1, T_2].$$

#### 4. Proof of Theorem 2.2

In this section  $\varepsilon > 0$  and  $\mu > 0$  are fixed. We rewrite system (1.10)

$$\partial_t w + A \partial_x w = h(w)$$

where

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & -\mu & 0 \end{pmatrix} \text{ and } h(w) = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\varepsilon}(p(u_1) - v) \end{pmatrix}$$

and by (1.3) and (1.9)  $p$  is globally lipschitz. As zero is an eigenvalue of the matrix  $A$ , the boundary  $\{x=0\}$  is characteristic, so for completeness we give the proof of the global existence. Using (2.3) it is sufficient to prove that the solution  $w$  is bounded on any domain  $[0, T] \times \mathbb{R}^+$ . In a first step we lift the boundary condition (1.12). We set  $\omega(t, x) = \varphi(t)\eta(x)$  where  $\eta$  is a smooth function compactly supported with  $\eta(0) = 1$ . We replace  $u_2$  by  $u_2 - \omega$  and we obtain the following initial-boundary value problem

$$\begin{cases} \partial_t w + A \partial_x w = h(w) + \begin{pmatrix} \partial_x \omega \\ -\partial_t \omega \\ \mu \partial_x \omega \end{pmatrix}, \\ w(0, x) = 0, x \in \mathbb{R}^+, \\ u_2(t, 0) = 0, t \in \mathbb{R}^+. \end{cases} \quad (4.1)$$

We diagonalize the matrix  $A$  by the matrix  $P$ :  $w = PW$  with

$$P = \begin{pmatrix} 1 & 1 & 1 \\ \sqrt{\mu} & 0 & -\sqrt{\mu} \\ \mu & 0 & \mu \end{pmatrix}.$$

We obtain

$$\begin{cases} \partial_t W + \begin{pmatrix} -\sqrt{\mu} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{\mu} \end{pmatrix} \partial_x W = H(W) + \Phi, \\ W(0, x) = 0, x \in \mathbb{R}^+, \\ W_1(t, 0) - W_3(t, 0) = 0, t \in \mathbb{R}^+. \end{cases} \quad (4.2)$$

We have  $H(W) = P^{-1}h(PW)$  so  $H$  is globally lipschitz

$$\exists K > 0, |\partial_W H| \leq K. \quad (4.3)$$

In addition,  $\Phi$  is given by

$$\Phi = P^{-1} \begin{pmatrix} \partial_x \omega \\ -\partial_t \omega \\ \mu \partial_x \omega \end{pmatrix}.$$

We denote by  $T^*$  the lifespan of the solution  $W$  for system (4.2) and we assume that  $T^* < +\infty$ . We will prove that  $\|W\|_{L^\infty([0, T^*] \times \mathbb{R}^+)} < +\infty$  so that by (2.3) we obtain a contradiction.

### $L^2$ estimate

We take the inner product of the first equation in (4.2) by  $W$  and we obtain

$$\frac{1}{2} \frac{d}{dt} \|W\|_{L^2(\mathbb{R}^+)}^2 + \int_{\mathbb{R}^+} \sqrt{\mu} (-W_1 \partial_x W_1 + W_3 \partial_x W_3) dx = \int_{\mathbb{R}^+} H(W) W dx + \int_{\mathbb{R}^+} \Phi W dx.$$

Using the third equation in (4.2) and (4.3) we obtain

$$\frac{1}{2} \frac{d}{dt} \|W\|_{L^2(\mathbb{R}^+)}^2 \leq C(1 + \|W\|_{L^2(\mathbb{R}^+)}^2). \quad (4.4)$$

### $H^1$ estimate

We derivate system (4.2) with respect to  $t$  and with similar computations we obtain that

$$\frac{1}{2} \frac{d}{dt} \|\partial_t W\|_{L^2(\mathbb{R}^+)}^2 \leq C(1 + \|\partial_t W\|_{L^2(\mathbb{R}^+)}^2). \quad (4.5)$$

By Gronwall lemma we obtain from (4.4) and (4.5) that

$$\|W\|_{L^\infty([0, T^*]; L^2(\mathbb{R}^+))} + \|\partial_t W\|_{L^\infty([0, T^*]; L^2(\mathbb{R}^+))} \leq C(T^*). \quad (4.6)$$

So using the first equation in (4.2) we have

$$\|\partial_x W_1\|_{L^\infty([0, T^*]; L^2(\mathbb{R}^+))} + \|\partial_x W_3\|_{L^\infty([0, T^*]; L^2(\mathbb{R}^+))} \leq C(T^*), \quad (4.7)$$

In addition we have

$$\partial_t \partial_x W_2 - \partial_{W_2} H_2(W) \partial_x W_2 = \mathcal{H}(t, x),$$

where

$$\mathcal{H} = \partial_{W_1} H_2(W) \partial_x W_1 + \partial_{W_3} H_2(W) \partial_x W_3 + \partial_x \Phi_2.$$

By (4.3) and (4.7) we have

$$\|\mathcal{H}\|_{L^\infty([0, T^*]; L^2(\mathbb{R}^+))} \leq C(T^*),$$

and since

$$\partial_x W_2(t, x) = \int_0^t \left( \exp \int_s^t \partial_{W_2} H_2(W(\tau, x)) d\tau \right) \mathcal{H}(s, x) ds,$$

we conclude that

$$\|\partial_x W_2\|_{L^\infty([0, T^*]; L^2(\mathbb{R}^+))} \leq C(T^*).$$

By Sobolev injections we can apply the continuation principle and we conclude the proof of Theorem 2.2.



