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Detecting abrupt changes of the long-range dependence or the self-similarity of a Gaussian process

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Abstract

In this paper, an estimator of m instants (m is supposed to be known) of abrupt changes of the parameter of long-range dependence or self-similarity is proved to satisfy a limit theorem with an explicit convergence rate for a sample of a Gaussian process. In each estimated zone where the parameter is supposed not to change, a central limit theorem is established for the parameter's (of long-range dependence, self-similarity) estimator and a goodness-of-fit test is also built. *To cite this article: J.M. Bardet, I. Kamoun, C. R. Acad. Sci. Paris, Ser. I 340 (2007).*

Résumé

Détection de ruptures du paramètre de longue mémoire, d'autosimilarité pour des processus gaussiens. Dans ce papier, pour une trajectoire d'un processus gaussien, un estimateur des m points de ruptures (m est supposé connu) du paramètre de longue mémoire ou d'autosimilarité est construit et on montre qu'il vérifie un théorème limite avec une vitesse de convergence explicite. Dans chaque zone (estimée) où ce paramètre est constant, un estimateur de ce paramètre vrifie un théorème limite centrale et un test d'ajustement est galement mis en place. *Pour citer cet article : J.M. Bardet, I. Kammoun, C. R. Acad. Sci. Paris, Ser. I 340 (2007).*

Version française abrégée Le problème de détection de points de ruptures moyennant la minimisation d'une fonction de contraste donnée a été étudié depuis le milieu des années 1990 dans le cadre de processus à longue mémoire (voir par exemple [9], [11], [12]). De ces approches, certaines ont été associées à un cadre paramétrique, telle que la détection de rupture selon la moyenne et/ou la variance, d'autres traitées dans un cadre non paramétrique (comme la détection de ruptures selon la distribution ou le spectre). Dans la littérature, différents auteurs ont également proposé des statistiques de test de l'hypothèse que le paramètre est inchangé contre le fait que le paramètre de longue mémoire varie en fonction du temps (voir

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par exemple [2], [7], [10]). À notre connaissance, le cadre semi-paramétrique de détection de changements en longue mémoire ou en autosimilarité est traité ici pour la première fois.

Dans ce papier, un estimateur des m points de ruptures ($m \in \mathbb{N}^*$) de la longue mémoire ou d'auto-similarité est conçu pour un échantillon de processus gaussien en se basant sur l'analyse par ondelettes. Pour ce type de processus, cette méthode a été proposée pour la première fois par Flandrin [8], puis développée par exemple dans Abry, *et al.* [1]. La convergence des estimateurs basés sur les ondelettes a été étudiée dans le cas du mouvement brownien fractionnaire (FBM) dans [3], et dans le cadre général de processus semi-paramétrique gaussiens stationnaires à longue mémoire par Moulines *et al.* [13] et Bardet *et al.* [5].

Dans notre approche, le principe de l'estimation du paramètre de longue mémoire ou d'auto-similarité est le suivant : dans chaque zone où il n'y a pas de changement, ce paramètre peut être estimé à partir d'une log-log régression de la variance des coefficients d'ondelettes sur plusieurs échelles choisies (voir (1)). Une fonction de contraste définie par la somme des carrés des distances entre ces points et les droites d'ajustement, dans les $m + 1$ zones possibles détectées, est minimisée (voir (2)), donnant un estimateur des points de ruptures (voir (3)). Sous certaines hypothèses générales, on montre qu'il vérifie un théorème limite avec une vitesse de convergence explicite (voir Theorem 1.1). Dans chacune des zones détectées, les paramètres de longue mémoire, (ou d'auto-similarité) peuvent être estimés, tout d'abord avec la régression des moindres carrés ordinaires (OLS), puis par une régression des moindres carrés pseudo-généralisés (FGLS). Un théorème de la limite centrale est établi pour chacun des deux estimateurs (voir Theorem 1.2 et Proposition 1.1 ci-dessous) et des intervalles de confiance peuvent être calculés. L'estimateur FGLS offre deux avantages : d'une part, sa variance asymptotique est plus petite que celle de l'estimateur OLS, et d'autre part, il permet la construction d'un test d'ajustement très simple (test de type Chi-deux) basé sur le carré des distances entre les points (d'abscisse, le logarithme d'une échelle choisie et d'ordonnée, le logarithme de la variance empirique des coefficients d'ondelettes pour cette échelle) et les droites de régression pseudo-généralisée correspondantes (voir (8)). La convergence vers une distribution du Chi-deux de ce test est établie dans le Theorem 1.3.

