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SEMICLASSICAL ANALYSIS FOR HARTREE EQUATION

RÉMI CARLES AND SATOSHI MASAKI

ABSTRACT. We justify WKB analysis for Hartree equation in space dimension at least three, in a régime which is supercritical as far as semiclassical analysis is concerned. The main technical remark is that the nonlinear Hartree term can be considered as a semilinear perturbation. This is in contrast with the case of the nonlinear Schrödinger equation with a local nonlinearity, where quasilinear analysis is needed to treat the nonlinearity.

1. INTRODUCTION

We consider the semiclassical limit $\varepsilon \rightarrow 0$ for the Hartree equation

$$(1.1) \quad i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = \lambda(|x|^{-\gamma} * |u^\varepsilon|^2)u^\varepsilon, \quad \gamma > 0, \quad \lambda \in \mathbb{R}, \quad x \in \mathbb{R}^n,$$

in space dimension $n \geq 3$. We consider initial data of WKB type,

$$(1.2) \quad u^\varepsilon(0, x) = a_0^\varepsilon(x)e^{i\phi_0(x)/\varepsilon},$$

where a_0^ε typically has an asymptotic expansion as $\varepsilon \rightarrow 0$,

$$a_0^\varepsilon \underset{\varepsilon \rightarrow 0}{\sim} a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \dots, \quad a_j \text{ independent of } \varepsilon \in]0, 1].$$

The approach that we follow is closely related to the pioneering works of P. Gérard [12], and E. Grenier [15], for the nonlinear Schrödinger equation with local nonlinearity:

$$(1.3) \quad i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = f(|u^\varepsilon|^2)u^\varepsilon,$$

where the function f is smooth, and real-valued. In [15], the assumption $f' > 0$ is necessary for the arguments of the proof. More recently, this assumption was relaxed in [3], allowing to consider the case $f(y) = +y^\sigma$, $\sigma \in \mathbb{N}$. Moreover, it is noticed in [3] that to carry out a WKB analysis in Sobolev spaces for (1.3), the assumption $f' \geq 0$ is essentially necessary. Typically, in the case $f' < 0$, working with analytic data is necessary, and sufficient as shown in [12, 21]. The reason is that the local nonlinearity is analyzed through *quasilinear* arguments in [15, 3], and f' determines the velocity of a wave equation: if $f' > 0$, then the wave equation is hyperbolic, and $f' < 0$, the underlying operator becomes elliptic.

The above discussion is altered for the Hartree type nonlinearity. Typically, no assumption is made on the sign of λ here. As noticed in [1] in the special case of the Schrödinger–Poisson system, the nonlocal nonlinearity in (1.1) can be handled by *semilinear* arguments. However, a quasilinear analysis is needed to handle the convective coupling.

There are at least two motivations to study this question, besides the general picture of justifying approximations motivated by physics. As remarked in [6] in the case of a local nonlinearity, WKB analysis and a geometrical transform can help understand the behavior of a wave function near a focal point, in a supercritical régime. In [20], other informations were obtained thanks to a different approach, in the case of a Hartree type nonlinearity. The approach of [20] and the results of the present paper will certainly be helpful to improve the understanding of the focusing phenomenon in semiclassical analysis. Another application of the WKB analysis for (1.1) concerns the Cauchy problem for the Hartree equation, that is (1.1) with $\varepsilon = 1$. Following the approach initiated in [4, 5, 9, 10, 18, 19], we can prove an ill-posedness result, together with a loss of regularity; see Corollary 1.9.

Assumption 1.1. *Let $n \geq 3$ and $\max(n/2 - 2, 0) < \gamma \leq n - 2$. We suppose the following conditions with some $s > n/2 + 1$:*

- *The initial amplitude $a_0^\varepsilon \in H^s(\mathbb{R}^n)$, uniformly for $\varepsilon \in [0, 1]$: there exists a constant C independent of ε such that $\|a_0^\varepsilon\|_{H^s} \leq C$.*
- *The initial phase ϕ_0 satisfies*

$$|\phi_0(x)| + |\nabla\phi_0(x)| \xrightarrow{|x| \rightarrow \infty} 0,$$

and $\nabla^2\phi_0 \in H^s(\mathbb{R}^n)$. Moreover, there exists $q_0 \in]n/(\gamma + 1), n[$ such that $\nabla\phi_0 \in L^{q_0}(\mathbb{R}^n)$.

Remark 1.2. We will see that the above assumption implies that ϕ_0 and $\nabla\phi_0$ are bounded, and enjoy some extra integrability properties. See Remark 3.3.

Remark 1.3. By employing the geometrical reduction made in [1], we can relax the assumption $|\phi_0(x)| + |\nabla\phi_0(x)| \rightarrow 0$ as $|x| \rightarrow \infty$ in a sense. Indeed, we can replace the initial phase ϕ_0 with $\phi_0 + \phi_{\text{quad}}$, where $\phi_{\text{quad}} \in C^\infty(\mathbb{R}^n)$ is a polynomial of degree at most two.

For $s > n/2$, we denote by $X^s(\mathbb{R}^n)$ the Zhidkov space

$$X^s(\mathbb{R}^n) = \{u \in L^\infty(\mathbb{R}^n) \mid \nabla u \in H^{s-1}(\mathbb{R}^n)\}.$$

This space was introduced in [22] (see also [23]) in the case $n = 1$, and its study was generalized to the multidimensional case in [11]. We denote

$$\|u\|_{X^s} := \|u\|_{L^\infty} + \|\nabla u\|_{H^{s-1}}.$$

We write $H^s = H^s(\mathbb{R}^n)$ and $X^s = X^s(\mathbb{R}^n)$.

Theorem 1.4. *Let Assumption 1.1 be satisfied. There exists $T > 0$ independent of ε and $s > 1 + n/2$, and a unique solution $u^\varepsilon \in C([0, T]; H^s)$ to the equation (1.1)–(1.2). Moreover, it can be written in the form $u^\varepsilon = a^\varepsilon e^{i\phi^\varepsilon/\varepsilon}$, where a^ε is complex-valued, ϕ^ε is real-valued, with*

$$a^\varepsilon \in C([0, T]; H^s), \quad \phi^\varepsilon \in C\left([0, T]; L^\infty \cap L^{\frac{nq_0}{n-q_0}}\right),$$

$$\text{and } \nabla\phi^\varepsilon \in C([0, T]; X^{s+1} \cap L^{q_0}).$$

Moreover, if $\phi_0 \in L^{p_0}$ for some $p_0 \in]n/\gamma, nq_0/(n-q_0)[$ then $\phi^\varepsilon \in C([0, T]; L^{p_0})$.

Note that obtaining a local existence time T which is independent of $\varepsilon \in]0, 1]$ is already a non-trivial information, at least for a focusing nonlinearity $\lambda < 0$. Using classical results on the Cauchy problem for Hartree equation [14], and a scaling argument, would yield an existence time that goes to zero with ε . Taking $q_0 = 2$, we immediately obtain the following corollary:

Corollary 1.5. *Let Assumption 1.1 be satisfied. Let $u^\varepsilon = a^\varepsilon e^{i\phi^\varepsilon/\varepsilon}$ be the solution given in Theorem 1.4. If $\gamma \in]n/2 - 1, n - 2]$ and $\nabla\phi_0 \in H^{s+1}$, then*

$$\phi^\varepsilon \in C\left([0, T]; X^{s+2} \cap L^{\frac{2n}{n-2}}\right).$$

With this local existence result, we can justify a WKB expansion, provided that the initial data have a suitable expansion as $\varepsilon \rightarrow 0$.

Assumption 1.6. *Let N be a positive integer. We suppose Assumption 1.1 with some $s > n/2 + 2N + 1$. Moreover, the initial amplitude a_0^ε writes*

$$(1.4) \quad a_0^\varepsilon = a_0 + \sum_{j=1}^N \varepsilon^j a_j + \varepsilon^N r_N^\varepsilon,$$

where $a_j \in H^s$ ($0 \leq j \leq N$) and $\|r_N^\varepsilon\|_{H^s} \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Theorem 1.7. *Let Assumption 1.6 be satisfied. Let $u^\varepsilon = a^\varepsilon e^{i\phi^\varepsilon/\varepsilon}$ be the unique solution given in Theorem 1.4. Then, there exist $(b_j, \varphi_j)_{0 \leq j \leq N}$, with*

$$\begin{aligned} b_j &\in C([0, T]; H^{s-2j}), \\ \varphi_j &\in C\left([0, T]; L^\infty \cap L^{\frac{nq_0}{n-q_0}}\right) \text{ and } \nabla\varphi_j \in C\left([0, T]; X^{s-2j+1} \cap L^{q_0}\right), \end{aligned}$$

such that:

$$\begin{aligned} a^\varepsilon &= b_0 + \sum_{j=1}^N \varepsilon^j b_j + o(\varepsilon^N) \quad \text{in } C([0, T]; H^{s-2N}), \\ \phi^\varepsilon &= \varphi_0 + \sum_{j=1}^{N-1} \varepsilon^j \varphi_j + o(\varepsilon^N) \quad \text{in } C([0, T]; L^\infty \cap L^{\frac{nq_0}{n-q_0}}), \\ \nabla\phi^\varepsilon &= \nabla\varphi_0 + \sum_{j=1}^N \varepsilon^j \nabla\varphi_j + o(\varepsilon^N) \quad \text{in } C([0, T]; X^{s-2N+1} \cap L^{q_0}). \end{aligned}$$

Moreover, for $j \geq 1$, $\varphi_j \in L^p$ for all $p > n/\gamma$, and $\nabla\varphi_j \in L^q$ for all $q > n/(\gamma + 1)$.