### 5. Proof of Theorem 2.3

We denote by  $T^*$  the lifespan of the smooth solution  $u^0 = (u_1^0, u_2^0)$  of system (1.13)-(1.14)-(1.15). Since the boundary data  $\varphi$  belongs to  $H^3(\mathbb{R})$  we have

$$\partial_t^i u^0 \in C^0([0, T^*]; H^{3-i}(\mathbb{R}^+)), \quad i = 0, 1, 2, 3. \quad (5.1)$$

We define the profile  $w^0$  by

$$w^0 = (u^0, v^0) = ((u_1^0, u_2^0), p(u_1^0)). \quad (5.2)$$

We denote

$$\gamma(t, x) = p'(u_1^0(t, x)), \quad t < T^*, x > 0, \quad (5.3)$$

$$\Gamma = \sup_{(t, x) \in [0, T^*] \times \mathbb{R}^+} \gamma(t, x), \quad (5.4)$$

and by (2.2),  $\Gamma < +\infty$ . We fix  $\mu$  such that

$$\mu > \Gamma. \quad (5.5)$$

We will construct the solution  $w^\varepsilon$  of the relaxation problem (1.10)-(1.11)-(1.12) writing

$$w^\varepsilon = w^0 + \varepsilon \begin{pmatrix} 0 \\ 0 \\ v^1 \end{pmatrix} + \varepsilon r, \quad (5.6)$$

where

$$v^1 = -\partial_t v^0 + \mu \partial_x u_2^0, \quad (5.7)$$

so that  $r$  satisfies the following system

$$\begin{cases} \partial_t r_1 - \partial_x r_2 = 0, \\ \partial_t r_2 - \partial_x r_3 = \partial_x v^1, \\ \partial_t r_3 - \mu \partial_x r_2 = \frac{1}{\varepsilon} (p'(u_1^0) r_1 - r_3) + F(t, x, \varepsilon r_1) (r_1)^2 - \partial_t v^1, \end{cases} \quad (5.8)$$

for  $(t, x) \in [0, T^*] \times \mathbb{R}^+$ , with the initial-boundary conditions

$$\begin{cases} r(0, x) = 0, x \in \mathbb{R}^+, \\ r_2(t, 0) = 0, 0 \leq t < T^*. \end{cases} \quad (5.9)$$

The function  $F$  is defined by

$$F(t, x, \xi) = \int_0^1 (1-s) p''(u_1^0(t, x) + s\xi) ds. \quad (5.10)$$

**First step:** we want to construct a suitable symmetrization for system (5.8). We denote by  $A$  and  $B$  the matrices

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & -\mu & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \gamma(t, x) & 0 & -1 \end{pmatrix}.$$

With this object, we will use the conservative-dissipative form introduced in [2]. We first need a symmetric positive definite matrix  $A_0$  such that  $AA_0$  is a symmetric matrix, and such that

$$BA_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -d \end{pmatrix} \text{ with } d > 0.$$

Following [7], such a matrix can be constructed using the entropic variables. For the special case of the Suliciu model we have

$$A_0(t, x) = \begin{pmatrix} (\gamma(t, x))^{-1} & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & \mu \end{pmatrix} = \begin{pmatrix} A_{0,11} & A_{0,12} \\ A_{0,21} & A_{0,22} \end{pmatrix}.$$

We obtain

$$AA_0 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -\mu \\ 0 & -\mu & 0 \end{pmatrix}, \quad BA_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma - \mu \end{pmatrix},$$

and we remark that with (5.5), we have  $\mu - \gamma \geq \mu - \Gamma > 0$ . Finally we can apply Proposition 2.7 in [2]: the conservative-dissipative variables  $\rho$  is defined by  $\rho = P(t, x)r$  with

$$P(t, x) = \begin{pmatrix} (A_{0,11})^{-\frac{1}{2}} & 0 \\ ((A_0^{-1})_{22})^{-\frac{1}{2}}(A_0^{-1})_{21} & ((A_0^{-1})_{22})^{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} \gamma^{\frac{1}{2}} & 0 & 0 \\ 0 & 1 & 0 \\ -\gamma(\mu - \gamma)^{-\frac{1}{2}} & 0 & (\mu - \gamma)^{-\frac{1}{2}} \end{pmatrix}.$$

In these variables, system (5.8) is equivalent to

$$\partial_t \rho + A_1 \partial_x \rho + L \rho = -\frac{1}{\varepsilon} \begin{pmatrix} 0 \\ 0 \\ \rho_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ F_1(t, x, \varepsilon \rho_1) \rho_1^2 \end{pmatrix} + H, \quad (5.11)$$

for  $(t, x) \in [0, T^*] \times \mathbb{R}^+$ , with the initial-boundary conditions

$$\rho(0, x) = 0 \text{ for } x \in \mathbb{R}^+ \text{ and } \rho_2(t, 0) = 0 \text{ for } t \in [0, T^*]. \quad (5.12)$$

