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# The Resource-Constrained Activity Insertion Problem with Minimum and Maximum Time Lags

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## Abstract

We define the resource-constrained activity insertion problem with minimum and maximum time lags. The problem aims at inserting a single activity in a partial schedule while preserving its structure represented through resource flow networks and minimizing the makespan increase caused by the insertion. We show that finding a feasible insertion position that minimizes the project duration is NP-hard in the general case. When only minimum time lags are considered and when activity durations are strictly positive, we show the problem is polynomially solvable, generalizing previously established results on activity insertion for the standard resource-constrained project scheduling problem.

*Key words:* Resource-constrained project scheduling, minimum and maximum time lags, activity insertion problem, complexity.

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## 1 Introduction

In the standard resource-constrained project scheduling problem (RCPSP), the precedence relations are simple: an activity cannot start before the end of all its predecessors. In other words, between the start time of an activity  $A_i$  and the start time of its successor  $A_j$  there is a minimal distance (or minimum time lag) equal to the duration of  $A_i$ . The RCPSP with minimum and maximum time lags (RCPSP/max) involves generalized precedence relations where the

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minimum time lag between  $A_i$  and  $A_j$  can be any non-negative value and where there is possibly a maximum allowed time lag between the start time of  $A_i$  and the start time of  $A_j$ .

Independently of makespan minimization, simply finding a resource-feasible schedule that respects both the minimum and maximum time lags is NP-hard [3] whereas, for the standard RCPSP, this problem is polynomial. For that reason, the RCPSP/max problem has been extensively studied in the scheduling literature. For a thorough analysis of this problem, we recommend the book of Neumann, Schwindt and Zimmermann [9]. In this paper, unlike most approaches that focused on finding a global schedule that minimizes the project duration, we illustrate the difficulties brought by the maximum time lags by considering the apparently simple problem of inserting a single activity inside an existing schedule.

This problem aims at finding an insertion position satisfying the minimum/maximum time lags and resource constraints and preserving the partial schedule structure. The objective is to minimize the project duration increase. Such a problem arises in local search procedures for reinsertion neighborhoods that un-schedule an activity and insert it at another position [2,4,10] and also for reactive insertion of unexpected activities [2]. The interest of preserving the structure of the partial schedule is twofold. First, it makes it possible to decrease the size of the search space, which is necessary for both neighborhood search and reactive scheduling. Second, this policy tends to minimize the disturbances during the on-line project execution phase, which is referred to as ensuring the schedule stability.

In [4,10] among others, operation insertion problems are solved in a generalized job-shop. In [7], the job insertion problem in job-shop scheduling is studied. In [6], insertion problems in a general disjunctive scheduling framework capturing a variety of job shop scheduling problems and insertion types are considered and lower and upper bound procedures are developed. Since disjunctive and job-shop scheduling are particular cases of the RCPSP where all resources have unit availability, the proposed procedures do not apply to the problem considered in this paper.

For the RCPSP and the RCPSP/max, we can define the structure of the schedule by means of resource flow networks, which induce additional precedence constraints preventing resource over-subscription. With such a structure, the activity insertion problem has been addressed already for the standard RCPSP in [1,2]. An  $O(n^2m)$  optimal insertion procedure has been proposed where  $n$  is the number of activities and  $m$  is the number of resources. In [1], a tabu search procedure based on the reinsertion of a critical activity in the current resource flow network has been designed and obtained good results to solve the standard RCPSP. Hence, an important issue is to establish whether this

approach can be extended to minimum and maximum time lags for proposing a new class of neighborhood search method for the RCPSP/max.

In this paper, we show that the considered insertion problem is unfortunately NP-hard in the general case. However, when maximum time lags are ignored, we propose a new polynomial algorithm, generalizing the results obtained in [1,2] to the RCPSP with minimum time lags.

## 2 Problem statement

Activities constituting the project are identified by set  $V = \{A_0, \dots, A_{n+1}\}$ . Activity  $A_0$  represents by convention the start of the schedule and activity  $A_{n+1}$  represents symmetrically the end of the schedule. The set of non-dummy activities is identified by  $\mathcal{A} = \{A_1, \dots, A_n\}$ .  $p_i$  denotes the duration of activity  $i$  with  $p_0 = p_{n+1} = 0$ . We assume in this paper that  $p_i > 0, \forall A_i \in \mathcal{A}$ .

