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MOMENTS OF THE WEIGHTED NON-CENTRAL CHI-SQUARE DISTRIBUTION

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Abstract :

The weighted non-central chi-square distribution with 1 degree of freedom, introduced by Chen is generalized to the case of ν degrees of freedom. Thus we obtain the non central moments as well as the central moments in specific cases.

Key words : weighted chi-square, central moment, non-central moment.

1. Introduction

In any production process, it is essential to supervise a certain number of variables of interest so that the manufactured goods are in conformity with the specifications defined by the research department. In order to assess the aptitude of a process to respect these specifications, the manufacturers commonly use various indices called capability indices. As the capability of the process is only known through a sample taken from the production, it is necessary to know the distribution of the estimators of these indices in order to estimate them by confidence intervals or to carry out statistical tests on their values. In most of the processes, the target aimed at by the variable of interest is located at the centre of the specifications, and the first indices used relate to this situation. The first indices relating to the asymmetrical case where the target is not located at the centre of the specifications appear in the nineties. However, faced with the difficulty to obtain the properties of the

estimators which are in the form of random variables quotients, Greenwich and Jahr-Schaffrath (1995) suggest to use rather an incapability index. Chen (1998) continues these works by improving that incapability index. To study the properties of the estimator of this new index, he takes an interest in the distribution of the random variable $Z = \text{Max}^2(\alpha_1 N(\delta, 1), -\alpha_2 N(\delta, 1))$, where α_1 and α_2 are two positive numbers, the density of which he gives as an infinite sum of central chi-square distributions in the following form

$$f_1(z) = \frac{e^{-\delta^2/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{2^{j/2} \Gamma((j+1)/2)}{\Gamma(j+1)} \delta^j \left[\alpha_1^{-2} f_{\chi_{1+j}^2}(\alpha_1^{-2} z) + (-1)^j \alpha_2^{-2} f_{\chi_{1+j}^2}(\alpha_2^{-2} z) \right].$$

However, Chen does not study the moments of that variable Z , but he only uses this decomposition in order to obtain the moments of the estimator of its incapability index in the form of an infinite sum. Soon, Chen, Pearn, and Lin (1999), then Pearn, Lin, and Chen (1999) use the density of Z to obtain the moments of the capability indices specific to the asymmetrical case, in the form of an infinite sum. More recently Grau (2006) uses the same principle to extend these results to a more general family of indices including the previous ones. It thus appears that the knowledge of the moments of Z , which is the main purpose of this paper, could allow a different approach to obtain the moments of the estimators of the capability indices, possibly by releasing itself from the infinite sums. By introducing the previous variable Z , Chen (1995) calls it weighted non-central chi-square distribution with one degree of freedom (d.f.). This terminology is justified by the usual presentation of the non-central chi-square distribution. Indeed, if Y is a $N(\delta, 1)$ distribution, then $Y^2 = (N(\delta, 1))^2 = \text{Max}^2(N(\delta, 1), -N(\delta, 1))$ is a non-central chi-square distribution with 1 d.f. and non-centrality parameter δ^2 . In this work, we intend to generalize this distribution to the case of any number ν of d.f., which we will note $\chi_{\nu}^2(\delta^2, \alpha_1, \alpha_2)$, the density of which is defined by

$$f_{\nu}(z) = \frac{e^{-\delta^2/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{2^{j/2} \Gamma((j+1)/2)}{\Gamma(j+1)} \delta^j \left[\alpha_1^{-2} f_{\chi_{\nu+j}^2}(\alpha_1^{-2} z) + (-1)^j \alpha_2^{-2} f_{\chi_{\nu+j}^2}(\alpha_2^{-2} z) \right].$$

Obviously, for the case where $\alpha_1 = \alpha_2 = 1$, this reduces to $\chi_{\nu}^2(\delta^2, 1, 1) = \chi_{\nu}^2(\delta^2)$, the probability density function of a non-central chi-square distribution with ν d.f. and non-centrality

parameter δ^2 . In section 2, we obtain the exact expression of the r -th moment of this distribution. Then, in section 3, we are interested in the central moments in some specific cases. This allows in particular to give a general expression of the central moments of a $\chi_v^2(\delta^2)$ distribution, which are usually obtained in a recursive way from cumulants.