Deux cas particuliers de processus gaussiens sont ensuite étudiés dans la section 2. En premier lieu, on s'intéresse aux séries chronologiques stationnaires longue mémoire avec un paramètre de Hurst constant par morceaux. On se place dans un cadre semi-paramétrique contenant par exemple les FGN et les processus FARIMA (voir Figure 1). En second lieu, le cas d'un processus à accroissements stationnaires et autosimilaire par morceaux est traité, ce qui revient à considérer des successions de FBM ayant des exposants de Hurst distincts (voir Figure 1). Pour ces deux exemples, les vitesses de convergence explicites des différents estimateurs et tests sont données et des simulations montrent leurs qualités (voir Table 1). D'autres simulations, preuves des thormes ainsi qu'un exemple plus gnral de dtection de ruptures dans le cadre de processus gaussien localement fractionnaire sont dtails dans [6].

1. Assumptions and main results

Let $(X_t)_{t \in \mathbb{N}}$ be a zero-mean Gaussian process and assume that (X_0, X_1, \dots, X_N) is known. as data are modelled by a time series ($T = \mathbb{N}$) or a continuous time process ($T = \mathbb{R}$). In the sequel, X will be a piecewise stationary long memory time series or a piecewise self-similar time series having stationary increments. Consider $\psi : \mathbb{R} \rightarrow \mathbb{R}$ a function called "the mother wavelet". For $(a, b) \in \mathbb{R}_+^* \times \mathbb{R}$, the wavelet coefficient of X for the scale a and the shift b is $d_X(a, b) := \frac{1}{\sqrt{a}} \int_{\mathbb{R}} \psi(\frac{t-b}{a})X(t)dt$. When only a discretized path of X is available, approximations $e_X(a, b)$ are only computable: ,

$$e_X(a, b) := \frac{1}{\sqrt{a}} \sum_{p=1}^N \psi\left(\frac{p-b}{a_N}\right)X_p \quad \text{for } (a, b) \in \mathbb{R}_+^* \times \mathbb{N}.$$

Assume that there exist $m \in \mathbb{N}$ (the number of abrupt changes) and

- $0 = \tau_0^* < \tau_1^* < \dots < \tau_m^* < \tau_{m+1}^* = 1$ (unknown parameters);
- two families $(\alpha_j^*)_{0 \leq j \leq m} \in \mathbb{R}^{m+1}$ and $(\beta_j^*)_{0 \leq j \leq m} \in (0, \infty)^{m+1}$ (unknown parameters);
- a sequence of "scales" $(a_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ (chosen) satisfying $a_n \geq a_{min}$, with $a_{min} > 0$,

such that for $j = 0, 1, \dots, m$ and $k \in D_N^*(j) \subset [[N\tau_j^*], [N\tau_{j+1}^*]]$,

$$\mathbb{E}[e_X^2(a_N, k)] \sim \beta_j^*(a_N)^{\alpha_j^*} \text{ when } N \rightarrow \infty.$$

A piecewise sample variance can be the appropriated estimator of such power law. Thus, define

$$S_k^{k'}(a_N) := \frac{a_N}{k' - k} \sum_{p=[k/a_N]}^{[k'/a_N]-1} e_X^2(a_N, a_N p) \text{ for } 0 \leq k < k' \leq N.$$