Corollary 1.8. *Let Assumption 1.6 be satisfied. The solution u^ε given in Theorem 1.4 has the following asymptotic expansion, as $\varepsilon \rightarrow 0$:*

$$u^\varepsilon = e^{i\varphi_0/\varepsilon} \left(\beta_0 + \varepsilon\beta_1 + \dots + \varepsilon^{N-1}\beta_{N-1} + \varepsilon^N \rho^\varepsilon \right), \quad \|\rho^\varepsilon\|_{H^{s-2N+2}} \xrightarrow{\varepsilon \rightarrow 0} 0,$$

where φ_0 is given by Theorem 1.7, and $\beta_j \in C([0, T]; H^{s-2j})$ is a smooth function of $(b_k, \varphi_{k+1})_{0 \leq k \leq j}$. For instance,

$$\beta_0 = b_0 e^{i\varphi_1} \quad ; \quad \beta_1 = b_1 e^{i\varphi_1} + i\varphi_2 b_0 e^{i\varphi_1}.$$

Corollary 1.9. *Let $n \geq 5$, $\lambda \in \mathbb{R}$, $\max(n/2 - 2, 2) < \gamma \leq n - 2$ and $0 < s < s_c = \gamma/2 - 1$. There exists a sequence of initial data*

$$(\psi_0^h)_{0 < h \leq 1}, \quad \psi_0^h \in \mathcal{S}(\mathbb{R}^n), \quad \|\psi_0^h\|_{H^s} \xrightarrow{h \rightarrow 0} 0,$$

a sequence of times $t^h \rightarrow 0$, such that the solution to

$$i\partial_t \psi^h + \frac{1}{2} \Delta \psi^h = \lambda \left(|x|^{-\gamma} * |\psi^h|^2 \right) \psi^h \quad ; \quad \psi|_{t=0} = \psi_0^h$$

satisfies

$$\left\| \psi^h(t^h) \right\|_{H^k} \xrightarrow{h \rightarrow 0} +\infty, \quad \forall k > \frac{s}{1 + s_c - s} = \frac{s}{\gamma/2 - s}.$$

Note that unlike in the case of Schrödinger equations with local nonlinearity, considering a large space dimension is necessary to observe this phenomenon: in low space dimensions, Hartree equations are locally well-posed in Sobolev spaces of positive regularity (see e.g. [8, 14]).

Using Sobolev embedding, one could infer a loss of regularity at the level of the energy space (consider $s > 1$ and $\frac{s}{\gamma/2 - s} = 1$, hence $\gamma = 4s > 1$), in the spirit of [19] (see also [2, 21] for Schrödinger equations), provided that the space dimension is $n \geq 7$.

The rest of this paper is organized as follows. In the next paragraph, we present the general strategy adopted in this paper. In §3, we collect some technical estimates. Theorem 1.4 is proved in §4, and Theorem 1.7 is proved in §5, as well as Corollary 1.8. Finally, Corollary 1.9 is inferred in §6.

2. GENERAL STRATEGY

To prove Theorem 1.4 and Theorem 1.7, we follow the same strategy as in [15]. Seek a solution u^ε to (1.1)–(1.2) represented as

$$(2.1) \quad u^\varepsilon(t, x) = a^\varepsilon(t, x) e^{i\phi^\varepsilon(t, x)/\varepsilon},$$

with a complex-valued space-time function a^ε and a real-valued space-time function ϕ^ε . Note that a^ε is expected to be complex-valued, even if its initial value a_0^ε is real-valued. We remark that the phase function ϕ^ε also depends on the parameter ε . Substituting the form (2.1) into (1.1), we obtain

$$\begin{aligned} i\varepsilon \left(\partial_t a^\varepsilon + a^\varepsilon i \frac{\partial_t \phi^\varepsilon}{\varepsilon} \right) + \frac{\varepsilon^2}{2} \left(\Delta a^\varepsilon + 2 \left(\nabla a^\varepsilon \cdot i \frac{\nabla \phi^\varepsilon}{\varepsilon} \right) - a^\varepsilon \frac{|\nabla \phi^\varepsilon|^2}{\varepsilon^2} + a^\varepsilon i \frac{\Delta \phi^\varepsilon}{\varepsilon} \right) \\ = \lambda (|x|^{-\gamma} * |a^\varepsilon|^2) a^\varepsilon. \end{aligned}$$

To obtain a solution of the above equation (hence, of (1.1)), we choose to consider the following system:

$$(2.2) \quad \begin{cases} \partial_t a^\varepsilon + \nabla a^\varepsilon \cdot \nabla \phi^\varepsilon + \frac{1}{2} a^\varepsilon \Delta \phi^\varepsilon = i \frac{\varepsilon}{2} \Delta a^\varepsilon \\ \partial_t \phi^\varepsilon + \frac{1}{2} |\nabla \phi^\varepsilon|^2 + \lambda (|x|^{-\gamma} * |a^\varepsilon|^2) = 0. \end{cases}$$

This choice is essentially the same as the one introduced by E. Grenier [15]. We consider this with the initial data

$$(2.3) \quad a|_{t=0} = a_0^\varepsilon, \quad \phi|_{t=0} = \phi_0.$$

From now on, we work only on (2.2)–(2.3). We first prove that it admits a unique solution with suitable regularity (see Theorem 1.4), hence providing a solution to (1.1)–(1.2). The asymptotic expansion (Theorem 1.7) then follows by the similar arguments.

To conclude this paragraph, we remark that uniqueness for (1.1)–(1.2) in the class $C([0, T]; H^s)$, $s > n/2 + 1$, is a straightforward consequence of [14] (see also [8]) in space dimension $3 \leq n \leq 5$, since the parameter $\varepsilon \in]0, 1]$ can be considered fixed. Indeed, in that case, one has $x \mapsto |x|^{-\gamma} \in L^p + L^\infty$ for some $p \geq 1$ and $p > n/4$, since $\gamma \leq n - 2 < 4$. Uniqueness is actually obtained in the weaker class of finite energy solutions. Since we work at a higher degree of regularity, we can simply notice that the nonlinear potential $|x|^{-\gamma} * |u^\varepsilon|^2$ is bounded in $L^\infty([0, T] \times \mathbb{R}^n)$: for $\chi \in C_0^\infty(\mathbb{R}^n; [0, 1])$, $\chi = 1$ near $x = 0$, we have from Young's inequality,

$$\begin{aligned} \||x|^{-\gamma} * |u^\varepsilon|^2\|_{L^\infty(\mathbb{R}^n)} &\leq \|\chi|x|^{-\gamma}\|_{L^1(\mathbb{R}^n)} \|u^\varepsilon\|_{L^\infty(\mathbb{R}^n)}^2 \\ &\quad + \|(1 - \chi)|x|^{-\gamma}\|_{L^\infty(\mathbb{R}^n)} \|u^\varepsilon\|_{L^2(\mathbb{R}^n)}^2. \end{aligned}$$

Since we work with an H^s regularity, $s > n/2 + 1$, the above right hand side is bounded, and uniqueness follows from standard energy estimates in L^2 .

3. PRELIMINARY ESTIMATES

We first recall a consequence of the Hardy-Littlewood-Sobolev inequality, which can be found in [16, Th. 4.5.9] or [13, Lemma 7]:

Lemma 3.1. *If $\varphi \in \mathcal{D}'(\mathbb{R}^n)$ is such that $\nabla\varphi \in L^p(\mathbb{R}^n)$ for some $p \in]0, n[$, then there exists a constant γ such that $\varphi - \gamma \in L^q(\mathbb{R}^n)$, with $1/p = 1/q + 1/n$.*

Remark 3.2. The limiting case $\gamma = n - 2$ corresponds to the Schrödinger–Poisson system considered in [1], with suitable conditions at infinity to integrate the Poisson equation.