2. Moments of the $\chi_v^2(\delta^2, \alpha_1, \alpha_2)$ distribution

In order to simplify the writing, we note

$$f_v(z) = \sum_{j=0}^{\infty} a_j \left[\alpha_1^{-2} f_{\chi_{v+j}^2}(\alpha_1^{-2} z) + (-1)^j \alpha_2^{-2} f_{\chi_{v+j}^2}(\alpha_2^{-2} z) \right], \text{ where } a_j = \frac{e^{-\delta^2/2}}{2\sqrt{\pi}} \frac{2^{j/2} \Gamma((j+1)/2)}{\Gamma(j+1)} \delta^j.$$

We thus have $E[Z^r] = (\alpha_1^{2r} + \alpha_2^{2r}) A_r + (\alpha_1^{2r} - \alpha_2^{2r}) B_r$, where

$$A_r = \sum_{j=0}^{\infty} a_{2j} E[(\chi_{v+2j}^2)^r] = \frac{e^{-\delta^2/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} 2^{r+j} \delta^{2j} \frac{\Gamma(j+1/2)}{\Gamma(2j+1)} \frac{\Gamma(r+j+v/2)}{\Gamma(j+v/2)} = \frac{e^{-\delta^2/2}}{2} \alpha_r(\delta), \text{ with}$$

$$\alpha_r(\delta) = \sum_{j=0}^{\infty} 2^{r+j} \delta^{2j} \frac{\Gamma(j+1/2) \Gamma(r+j+v/2)}{\Gamma(2j+1) \Gamma(j+v/2) \Gamma(1/2)},$$

$$\text{and } B_r = \sum_{j=0}^{\infty} a_{2j+1} E[(\chi_{v+2j+1}^2)^r] = \frac{e^{-\delta^2/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{2^{j+1/2} \Gamma(j+1)}{\Gamma(2j+2)} \delta^{2j+1} 2^r \frac{\Gamma(r+j+(v+1)/2)}{\Gamma(j+(v+1)/2)}$$

$$= e^{-\delta^2/2} \sum_{j=0}^{\infty} \frac{2^r \delta^{2j+1}}{2^{j+3/2} \Gamma(j+3/2)} \frac{\Gamma(r+j+(v+1)/2)}{\Gamma(j+(v+1)/2)} = e^{-\delta^2/2} \beta_r(\delta), \text{ with}$$

$$\beta_r(\delta) = \sum_{j=0}^{\infty} 2^{r-j-3/2} \delta^{2j+1} \frac{\Gamma(r+j+(v+1)/2)}{\Gamma(j+3/2) \Gamma(j+(v+1)/2)}.$$

The five following lemmas allow to give expressions of $\alpha_r(\delta)$ and $\beta_r(\delta)$ in form of finite sums.

Lemma 1 : $\beta_{r+1}(\delta) = \delta \beta_r'(\delta) + (2r+v) \beta_r(\delta)$.

Proof

$$\text{We have } \delta^{2r+v} \beta_r(\delta) = \sum_{j=0}^{\infty} 2^{r-j-3/2} \delta^{2r+v+2j+1} \frac{\Gamma(r+j+(v+1)/2)}{\Gamma(j+3/2) \Gamma(j+(v+1)/2)}.$$

By derivation with respect to δ ,

$$(2r + \nu)\delta^{2r+\nu-1}\beta_r(\delta) + \delta^{2r+\nu}\beta_r'(\delta) = \sum_{j=0}^{\infty} 2^{r+1-j-3/2} \delta^{2r+\nu+2j} \frac{\Gamma(r+1+j+(\nu+1)/2)}{\Gamma(j+3/2)\Gamma(j+(\nu+1)/2)} = \delta^{2r+\nu-1}\beta_{r+1}(\delta),$$

thus the lemma.

Lemma 2 : $\beta_r''(\delta) = \beta_{r+1}(\delta) - (\nu-1)\delta^{-1}\beta_r'(\delta) + (\nu-1)\delta^{-1} \frac{2^r \Gamma(r+(\nu+1)/2)}{\sqrt{2\pi}\Gamma((\nu+1)/2)}.$

Proof

We have $\beta_r'(\delta) = \sum_{j=0}^{\infty} 2^{r-j-3/2} \delta^{2j} \frac{(2j+1)\Gamma(r+j+(\nu+1)/2)}{\Gamma(j+3/2)\Gamma(j+(\nu+1)/2)}$

$$= \frac{2^r \Gamma(r+(\nu+1)/2)}{\sqrt{2\pi}\Gamma((\nu+1)/2)} + \sum_{j=1}^{\infty} 2^{r-j-3/2} \delta^{2j} \frac{(2j+1)\Gamma(r+j+(\nu+1)/2)}{\Gamma(j+3/2)\Gamma(j+(\nu+1)/2)}$$

$$= \frac{2^r \Gamma(r+(\nu+1)/2)}{\sqrt{2\pi}\Gamma((\nu+1)/2)} + \sum_{j=0}^{\infty} 2^{r-j-1/2} \delta^{2j+2} \frac{\Gamma(r+1+j+(\nu+1)/2)}{(2j+\nu+1)\Gamma(j+3/2)\Gamma(j+(\nu+1)/2)}.$$

By derivation of $\delta^{\nu-1}\beta_r'(\delta)$ with respect to δ ,

$$(\nu-1)\delta^{\nu-2}\beta_r'(\delta) + \delta^{\nu-1}\beta_r''(\delta) = (\nu-1)\delta^{\nu-2} \frac{2^r \Gamma(r+(\nu+1)/2)}{\sqrt{2\pi}\Gamma((\nu+1)/2)} + \delta^{\nu-1}\beta_{r+1}(\delta), \text{ thus the lemma.}$$

Lemma 3 : $\beta_1(\delta) = \frac{\delta}{\sqrt{2\pi}} + \frac{e^{\delta^2/2} \text{erf}(\delta/\sqrt{2})}{2}(\nu + \delta^2)$, where $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$.