The matrix  $A_1 = PAP^{-1}$  is symmetric

$$A_1(t, x) = \begin{pmatrix} 0 & -\gamma^{\frac{1}{2}} & 0 \\ -\gamma^{\frac{1}{2}} & 0 & -(\mu - \gamma)^{\frac{1}{2}} \\ 0 & -(\mu - \gamma)^{\frac{1}{2}} & 0 \end{pmatrix}.$$

The matrix  $L$  is given by  $L(t, x) = P \partial_t P^{-1} + PA \partial_x P^{-1}$ . In addition,  $F_1$  and  $H$  are given by

$$F_1(t, x, \xi) = \gamma^{-1}(\mu - \gamma)^{-\frac{1}{2}} F(t, x, \gamma^{-\frac{1}{2}} \xi), \quad (5.13)$$

$$H(t, x) = \begin{pmatrix} 0 \\ \partial_x v^1 \\ -(\mu - \gamma)^{-\frac{1}{2}} \partial_t v^1 \end{pmatrix}.$$

From (5.1) we have

$$\partial_t^i \gamma \in \mathcal{C}^0([0, T^*]; H^{3-i}(\mathbb{R}^+)), i = 0, 1, 2, 3, \quad (5.14)$$

and using (2.2) there exists  $\alpha > 0$  such that

$$\gamma(t, x) \geq \alpha \text{ for } (t, x) \in [0, T^*] \times \mathbb{R}^+. \quad (5.15)$$

Using (5.14), (5.15) and (5.5) we have

$$A_1, \partial_t A_1, \partial_x A_1 \in \mathcal{C}^0([0, T^*]; L^\infty(\mathbb{R}^+)), \quad (5.16)$$

$$L, \partial_t L, \partial_x L \in \mathcal{C}^0([0, T^*]; L^\infty(\mathbb{R}^+)). \quad (5.17)$$

Using (5.1) and (5.7) we have

$$\partial_t^i H \in \mathcal{C}^0([0, T^*]; H^{1-i}(\mathbb{R}^+)), i = 0, 1. \quad (5.18)$$

We recall that by (5.10) and (5.13) we have

$$F_1(t, x, \xi) = \gamma^{-1}(t, x) (\mu - \gamma(t, x))^{-\frac{1}{2}} \int_0^1 (1-s) p''(u_1^0(t, x) + s \gamma^{-\frac{1}{2}}(t, x) \xi) ds,$$

so, by (5.14), (5.15) and (5.5) we have

$$F_1, \partial_t F_1, \partial_x F_1, \partial_\xi F_1 \in \mathcal{C}^0([0, T^*]; L^\infty(\mathbb{R}^+ \times [-1, 1])). \quad (5.19)$$

Now we fix  $T < T^*$  and we introduce  $T_\varepsilon$  defined by

$$T_\varepsilon = \sup \left\{ t \leq T, \|\rho\|_{L^\infty([0, t] \times \mathbb{R}^+)} \leq \frac{1}{\varepsilon} \right\}. \quad (5.20)$$

We will prove that, for  $\varepsilon$  small enough,  $T_\varepsilon = T$  and that there exists  $K$  such that for all  $\varepsilon$  small enough,

$$\|\rho\|_{L^\infty([0, T]; H^1(\mathbb{R}^+))} + \|\partial_t \rho\|_{L^\infty([0, T]; L^2(\mathbb{R}^+))} \leq K. \quad (5.21)$$

First, by variational methods, we obtain  $L^2$ -estimates on  $\rho$  and  $\partial_t \rho$ . To obtain  $L^2$ -estimates on  $\partial_x \rho$  we use the equations taking into account that the boundary  $\{x=0\}$  is characteristic.

### Second step: variational estimates

We take the inner product of system (5.11) by  $\rho$  and we obtain that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\rho\|_{L^2(\mathbb{R}^+)}^2 + \int_{\mathbb{R}^+} A_1 \partial_x \rho \cdot \rho dx + \int_{\mathbb{R}^+} L \rho \cdot \rho dx + \frac{1}{\varepsilon} \int_{\mathbb{R}^+} \rho_3^2 dx &= \int_{\mathbb{R}^+} F_1(t, x, \varepsilon \rho_1) \rho_1^2 \rho_3 \\ &+ \int_{\mathbb{R}^+} H \cdot \rho dx. \end{aligned}$$

Using (5.12) we obtain that

$$\int_{\mathbb{R}^+} A_1 \partial_x \rho \cdot \rho dx = -\frac{1}{2} \int_{\mathbb{R}^+} (\partial_x A_1) \rho \cdot \rho dx.$$

With the estimates (5.16),..., (5.19) and since  $\varepsilon|\rho| \leq 1$  on  $[0, T_\varepsilon] \times \mathbb{R}^+$ , there exists a constant  $C > 0$  such that, for  $t \leq T_\varepsilon$ ,

$$\frac{1}{2} \frac{d}{dt} \|\rho\|_{L^2(\mathbb{R}^+)}^2 + \frac{1}{\varepsilon} \int_{\mathbb{R}^+} \rho_3^2 dx \leq C(1 + \|\rho\|_{L^2(\mathbb{R}^+)}^2 + \|\rho_1\|_{L^\infty(\mathbb{R}^+)} \|\rho_1\|_{L^2(\mathbb{R}^+)} \|\rho_3\|_{L^2(\mathbb{R}^+)}).$$