Remark 3.3. By Lemma 3.1 and Sobolev inequality, Assumption 1.1 implies $\phi_0 \in L^{\frac{nq_0}{n-q_0}} \cap L^\infty$, and $\nabla\phi_0 \in L^{q_0} \cap X^{s+1}$. Note that $2n/(n-2) < n$ if $n \geq 5$. Therefore, in this case, we can always find q_0 in $]n/(\gamma+1), n[$ such that $\nabla\phi_0 \in L^{q_0}$.

The next two lemmas can be found in [17]:

Lemma 3.4 (Commutator estimate). *Let $s \geq 0$ and $1 < p < \infty$. Set $\Lambda = (1 - \Delta)^{1/2}$. Then, it holds that*

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \leq c(\|\nabla f\|_{L^\infty} \|\Lambda^{s-1} g\|_{L^p} + \|\Lambda^s f\|_{L^p} \|g\|_{L^\infty}).$$

Lemma 3.5. *Let $s > 0$ and $1 < p < \infty$. There exists $C > 0$ such that*

$$\|\Lambda^s(fg)\|_{L^p} \leq C(\|\Lambda^s f\|_{L^p} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|\Lambda^s g\|_{L^p}), \quad \forall f, g \in W^{s,p} \cap L^\infty.$$

The following lemma is crucial for our analysis:

Lemma 3.6. *Let $n \geq 3$, $k \geq 0$, and $s_1, s_2 \in \mathbb{R}$. Let $\gamma > 0$ satisfying $n/2 - k < \gamma \leq n - k - s_1 + s_2$. Then, there exists C_s such that*

$$\left\| |\nabla|^k (|x|^{-\gamma} * f) \right\|_{H^{s_1}} \leq C_s (\|f\|_{H^{s_2}} + \|f\|_{L^1}), \quad \forall f \in L^1 \cap H^{s_2}.$$

Proof. Since $\mathcal{F}|x|^{-\gamma} = C|\xi|^{-n+\gamma}$, it holds that

$$\left\| |\nabla|^k (|x|^\gamma * f) \right\|_{H^{s_1}} = C \left\| \langle \xi \rangle^{s_1} |\xi|^{-n+\gamma+k} \mathcal{F}f \right\|_{L^2}.$$

The high frequency part ($|\xi| > 1$) is bounded by $C \|f\|_{H^{s_2}}$ if $-n + \gamma + k + s_1 - s_2 \leq 0$. On the other hand, the low frequency part ($|\xi| \leq 1$) is bounded by

$$C \|\mathcal{F}f\|_{L^\infty} \int_{|\xi| \leq 1} |\xi|^{2(-n+\gamma+k)} d\xi \leq C \|f\|_{L^1}$$

if $2(-n + \gamma + k) > -n$, that is, if $\gamma > n/2 - k$. \square

4. EXISTENCE RESULT: PROOF OF THEOREM 1.4

Operating ∇ to the equation for ϕ^ε in (2.2) and putting $v^\varepsilon := \nabla \phi^\varepsilon$, we obtain the following system:

$$(4.1) \quad \begin{cases} \partial_t a^\varepsilon + v^\varepsilon \cdot \nabla a^\varepsilon + \frac{1}{2} a^\varepsilon \nabla \cdot v^\varepsilon = i \frac{\varepsilon}{2} \Delta a^\varepsilon, & a^\varepsilon|_{t=0} = a_0^\varepsilon, \\ \partial_t v^\varepsilon + v^\varepsilon \cdot \nabla v^\varepsilon + \lambda \nabla (|x|^{-\gamma} * |a^\varepsilon|^2) = 0, & v^\varepsilon|_{t=0} = \nabla \phi_0. \end{cases}$$

We first construct the solution $(a^\varepsilon, v^\varepsilon)$ to the system (4.1).

Proposition 4.1. *Let Assumption 1.1 be satisfied. There exists $T > 0$ independent of ε and s , such that for all $\varepsilon \in [0, 1]$, (4.1) has a unique solution*

$$(a^\varepsilon, v^\varepsilon) \in C \left([0, T]; H^s \times \left(X^{s+1} \cap L^{\frac{nq_0}{n-q_0}} \right) \right).$$

Moreover, the norm of $(a^\varepsilon, v^\varepsilon)$ is bounded uniformly for $\varepsilon \in]0, 1]$.

4.1. Regularized system. We shall prove the existence of the solution to the system (4.1) by taking the limit of the solutions to the corresponding regularized system. We take $\varphi \in C_0^\infty(\mathbb{R}^n)$, with $\int_{\mathbb{R}^n} \varphi(x) dx = 1$ and $\varphi \geq 0$ and set

$$(4.2) \quad J_\delta f = \varphi_\delta * f$$

where $\varphi_\delta = \delta^{-n} \varphi(x/\delta)$. We first treat the following regularized system:

$$(4.3) \quad \begin{cases} \partial_t a_\delta^\varepsilon + J_\delta(v_\delta^\varepsilon \cdot \nabla J_\delta a_\delta^\varepsilon) + \frac{1}{2} a_\delta^\varepsilon \nabla \cdot J_\delta v_\delta^\varepsilon = i \frac{\varepsilon}{2} \Delta J_\delta^2 a_\delta^\varepsilon; & a_\delta^\varepsilon|_{t=0} = a_0^\varepsilon. \\ \partial_t v_\delta^\varepsilon + J_\delta(v_\delta^\varepsilon \cdot \nabla J_\delta v_\delta^\varepsilon) + \lambda \nabla J_\delta (|x|^{-\gamma} * |a_\delta^\varepsilon|^2) = 0; & v_\delta^\varepsilon|_{t=0} = \nabla \phi_0. \end{cases}$$

The point is that the regularized equations (4.3) have been chosen so that the Cauchy problem can be solved as in the standard framework of Sobolev and Zhidkov spaces:

Lemma 4.2. *Let Assumption 1.1 be satisfied. For all $\varepsilon \in [0, 1]$ and $\delta \in]0, 1]$, there exists $T_\delta^\varepsilon > 0$ such that the Cauchy problem (4.3) has a unique solution $(a_\delta^\varepsilon, v_\delta^\varepsilon) \in C^1([0, T_\delta^\varepsilon], H^{s+1} \times X^{s+2} \cap L^{\frac{2n}{n-2}})$.*

Proof. The proof is based on the usual theorem for ordinary differential equations. We use the following estimates

$$\begin{aligned} \|J_\delta(v_\delta^\varepsilon \cdot \nabla J_\delta a_\delta^\varepsilon)\|_{H^{s+1}} &\leq C \|v_\delta^\varepsilon\|_{H^{s+1}} \|\nabla J_\delta a_\delta^\varepsilon\|_{H^{s+1}} \\ &\leq C \delta^{-1} \|v_\delta^\varepsilon\|_{H^{s+2}} \|a_\delta^\varepsilon\|_{H^{s+1}}, \end{aligned}$$

and

$$\begin{aligned}
\|a_\delta^\varepsilon \nabla \cdot J_\delta v_\delta^\varepsilon\|_{H^{s+1}} &\leq C\delta^{-1} \|a_\delta^\varepsilon\|_{H^{s+1}} \|v_\delta^\varepsilon\|_{X^{s+2}}, \\
\|\Delta J_\delta^2 a_\delta^\varepsilon\|_{H^{s+1}} &\leq C\delta^{-2} \|a_\delta^\varepsilon\|_{H^{s+1}}, \\
\|J_\delta(v_\delta^\varepsilon \cdot \nabla J_\delta v_\delta^\varepsilon)\|_{X^{s+2}} &\leq C\delta^{-1} \|v_\delta^\varepsilon\|_{X^{s+2}}^2, \\
\|\nabla J_\delta(|x|^{-\gamma} * |a_\delta^\varepsilon|^2)\|_{X^{s+2}} &\leq C \|\Delta(|x|^{-\gamma} * |a_\delta^\varepsilon|^2)\|_{H^{s+1}} \\
&\leq C \|a_\delta^\varepsilon\|_{H^{s+1}}^2.
\end{aligned}$$

We have applied Lemma 3.6 with $k = 2$ and $s_1 = s_2 = s + 1$. We note that the space $X^{s+2} \cap L^{\frac{2n}{n-2}}$ with norm $\|\cdot\|_{X^{s+2}}$ is complete. \square

4.2. Uniform bound. We shall establish an upper bound for the H^s norm and X^{s+1} norm of a_δ^ε and v_δ^ε for $s > n/2 + 1$, respectively. We first estimate the H^s norm of a_δ^ε . We use the following convention for the scalar product in L^2 :

$$\langle \varphi, \psi \rangle := \int_{\mathbb{R}^n} \varphi(x) \overline{\psi(x)} dx.$$