A valuated activity-on-node graph  $G(V, E, l)$  is defined where nodes correspond to activities and arcs correspond to precedence relations. Each arc  $(A_i, A_j) \in E$  is valuated by an integer time lag  $l_{ij}$ .  $l_{ij} \geq 0$  corresponds to a minimum time lag of  $l_{ij}$  units stating that  $A_j$  has to start at least  $l_{ij}$  time units after the start time of  $A_i$ . In the standard RCPSP, only minimum time lags are considered and  $l_{ij} = p_i$  for each arc  $(A_i, A_j) \in E$ .  $l_{ij} \leq 0$  corresponds to a time lag of  $-l_{ij}$  units stating that  $A_i$  has to start at the latest  $l_{ij}$  time units after the start time of  $A_j$ . We assume that we have at least the following arcs in  $E$ : an arc  $(A_0, A_i)$  per activity  $A_i \in \mathcal{A}$  valuated by  $l_{0i} \geq 0$  and an arc  $(A_i, A_{n+1})$  per activity  $A_i \in \mathcal{A}$  valuated by  $l_{i(n+1)} = p_i$ . Resource constraints are defined as in the standard RCPSP:  $\mathcal{R} = \{R_1, \dots, R_m\}$  denotes the set of  $m$  resources.  $B_k$  denotes the availability of  $R_k$ .  $b_{ik}$  represents the amount of resource  $R_k$  used during the execution of  $i$ .

A solution  $S$  is a schedule giving the start time  $S_i$  of each activity  $A_i$ .  $S$  is feasible if it is compatible with the generalized precedence constraints and the resource constraints which are defined below, where  $\mathcal{A}_t = \{i \in \mathcal{A} \mid S_i \leq t < S_i + p_i\}$  represents the set of non-dummy activities in process at time  $t$ .

$$S_j - S_i \geq l_{ij} \quad \forall (A_i, A_j) \in E \quad (1)$$

$$\sum_{A_i \in \mathcal{A}_t} b_{ik} \leq B_k \quad \forall R_k \in \mathcal{R}, \forall t \geq 0 \quad (2)$$

The makespan of a schedule  $S$  is equal to  $S_{n+1}$ , the start time of the end activity. The above-defined set  $\mathcal{A}_t$  and constraints state that an activity cannot be interrupted once it is started. This is referred to as not allowing *preemption*.

The RCPSP with minimum and maximum time lags is the problem (P) of finding a non-preemptive schedule  $S$  of minimal makespan  $S_{n+1}$  such that  $S_0 = 0$  and subject to precedence and resource constraints. This problem does not necessarily have a solution. This can be due to an inconsistency of the precedence constraints, independently of the resource constraints. To verify the consistency of the precedence constraints  $E$ , a distance matrix  $(\delta_{ij})_{i,j \in V^2}$  can be computed where  $\delta_{ij}$  is the length of the longest path between  $i$  and  $j$  in  $G(V, E, l)$ .  $(\delta_{ij})$  can be computed in  $\mathcal{O}(n^3)$  by the Floyd-Warshall algorithm. More precisely, there is no solution if there is a cycle of positive length in  $G(V, E, l)$ . In the case the temporal constraints are consistent, the problem does not necessarily have a solution neither and the corresponding decision problem is NP-complete [3].

We use the concept of resource-flow network to represent the solutions to (P) [1,2,5,8,9]. A resource flow  $f$  is a  $(n+2) \times (n+2) \times m$  matrix verifying equations (3-5) defined below:

$$f_{ijk} \geq 0 \quad \forall A_i \in \mathcal{A} \cup \{A_0\}, \forall A_j \in \mathcal{A} \cup \{A_{n+1}\}, \forall R_k \in \mathcal{R} \quad (3)$$

$$\sum_{A_i \in \mathcal{A} \cup \{A_{n+1}\}} f_{0ik} = \sum_{A_i \in \mathcal{A} \cup \{A_0\}} f_{i(n+1)k} = B_k \quad \forall R_k \in \mathcal{R} \quad (4)$$

$$\sum_{A_j \in \mathcal{A} \cup \{A_{n+1}\}} f_{ijk} = \sum_{A_j \in \mathcal{A} \cup \{A_0\}} f_{jik} = b_{ik} \quad \forall A_i \in \mathcal{A}, \forall R_k \in \mathcal{R} \quad (5)$$