Proof

Note $L(f)$ the Laplace transform of the function f . We have

$$L\left(\sum_{j=0}^{\infty} \frac{(\lambda/2)^j}{\Gamma(j+1/2)}\right) = \sqrt{\lambda} L\left(\sum_{j=0}^{\infty} \frac{\lambda^{j-1/2}}{2^j \Gamma(j+1/2)}\right) = \sqrt{\lambda} \sum_{j=0}^{\infty} \frac{1}{2^j p^{j+1/2}} = \sqrt{\lambda} \left(\frac{1}{\sqrt{p}} + \frac{1}{2\sqrt{p}(p-1/2)} \right)$$

$$= \sqrt{\lambda} \left[L\left(\frac{1}{\sqrt{\pi\lambda}}\right) + \frac{1}{2} L\left(\frac{e^{\lambda/2}}{\sqrt{1/2}} \text{erf}(\sqrt{\lambda/2})\right) \right] = L\left(\frac{1}{\sqrt{\pi}} + e^{\lambda/2} \sqrt{\frac{\lambda}{2}} \text{erf}(\sqrt{\lambda/2})\right),$$

and then for $\lambda = \delta^2$, $\sum_{j=0}^{\infty} \frac{\delta^{2j}}{2^j \Gamma(j+1/2)} = \frac{1}{\sqrt{\pi}} + \frac{e^{\delta^2/2}}{\sqrt{2}} \delta \text{erf}(\delta/\sqrt{2})$.

In addition, since $\beta_1(\delta)$ can be written in the following form,

$$\beta_1(\delta) = \frac{1}{\sqrt{2}\delta^{\nu-1}} \frac{\partial}{\partial \delta} \left(\sum_{j=0}^{\infty} \frac{\delta^{2j+\nu-1}}{2^j \Gamma(j+1/2)} \right) - \frac{(\nu-1)\delta^{-1}}{\sqrt{2\pi}},$$

we have $\beta_1(\delta) = \frac{1}{\sqrt{2}\delta^{\nu-1}} \frac{\partial}{\partial \delta} \left(\delta^{\nu-1} \left(\frac{1}{\sqrt{\pi}} + \frac{e^{\delta^2/2}}{\sqrt{2}} \delta \operatorname{erf}(\delta/\sqrt{2}) \right) \right) - \frac{(\nu-1)\delta^{-1}}{\sqrt{2\pi}}$

$$= \frac{1}{\sqrt{2}\delta^{\nu-1}} \left((\nu-1+\delta^2) \frac{\delta^{\nu-2}}{\sqrt{\pi}} + \delta^{\nu-1} \frac{e^{\delta^2/2}}{\sqrt{2}} \operatorname{erf}(\delta/\sqrt{2}) (\nu+\delta^2) \right) - \frac{(\nu-1)\delta^{-1}}{\sqrt{2\pi}}, \text{ thus the lemma.}$$

Lemma 4 : $\alpha_r''(\delta) = \alpha_{r+1}(\delta) - (\nu-1)\delta^{-1}\alpha_r'(\delta).$

Proof

We have $\alpha_r(\delta) = \sum_{j=0}^{\infty} 2^{r+j} \delta^{2j} \frac{\Gamma(j+1/2)\Gamma(r+j+\nu/2)}{\Gamma(2j+1)\Gamma(j+\nu/2)\Gamma(1/2)}.$

By derivation with respect to δ ,

$$\alpha_r'(\delta) = \sum_{j=1}^{\infty} 2^{r+j} 2j \delta^{2j-1} \frac{\Gamma(j+1/2)\Gamma(r+j+\nu/2)}{\Gamma(2j+1)\Gamma(j+\nu/2)\Gamma(1/2)} = \sum_{j=0}^{\infty} 2^{r+1+j} \frac{\delta^{2j+1}}{(2j+\nu)} \frac{\Gamma(j+1/2)\Gamma(r+1+j+\nu/2)}{\Gamma(2j+1)\Gamma(j+\nu/2)\Gamma(1/2)},$$

and $(\delta^{\nu-1}\alpha_r'(\delta))' = \sum_{j=0}^{\infty} 2^{r+1+j} \delta^{2j+\nu-1} \frac{\Gamma(j+1/2)\Gamma(r+1+j+\nu/2)}{\Gamma(2j+1)\Gamma(j+\nu/2)\Gamma(1/2)},$ thus

$$(\nu-1)\delta^{\nu-2}\alpha_r'(\delta) + \delta^{\nu-1}\alpha_r''(\delta) = \delta^{\nu-1}\alpha_{r+1}(\delta), \text{ and the lemma.}$$