Set $\Lambda = (I - \Delta)^{1/2}$. We shall estimate

$$\frac{d}{dt} \|a_\delta^\varepsilon\|_{H^s}^2 = 2 \operatorname{Re} \langle \partial_t \Lambda^s a_\delta^\varepsilon, \Lambda^s a_\delta^\varepsilon \rangle.$$

Since $[\Lambda^s, \nabla] = 0$ and $[\Lambda^s, J_\delta] = 0$, by commuting Λ^s with the equation for a_δ^ε , we find:

$$(4.4) \quad \partial_t \Lambda^s a_\delta^\varepsilon + J_\delta \Lambda^s (v_\delta^\varepsilon \cdot \nabla J_\delta a_\delta^\varepsilon) + \frac{1}{2} \Lambda^s (a_\delta^\varepsilon \nabla J_\delta \cdot v_\delta^\varepsilon) - i \frac{\varepsilon}{2} J_\delta \Delta J_\delta \Lambda^s a_\delta^\varepsilon = 0.$$

The coupling of the second term and $\Lambda^s a_\delta^\varepsilon$ is written as

$$\begin{aligned}
\langle \Lambda^s (v_\delta^\varepsilon \cdot \nabla J_\delta a_\delta^\varepsilon), J_\delta \Lambda^s a_\delta^\varepsilon \rangle &= \langle v_\delta^\varepsilon \cdot \nabla J_\delta \Lambda^s a_\delta^\varepsilon, J_\delta \Lambda^s a_\delta^\varepsilon \rangle \\
&\quad + \langle [\Lambda^s, v_\delta^\varepsilon] \cdot \nabla J_\delta a_\delta^\varepsilon, J_\delta \Lambda^s a_\delta^\varepsilon \rangle,
\end{aligned}$$

where we have use the fact that $\langle J_\delta f, g \rangle = \langle f, J_\delta g \rangle$ for any f and g . We see from the integration by parts that

$$(4.5) \quad |\operatorname{Re} \langle v_\delta^\varepsilon \cdot \nabla J_\delta \Lambda^s a_\delta^\varepsilon, J_\delta \Lambda^s a_\delta^\varepsilon \rangle| \leq \frac{1}{2} \|\nabla v_\delta^\varepsilon\|_{L^\infty} \|J_\delta \Lambda^s a_\delta^\varepsilon\|_{L^2}^2.$$

Moreover, the commutator estimate shows that

$$\begin{aligned}
(4.6) \quad |\operatorname{Re} \langle [\Lambda^s, v_\delta^\varepsilon] \cdot \nabla J_\delta a_\delta^\varepsilon, J_\delta \Lambda^s a_\delta^\varepsilon \rangle| \\
\leq C (\|\nabla v_\delta^\varepsilon\|_{H^{s-1}} \|J_\delta \nabla a_\delta^\varepsilon\|_{L^\infty} + \|\nabla v_\delta^\varepsilon\|_{L^\infty} \|J_\delta \nabla a_\delta^\varepsilon\|_{H^{s-1}}) \|J_\delta \Lambda^s a_\delta^\varepsilon\|_{L^2}
\end{aligned}$$

We estimate the third term of (4.4) by the Kato–Ponce inequality as

$$\begin{aligned}
(4.7) \quad |\operatorname{Re} \langle \Lambda^s (a_\delta^\varepsilon \nabla J_\delta \cdot v_\delta^\varepsilon), \Lambda^s a_\delta^\varepsilon \rangle| \\
\leq C (\|a_\delta^\varepsilon\|_{L^\infty} \|J_\delta \nabla v_\delta^\varepsilon\|_{H^s} + \|a_\delta^\varepsilon\|_{H^s} \|J_\delta \nabla v_\delta^\varepsilon\|_{L^\infty}) \|\Lambda^s a_\delta^\varepsilon\|_{L^2}
\end{aligned}$$

and the last term vanishes since

$$(4.8) \quad \operatorname{Re} \langle -i \Delta J_\delta \Lambda^s a_\delta^\varepsilon, J_\delta \Lambda^s a_\delta^\varepsilon \rangle = \operatorname{Re} i \|\nabla J_\delta \Lambda^s a_\delta^\varepsilon\|_{L^2}^2 = 0.$$

Therefore, summarizing (4.4)–(4.8), we end up with

$$\frac{d}{dt} \|a_\delta^\varepsilon\|_{H^s}^2 \leq C (\|a_\delta^\varepsilon\|_{W^{1,\infty}} + \|\nabla v_\delta^\varepsilon\|_{L^\infty}) (\|a_\delta^\varepsilon\|_{H^s} + \|v_\delta^\varepsilon\|_{X^{s+1}}) \|a_\delta^\varepsilon\|_{H^s},$$

hence

$$(4.9) \quad \frac{d}{dt} \|a_\delta^\varepsilon\|_{H^s}^2 \leq C(\|a_\delta^\varepsilon\|_{W^{1,\infty}} + \|\nabla v_\delta^\varepsilon\|_{L^\infty})(\|a_\delta^\varepsilon\|_{H^s}^2 + \|v_\delta^\varepsilon\|_{X^{s+1}}^2).$$

Let us proceed to the estimate of v_δ^ε . We denote the operator $\Lambda^s \nabla$ by Q . From the equation for v_δ^ε , we have

$$(4.10) \quad \partial_t Q v_\delta^\varepsilon + J_\delta Q (v_\delta^\varepsilon \cdot \nabla J_\delta v_\delta^\varepsilon) + Q \nabla \Lambda^s J_\delta (|x|^{-\gamma} * |a_\delta^\varepsilon|^2) = 0$$

We consider the coupling of this equation and $Q v_\delta^\varepsilon$. The second term can be written as

$$\begin{aligned} \langle Q (v_\delta^\varepsilon \cdot \nabla J_\delta v_\delta^\varepsilon), J_\delta Q v_\delta^\varepsilon \rangle &= \langle v_\delta^\varepsilon \cdot \nabla J_\delta Q v_\delta^\varepsilon, J_\delta Q v_\delta^\varepsilon \rangle \\ &\quad + \langle [Q, v_\delta^\varepsilon] \cdot \nabla J_\delta v_\delta^\varepsilon, J_\delta Q \nabla v_\delta^\varepsilon \rangle. \end{aligned}$$

As the previous case, integration by parts shows

$$(4.11) \quad |\operatorname{Re} \langle v_\delta^\varepsilon \cdot \nabla J_\delta Q v_\delta^\varepsilon, J_\delta Q v_\delta^\varepsilon \rangle| \leq \frac{1}{2} \|\nabla \cdot v_\delta^\varepsilon\|_{L^\infty} \|J_\delta Q v_\delta^\varepsilon\|_{L^2},$$

and the commutator estimate also shows

$$(4.12) \quad |\operatorname{Re} \langle [Q, v_\delta^\varepsilon] \cdot \nabla J_\delta v_\delta^\varepsilon, J_\delta Q v_\delta^\varepsilon \rangle| \leq C(\|\nabla v_\delta^\varepsilon\|_{H^s} \|J_\delta \nabla v_\delta^\varepsilon\|_{L^\infty} + \|\nabla v_\delta^\varepsilon\|_{L^\infty} \|J_\delta \nabla v_\delta^\varepsilon\|_{H^s}) \|J_\delta Q v_\delta^\varepsilon\|_{L^2}.$$

For the estimate of the Hartree nonlinearity, we use Lemma 3.6 with $k = 2$ and $s_1 = s_2 = s$, to obtain

$$(4.13) \quad \begin{aligned} \|\lambda J_\delta \Lambda^s \nabla^2 (|x|^{-\gamma} * |a_\delta^\varepsilon|^2)\|_{L^2} &\leq C(\| |a_\delta^\varepsilon|^2 \|_{H^s} + \| |a_\delta^\varepsilon|^2 \|_{L^1}) \\ &\leq C(\|a_\delta^\varepsilon\|_{H^s} \|a_\delta^\varepsilon\|_{L^\infty} + \|a_\delta^\varepsilon\|_{L^2}^2). \end{aligned}$$

Summarizing (4.10)–(4.13), we deduce that

$$\frac{d}{dt} \|\nabla v_\delta^\varepsilon\|_{H^s}^2 \leq C(\|a_\delta^\varepsilon\|_{L^2} + \|a_\delta^\varepsilon\|_{L^\infty} + \|\nabla v_\delta^\varepsilon\|_{L^\infty})(\|a_\delta^\varepsilon\|_{H^s} + \|\nabla v_\delta^\varepsilon\|_{H^s}) \|\nabla v_\delta^\varepsilon\|_{H^s},$$

and so that

$$(4.14) \quad \frac{d}{dt} \|\nabla v_\delta^\varepsilon\|_{H^s}^2 \leq C(\|a_\delta^\varepsilon\|_{L^2} + \|a_\delta^\varepsilon\|_{L^\infty} + \|\nabla v_\delta^\varepsilon\|_{L^\infty})(\|a_\delta^\varepsilon\|_{H^s}^2 + \|\nabla v_\delta^\varepsilon\|_{H^s}^2).$$

Using Lemma 3.1, we see that the above estimate yields an $L^{2n/(n-2)}$ estimate for v_δ^ε . Interpolating with a suitable \dot{H}^k norm shows that the L^∞ norm of v_δ^ε is estimated as above. Alternatively, integrating the second equation of (4.3) with respect to time, Sobolev embedding directly yields a similar estimate for v_δ^ε in $L^\infty(\mathbb{R}^n)$.