$f_{ijk}$  denotes the number of resource  $R_k$  units transferred from activity  $A_i$  to activity  $A_j$ . We define  $E(f)$  as the set of arcs induced by a resource flow  $f$ , i.e.  $E(f) = \{(A_i, A_j) \in V^2 \mid \exists R_k \in \mathcal{R}, f_{ijk} > 0\}$ .  $E(f)$  can be seen as the set of precedence constraints represented by the flow. Consider graph  $G^f = G(V, E \cup E(f), l(f))$  where  $l(f)$  is defined as follows. Each arc  $(i, j) \in E$  with  $(i, j) \notin E(f)$  is valued by  $l_{ij}$ . Each arc  $(i, j) \in E(f)$  with  $(i, j) \notin E$  is valued by  $p_i$ . Each arc  $(i, j) \in E \cap E(f)$  is valued by  $\max(p_i, l_{ij})$ .

We say that  $A_j$  is a resource successor (predecessor) of  $A_i$  if there is a resource  $R_k \in \mathcal{R}$  such that  $f_{ijk} > 0$  ( $f_{jik} > 0$ ).

Let  $(\delta_{ij}^f)_{A_i, A_j \in V}$  denotes the distance matrix associated with flow  $f$ .  $\delta_{ij}^f$  is the length of the longest path from  $A_i$  to  $A_j$  in  $G^f$ . The precedence constraints induced by the flow are consistent if and only if there is no cycle of positive length in  $G^f$ . In particular,  $\delta_{0i}^f, \forall A_i \in V$  denotes the earliest start schedule associated with the flow  $f$ . We set  $\delta_{ij}^f = -\infty$  if there is no path from  $A_i$  to  $A_j$  in  $G^f$ .

**Proposition 1** (P) can be defined as the problem of finding a resource flow  $f$  verifying (3-5), such that  $G^f$  contains no cycle of positive length and such that  $\delta_{0(n+1)}^f$  is minimal.

**Proof** The arguments used in Theorem 1 and 2 in [8] hold. See also Section 2.13 in [9]  $\square$

Hence a flow verifying (3-5), such that  $G^f$  contains no cycle of positive length is said to be feasible and is considered in what follows as a solution to (P).

To define the resource constrained activity insertion problem (RCAIP), we consider a partial solution in which all activities but one, denoted by activity  $A_x$  ( $0 < x < n + 1$ ), have been scheduled. This amounts to considering a complete solution to problem  $(P_x)$  identical to (P) except that  $b_{xk} = 0, \forall R_k \in \mathcal{R}$ .

Consider an example issued from [9], comprising five real activities and a single resource of three units. Durations and resource demands are given in Table 1. Minimum and maximum time lags and the corresponding the project network are displayed on the left side of Figure 1.

$A_i$	0	1	2	3	4	5	6
$p_i$	0	6	4	2	4	2	0
$b_i$	0	1	2	2	2	3	0

Table 1  
Durations and resource demands

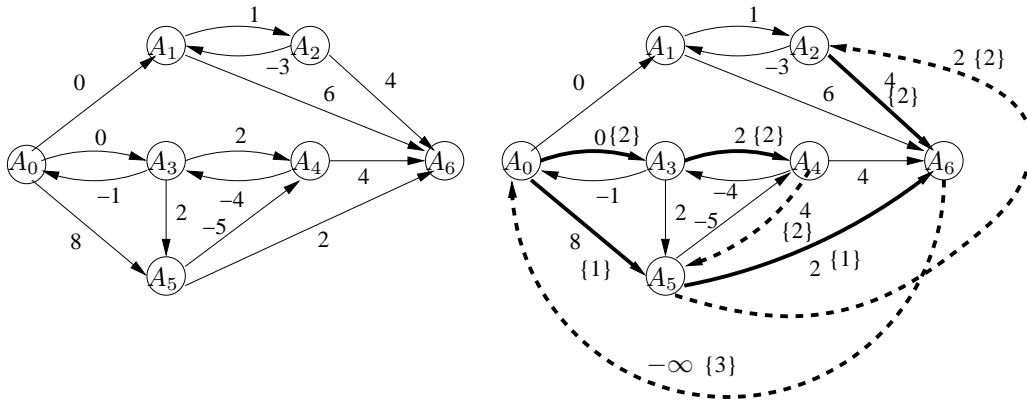


Fig. 1. Minimum/maximum time lags (left side) and flow  $f$  solution to  $(P_1)$  (right side)