Now set $0 < r_1 < \dots < r_\ell$ with $\ell \in \mathbb{N}^*$, and assume that a multidimensional central limit theorem can be established for $\left(\log(S_k^{k'}(r_i a_N))\right)_{1 \leq i \leq \ell}$, when $[N\tau_j^*] \leq k < k' \leq [N\tau_{j+1}^*]$, *i.e.*

$$\sqrt{\frac{k' - k}{a_N}} \left(\log(S_k^{k'}(r_i a_N)) - \log(\beta_j^*) - \alpha_j^* \log(r_i a_N) \right)_{1 \leq i \leq \ell} \xrightarrow[N \rightarrow \infty]{|k' - k| \rightarrow \infty} \mathcal{N}(0, \Gamma^{(j)}(\alpha_j^*, r_1, \dots, r_\ell)), \quad (1)$$

with $\Gamma^{(j)}(\alpha_j^*, r_1, \dots, r_\ell) = (\gamma_{pq}^{(j)})_{1 \leq p, q \leq \ell}$ a $(\ell \times \ell)$ matrix not depending on N such that $\alpha \mapsto \Gamma^{(j)}(\alpha, r_1, \dots, r_\ell)$ is a continuous function and a positive matrix for all α . Define a contrast function

$$U_N((\alpha_j)_{0 \leq j \leq m}, (\beta_j)_{0 \leq j \leq m}, (k_j)_{1 \leq j \leq m}) = \sum_{j=0}^m \sum_{i=1}^{\ell} \left(\log(S_{k_j}^{k_{j+1}}(r_i a_N)) - (\alpha_j \log(r_i a_N) + \log \beta_j) \right)^2 \quad (2)$$

with $(\alpha_j)_{0 \leq j \leq m} \in A^{m+1} \subset \mathbb{R}^{m+1}$, $(\beta_j)_{0 \leq j \leq m} \in B^{m+1} \subset (0, \infty)^{m+1}$, $0 = k_0 < k_1 < \dots < k_m < k_{m+1} = N$, $(k_j)_{1 \leq j \leq m} \in K_m(N) \subset \mathbb{N}^m$. The vector of estimated parameters $\hat{\alpha}_j$, $\hat{\beta}_j$ and \hat{k}_j (and therefore $\hat{\tau}_j$) is the vector which minimizes this contrast function in $A^{m+1} \times B^{m+1} \times K_m(N)$, *i.e.*,

$$((\hat{\alpha}_j)_{0 \leq j \leq m}, (\hat{\beta}_j)_{0 \leq j \leq m}, (\hat{k}_j)_{1 \leq j \leq m}) := \text{Argmin} \left\{ U_N((\alpha_j)_{0 \leq j \leq m}, (\beta_j)_{0 \leq j \leq m}, (k_j)_{1 \leq j \leq m}) \right\} \quad (3)$$

$$\hat{\tau}_j := \hat{k}_j / N \text{ for } 1 \leq j \leq m. \quad (4)$$

For a given $(k_j)_{1 \leq j \leq m}$, it is obvious that $(\hat{\alpha}_j)_{0 \leq j \leq m}$ and $(\log \hat{\beta}_j)_{0 \leq j \leq m}$ are obtained from a log-log regression of $(S_{k_j}^{k_{j+1}}(r_i a_N))_i$ onto $(r_i a_N)_i$, *i.e.*

$$\begin{pmatrix} \hat{\alpha}_j \\ \log \hat{\beta}_j \end{pmatrix} = (L'_1 \cdot L_1)^{-1} L'_1 \cdot Y_{k_j}^{k_{j+1}} \text{ with } Y_{k_j}^{k_{j+1}} := (\log(S_{k_j}^{k_{j+1}}(r_i \cdot a_N)))_{1 \leq i \leq \ell}, \quad L_{a_N} := \begin{pmatrix} \log(r_1 a_N) & 1 \\ \vdots & \vdots \\ \log(r_\ell a_N) & 1 \end{pmatrix}.$$

Therefore $(\hat{k}_j)_{1 \leq j \leq m} = \text{Argmin} \left\{ U_N((\hat{\alpha}_j)_{0 \leq j \leq m}, (\hat{\beta}_j)_{0 \leq j \leq m}, (k_j)_{1 \leq j \leq m}), (k_j)_{1 \leq j \leq m} \in K_m(N) \right\}$. In this paper, parameters (α_j^*) are supposed to satisfied abrupt changes:

Assumption C : Parameters (α_j^*) are such that $|\alpha_{j+1}^* - \alpha_j^*| \neq 0$ for all $j = 0, 1, \dots, m - 1$.