Therefore we obtain that for  $t \leq T_\varepsilon$ ,

$$\frac{d}{dt} \|\rho\|_{L^2(\mathbb{R}^+)}^2 + \frac{1}{\varepsilon} \int_{\mathbb{R}^+} \rho_3^2 dx \leq C(1 + \|\rho\|_{L^2(\mathbb{R}^+)}^2 + \varepsilon \|\rho_1\|_{L^\infty(\mathbb{R}^+)}^2 \|\rho_1\|_{L^2(\mathbb{R}^+)}^2). \quad (5.22)$$

We can derivate (5.11)-(5.12) with respect to  $t$

$$\begin{aligned} \partial_t \partial_t \rho + A_1 \partial_x \partial_t \rho + L \partial_t \rho + \frac{1}{\varepsilon} \begin{pmatrix} 0 \\ 0 \\ \partial_t \rho_3 \end{pmatrix} &= -\partial_t A_1 \partial_x \rho - \partial_t L \rho + \begin{pmatrix} 0 \\ 0 \\ \partial_t F_1(t, x, \varepsilon \rho_1) \rho_1^2 \end{pmatrix} \\ &+ \begin{pmatrix} 0 \\ 0 \\ \varepsilon \partial_\xi F_1(t, x, \varepsilon \rho_1) \partial_t \rho_1 \rho_1^2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2F_1(t, x, \varepsilon \rho_1) \rho_1 \partial_t \rho_1 \end{pmatrix} + \partial_t H. \end{aligned}$$

With the same arguments as before we obtain that there exists  $C > 0$  such that for  $t \leq T_\varepsilon$ ,

$$\begin{aligned} \frac{d}{dt} \|\partial_t \rho\|_{L^2(\mathbb{R}^+)}^2 + \frac{1}{\varepsilon} \int_{\mathbb{R}^+} (\partial_t \rho_3)^2 dx &\leq C(1 + \|\rho\|_{L^2(\mathbb{R}^+)}^2 + \|\partial_t \rho\|_{L^2(\mathbb{R}^+)}^2 + \|\partial_x \rho\|_{L^2(\mathbb{R}^+)}^2) \\ &+ C\varepsilon \|\rho_1\|_{L^\infty(\mathbb{R}^+)}^2 (\|\rho_1\|_{L^2(\mathbb{R}^+)}^2 + \|\partial_t \rho_1\|_{L^2(\mathbb{R}^+)}^2). \end{aligned} \quad (5.23)$$

We define  $\psi$  by

$$\psi(t) = \left( \|\rho(t)\|_{L^2(\mathbb{R}^+)}^2 + \|\partial_t \rho(t)\|_{L^2(\mathbb{R}^+)}^2 \right)^{\frac{1}{2}}, \quad (5.24)$$

so we obtain by (5.22) and (5.23) the  $L^2$ -estimate: there exists  $C > 0$  such that for  $t \leq T_\varepsilon$ ,

$$\begin{aligned} \frac{d}{dt} (\psi(t))^2 + \frac{1}{\varepsilon} (\|\rho_3\|_{L^2(\mathbb{R}^+)}^2 + \|\partial_t \rho_3\|_{L^2(\mathbb{R}^+)}^2) &\leq C(1 + (\psi(t))^2) \\ &+ \varepsilon \|\rho_1\|_{L^\infty(\mathbb{R}^+)}^2 (\psi(t))^2 + \|\partial_x \rho\|_{L^2(\mathbb{R}^+)}^2. \end{aligned} \quad (5.25)$$

### Third step

We now estimate  $\partial_x \rho$  using the equations

$$\begin{cases} \partial_t \rho_1 - \gamma^{\frac{1}{2}} \partial_x \rho_2 + (L\rho)_1 = 0, \\ \partial_t \rho_2 - \gamma^{\frac{1}{2}} \partial_x \rho_1 - (\mu - \gamma)^{\frac{1}{2}} \partial_x \rho_3 + (L\rho)_2 = H_2, \\ \partial_t \rho_3 - (\mu - \gamma)^{\frac{1}{2}} \partial_x \rho_2 + (L\rho)_3 + \frac{1}{\varepsilon} \rho_3 = F_1(t, x, \varepsilon \rho_1) \rho_1^2 + H_3. \end{cases} \quad (5.26)$$

From the first equation in (5.26), and with (5.15) and (5.17) we have for  $t \in [0, T_\varepsilon]$

$$\|\partial_x \rho_2\|_{L^2(\mathbb{R}^+)} \leq C\psi. \quad (5.27)$$

Let us introduce  $\tilde{\rho}_1 = \rho_1 + \gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}}\rho_3$ . From the second equation in (5.26) we have

$$\partial_t \rho_2 - \gamma^{\frac{1}{2}} \partial_x \tilde{\rho}_1 + \gamma^{\frac{1}{2}} \partial_x (\gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}})\rho_3 + (L\rho)_2 = H_2,$$

so, by (5.15), (5.14), (5.17) and (5.18) we obtain that

$$\|\partial_x \tilde{\rho}_1\|_{L^2(\mathbb{R}^+)} \leq C(1 + \psi). \quad (5.28)$$