Now, putting $M_\delta^\varepsilon(t) := \|a_\delta^\varepsilon(t)\|_{H^s}^2 + \|\nabla v_\delta^\varepsilon(t)\|_{H^s}^2 + \|v_\delta^\varepsilon(t)\|_{L^\infty}^2$, we conclude from (4.9) and (4.14) that

$$(4.15) \quad M_\delta^\varepsilon \leq C + C \int_0^t (\|a_\delta^\varepsilon\|_{L^2} + \|a_\delta^\varepsilon\|_{W^{1,\infty}} + \|\nabla v_\delta^\varepsilon\|_{L^\infty}) M_\delta^\varepsilon d\tau.$$

We obtain the following Lemma.

Lemma 4.3. *Let Assumption 1.1 be satisfied with $s > n/2 + 1$. There exists T independent of δ and ε such that the solution $(a_\delta^\varepsilon, v_\delta^\varepsilon)$ is bounded in $C([0, T]; H^s \times X^{s+1})$ uniformly in $\delta \in]0, 1[$.*

Proof. We only estimate the above $M_\delta^\varepsilon(t)$. Note that v_δ^ε vanishes at spatial infinity. It implies that $\|v_\delta^\varepsilon\|_{L^\infty}$ is bounded by $\|\nabla v_\delta^\varepsilon\|_{H^s}$ with some constant, since $n \geq 3$. and $s > n/2 + 1$. Sobolev embedding and (4.15) yield

$$(4.16) \quad M_\delta^\varepsilon(t) \leq C + C \int_0^t (M_\delta^\varepsilon(\tau))^{3/2} d\tau.$$

Therefore, there exists $T_\delta^\varepsilon > 0$ depending only on $M_\delta^\varepsilon(0)$ such that $M_\delta^\varepsilon(t)$ is bounded by constant times $M_\delta^\varepsilon(0)$ uniformly in $t \in [0, T_\delta^\varepsilon]$. Since $M_\delta^\varepsilon(0)$ is bounded independent of δ and ε by assumption, T_δ^ε can be taken independent of δ and ε , as well as the upper bound of $M_\delta^\varepsilon(t)$. \square

4.3. Existence of the solution to the nonlinear hyperbolic system.

Next we prove the existence of the solution to (4.1). From Lemma 4.3, we see that the sequences $\{a_\delta^\varepsilon\}_\delta$ and $\{v_\delta^\varepsilon\}_\delta$ are uniformly bounded in $C([0, T]; H^s)$ and $C([0, T]; X^{s+1} \cap L^{\frac{2n}{n-2}})$, respectively. Therefore, from Ascoli–Arzela’s theorem, for a subsequence δ' of δ ,

$$\begin{aligned} a_{\delta'}^\varepsilon &\rightharpoonup a^\varepsilon \quad \text{weakly in } C([0, T]; H^s), \\ v_{\delta'}^\varepsilon &\rightharpoonup v^\varepsilon \quad \text{weakly in } C([0, T]; L^{\frac{2n}{n-2}}), \\ \nabla v_{\delta'}^\varepsilon &\rightharpoonup \nabla v^\varepsilon \quad \text{weakly in } C([0, T]; H^s) \end{aligned}$$

as $\delta' \rightarrow 0$. Moreover, we have $(a^\varepsilon, v^\varepsilon) \in C_w([0, T]; H^s \times X^{s+1} \cap L^{\frac{2n}{n-2}})$.

We shall show that $(a^\varepsilon, v^\varepsilon)$ satisfies (4.1) in $\mathcal{D}'([0, T] \times \mathbb{R}^n)$. We fix some t . We choose some s' so that $s > s' > n/2 + 1$. Then, the above convergences imply $(a_{\delta'}^\varepsilon, \nabla v_{\delta'}^\varepsilon) \rightarrow (a^\varepsilon, \nabla v^\varepsilon)$ strongly in $C([0, T]; H_{\text{loc}}^{s'} \times H_{\text{loc}}^{s'})$ as $\delta' \rightarrow 0$. Then, we deduce that $a_{\delta'}^\varepsilon$, $\nabla a_{\delta'}^\varepsilon$, $v_{\delta'}^\varepsilon$, and $\nabla v_{\delta'}^\varepsilon$ converge uniformly in any compact subset of \mathbb{R}^n , since $s > n/2 + 1$ and $\|v_{\delta'}^\varepsilon - v^\varepsilon\|_{L^\infty}$ is bounded by $\|\nabla v_{\delta'}^\varepsilon - \nabla v^\varepsilon\|_{H^{s'}}$ with some constant. Then, we can pass to the limit in all the terms in (4.1), except possibly the Hartree term. Since $(f * g) * h = f * (g * h)$ and $\langle f * g, h \rangle = \langle f, \check{g} * h \rangle$ with $\check{g}(x) = \bar{g}(-x)$, the Hartree term can be rewritten as

$$\langle \lambda \nabla J_{\delta'}(|x|^{-\gamma} * |a_{\delta'}^\varepsilon|^2), \varphi \rangle = -\lambda \langle J_{\delta'} |a_{\delta'}^\varepsilon|^2, |x|^{-\gamma} * \nabla \varphi \rangle.$$

The function $|x|^{-\gamma} * \nabla \varphi$ is not compactly supported, but an $\varepsilon/3$ -argument shows that the right hand side tends to $-\lambda \langle |a^\varepsilon|^2, |x|^{-\gamma} * \nabla \varphi \rangle$. Thus, we obtain the solution $(a^\varepsilon, v^\varepsilon) \in C_w([0, T]; H^s \times X^{s+1} \cap L^{\frac{2n}{n-2}})$. We now claim that this solution is strongly continuous in time. To prove this, we only have to show that the solution is norm continuous, that is, the function $M^\varepsilon(t) := \|a^\varepsilon(t)\|_{H^s}^2 + \|\nabla v^\varepsilon(t)\|_{H^s}^2$ is continuous in time. In the same way as (4.15), we have

$$(4.17) \quad \frac{d}{dt} M^\varepsilon \leq C (\|a^\varepsilon\|_{L^2} + \|a^\varepsilon\|_{W^{1,\infty}} + \|\nabla v^\varepsilon\|_{L^\infty}) M^\varepsilon.$$

Since the right hand side is bounded, M^ε is upper semi-continuous. Weak continuities of a^ε and v^ε imply the lower semi-continuity of M^ε . Hence, M^ε is continuous.