Theorem 1.1 Define $\underline{\tau}^* := (\tau_1^*, \dots, \tau_m^*)$, $\widehat{\underline{\tau}} := (\widehat{\tau}_1, \dots, \widehat{\tau}_m)$ and $\|\underline{\tau}\|_m := \max(|\tau_1|, \dots, |\tau_m|)$. Let $\ell \in \mathbb{N} \setminus \{0, 1, 2\}$. If Assumption C and relation (1) holds with $(\alpha_j^*)_{0 \leq j \leq m}$ such that $\alpha_j^* \in [a, a']$ and $a < a'$ for all $j = 0, \dots, m$, then if $a_N^{1+2(a'-a)} N^{-1} \xrightarrow{N \rightarrow \infty} 0$, for all $(v_n)_n$ satisfying $v_N \cdot a_N^{1+2(a'-a)} N^{-1} \xrightarrow{N \rightarrow \infty} 0$,

$$\mathbb{P}\left(v_N \|\underline{\tau}^* - \widehat{\underline{\tau}}\|_m \geq \eta\right) \xrightarrow{N \rightarrow \infty} 0 \quad \text{for all } \eta > 0. \quad (5)$$

The proof of this result is provided in [6]. For $j = 0, 1, \dots, m$, the log-log regression of $(S_{\widehat{k}_j}^{\widehat{k}_{j+1}}(r_i a_N))_{1 \leq i \leq \ell}$ onto $(r_i a_N)_{1 \leq i \leq \ell}$ provides estimators of α_j^* and β_j^* . However, even if τ_j converges to τ_j^* , $\widehat{k}_j = N \cdot \widehat{\tau}_j$ does not converge to k_j^* , and therefore $\mathbb{P}([\widehat{k}_j, \widehat{k}_{j+1}] \subset [k_j^*, k_{j+1}^*])$ does not tend to 1. So, define \tilde{k}_j and \tilde{k}'_j such that $\tilde{k}_j = \widehat{k}_j + \frac{N}{v_N}$ and $\tilde{k}'_j = \widehat{k}_{j+1} - \frac{N}{v_N}$. From (5) with $\eta = 1/2$, $\mathbb{P}([\tilde{k}_j, \tilde{k}'_j] \subset [k_j^*, k_{j+1}^*]) \xrightarrow{N \rightarrow \infty} 1$. Then,

Theorem 1.2 Let $\Theta_j^* := \begin{pmatrix} \alpha_j^* \\ \log \beta_j^* \end{pmatrix}$ and $\tilde{\Theta}_j := (L_1' L_1)^{-1} L_1' Y_{\tilde{k}_j}^{\tilde{k}'_j} = \begin{pmatrix} \tilde{\alpha}_j \\ \log \tilde{\beta}_j \end{pmatrix}$. Under the same assumptions as in Theorem 1.1, for $j = 0, \dots, m$, with $\Sigma^{(j)}(\alpha_j^*, r_1, \dots, r_\ell) := (L_1' L_1)^{-1} L_1' \Gamma^{(j)}(\alpha_j^*, r_1, \dots, r_\ell) L_1 (L_1' L_1)^{-1}$,

$$\sqrt{\frac{N(\tau_{j+1}^* - \tau_j^*)}{a_N}} (\tilde{\Theta}_j - \Theta_j^*) \xrightarrow{N \rightarrow \infty} \mathcal{N}(0, \Sigma^{(j)}(\alpha_j^*, r_1, \dots, r_\ell)) \quad (6)$$