We cannot estimate  $\partial_x \rho_1$  or  $\partial_x \rho_3$  by the same method because the boundary  $\{x=0\}$  is characteristic. We rewrite the third equation in (5.26)

$$\partial_t \rho_3 + \frac{1}{\varepsilon} \rho_3 = \gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}}(\partial_t \rho_1 + (L\rho)_1) - (L\rho)_3 + F_1(t, x, \varepsilon \rho_1) \rho_1^2 + H_3.$$

So eliminating  $\rho_1$  we obtain

$$\begin{aligned} \mu \gamma^{-1} \partial_t \rho_3 + \frac{1}{\varepsilon} \rho_3 = & \gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}} [\partial_t \tilde{\rho}_1 - \partial_t (\gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}})\rho_3] + M_1(t, x) \tilde{\rho}_1 + M_2(t, x) \rho_2 \\ & + M_3(t, x) \rho_3 + H_3 + F_1(t, x, \varepsilon \rho_1) \rho_1^2, \end{aligned} \quad (5.29)$$

with  $\rho_1 = \tilde{\rho}_1 - \gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}}\rho_3$ . We derivate (5.29) with respect to  $x$  and we obtain the equation satisfied by  $\partial_x \rho_3$

$$\partial_t \partial_x \rho_3 + \tau(t, x) \partial_x \rho_3 = \sum_{i=1}^6 T_i, \quad (5.30)$$

with

$$\begin{aligned} \tau = & \mu^{-1} \gamma \left( \frac{1}{\varepsilon} + \gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}} \partial_t (\gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}}) + \varepsilon \partial_\xi F_1(t, x, \varepsilon \rho_1) \gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}} \rho_1^2 \right. \\ & \left. + 2F_1(t, x, \varepsilon \rho_1) \rho_1 \gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}} - M_3(t, x) \right), \end{aligned}$$

$$T_1 = \mu^{-1} \gamma^{\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}} \partial_t \partial_x \tilde{\rho}_1,$$

$$\begin{aligned} T_2 = & \mu^{-1} \gamma \left( \partial_x (\gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}}) \partial_t \tilde{\rho}_1 - \partial_x (\gamma^{-1} \mu) \partial_t \rho_3 \right. \\ & \left. - \partial_x (\gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}} \partial_t (\gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}})) \rho_3 \right. \\ & \left. + (\partial_x M_1) \tilde{\rho}_1 + (\partial_x M_2) \rho_2 + (\partial_x M_3) \rho_3 \right), \end{aligned}$$

$$T_3 = \mu^{-1} \gamma \partial_x H_3,$$

$$T_4 = \mu^{-1} \gamma (M_1 \partial_x \tilde{\rho}_1 + M_2 \partial_x \rho_2),$$

$$\begin{aligned} T_5 = & \mu^{-1} \gamma \left( \partial_x F_1(t, x, \varepsilon \rho_1) \rho_1^2 - \varepsilon \partial_\xi F_1(t, x, \varepsilon \rho_1) \partial_x (\gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}}) \rho_1^2 \rho_3 \right. \\ & \left. - 2F_1(t, x, \varepsilon \rho_1) \partial_x (\gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}}) \rho_1 \rho_3 \right), \end{aligned}$$

$$T_6 = \mu^{-1} \gamma (\varepsilon \partial_\xi F_1(t, x, \varepsilon \rho_1) \rho_1^2 \partial_x \tilde{\rho}_1 + 2F_1(t, x, \varepsilon \rho_1) \rho_1 \partial_x \tilde{\rho}_1).$$

For  $t \in [0, T_\varepsilon]$ , using (5.5), (5.14) (5.15) and (5.19) we obtain that

$$\left| \tau(t, x) - \frac{\mu^{-1} \gamma}{\varepsilon} \right| \leq C + C_0 \|\rho_1\|_{L^\infty(\mathbb{R}^+)}.$$

We define  $T_\varepsilon^1 \leq T_\varepsilon$  by

$$T_\varepsilon^1 = \max \left\{ t \leq T_\varepsilon, \|\rho_1\|_{L^\infty([0,t] \times \mathbb{R}^+)} \leq \frac{1}{2C_0\varepsilon} \right\}, \quad (5.31)$$

so there exists  $\tau_1 > 0$  and  $\tau_2 > 0$  such that

$$\forall t \leq T_\varepsilon^1, \forall x > 0, \frac{\tau_1}{\varepsilon} \leq \tau(t, x) \leq \frac{\tau_2}{\varepsilon}. \quad (5.32)$$

We solve Equation (5.30) by Duhamel formula

$$\partial_x \rho_3 = \sum_{i=1}^6 \mathcal{T}_i, \quad (5.33)$$

with

$$\mathcal{T}_i(t, x) = \int_0^t \exp\left(-\int_s^t \tau(\sigma, x) d\sigma\right) \mathcal{T}_i(s, x) ds.$$