Lemma 5 : $A_1 = \frac{\nu + \delta^2}{2}.$

Proof

We have

$$A_1 = \frac{e^{-\delta^2/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} 2^{1+j} \delta^{2j} \frac{\Gamma(j+1/2)}{\Gamma(2j+1)} \frac{\Gamma(1+j+\nu/2)}{\Gamma(j+\nu/2)} = \frac{e^{-\delta^2/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} 2^{1+j} \delta^{2j} \frac{\Gamma(2j)\sqrt{2\pi}}{2^{2j-1/2}\Gamma(j)\Gamma(2j+1)} \frac{\Gamma(1+j+\nu/2)}{\Gamma(j+\nu/2)}$$

$$= \frac{e^{-\delta^2/2}}{2} \sum_{j=0}^{\infty} \frac{\delta^{2j}}{2^j} \frac{(\nu+2j)}{j!} = \frac{\nu}{2} + \frac{e^{-\delta^2/2}}{2} \delta \frac{\partial}{\partial \delta} \sum_{j=0}^{\infty} \frac{\delta^{2j}}{2^j j!} = \frac{\nu + \delta^2}{2}, \text{ thus the lemma.}$$

Proposition 1 : $\beta_r(\delta) = \frac{1}{\sqrt{2\pi}} P_r + \frac{e^{\delta^2/2} \operatorname{erf}(\delta/\sqrt{2})}{2} Q_r,$ where $Q_r = \sum_{i=0}^r a_{2i} \delta^{2i}$, and $P_r = \sum_{i=1}^r b_{2i-1} \delta^{2i-1}.$

Proof

This result is true for $r = 1$ according to lemma 3. Let us suppose that it is true for the rank r .

We have $\beta_r'(\delta) = \frac{1}{\sqrt{2\pi}}(P_r' + Q_r) + \frac{e^{\delta^2/2} \operatorname{erf}(\delta/\sqrt{2})}{2}(\delta Q_r + Q_r')$,

$$\beta_r''(\delta) = \frac{1}{\sqrt{2\pi}}(P_r'' + \delta Q_r + 2Q_r') + \frac{e^{\delta^2/2} \operatorname{erf}(\delta/\sqrt{2})}{2}((1 + \delta^2)Q_r + 2\delta Q_r' + Q_r''), \quad (1)$$

from where according to lemma 2,

$$\begin{aligned} \beta_{r+1}(\delta) = \frac{1}{\sqrt{2\pi}} & \left(P_r'' + \delta Q_r + 2Q_r' + (\nu - 1)\delta^{-1}(P_r' + Q_r) - (\nu - 1)\delta^{-1} \frac{2^r \Gamma(r + (\nu + 1)/2)}{\Gamma((\nu + 1)/2)} \right) \\ & + \frac{e^{\delta^2/2} \operatorname{erf}(\delta/\sqrt{2})}{2} ((\nu + \delta^2)Q_r + 2\delta Q_r' + Q_r'' + (\nu - 1)\delta^{-1}Q_r'). \end{aligned}$$

Let $P_{r+1} = P_r'' + \delta Q_r + 2Q_r' + (\nu - 1)\delta^{-1}(P_r' + Q_r) - (\nu - 1)\delta^{-1} \frac{2^r \Gamma(r + (\nu + 1)/2)}{\Gamma((\nu + 1)/2)}$,

and $Q_{r+1} = (\nu + \delta^2)Q_r + 2\delta Q_r' + Q_r'' + (\nu - 1)\delta^{-1}Q_r'$. (2)

Q_{r+1} is obviously an even polynomial of $2(r+1)$ degrees.

For P_{r+1} , we have

$$\begin{aligned} (\nu - 1)\delta^{-1}(P_r' + Q_r) &= (\nu - 1)\delta^{-1} \left(\sum_{i=1}^r b_{2i-1}(2i-1)\delta^{2i-2} + \sum_{i=0}^r a_{2i}\delta^{2i} \right) \\ &= (\nu - 1)\delta^{-1} (P_r'(0) + Q_r(0)) + (\nu - 1) \left(\sum_{i=1}^{r-1} (b_{2i+1}(2i+1) + a_{2i})\delta^{2i-1} + a_{2r}\delta^{2r-1} \right). \end{aligned}$$

Now $\beta_r'(0) = 2^{r-1/2} \frac{\Gamma(r + (\nu + 1)/2)}{\sqrt{\pi}\Gamma((\nu + 1)/2)}$ according to its definition, and $\beta_r'(0) = \frac{1}{\sqrt{2\pi}}(P_r'(0) + Q_r(0))$

according to the assumption of recurrence, therefore $P_r'(0) + Q_r(0) = 2^r \frac{\Gamma(r + (\nu + 1)/2)}{\Gamma((\nu + 1)/2)}$, and

consequently $P_{r+1} = P_r'' + \delta Q_r + 2Q_r' + (\nu - 1) \left(\sum_{i=1}^{r-1} (b_{2i+1}(2i+1) + a_{2i})\delta^{2i-1} + a_{2r}\delta^{2r-1} \right)$, which is

definitely an odd polynomial of $2r+1$ degrees.