A second estimator of Θ_j^* can be obtained from feasible generalized least squares (FGLS) estimation. Indeed, the asymptotic covariance matrix $\Gamma^{(j)}(\alpha_j^*, r_1, \dots, r_\ell)$ can be estimated by the matrix $\tilde{\Gamma}^{(j)} := \Gamma^{(j)}(\tilde{\alpha}_j, r_1, \dots, r_\ell)$ and $\tilde{\Gamma}^{(j)} \xrightarrow{N \rightarrow \infty} \Gamma^{(j)}(\alpha_j^*, r_1, \dots, r_\ell)$. Then, the FGLS estimator $\bar{\Theta}_j$ of Θ_j^* is defined from the minimization for all Θ of the following squared distance,

$$\|Y_{\tilde{k}_j}^{\tilde{k}'_j} - L_{a_N} \cdot \Theta\|_{\tilde{\Gamma}^{(j)}}^2 = (Y_{\tilde{k}_j}^{\tilde{k}'_j} - L_{a_N} \Theta)' \cdot (\tilde{\Gamma}^{(j)})^{-1} \cdot (Y_{\tilde{k}_j}^{\tilde{k}'_j} - L_{a_N} \Theta).$$

and therefore define $\bar{\Theta}_j := (L_1' (\tilde{\Gamma}^{(j)})^{-1} L_1)^{-1} L_1' (\tilde{\Gamma}^{(j)})^{-1} Y_{\tilde{k}_j}^{\tilde{k}'_j}$.

Proposition 1.1 Under the same assumptions as in Theorem 1.2, for $j = 0, \dots, m$

$$\sqrt{\frac{N(\tau_{j+1}^* - \tau_j^*)}{a_N}} (\bar{\Theta}_j - \Theta_j^*) \xrightarrow{N \rightarrow \infty} \mathcal{N}(0, M^{(j)}(\alpha_j^*, r_1, \dots, r_\ell)) \quad (7)$$

with $M^{(j)}(\alpha_j^*, r_1, \dots, r_\ell) := (L_1' (\Gamma^{(j)}(\alpha_j^*, r_1, \dots, r_\ell))^{-1} L_1)^{-1} \leq \Sigma^{(j)}(\alpha_j^*, r_1, \dots, r_\ell)$ (for the order's relation between positive symmetric matrix).

Therefore $\bar{\alpha}_j$ is more accurate than $\tilde{\alpha}_j$ for estimating α_j^* when N is large enough. For $j = 0, \dots, m$, let $T^{(j)}$ be the FGLS distance between points $(\log(r_i a_N), \log(S_{\tilde{k}_j}^{\tilde{k}'_j}))_{1 \leq i \leq \ell}$ and the FGLS regression line. The following theorem describes the asymptotic behavior of a goodness-of-fit test on each segment $[\tilde{k}_j, \tilde{k}'_j]$:

Theorem 1.3 Under the same assumptions as in Theorem 1.1, for $j = 0, \dots, m$

$$T^{(j)} = \frac{N(\tau_{j+1}^* - \tau_j^*)}{a_N} \|Y_{\tilde{k}_j}^{\tilde{k}'_j} - L_{a_N} \bar{\Theta}_j\|_{\tilde{\Gamma}^{(j)}}^2 \xrightarrow{N \rightarrow \infty} \chi^2(\ell - 2). \quad (8)$$

2. Applications

2.1. Piecewise long memory Gaussian processes

Assume that the process $X = (X_t)_{t \in \mathbb{N}}$ is a Gaussian piecewise long-range dependent (LRD) process, *i.e.* there exists $(D_j^*)_{0 \leq j \leq m} \in (0, 1)^{m+1}$ and for all $j = 0, \dots, m$ and $k \in \{[N\tau_j^*], [N\tau_j^*] + 1, \dots, [N\tau_{j+1}^*] - 1\}$, $X_k = X_{k - [N\tau_j^*]}^{(j)}$, where $X^{(j)} = (X_t^{(j)})_{t \in \mathbb{N}}$ satisfies the following Assumption LRD(D_j^*).

Assumption LRD(D): Y is a centered stationary Gaussian process with spectral density f such that $f(\lambda) = |\lambda|^{-D} \cdot f^*(\lambda)$ for all $\lambda \in [-\pi, \pi] \setminus \{0\}$ with $f^*(0) > 0$ and with $C_2 > 0$, $|f^*(\lambda) - f^*(0)| \leq C_2 \cdot |\lambda|^2$ for all $\lambda \in [-\pi, \pi]$.