Lemma 4.4. *Let Assumption 1.1 be satisfied. Suppose $s > n/2 + 1$. Let T be given in Lemma 4.3. For all $\varepsilon \in [0, 1]$, there exists $(a^\varepsilon, v^\varepsilon) \in C([0, T]; H^s \times X^{s+1} \cap L^{\frac{2n}{n-2}})$ which solves (4.1) in \mathcal{D}' .*

4.4. Uniqueness. We next prove the uniqueness of the solution $(a^\varepsilon, v^\varepsilon)$ by showing that if $(a_1^\varepsilon, v_1^\varepsilon)$ and $(a_2^\varepsilon, v_2^\varepsilon)$ are solutions to (4.1), in the class $C([0, T]; H^s \times X^{s+1} \cap L^{2n/(n-2)})$ for some $s > n/2 + 1$, then the distance $(a_1^\varepsilon - a_2^\varepsilon, v_1^\varepsilon - v_2^\varepsilon)$ is equal to zero in $L^\infty([0, T]; L^2 \times \dot{H}^1)$ sense. Denote $(d_a^\varepsilon, d_v^\varepsilon) := (a_1^\varepsilon - a_2^\varepsilon, v_1^\varepsilon - v_2^\varepsilon)$. Then, from (4.1), the system for $(d_a^\varepsilon, d_v^\varepsilon)$ is rewritten as

$$(4.18) \quad \partial_t d_a^\varepsilon + d_v^\varepsilon \cdot \nabla a_1^\varepsilon + v_2^\varepsilon \cdot \nabla d_a^\varepsilon + \frac{1}{2} d_a^\varepsilon \cdot \nabla v_1^\varepsilon + \frac{1}{2} a_2^\varepsilon \cdot \nabla d_v^\varepsilon = i \frac{\varepsilon}{2} \Delta d_a^\varepsilon,$$

$$(4.19) \quad \partial_t d_v^\varepsilon + d_v^\varepsilon \cdot \nabla v_1^\varepsilon + v_2^\varepsilon \cdot \nabla d_v^\varepsilon + \lambda \nabla (|x|^{-\gamma} * (d_a^\varepsilon \overline{a_1^\varepsilon} + a_2^\varepsilon \overline{d_a^\varepsilon})) = 0.$$

Now estimate the L^2 norm of d_a^ε . From the equation in (4.18), it holds that

$$\begin{aligned} \frac{d}{dt} \|d_a^\varepsilon\|_{L^2}^2 &= 2 \operatorname{Re} \langle \partial_t d_a^\varepsilon, d_a^\varepsilon \rangle \\ &\leq C |\operatorname{Re} \langle d_v^\varepsilon \cdot \nabla a_1^\varepsilon, d_a^\varepsilon \rangle| + C |\operatorname{Re} \langle v_2^\varepsilon \cdot \nabla d_a^\varepsilon, d_a^\varepsilon \rangle| + C |\operatorname{Re} \langle d_a^\varepsilon \cdot \nabla v_1^\varepsilon, d_a^\varepsilon \rangle| \\ &\quad + C |\operatorname{Re} \langle a_2^\varepsilon \cdot \nabla d_v^\varepsilon, d_a^\varepsilon \rangle| + |\operatorname{Re} \langle i \Delta d_a^\varepsilon, d_a^\varepsilon \rangle|. \end{aligned}$$

Now, Hölder's inequality and integration by parts show that

$$\begin{aligned} |\operatorname{Re} \langle a_2^\varepsilon \cdot \nabla d_v^\varepsilon, d_a^\varepsilon \rangle| &\leq \|a_2^\varepsilon\|_{L^\infty} \|\nabla d_v^\varepsilon\|_{L^2} \|d_a^\varepsilon\|_{L^2}, \\ |\operatorname{Re} \langle v_2^\varepsilon \cdot \nabla d_a^\varepsilon, d_a^\varepsilon \rangle| + |\operatorname{Re} \langle d_a^\varepsilon \cdot \nabla v_1^\varepsilon, d_a^\varepsilon \rangle| &\leq (\|\nabla v_1^\varepsilon\|_{L^\infty} + \|\nabla v_2^\varepsilon\|_{L^\infty}) \|d_a^\varepsilon\|_{L^2}^2, \\ \operatorname{Re} \langle i \Delta d_a^\varepsilon, d_a^\varepsilon \rangle &= 0. \end{aligned}$$

Another use of Hölder's and Sobolev inequalities shows

$$\begin{aligned} |\langle d_v^\varepsilon \cdot \nabla a_1^\varepsilon, d_a^\varepsilon \rangle| &\leq \|d_v^\varepsilon\|_{L^{\frac{2n}{n-2}}} \|\nabla a_1^\varepsilon\|_{L^n} \|d_a^\varepsilon\|_{L^2} \\ &\leq C \|\nabla a_1^\varepsilon\|_{H^{\frac{n}{2}-1}} \|\nabla d_v^\varepsilon\|_{L^2} \|d_a^\varepsilon\|_{L^2} \end{aligned}$$

Thus, we end up with the estimate

$$(4.20) \quad \frac{d}{dt} \|d_a^\varepsilon\|_{L^2}^2 \leq C \left(\|d_a^\varepsilon\|_{L^2}^2 + \|\nabla d_v^\varepsilon\|_{L^2}^2 \right),$$

where the constant C depends on $\|a_1^\varepsilon\|_{H^{n/2}}$, $\|a_2^\varepsilon\|_{L^\infty}$, and $\|\nabla v_k^\varepsilon\|_{L^2}$ ($k = 1, 2$). Similarly, for all $1 \leq i, j \leq n$, we have the estimates for $\partial_i d_{v,j}^\varepsilon$:

$$\begin{aligned} |\langle ((\partial_i d_v^\varepsilon) \cdot \nabla) v_{1,j}^\varepsilon, \partial_i d_{v,j}^\varepsilon \rangle| &\leq C \|\nabla v_{1,j}^\varepsilon\|_{L^\infty} \|\partial_i d_{v,j}^\varepsilon\|_{L^2}^2, \\ |\langle (d_v^\varepsilon \cdot \nabla) \partial_i v_{1,j}^\varepsilon, \partial_i d_{v,j}^\varepsilon \rangle| &\leq C \|\nabla \partial_i v_{2,j}^\varepsilon\|_{L^n} \|d_v^\varepsilon\|_{L^{\frac{2n}{n-2}}} \|\partial_i d_{v,j}^\varepsilon\|_{L^2}, \\ |\langle ((\partial_i v_2^\varepsilon) \cdot \nabla) d_{v,j}^\varepsilon, \partial_i d_{v,j}^\varepsilon \rangle| &\leq C \|\partial_i v_2^\varepsilon\|_{L^\infty} \|\nabla d_{v,j}^\varepsilon\|_{L^2}^2, \\ |\langle (v_2^\varepsilon \cdot \nabla) \partial_i d_{v,j}^\varepsilon, \partial_i d_{v,j}^\varepsilon \rangle| &\leq C \|\nabla v_1^\varepsilon\|_{L^\infty} \|\partial_i d_{v,j}^\varepsilon\|_{L^2}^2, \\ |\langle \partial_i \partial_j (|x|^{-\gamma} * (d_a^\varepsilon \overline{a_1^\varepsilon})), \partial_i d_{v,j}^\varepsilon \rangle| &\leq C (\|a_1^\varepsilon\|_{L^\infty} + \|a_1^\varepsilon\|_{L^2}) \|d_a^\varepsilon\|_{L^2} \|\partial_i d_{v,j}^\varepsilon\|_{L^2}, \\ |\langle \partial_i \partial_j (|x|^{-\gamma} * (a_2^\varepsilon \overline{d_a^\varepsilon})), \partial_i d_{v,j}^\varepsilon \rangle| &\leq C (\|a_2^\varepsilon\|_{L^\infty} + \|a_2^\varepsilon\|_{L^2}) \|d_a^\varepsilon\|_{L^2} \|\partial_i d_{v,j}^\varepsilon\|_{L^2}, \end{aligned}$$

where $v_{1,j}^\varepsilon$ and $d_{v,j}^\varepsilon$ denote the j -th components of v_1^ε and d_v^ε , respectively. Summing up over i and j , we obtain

$$(4.21) \quad \frac{d}{dt} \|\nabla d_v^\varepsilon\|_{L^2}^2 \leq C \left(\|d_a^\varepsilon\|_{L^2}^2 + \|\nabla d_v^\varepsilon\|_{L^2}^2 \right).$$

Denote $D(t) := \|d_a^\varepsilon(t)\|_{L^2}^2 + \|\nabla d_v^\varepsilon(t)\|_{L^2}^2$: since $D(0) = 0$, Gronwall lemma shows that $D(t) = 0$ for all $t \in [0, T]$.

Lemma 4.5. *The solution $(a^\varepsilon, v^\varepsilon) \in C([0, T]; H^s \times X^{s+1} \cap L^{\frac{2n}{n-2}})$ to the system (4.1) given in Lemma 4.4 is unique.*

4.5. Completion of the proof of Proposition 4.1. Now we complete the proof of existence result.

Proof of Proposition 4.1. We have already shown that the system (4.1) has a unique solution $(a^\varepsilon, v^\varepsilon) \in C([0, T]; H^s \times X^{s+1} \cap L^{\frac{2n}{n-2}})$.