Proposition 2: $Q_r = \sum_{k=0}^r \frac{2^r \Gamma(r + \nu/2)}{2^k \Gamma(k + \nu/2)} \binom{r}{k} \delta^{2k} = E\left[(\chi_\nu^2(\delta^2))^r\right].$

Proof

According to lemmas 1 and 2 we have

$$\begin{aligned}\beta_r''(\delta) &= \delta\beta_r'(\delta) + (2r + \nu)\beta_r(\delta) - (\nu - 1)\delta^{-1}\beta_r'(\delta) + (\nu - 1)\delta^{-1} \frac{2^r \Gamma(r + (\nu + 1)/2)}{\sqrt{2\pi}\Gamma((\nu + 1)/2)} \\ &= \frac{1}{\sqrt{2\pi}} \left[(\delta - (\nu - 1)\delta^{-1})(P_r' + Q_r) + (2r + \nu)P_r + (\nu - 1)\delta^{-1} \frac{2^r \Gamma(r + (\nu + 1)/2)}{\Gamma((\nu + 1)/2)} \right] \\ &\quad + \frac{e^{\delta^2/2} \operatorname{erf}(\delta/\sqrt{2})}{2} \left[(\delta - (\nu - 1)\delta^{-1})Q_r' + (\delta^2 + 2r + 1)Q_r \right].\end{aligned}$$

Taking into account the fact that $e^{\delta^2/2} \operatorname{erf}(\delta/\sqrt{2})$ is neither polynomial, nor fractional, from (1)

$$\text{we deduce } P_r'' + \delta Q_r' + 2Q_r' = (\delta - (\nu - 1)\delta^{-1})(P_r' + Q_r) + (2r + \nu)P_r + (\nu - 1)\delta^{-1} \frac{2^r \Gamma(r + (\nu + 1)/2)}{\Gamma((\nu + 1)/2)},$$

$$\text{and } ((1 + \delta^2)Q_r' + 2\delta Q_r' + Q_r'') = (\delta - (\nu - 1)\delta^{-1})Q_r' + (\delta^2 + 2r + 1)Q_r, \text{ or}$$

$$P_r'' + 2Q_r' = (\delta - (\nu - 1)\delta^{-1})P_r' + (2r + \nu)P_r - (\nu - 1)\delta^{-1}Q_r + (\nu - 1)\delta^{-1} \frac{2^r \Gamma(r + (\nu + 1)/2)}{\Gamma((\nu + 1)/2)}, \quad (3)$$

$$\text{and } Q_r'' + (\delta + (\nu - 1)\delta^{-1})Q_r' = 2rQ_r. \quad (4)$$

Taking $\delta = 0$ in (3), we have $Q_r''(0) + 2(\nu - 1)a_2 = 2rQ_r(0)$, then in (2),

$$Q_{r+1}(0) = \nu Q_r(0) + Q_r''(0) + 2(\nu - 1)a_2 = (2r + \nu)Q_r(0), \text{ whence}$$

$$Q_{r+1}(0) = (2r + \nu)(2r + \nu - 2) \dots (\nu + 2)\nu, \text{ since } Q_1(0) = \nu.$$

In (4), by equalizing the terms of rank $2k - 2$ ($k \neq r + 1$), we have

$$(2k)(2k - 1)a_{2k} + 2(k - 2)a_{2k-2} + 2k(\nu - 1)a_{2k} = 2ra_{2k-2}, \text{ thus}$$

$$a_{2k} = \frac{(r - k + 1)}{k(2k + \nu - 2)} a_{2k-2} = \dots = \frac{(r - k + 1)(r - k + 2) \dots r}{k(k - 1) \dots 3.2.1(2k + \nu - 2)(2k + \nu - 4) \dots (\nu + 2)\nu} a_0.$$

Since $a_0 = Q_r(0) = (2r + \nu - 2)(2r + \nu - 4) \dots (\nu + 2)\nu$, we have

$$a_{2k} = \binom{r}{k} \frac{(2r+\nu-2)(2r+\nu-4)\dots(\nu+2)\nu}{(2k+\nu-2)(2k+\nu-4)\dots(\nu+2)\nu} = \binom{r}{k} \frac{2^r}{2^k} \frac{\Gamma(r+\nu/2)}{\Gamma(k+\nu/2)}, \text{ thus}$$

$$Q_r = \sum_{k=0}^r \frac{2^r \Gamma(r+\nu/2)}{2^k \Gamma(k+\nu/2)} \binom{r}{k} \delta^{2k} = E \left[\left(\chi_\nu^2(\delta^2) \right)^r \right], \text{ (e.g. Johnson et al, 1994, p. 448), and the}$$

proposition for $0 \leq k < r$.