Following [5], if the mother wavelet is supposed to be included in a Sobolev ball, then

Corollary 1 *Let X be a Gaussian piecewise LRD process defined as above and $\psi : \mathbb{R} \mapsto \mathbb{R}$ be $[0, 1]$ -supported with $\psi(0) = \psi(1) = 0$ and $\int_0^1 \psi(t) dt = 0$ and such that there exists sequence $(\psi_\ell)_{\ell \in \mathbb{Z}}$ satisfying $\psi(\lambda) = \sum_{\ell \in \mathbb{Z}} \psi_\ell e^{2\pi i \ell \lambda} \in \mathbb{L}^2([0, 1])$ and $\sum_{\ell \in \mathbb{Z}} (1 + |\ell|)^{5/2} |\psi_\ell| < \infty$. Under Assumption C, for all $0 < \kappa < 2/15$, if $a_N = N^{\kappa+1/5}$ and $v_N = N^{2/5-3\kappa}$ then (5), (6), (7) and (8) hold.*

Thus, the rate of convergence of $\widehat{\tau}$ to τ^* (in probability) is $N^{2/5-3\kappa}$ for $0 < \kappa$ as small as one wants. Estimators \widehat{D}_j and $\widehat{\overline{D}}_j$ converge to the parameters D_j^* following a central limit theorem with a rate of convergence $N^{2/5-\kappa/2}$ for $0 < \kappa$ as small as one wants. Convincing results of simulations can be observed in Table 1 and Figure 1.

N = 20000			N = 5000					N = 10000				
$\tau_1:0.75 \ D_0:0.2 \ D_1:0.8$			$\tau_1:0.3$		$\tau_2:0.78$	$H_0:0.6$		$H_1:0.8$		$H_2:0.5$		
$\widehat{\tau}_1$	\widehat{D}_0	\widehat{D}_1	$\widehat{\tau}_1$	$\widehat{\tau}_2$	\widehat{H}_0	\widehat{H}_1	\widehat{H}_2	$\widehat{\tau}_1$	$\widehat{\tau}_2$	\widehat{H}_0	\widehat{H}_1	\widehat{H}_2
0.7540	0.1902	0.7926	0.3465	0.7942	0.5578	0.7272	0.4395	0.3086	0.7669	0.5597	0.7633	0.4993
$\widehat{\sigma}_{\tau_1}$	$\widehat{\sigma}_{D_0}$	$\widehat{\sigma}_{D_1}$	$\widehat{\sigma}_{\tau_1}$	$\widehat{\sigma}_{\tau_2}$	$\widehat{\sigma}_{H_0}$	$\widehat{\sigma}_{H_1}$	$\widehat{\sigma}_{H_2}$	$\widehat{\sigma}_{\tau_1}$	$\widehat{\sigma}_{\tau_2}$	$\widehat{\sigma}_{H_0}$	$\widehat{\sigma}_{H_1}$	$\widehat{\sigma}_{H_2}$
0.0215	0.0489	0.0761	0.1212	0.1322	0.0595	0.0837	0.0643	0.0893	0.0675	0.0449	0.0813	0.0780
\sqrt{MSE}	\sqrt{MSE}	\sqrt{MSE}	\sqrt{MSE}	\sqrt{MSE}	\sqrt{MSE}	\sqrt{MSE}	\sqrt{MSE}	\sqrt{MSE}	\sqrt{MSE}	\sqrt{MSE}	\sqrt{MSE}	\sqrt{MSE}
0.0218	0.0499	0.0764	0.1298	0.1330	0.0730	0.1110	0.0883	0.0897	0.0687	0.0604	0.0892	0.0780

Table 1

Left: Estimation of τ_1 , D_0 and D_1 in the case of piecewise FARIMA(0, d_j , 0) ($d_1 = 0.1$ and $d_2 = 0.4$) with one change point when $N = 20000$ (50 realizations). Right: Estimation of τ_1 , τ_2 , H_0 , H_1 and H_2 in the case of piecewise FBM with two change points when $N = 5000$ and $N = 10000$ (50 realizations)