We define  $\Psi$  by

$$\Psi(t) = \sup_{[0,t]} \psi(s), \quad (5.34)$$

where  $\psi$  is given by (5.24). Integrating by parts in  $\mathcal{T}_1$  we obtain

$$\begin{aligned} \mathcal{T}_1(t, x) = & - \int_0^t \mu^{-1} \gamma^{\frac{1}{2}} (\mu - \gamma)^{\frac{1}{2}} \tau(s, x) \exp\left(-\int_s^t \tau(\sigma, x) d\sigma\right) \partial_x \tilde{\rho}_1(s, x) ds \\ & - \int_0^t \exp\left(-\int_s^t \tau(\sigma, x) d\sigma\right) \partial_s (\mu^{-1} \gamma^{\frac{1}{2}} (\mu - \gamma)^{\frac{1}{2}})(s, x) \partial_x \tilde{\rho}_1(s, x) ds \\ & + \mu^{-1} \gamma^{\frac{1}{2}} (\mu - \gamma)^{\frac{1}{2}} \partial_x \tilde{\rho}_1(t, x). \end{aligned}$$

Using (5.32), (5.5), (5.14), (5.15) and (5.28) we have

$$\|\mathcal{T}_1(t, \cdot)\|_{L^2(\mathbb{R}^+)} \leq \int_0^t \exp\left(-\frac{\tau_1}{\varepsilon}(t-s)\right) C(\psi(s) + 1) \left(1 + \frac{\tau_2}{\varepsilon}\right) ds + C(\psi(t) + 1),$$

and we obtain that

$$\forall t \leq T_\varepsilon^1, \|\mathcal{T}_1\|_{L^2(\mathbb{R}^+)} \leq C(1 + \Psi(t)). \quad (5.35)$$

Using (5.5) (5.14) (5.15) (5.24) (5.34) and also (5.18) for  $T_3$  and (5.27) and (5.28) for  $T_4$ , we obtain

$$\forall t \leq T_\varepsilon^1, \|\mathcal{T}_2\|_{L^2(\mathbb{R}^+)} + \|\mathcal{T}_3\|_{L^2(\mathbb{R}^+)} + \|\mathcal{T}_4\|_{L^2(\mathbb{R}^+)} \leq C\varepsilon(1 + \Psi(t)). \quad (5.36)$$

For the nonlinear terms  $T_5$  and  $T_6$  we use in addition (5.19) (5.20) and we obtain

$$\forall t \leq T_\varepsilon^1, \|\mathcal{T}_5\|_{L^2(\mathbb{R}^+)} + \|\mathcal{T}_6\|_{L^2(\mathbb{R}^+)} \leq C(1 + \Psi(t)). \quad (5.37)$$

Therefore we obtain the following estimation for  $\partial_x \rho$  using (5.27) (5.28) (5.33) (5.35) (5.36) (5.37)

$$\forall t \leq T_\varepsilon^1, \|\partial_x \rho\|_{L^2(\mathbb{R}^+)} \leq C(1 + \Psi(t)), \quad (5.38)$$

so we have

$$\forall t \leq T_\varepsilon^1, \|\rho\|_{L^\infty(\mathbb{R}^+)} \leq C_1(1 + \Psi(t)). \quad (5.39)$$

#### Fourth step

By a comparison method we estimate  $\Psi$ . For  $t \leq T_\varepsilon^1$ , integrating (5.25) from 0 to  $t$ , using (5.38) and (5.39) we obtain that

$$(\Psi(t))^2 \leq C_2 \int_0^t (1 + (\Psi(s))^2 + \varepsilon(\Psi(s))^4) ds. \quad (5.40)$$

We introduce the differential equation

$$y'_\varepsilon = C_2(1 + y_\varepsilon + \varepsilon y_\varepsilon^2), y_\varepsilon(0) = 0. \quad (5.41)$$

There exists  $\varepsilon_0 > 0$  such that, for  $\varepsilon \leq \varepsilon_0$ , the lifespan of  $y_\varepsilon$  is greater than  $T$ . So we have

$$\forall \varepsilon \leq \varepsilon_0, \forall t \leq T, y_\varepsilon(t) \leq y_{\varepsilon_0}(t) \leq y_{\varepsilon_0}(T) = C_3.$$

By comparison principle we deduce from (5.40) that

$$\forall \varepsilon \leq \varepsilon_0, \forall t \leq T_\varepsilon^1, (\Psi(t))^2 \leq C_3,$$

and from (5.39),

$$\forall \varepsilon \leq \varepsilon_0, \forall t \leq T_\varepsilon^1, \|\rho\|_{L^\infty(\mathbb{R}^+)} \leq C_1(1 + \sqrt{C_3}).$$

Let  $\varepsilon_1 > 0$  such that  $\varepsilon_1 \leq \varepsilon_0$  such that

$$\forall \varepsilon \leq \varepsilon_1, C_1(1 + \sqrt{C_3}) \leq \frac{1}{2C_0\varepsilon}.$$

So, by (5.20) and (5.31), we have for  $\varepsilon \leq \varepsilon_1$ ,  $T_\varepsilon^1 = T_\varepsilon = T$  and we conclude the proof by the estimate

$$\exists K > 0, \forall \varepsilon \leq \varepsilon_1, \|\rho\|_{L^\infty([0,T];H^1(\mathbb{R}^+))} + \|\partial_t \rho\|_{L^\infty([0,T];L^2(\mathbb{R}^+))} \leq K.$$