In (4), by equalizing the terms of rank $2r-2$, we have

$$(2r)(2r-1)a_{2r} + (2r-2)a_{2r-2} + 2r(\nu-1)a_{2r} = 2ra_{2r-2}, \text{ thus } a_{2r} = \frac{a_{2r-2}}{r(2r+\nu-2)}. \text{ Since}$$

$$a_{2r-2} = \binom{r}{r-1} \frac{2^r}{2^{r-1}} \frac{\Gamma(r+\nu/2)}{\Gamma(r-1+\nu/2)} = 2r(r-1+\nu/2), \text{ we have } a_{2r-2} = 1, \text{ and the proposition for } k = r.$$

Proposition 3 :
$$P_r = \sum_{k=1}^r \frac{\Gamma(k+r+(\nu-1)/2)\Gamma(r+\nu/2)}{\Gamma(k+(\nu-1)/2)\Gamma(k+1/2)} 2^{r-k} \\ \times \sum_{i=0}^{r-k} \frac{\Gamma(k+i+(\nu-1)/2)\Gamma(k+i+1/2)}{\Gamma(k+r+i+1+(\nu-1)/2)\Gamma(k+i+\nu/2)} \binom{r}{k+i} (2(k+i)+(\nu-1)/2) \delta^{2k-1}.$$

Proof

In (3), by equalizing the terms of rank $2r-1$, we have

$$4ra_{2r} = (2r-1)b_{2r-1} + (2r+\nu)b_{2r-1} - (\nu-1)a_{2r}, \text{ thus } b_{2r-1} = a_{2r} = 1, \text{ and by equalizing the terms of rank } 2k-1 \text{ (} k \neq r \text{), we have}$$

$$(2k)(2k+1)b_{2k+1} + 2(2k)a_{2k} = (2k-1)b_{2k-1} - (\nu-1)(2k+1)b_{2k+1} + (2r+\nu)b_{2k-1} - (\nu-1)a_{2k}, \text{ thus}$$

$$b_{2k-1} = \frac{(2k+\nu-1)(2k+1)}{(2k+2r+\nu-1)} b_{2k+1} + \frac{2(2k)+(\nu-1)}{(2k+2r+\nu-1)} a_{2k} \\ = 2 \frac{\Gamma(k+1+(\nu-1)/2)}{\Gamma(k+(\nu-1)/2)} \frac{\Gamma(k+1+1/2)}{\Gamma(k+1/2)} \frac{\Gamma(k+r+(\nu-1)/2)}{\Gamma(k+r+1+(\nu-1)/2)} b_{2k+1} \\ + (2(2k)+(\nu-1)) \frac{\Gamma(k+r+(\nu-1)/2)}{2\Gamma(k+r+1+(\nu-1)/2)} a_{2k}$$

=..., and in a recursive way we obtain the proposition by using the previous expression of

$$a_{2k}.$$

Proposition 4 : $A_r = \frac{Q_r}{2}$.

Proof

This result is true for $r = 1$ according to lemma 5 and proposition 2. Let us suppose that it is

true for rank r . We have $A_r = \frac{e^{-\delta^2/2}}{2} \alpha_r(\delta) = \frac{Q_r}{2}$, thus $\alpha_r(\delta) = e^{\delta^2/2} Q_r$, $\alpha'_r(\delta) = e^{\delta^2/2} (\delta Q_r + Q'_r)$,

and $\alpha''_r(\delta) = e^{\delta^2/2} ((1 + \delta^2) Q_r + 2\delta Q'_r + Q''_r)$.

From lemma 4, $\alpha_{r+1}(\delta) = e^{\delta^2/2} ((1 + \delta^2) Q_r + 2\delta Q'_r + Q''_r) + (\nu - 1) \delta^{-1} \alpha'_r(\delta)$

$= e^{\delta^2/2} ((\nu + \delta^2) Q_r + 2\delta Q'_r + Q''_r + (\nu - 1) \delta^{-1} Q'_r) = e^{\delta^2/2} Q_{r+1}$ according to (2), thus $A_{r+1} = \frac{Q_{r+1}}{2}$.

Theorem 1 : If Z is a weighted non-central chi-square distribution $\chi^2_\nu(\delta^2, \alpha_1, \alpha_2)$, then

$$\mu'_r = E[Z^r] = (\alpha_1^{2r} + \alpha_2^{2r}) A_r + (\alpha_1^{2r} - \alpha_2^{2r}) \left(P_r \frac{e^{-\delta^2/2}}{\sqrt{2\pi}} + A_r \operatorname{erf} \left(\frac{\delta}{\sqrt{2}} \right) \right),$$

$$\text{where } A_r = \frac{1}{2} \sum_{k=0}^r \frac{2^{r-k} \Gamma(r + \nu/2)}{\Gamma(k + \nu/2)} \binom{r}{k} \delta^{2k} = \frac{1}{2} E \left[(\chi^2_\nu(\delta^2))^r \right],$$

$$\text{and } P_r = \sum_{k=1}^r \frac{\Gamma(k + r + (\nu - 1)/2) \Gamma(r + \nu/2)}{\Gamma(k + (\nu - 1)/2) \Gamma(k + 1/2)} 2^{r-k}$$

$$\times \sum_{i=0}^{r-k} \frac{\Gamma(k + i + (\nu - 1)/2) \Gamma(k + i + 1/2)}{\Gamma(k + r + i + 1 + (\nu - 1)/2) \Gamma(k + i + \nu/2)} \binom{r}{k+i} (2(k+i) + (\nu - 1)/2) \delta^{2k-1}.$$