We show that the existence time T is independent of s , thanks to tame estimates. In the above proof, the existence time T depends on s . However, once we show the existence of the solution in $[0, T_{s_0}]$ for some $s_0 > n/2 + 1$, then, for any $s_1 > n/2 + 1$, we deduce from (4.17) that

$$\begin{aligned} M_{s_1}^\varepsilon(t) &\leq M_{s_1}^\varepsilon(0) \exp\left(Ct \sup_{0 \leq \tau \leq t} (\|a^\varepsilon(\tau)\|_{L^2} + \|a^\varepsilon(\tau)\|_{W^{1,\infty}} + \|\nabla v^\varepsilon(\tau)\|_{L^\infty})\right) \\ &\leq M_{s_1}^\varepsilon(0) \exp(Ct \sup M_{s_0}^\varepsilon), \end{aligned}$$

and so that $M_{s_1}^\varepsilon(t) < \infty$ holds for $t \in [0, T_{s_0}]$. It means that the solution $(a^\varepsilon, v^\varepsilon)$ extends to time T_{s_0} as a $H^{s_1} \times X^{s_1+1} \cap L^{\frac{2n}{n-2}}$ -valued function, that is, $T_{s_1} \geq T_{s_0}$. The same argument also shows $T_{s_0} \geq T_{s_1}$. Therefore, T does not depend on s . \square

4.6. Construction of ϕ^ε . We finally construct ϕ^ε from v^ε defined in Proposition 4.1. Since v^ε is known, in view of (2.2), it is natural to define ϕ^ε as

$$(4.22) \quad \phi^\varepsilon(t, x) = \phi_0(x) - \int_0^t \left(\frac{1}{2} |v^\varepsilon(\tau, x)|^2 + \lambda (|x|^{-\gamma} * |a^\varepsilon|^2)(\tau, x) \right) d\tau.$$

By Assumption 1.1 and Lemma 3.1, $\phi_0 \in L^\infty \cap L^{\frac{nq_0}{n-q_0}}$. Proposition 4.1 shows that

$$|v^\varepsilon|^2 \in C\left([0, T]; X^{s+1} \cap L^{\frac{nq_0}{n-q_0}}\right).$$

Lemma 3.6 with $k = 2$ and $s_1 = s_2 = s$ shows that

$$\nabla^2 (|x|^{-\gamma} * |a^\varepsilon|^2) \in C([0, T]; H^s).$$

By the Hardy–Littlewood–Sobolev inequality, for all $n/(\gamma + 1) < q < \infty$, it holds that

$$\|\nabla (|x|^{-\gamma} * |a^\varepsilon|^2)\|_{L^q} \leq C \|(|x|^{-\gamma-1} * |a^\varepsilon|^2)\|_{L^q} \leq C \|a^\varepsilon\|_{H^s}^2,$$

and $\nabla (|x|^{-\gamma} * |a^\varepsilon|^2) \in C([0, T]; H^{s+1})$. Moreover, the Sobolev inequality shows $|x|^{-\gamma} * |a^\varepsilon|^2 \in L^\infty$. Therefore, ϕ^ε has the regularity announced in Theorem 1.4. To conclude, we simply notice the identity

$$\partial_t (\nabla \phi^\varepsilon - v^\varepsilon) = \nabla \partial_t \phi^\varepsilon - \partial_t v^\varepsilon = 0,$$

so that $v^\varepsilon = \nabla \phi^\varepsilon$, and (4.22) yields the second equation in (2.2). This completes the proof of Theorem 1.4.

5. ASYMPTOTIC EXPANSION

Proof of Theorem 1.7. First order. Suppose that Assumption 1.6 is satisfied with $N \geq 1$. We already know that the equation (4.1) has a unique solution $(a^\varepsilon, v^\varepsilon) \in C([0, T]; H^s \times X^{s+1} \cap L^{q_0})$ for all $\varepsilon \in [0, 1]$. Denote $(b_0, w_0) := (a^\varepsilon, v^\varepsilon)|_{\varepsilon=0}$. We define

$$(5.1) \quad b^\varepsilon = \frac{a^\varepsilon - b_0}{\varepsilon}, \quad w^\varepsilon = \frac{v^\varepsilon - w_0}{\varepsilon}.$$

Substituting $a^\varepsilon = b_0 + \varepsilon b^\varepsilon$ and $v^\varepsilon = w_0 + \varepsilon w^\varepsilon$ into the system (4.1), we obtain the system for $(b^\varepsilon, w^\varepsilon)$:

$$(5.2) \quad \begin{cases} \partial_t b^\varepsilon + w^\varepsilon \cdot \nabla b_0 + w_0 \cdot \nabla b^\varepsilon + \frac{1}{2} b_0 \nabla \cdot w^\varepsilon + \frac{1}{2} b^\varepsilon \nabla \cdot w_0 \\ \quad + \varepsilon w^\varepsilon \cdot \nabla b^\varepsilon + \frac{\varepsilon}{2} b^\varepsilon \nabla \cdot w^\varepsilon - i \frac{1}{2} \Delta b_0 = i \frac{\varepsilon}{2} \Delta b^\varepsilon, \\ \partial_t w^\varepsilon + w^\varepsilon \cdot \nabla w_0 + w_0 \cdot \nabla w^\varepsilon + \lambda \nabla (|x|^{-\gamma} * 2 \operatorname{Re}(b_0 \bar{b}^\varepsilon)) \\ \quad + \varepsilon w^\varepsilon \cdot \nabla w^\varepsilon + \lambda \varepsilon \nabla (|x|^{-\gamma} * |b^\varepsilon|^2) = 0, \end{cases}$$

$$(5.3) \quad b_{|t=0}^\varepsilon = a_1 + \sum_{j=1}^{N-1} \varepsilon^j a_{j+1} + \varepsilon^{N-1} r_N^\varepsilon, \quad w_{|t=0}^\varepsilon = 0,$$

where we have used the fact that (b_0, w_0) is the solution to the system (4.1) with $\varepsilon = 0$ and the assumption that the initial data of a^ε is written as $a_0^\varepsilon = a_0 + \varepsilon a_1 + \sum_{j=2}^N \varepsilon^j a_j + \varepsilon^N r_N^\varepsilon$. Since we know that $b^\varepsilon \in C([0, T]; H^s)$ and $w^\varepsilon \in C([0, T]; X^{s+1} \cap L^{\frac{nq_0}{n-q_0}})$ for $\varepsilon > 0$, we just need to prove *a priori* estimates which are independent of ε . Mimicking the energy estimates (4.9) and (4.14), we obtain

$$(5.4) \quad \begin{aligned} & \frac{d}{dt} (\|b^\varepsilon\|_{H^{s-2}}^2 + \|\nabla w^\varepsilon\|_{H^{s-2}}^2) \\ & \leq C + C(1 + \varepsilon(\|b^\varepsilon\|_{L^2} + \|b^\varepsilon\|_{W^{1,\infty}} + \|w^\varepsilon\|_{L^\infty})) (\|b^\varepsilon\|_{H^{s-2}}^2 + \|\nabla w^\varepsilon\|_{H^{s-2}}^2), \end{aligned}$$

where the constant C depends on $\|b_0\|_{H^s}$ and $\|\nabla w_0\|_{H^s}$. Indeed, the quadratic terms of the system can be handled by the same way since they are exactly the same as those in the system (4.1) up to the constant ε . We estimate linear terms essentially by the same way. Note that the integration by parts does not work well, and so that we need the H^{s-1} -boundedness of b_0 and ∇w_0 . The term $i \frac{1}{2} \Delta b_0$ is also new. By the presence of this term, b_0 is required to be bounded in H^s .

Mimicking the proof of Proposition 4.1, we can show the existence of a unique solution $(b^\varepsilon, w^\varepsilon) \in C([0, T], H^{s-2} \times X^{s-1} \cap L^{\frac{2n}{n-2}})$ for all $\varepsilon \in [0, 1]$. Since $b^\varepsilon(0)$ is uniformly bounded in H^{s-2} by assumption, we see that the $H^{s-2} \times X^{s-1}$ -bound of $(b^\varepsilon, w^\varepsilon)$ is independent of ε . It proves $\|a^\varepsilon - b_0\|_{H^{s-2}} + \|\nabla v^\varepsilon - \nabla w_0\|_{H^{s-2}} = O(\varepsilon)$. Moreover, the existence time T is also independent of ε . Then, we see from (5.1) that, for $\varepsilon > 0$, the existence time for $(b^\varepsilon, w^\varepsilon)$ must be equal to that for $(a^\varepsilon, v^\varepsilon)$. Hence, we conclude that the existence time for $(b^\varepsilon, w^\varepsilon)$ with $\varepsilon = 0$ is also the same.