2.2. Piecewise fractional Brownian motions

Now, X will be called a piecewise fractional Brownian motion if there exist two families of parameters $(H_j^*)_{0 \leq j \leq m} \in (0, 1)^{m+1}$ and $(\sigma_j^{*2})_{0 \leq j \leq m} \in (0, \infty)^{m+1}$ such that for all $j = 0, \dots, m$ and $t \in [[N\tau_j^*], [N\tau_j^*] + 1, \dots, [N\tau_{j+1}^*] - 1]$, $X_t = X_{t - [N\tau_j^*]}^{(j)}$, where $X^{(j)} = (X_t^{(j)})_{t \in \mathbb{R}}$ is a FBM with parameters H_j^* and σ_j^{*2} . Following the results of [3], one obtains,

Corollary 2 *Let X be a piecewise FBM and $\psi : \mathbb{R} \rightarrow \mathbb{R}$ be a piecewise continuous and left (or right)-differentiable, such that $|\psi'(t^-)|$ is Riemann integrable with $\psi'(t^-)$ the left-derivative of ψ in t , with support included in $[0, 1]$ and $\int_{\mathbb{R}} t^p \psi(t) dt = \int_0^1 t^p \psi(t) dt = 0$ for $p = 0, 1$. Let $A := |\sup_j H_j^* - \inf_j H_j^*|$.*

If $A < 1/2$, under Assumption C, for all $0 < \kappa < \frac{1}{1+4A} - \frac{1}{3}$, if $a_N = N^{1/3+\kappa}$ and $v_N = N^{2/3(1-2A)-\kappa(2+4A)}$ then (5), (6), (7) and (8) hold.

Thus, the rate of convergence of $\widehat{\tau}$ to τ^* (in probability) can be $N^{2/3(1-2A)-\kappa'}$ for $0 < \kappa'$ as small as one wants when $a_N = N^{1/3+\kappa'/(2+4A)}$. Results of simulations can be observed in Table 1 and Figure 1 in a case where $A = 0.4 < 0.5$. However, Corollary 2 is quite surprising: the smaller A , *i.e.* the smaller the differences between the parameters H_j , the faster the convergence rates of estimators $\widehat{\tau}_j$ to τ_j^* . And if the difference between two successive parameters H_j is too large, the estimators $\widehat{\tau}_j$ do not seem to converge (simulations exhibit this paroxysm in [6]).

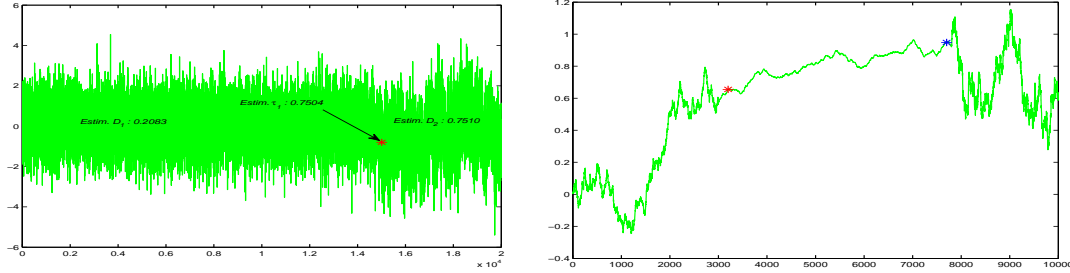


Figure 1. Left: Piecewise FARIMA(0, d_j ,0) (with $d_0 : 0.1$, $d_1 : 0.4$ and $\tau_1 : 0.75$). Right: Piecewise FBM(H_j) ($\tau_1 : 0.3$, $\tau_2 : 0.78$, $H_0 : 0.6$, $H_1 : 0.8$ and $H_2 : 0.5$), ($\widehat{\tau}_1 : 0.32$, $\widehat{\tau}_2 : 0.77$, $\widehat{H}_0 : 0.5608$, $\widehat{H}_1 : 0.7814$ and $\widehat{H}_2 : 0.4751$).

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