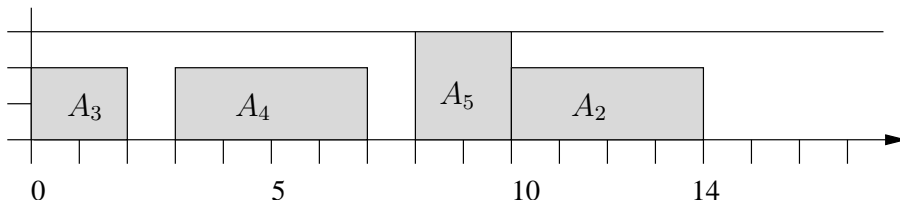


Fig. 2. Gantt chart of the solution to  $(P_1)$

We consider a partial schedule for (P) in which activity  $A_1$  is not scheduled, corresponding to a complete solution of  $(P_1)$ , displayed in Figure 2. The resource flow corresponding to this schedule is displayed on the right side of Figure 1. Thin arcs correspond to the original precedence constraints only while thick arcs are induced by the flow (the flow value is displayed between braces). Plain thick arcs belong to  $E \cup E(f)$  while thick dotted arcs are only induced by the flow.

The problem considered in this paper amounts to compute a solution to (P) by inserting activity  $A_x$  in the flow  $f$  associated with the solution of  $(P_x)$  in such a way that the resource flow assigned to  $A_x$  can only be taken from  $f$ . More formally we define the considered *resource-constrained activity insertion problem* (RCAIP $_{xf}$ ) as the problem to find  $q_{ijk}$  such that  $0 \leq q_{ijk} \leq f_{ijk}$  for all  $i, j \in V, \forall R_k \in \mathcal{R}$  and such that the flow  $f'$  defined by:

$$f'_{xj} = \sum_{A_i \in V} q_{ijk} \quad \forall A_j \in V, \forall R_k \in \mathcal{R} \quad (6)$$

$$f'_{ix} = \sum_{A_j \in V} q_{ijk} \quad \forall A_i \in V, \forall R_k \in \mathcal{R} \quad (7)$$

$$f'_{ij} = f_{ijk} - q_{ijk} \quad \forall A_i, A_j \in V, \forall R_k \in \mathcal{R} \quad (8)$$

verifies equations (3-5), does not create a cycle of positive length in  $G^{f'}$  and minimizes  $\delta_{0(n+1)}^{f'}$ . Note we have  $\delta_{0(n+1)}^{f'} \geq \delta_{0(n+1)}^f$  since by transitivity, any precedence constraint induced by  $f$  is also induced by  $f'$ .

In flow  $f$ , solution to  $(P_1)$ , presented in Figure 1, the non-zero flow amounts are  $f_{03} = 2, f_{05} = 1, f_{34} = 2, f_{45} = 2, f_{56} = 1, f_{52} = 2, f_{26} = 2$ . The activity candidate for insertion  $A_1$  is such that  $b_i = 1$ . Hence, to insert  $A_1$  into  $f$  according to the insertion scheme described above, we have to find activities  $A_i$  and  $A_j$  such that the amount of flow taken from  $f_{ij}$  and redirected to  $A_1$  is  $q_{ij} = 1$ . The reader may check that among the non zero flow amounts listed above, the only valid candidates are  $A_i = A_5$  and  $A_j = A_6$ . Indeed, inserting  $A_1$  before activity  $A_5$  or between  $A_2$  and  $A_5$  violates constraint  $S_2 \leq S_1 + 3$  while inserting  $A_1$  after  $A_2$  violates constraint  $S_2 \geq S_1 + 1$ . It follows that we obtain the solution to RCAIP $_{A_1f}$  displayed in Figure 3 of makespan 16.