## 6. Annex

Using the method in W.A. Yong [22] we show the convergence result for the Cauchy problem

$$\begin{cases} \partial_t u_1^\varepsilon - \partial_x u_2^\varepsilon = 0, \\ \partial_t u_2^\varepsilon - \partial_x v^\varepsilon = 0, \\ \partial_t v^\varepsilon - \mu \partial_x u_2^\varepsilon = \frac{1}{\varepsilon}(p(u_1^\varepsilon) - v^\varepsilon), \end{cases} \quad (6.1)$$

for  $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$  with the smooth initial data

$$w^\varepsilon(0, x) = w_0(x) = (u_0(x), v_0(x)) \text{ for } x \in \mathbb{R}. \quad (6.2)$$

Let us introduce  $u^0$  the smooth solution of the Cauchy problem

$$\begin{cases} \partial_t u_1^0 - \partial_x u_2^0 = 0, \\ \partial_t u_2^0 - \partial_x p(u_1^0) = 0, \end{cases} \quad (6.3)$$

with the initial data

$$u^0(0, x) = u_0(x). \quad (6.4)$$

As in Tzavaras [21] we assume that there exists  $\gamma > 0$  and  $\Gamma > 0$  such that

$$\forall \xi \in \mathbb{R}, \gamma \leq p'(\xi) \leq \Gamma < \mu, \quad (6.5)$$

so the problem (6.1)-(6.2) admits a global solution  $w^\varepsilon = (u^\varepsilon, v^\varepsilon)$  such that

$$w^\varepsilon \in \mathcal{C}^0(\mathbb{R}^+; H^s(\mathbb{R})) \cap \mathcal{C}^1(\mathbb{R}^+; H^{s-1}(\mathbb{R})).$$

We will prove the following convergence theorem.

**THEOREM 6.1.** *Under assumption (6.5), if  $w_0 \in H^s(\mathbb{R})$  with  $s \geq 2$ , then there exists  $T_1 > 0$  such that when  $\varepsilon$  tends to zero,  $u^\varepsilon$  tends to  $u^0$  in  $L^\infty([0, T_1]; H^s(\mathbb{R}))$ .*

**REMARK 6.1.** *It would be possible to relax hypothesis (6.5) as in Theorem 2.3; in this case, the lifespan of  $w^\varepsilon$  is uniformly greater than  $T_1$ .*

**REMARK 6.2.** *In fact it appears a boundary layer in time which affects only the third component of  $w^\varepsilon$ .*

### Sketch of the proof

**First step:** the stability assumption in [22] are satisfied. As in [21] and [7], we consider the strictly convex entropy function for the system (6.1)

$$\mathcal{E}(u_1, u_2, v) = \frac{1}{2}u_2^2 + u_1v - \frac{\mu}{2}u_1^2 - \int_0^{v-\mu u_1} h^{-1}(y)dy,$$

where  $h(\xi) = p(\xi) - \mu\xi$  which is strictly decreasing by (6.5). So  $A_0(w) = \mathcal{E}''(w)$  is a symmetrizer for the system. Denoting  $a = (h^{-1})'(v - \mu u_1)$  we obtain

$$A_0(w) = \begin{pmatrix} -\mu - \mu^2 a & 0 & 1 + \mu a \\ 0 & 1 & 0 \\ 1 + \mu a & 0 & -a \end{pmatrix},$$

and the system (6.1) is equivalent to the quasilinear symmetric system

$$A_0(w)\partial_t w + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \partial_x w = \frac{1}{\varepsilon}(p(u_1) - v) \begin{pmatrix} 1 + \mu a \\ 0 \\ -a \end{pmatrix}. \quad (6.6)$$

We denote

$$Q(w) = \begin{pmatrix} 0 \\ 0 \\ p(u_1) - v \end{pmatrix} \text{ and } P(w) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -p'(u_1) & 0 & 1 \end{pmatrix},$$

and we obtain

$$P(w)Q'(w)P^{-1}(w) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (6.7)$$



On the equilibrium manifold  $\mathcal{V} = \{v = p(u_1)\}$ , we have

$$A_0(w)Q'(w) + Q'(w)A_0(w) = \frac{2}{p'(u_1) - \mu} \begin{pmatrix} (p'(u_1))^2 & 0 & -p'(u_1) \\ 0 & 0 & 0 \\ -p'(u_1) & 0 & 1 \end{pmatrix}. \quad (6.8)$$

Using (6.6), (6.7) and (6.8) we obtain the stability conditions in [22].

**Second step:** we use Theorems 6.1 and 6.2 in [22]. We introduce the interior profile  $w^0 = ((u_1^0, u_2), p(u_1^0))$  and the boundary layer term  $I^0 = \tilde{I}^0 - w^0(0, x)$  where  $\tilde{I}^0$  is the solution of

$$\frac{d\tilde{I}_0}{d\tau} = Q(\tilde{I}_0), \quad \tilde{I}(\tau=0) = w_0(x).$$

We have  $I_1^0 = I_2^0 = 0$  and

$$I_3^0(\tau, x) = (v_0(x) - p(u_1, 0))e^{-\tau},$$

and we obtain

$$w^\varepsilon(t, x) = w^0(t, x) + I^0\left(\frac{t}{\varepsilon}, x\right) + \mathcal{O}(\varepsilon),$$

so we conclude the proof of Theorem 6.1.