Proof

We have $E[Z^r] = (\alpha_1^{2r} + \alpha_2^{2r}) A_r + (\alpha_1^{2r} - \alpha_2^{2r}) B_r$. From propositions 2 and 4,

$A_r = \frac{1}{2} E \left[(\chi^2_\nu(\delta^2))^r \right]$. Since $B_r = e^{-\delta^2/2} \beta_r(\delta)$, expression of P_r is obtained from propositions 1

and 4, then the theorem.

In particular, we have

$$A_1 = \frac{1}{2} (\nu + \delta^2), \quad P_1 = \delta; \quad A_2 = \frac{1}{2} [(2 + \nu)\nu + 2(2 + \nu)\delta^2 + \delta^4], \quad P_2 = (2\nu + 3)\delta + \delta^3;$$

$$A_3 = \frac{1}{2} \left[(4+\nu)(2+\nu)\nu + 3(4+\nu)(2+\nu)\delta^2 + 3(4+\nu)\delta^4 + \delta^6 \right],$$

$$P_3 = 3(\nu^2 + 5\nu + 5)\delta + (3\nu + 11)\delta^3 + \delta^5 ;$$

$$A_4 = \frac{1}{2} \left[(6+\nu)(4+\nu)(2+\nu)\nu + 4(6+\nu)(4+\nu)(2+\nu)\delta^2 + 6(6+\nu)(4+\nu)\delta^4 + 4(6+\nu)\delta^6 + \delta^8 \right],$$

$$P_4 = (4\nu^3 + 42\nu^2 + 128\nu + 105)\delta + (6\nu^2 + 56\nu + 123)\delta^3 + (4\nu + 23)\delta^5 + \delta^7.$$

In the particular case where $\nu = 1$, P_r is simplified and is equal to

$$P_r = \sum_{k=1}^r \frac{\Gamma(r+k)}{2^{r-k} \Gamma(2k)} \sum_{i=0}^{r-k} \binom{2r}{i} \delta^{2k-1}.$$

In the case where $\alpha_1 = \alpha_2 = 1$, Z is a chi-square distribution with ν d.f. , and non-centrality parameter δ^2 , and the previous formula gives again $\mu_r' = E \left[(\chi_\nu^2(\delta^2))^r \right]$.

In the case where $\delta = 0$, Z is a weighted chi-square distribution with ν d.f. , of which the density is $f(z) = \frac{1}{2} \left[\alpha_1^{-2} f_{\chi_\nu^2}(\alpha_1^{-2} z) + \alpha_2^{-2} f_{\chi_\nu^2}(\alpha_2^{-2} z) \right]$, and $\mu_r' = \frac{1}{2} (\alpha_1^{2r} + \alpha_2^{2r}) E \left[(\chi_\nu^2)^r \right]$, since P_r is odd.

3. Central moments of the $\chi_\nu^2(\delta^2, \alpha_1, \alpha_2)$ distribution

The r -th central moment is $\mu_r = E \left[(Z - E(Z))^r \right] = \sum_{k=0}^r \binom{r}{k} (-1)^{r-k} \mu_k' \mu_1'^{r-k}.$

In the general case, the complex expression of the non-central moments does not allow to obtain simple results for the central moments. Simplifications can only be obtained in the case of a non-weighted chi-square distribution or in the case of a central chi-square distribution

Théorème 2 : *If Z is a non-weighted and non-central chi-square distribution $\chi_\nu^2(\delta^2)$, then*

$$\mu_r = \sum_{i=0}^r \sum_{j=0}^i \sum_{k=i-j}^{r-j} \delta^{2(r-i)} \nu^{i-j} (-1)^k \binom{r}{r-k} \binom{k}{k+j-i} \binom{r-k}{r-k-j} 2^j \Gamma(r-k+\nu/2) / \Gamma(r-k-j+\nu/2).$$

Proof

For a non-weighted distribution, $\alpha_1 = \alpha_2 = 1$, thus $\mu_r' = E\left[(\chi_v^2(\delta^2))^r\right] = \sum_{k=0}^r \frac{2^{r-k} \Gamma(r+\nu/2)}{\Gamma(k+\nu/2)} \binom{r}{k} \delta^{2k}$,

and $\mu_1' = \nu + \delta^2$, so

$$\mu_r = \sum_{k=0}^r \binom{r}{k} (-1)^{r-k} 2^k \Gamma(k+\nu/2) \left(\sum_{i=0}^k \binom{k}{i} \frac{\delta^{2i}}{2^i \Gamma(i+\nu/2)} \right) (\nu + \delta^2)^{r-k},$$

whence the theorem after gathering the terms of the same rank.