Thus, putting $(b_1, w_1) = (b^\varepsilon, w^\varepsilon)|_{\varepsilon=0}$, we obtain the solution to the system

$$(5.5) \quad \begin{cases} \partial_t b_1 + w_1 \cdot \nabla b_0 + w_0 \cdot \nabla b_1 + \frac{1}{2} b_0 \nabla \cdot w_1 + \frac{1}{2} b_1 \nabla \cdot w_0 - i \frac{1}{2} \Delta b_0 = 0, \\ \partial_t w_1 + w_1 \cdot \nabla w_0 + w_0 \cdot \nabla w_1 + \lambda \nabla (|x|^{-\gamma} * 2 \operatorname{Re}(b_0 \bar{b}_1)) = 0, \end{cases}$$

$$(5.6) \quad b_1|_{t=0} = a_1, \quad w_1|_{t=0} = 0.$$

Since $w_1|_{t=0} \equiv 0 \in L^r$ for all r and $b_1 \in H^{s-2}$, we see that $(b_1, w_1) \in C([0, T]; H^{s-2} \times X^{s-1} \cap L^q)$ for all $q \in]n/(\gamma+1), \infty]$. By the similar way as the construction of ϕ^ε , we can construct ϕ_1 so that $\phi_1 \in L^p$ for all $p \in]n/\gamma, \infty]$ and $\nabla \phi_1 = v_1$.

Higher order. Let $N \geq 2$ and Assumption 1.6 be satisfied. Take $m \in [2, N]$ and assume that, for $1 \leq k \leq m-1$, the system

$$(5.7) \quad \begin{cases} \partial_t b_k + \sum_{i+j=k} w_i \cdot \nabla b_j + \frac{1}{2} \sum_{i+j=k} b_i \nabla \cdot w_j - i \frac{1}{2} \Delta b_{k-1} = 0, \\ \partial_t w_k + \sum_{i+j=k} w_i \cdot \nabla w_j + \lambda \sum_{i+j=k} \nabla (|x|^{-\gamma} * 2 \operatorname{Re}(b_i \bar{b}_j)) = 0, \end{cases}$$

$$(5.8) \quad b_k|_{t=0} = a_k, \quad w_k|_{t=0} = 0,$$

has a unique solution $(b_k, w_k) \in C([0, T]; H^{s-2k} \times X^{s-2k+1} \cap L^q)$, where $q \in]n/(\gamma+1), \infty]$. Denote

$$b^\varepsilon = \frac{a^\varepsilon - \sum_{j=0}^{m-1} \varepsilon^j b_j}{\varepsilon^m}, \quad w^\varepsilon = \frac{v^\varepsilon - \sum_{j=0}^{m-1} \varepsilon^j w_j}{\varepsilon^m}.$$

Then, $(b^\varepsilon, w^\varepsilon)$ satisfies the following system:

$$(5.9) \quad \left\{ \begin{array}{l} \partial_t b^\varepsilon + \sum_{\ell=0}^{m-1} \varepsilon^\ell \left(w^\varepsilon \cdot \nabla b_\ell + w_\ell \cdot \nabla b^\varepsilon + \frac{1}{2} b^\varepsilon \nabla \cdot w_\ell + \frac{1}{2} b_\ell \nabla \cdot w^\varepsilon \right) \\ \quad + \sum_{\ell=0}^{m-1} \varepsilon^\ell \sum_{i,j < m, i+j=m+\ell} \left(w_i \cdot \nabla b_j + \frac{1}{2} b_i \nabla \cdot w_j \right) \\ \quad + \varepsilon^m w^\varepsilon \cdot \nabla b^\varepsilon + \frac{\varepsilon^m}{2} b^\varepsilon \nabla \cdot w^\varepsilon - i \frac{1}{2} \Delta b_{m-1} = i \frac{\varepsilon}{2} \Delta b^\varepsilon, \\ \partial_t w^\varepsilon + \sum_{\ell=0}^{m-1} \varepsilon^\ell (w^\varepsilon \cdot \nabla w_\ell + w_\ell \cdot \nabla w^\varepsilon) + \lambda \nabla (|x|^{-\gamma} * 2 \operatorname{Re}(b_\ell \bar{b}^\varepsilon)) \\ \quad + \sum_{\ell=0}^{m-1} \varepsilon^\ell \sum_{i,j < m, i+j=m+\ell} (w_i \cdot \nabla w_j + \lambda \nabla (|x|^{-\gamma} * 2 \operatorname{Re}(b_i \bar{b}_j))) \\ \quad + \varepsilon^m w^\varepsilon \cdot \nabla w^\varepsilon + \lambda \varepsilon^m \nabla (|x|^{-\gamma} * |b^\varepsilon|^2) = 0, \end{array} \right.$$

$$(5.10) \quad b_{|t=0}^\varepsilon = a_m + \sum_{j=1}^{N-m} \varepsilon^j a_{j+m} + \varepsilon^{N-m} r_N^\varepsilon, \quad w_{|t=0}^\varepsilon = 0,$$

By induction on m , the system (5.9)–(5.10) has a unique solution

$$(b^\varepsilon, w^\varepsilon) \in C([0, T]; H^{s-2m} \times X^{s-2m+1}) \quad (s - 2m + 1 > n/2).$$

As the first order, we see that the upper bound of $(b^\varepsilon, w^\varepsilon)$ and the existence time T is independent of ε . Furthermore, T is the same one as for $(a^\varepsilon, v^\varepsilon)$. Note that we need the H^{s-2m+2} boundedness of Δb_{m-1} to solve the system (5.9)–(5.10). Denote $(b_m, w_m) := (b^\varepsilon, w^\varepsilon)|_{\varepsilon=0}$. Then, (b_m, w_m) satisfies (5.7)–(5.8) with $k = m$.

Since $w_m|_{t=0} \equiv 0 \in L^r$ for all r and $b_m \in H^{s-2m}$ with $s - 2m > n/2 + 2(N - m) + 1 \geq n/2 + 1$, we see that $w_m \in C([0, T]; X^{s-2m+1} \cap L^q)$ for all $q \in]n/(\gamma+1), \infty]$, and that there exists φ_m which satisfies $\varphi_m \in C([0, T], L^p)$ for all $p \in]n/\gamma, \infty]$ and $\nabla \varphi_m = w_m$. \square

Proof of Corollary 1.8. Corollary 1.8 is a straightforward consequence of Theorem 1.7, by considering the asymptotic expansion of

$$e^{i\varphi_0/\varepsilon + i\varphi_1} (b_0 + \varepsilon b_1 + \dots + \varepsilon^N b_N) e^{i\varepsilon\varphi_2 + i\varepsilon^2\varphi_3 + \dots + i\varepsilon^{N-1}\varphi_N}$$

in powers of ε . The first exponential is not modified, but one considers the asymptotic expansion of the last two terms in this product.

Note the shift in precision, between Theorem 1.7 and Corollary 1.8: the initial order of precision $o(\varepsilon^N)$ becomes $o(\varepsilon^{N-1})$ (in $L^2 \cap L^\infty$). This is because the phase ϕ^ε is divided by ε to go back to u^ε . This phenomenon has several consequences, see e.g. [7] for instabilities. \square

6. PROOF OF COROLLARY 1.9

To see that Corollary 1.9 is a consequence of Corollary 1.8, we resume the same approach as in [7]. Let $a_0 \in \mathcal{S}(\mathbb{R}^n)$ be non-trivial (independent of h), and consider

$$\psi_0^h(x) = h^{s-n/2} a_0 \left(\frac{x}{h} \right).$$

Let $\varepsilon = h^{s_c - s} = h^{\gamma/2 - 1 - s}$: h and ε go to zero simultaneously by assumption. Consider the change of unknown function

$$\psi^h(t, x) = h^{s-n/2} u^\varepsilon \left(\frac{t}{\varepsilon h^2}, \frac{x}{h} \right).$$

Then the Cauchy problem for ψ^h is *equivalent* to:

$$i\varepsilon \partial_t u^\varepsilon + \frac{\varepsilon^2}{2} \Delta u^\varepsilon = \lambda(|x|^{-\gamma} * |u^\varepsilon|^2) u^\varepsilon \quad ; \quad u^\varepsilon|_{t=0} = a_0.$$

This is (1.1)–(1.2) with $\phi_0 = 0$, and $a_0^\varepsilon = a_0$ independent of ε . By construction, the phase φ_0 provided in Theorem 1.7 is such that

$$\varphi_0|_{t=0} = 0 \quad ; \quad \partial_t \varphi_0|_{t=0} = -\lambda|x|^{-\gamma} * |a_0|^2,$$

where we have used the equation determining φ_0 , that is, (2.2) with $\varepsilon = 0$. Therefore, there exists $\tau > 0$ independent of ε such that $\varphi_0|_{t=\tau}$ is non-trivial on the support of a_0 : at time $t = \tau$, u^ε is exactly ε -oscillatory, from Corollary 1.8. Back to ψ^h , this yields Corollary 1.9, up to replacing a_0 with $|\log h|^{-1} a_0$ (this makes no trouble, since the competition in terms of h is logarithmic decay *vs.* algebraic decay).

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