REMARK 6.3. *If  $w_0$  belongs to the equilibrium manifold then the order zero boundary layer term vanishes.*

REMARK 6.4. *In fact using more precisely [22] and the appendix of [3] we can prove that  $T_1$  can be arbitrarily close to the lifespan of  $u^0$  as in Theorem 2.3.*

REMARK 6.5. *In this annex the matrix  $P$  introduced in [22] plays an analogous role as the matrix  $P$  in section 5.*

#### REFERENCES

- [1] Denise Aregba-Driollet and Vuk Milišić, *Kinetic approximation of a boundary value problem for conservation laws*, Numer. Math., 97 (4), 595–633, 2004.
- [2] Stefano Bianchini, Bernard Hanouzet and Roberto Natalini, *Asymptotic Behaviour of Smooth Solutions for Partially Dissipative Hyperbolic Systems with a Convex Entropy*, Preprint 2005, to appear in *Comm. Pure Appl. Math.*.
- [3] Y. Brenier and W.-A. Yong, *Derivation of particle, string, and membrane motions from the Born-Infeld electromagnetism*, J. Math. Phys., 46, 062305, 2005.
- [4] Gilles Carbou and Bernard Hanouzet, *Relaxation approximation of some nonlinear Maxwell initial-boundary value problem*, Comm. Math. Sci., 4(2), 331–344, 2006.
- [5] A. Chalabi and D. Seghir, *Convergence of relaxation schemes for initial boundary value problems for conservation laws*, Comput. Math. Appl., 43(8-9), 1079–1093, 2002.
- [6] Olivier Guès, *Problème mixte hyperbolique quasi-linéaire caractéristique*, Comm. Partial Differential Equations, 15(5), 595–645, 1990.
- [7] B. Hanouzet and R. Natalini, *Global existence of smooth solutions for partially dissipative hyperbolic systems with a convex entropy*, Arch. Ration. Mech. Anal., 169(2), 89–117, 2003.
- [8] L. Hsiao and R. Pan, *Zero relaxation limit to centered rarefaction waves for a rate-type viscoelastic system*, J. Differential Equations, 157(1), 20–40, 1999.
- [9] Wendy Kress, *Asymptotic behavior of hyperbolic boundary value problems with relaxation term*, In *Hyperbolic problems: theory, numerics, applications, Vol. I, II (Magdeburg, 2000)*, volume 141 of *Internat. Ser. Numer. Math.*, 140, pages 633–642. Birkhäuser, Basel, 2001.
- [10] H. Li and R. Pan, *Zero relaxation limit for piecewise smooth solutions to a rate-type viscoelastic system in the presence of shocks*, J. Math. Anal. Appl., 252(1), 298–324, 2000.

- [11] Hailiang Liu and Wen-An Yong, *Time-asymptotic stability of boundary-layers for a hyperbolic relaxation system*, Comm. Partial Differential Equations, 26(7-8), 1323–1343, 2001.
- [12] A. Majda, *Compressible fluid flow and systems of conservation laws in several space variables*. Applied Mathematical Sciences, 53. Springer-Verlag, New York, 1984.
- [13] A. Matsumura and M. Mei, *Convergence to travelling fronts of solutions of the  $p$ -system with viscosity in the presence of a boundary*, Arch. Ration. Mech. Anal., 146(1), 1–22, 1999.
- [14] Vuk Milišić, *Stability and convergence of discrete kinetic approximations to an initial-boundary value problem for conservation laws*, Proc. Amer. Math. Soc., 131(6), 1727–1737 (electronic), 2003.
- [15] Roberto Natalini and Andrea Terracina, *Convergence of a relaxation approximation to a boundary value problem for conservation laws*, Comm. Partial Differential Equations, 26(7-8), 1235–1252, 2001.
- [16] K. Nishihara and T. Yang, *Boundary effect on asymptotic behaviour of solutions to the  $p$ -system with linear damping*, J. Differential Equations, 156(2), 439–458, 1999.
- [17] D. Serre and J. Shearer, *Convergence with physical viscosity for nonlinear elasticity*, Preprint 1993.
- [18] F. Sueur, *Couches limites semilinéaires*, Ann. Fac. Sci. Toulouse Math. (6), 15(2), 323–380, 2006.
- [19] I. Suliciu, *On modelling phase transition by means of rate-type constitutive equations, shock wave structure*, Int. J. Ing. Sci., 28, 827–841, 1990.
- [20] S. Tang and H. Zhao, *Stability of Suliciu model for phase transitions*, Commun. Pure Appl. Anal., 3(4), 545–556, 2004.
- [21] A. E. Tzavaras, *Materials with internal variables and relaxation to conservation laws*, Arch. Ration. Mech. Anal., 146(2), 129–155, 1999.
- [22] Wen-An Yong, *Singular perturbations of first-order hyperbolic systems with stiff source terms*, J. Differential Equations, 155(1), 89–132, 1999.