Let us note that the summation on i from 0 to r can be restricted from $[(r+1)/2]$ to r , where $[]$

represents the integer part. Indeed $\mu_r = \sum_{i=0}^r \delta^{2(r-i)} \sum_{j=0}^i B_j$, where

$$\begin{aligned} B_j &= \sum_{k=i-j}^{r-j} \nu^{i-j} (-1)^k \binom{r}{r-k} \binom{k}{k+j-i} \binom{r-k}{r-k-j} 2^j \Gamma(r-k+\nu/2) / \Gamma(r-k-j+\nu/2) \\ &= \sum_{k=0}^r \nu^{i-j} (-1)^k \binom{r}{r-k} \binom{k}{k+j-i} \binom{r-k}{r-k-j} 2^j \Gamma(r-k+\nu/2) / \Gamma(r-k-j+\nu/2). \end{aligned}$$

Now $\nu^{i-j} \binom{k}{k+j-i} \binom{r-k}{r-k-j} 2^j \Gamma(r-k+\nu/2) / \Gamma(r-k-j+\nu/2)$ is a polynomial in k of degree

$(i+j)$. B_j is thus null for $r \geq i+j+1$, since $\sum_{k=0}^r (-1)^k \binom{r}{k} k^m = 0$ for $r \geq m+1$. Consequently

$\sum_{j=0}^i B_j$ will be null for $r \geq 2i+1$, that is to say for $i \leq (r-1)/2$.

It should be noted that the central moments of a non-central chi-square distribution are usually obtained in a recursive way from cumulants. The previous formula allows to find the central moments with a rank lower or equal to 4 (e.g. Johnson et al, 1994, p. 447),

$$\mu_2 = 2(\nu + 2\delta^2), \quad \mu_3 = 8(\nu + 3\delta^2), \quad \mu_4 = 12(\nu^2 + 4\nu(\delta^2 + 1) + 4\delta^2(\delta^2 + 4)).$$

Theorem 3 : *If Z is a weighted central chi-square distribution $\chi_v^2(0, \alpha_1, \alpha_2)$, then*

$$\mu_r = \sum_{k=0}^r (\alpha_1^{2k} + \alpha_2^{2k}) (\alpha_1^2 + \alpha_2^2)^{r-k} \binom{r}{k} (-1)^{r-k} 2^{2k-r-1} \nu^{r-k} \Gamma(k+\nu/2) / \Gamma(\nu/2).$$

Proof

Since $\delta = 0$, we have $\mu_r' = \frac{1}{2}(\alpha_1^{2r} + \alpha_2^{2r}) E[(\chi_\nu^2)^r]$, whence the theorem.

In particular, $\mu_2 = \frac{1}{4}\nu^2(\alpha_1^2 - \alpha_2^2)^2 + \nu(\alpha_1^4 + \alpha_2^4)$, $\mu_3 = \frac{3}{2}\nu^2(\alpha_1^6 - \alpha_1^4\alpha_2^2 - \alpha_1^2\alpha_2^4 + \alpha_2^6) + 4\nu(\alpha_1^6 + \alpha_2^6)$,

$$\begin{aligned} \mu_4 = & \frac{1}{16}\nu^4(\alpha_1^8 - 4\alpha_1^6\alpha_2^2 + 6\alpha_1^4\alpha_2^4 - 4\alpha_1^2\alpha_2^6 + \alpha_2^8) + \frac{1}{2}\nu^3(3\alpha_1^8 - 6\alpha_1^6\alpha_2^2 + 6\alpha_1^4\alpha_2^4 - 6\alpha_1^2\alpha_2^6 + 3\alpha_2^8) \\ & + 2\nu^2(7\alpha_1^8 - 4\alpha_1^6\alpha_2^2 - 4\alpha_1^2\alpha_2^6 + 7\alpha_2^8) + 24\nu(\alpha_1^8 + \alpha_2^8). \end{aligned}$$

Theorem 4 : *If Z is a non-weighted central chi-square distribution χ_ν^2 , then*

$$\mu_r = \sum_{k=0}^r \binom{r}{k} (-1)^{r-k} 2^k \nu^{r-k} \Gamma(k + \nu/2) / \Gamma(\nu/2).$$

Proof

For a non-weighted central chi-square distribution, $\delta = 0$ and $\alpha_1 = \alpha_2 = 1$. The theorem is then obtained from theorems 2 or 3.

Conclusion

We suggest an extension of the weighted non-central chi-square distribution introduced by Chen (1998) for 1 d.f., to the case of ν d.f.. This new distribution with 4 parameters is a generalization of the non central chi-square distribution. The non central moments are obtained. The complexity of the expressions does not allow to consider a simple relation for the central moments in the general case. Those are obtained for specific cases which allow in particular to give an expression for the non-central chi-square distribution without requiring the use of cumulants.

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