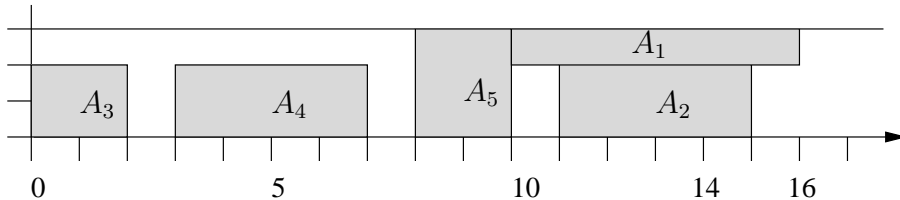


Fig. 3. Gantt chart of the solution to RCAIP $_{1f}$

### 3 Insertion positions

We define an insertion position as an ordered pair of activity sets  $(\alpha, \beta)$  such that  $(\alpha, \beta) \in V^2$ ,  $\alpha \cap \beta = \emptyset$  and verifying:

$$\sum_{A_i \in \alpha, A_j \in \beta} f_{ijk} \geq b_{xk}, \forall R_k \in \mathcal{R}. \quad (9)$$

An insertion position induces a set of arcs

$$E(\alpha, \beta) = \{(A_a, A_x)\}_{A_a \in \alpha} \cup \{(A_x, A_b)\}_{A_b \in \beta}.$$

We consider graph  $G^{f\alpha\beta} = G(V, E \cup E(f) \cup E(\alpha, \beta), l(f, \alpha, \beta))$  where  $l(f, \alpha, \beta)$  denotes the arc valuation induced by  $(\alpha, \beta)$ .  $l(f, \alpha, \beta)$  defines the same values as  $l(f)$  except for the following arcs. For each  $A_a \in \alpha$ ,  $(A_a, A_x)$  is valued by  $p_a$  if  $(A_a, A_x) \notin E$  or by  $\max(p_a, l_{ax})$  otherwise. For each  $A_b \in \beta$ ,  $(A_x, A_b)$  is valued by  $p_x$  if  $(A_x, A_b) \notin E$  or by  $\max(p_x, l_{xb})$  otherwise. Let  $\delta_{ij}^{f\alpha\beta}$  denotes the length of the longest path from  $A_i$  to  $A_j$  in  $G^{f\alpha\beta}$ .

**Theorem 1** (*RCAIP<sub>xf</sub>*) amounts to find a feasible insertion position  $(\alpha, \beta)$  such that there is no positive length cycle in  $G^{f\alpha\beta}$  and  $\delta_{0(n+1)}^{f\alpha\beta}$  is minimized.

**Proof** Given a feasible solution  $(q_{ijk})$  of the insertion problem and its corresponding flow  $f'$ , an insertion position can be defined by setting  $\alpha = \{A_i \in V \mid \exists A_j \in V, \exists R_k \in \mathcal{R}, q_{ijk} > 0\}$  and  $\beta = \{A_i \in V \mid \exists A_j \in V, \exists R_k \in \mathcal{R}, q_{jik} > 0\}$ . Equation (9) is obviously verified. Furthermore it holds that  $E(\alpha, \beta) \cup E(f) = E(f')$  and  $l(f, \alpha, \beta) = l(f')$ . We have consequently no positive length path in  $G^{f\alpha\beta}$  and  $\delta_{0(n+1)}^{f\alpha\beta} = \delta_{0(n+1)}^{f'}$ .

Conversely, consider an insertion position such that there is no positive length cycle in  $G^{f\alpha\beta}$ . Subset  $\alpha$  ( $\beta$ ) corresponds to the set of activities that can send (receive) resource-flow to (from)  $A_i$  and, consequently, that are possible resource predecessors (successors). For each resource  $R_k \in \mathcal{R}$ , let  $(q_{ijk})$  be computed by the following algorithm. For each resource  $R_k \in \mathcal{R}$ , set  $q_{ijk} = 0, \forall A_i, A_j \in V$ . Then for each  $R_k \in \mathcal{R}$  and for each  $A_i, A_j \in \alpha \times \beta$ , if  $\sum_{A_i, A_j \in \alpha \times \beta} q_{ijk} < b_k$ , update  $q_{ijk} \leftarrow \min(b_k - \sum_{A_i, A_j \in \alpha \times \beta} q_{ijk}, f_{ijk})$ . Flow  $f'$  generated with the so-computed  $(q_{ijk})$  verifies obviously equations (3-5). Furthermore, it is such that  $E(f') \subseteq E(\alpha, \beta) \cup E(f)$ . Note that we have not necessarily the equality since there may be  $A_a \in \alpha$  and  $A_b \in \beta$  such that  $q_{ijk} = 0, \forall R_k \in \mathcal{R}$ . We have for the same reason  $l(f')_{ij} \leq l(f, \alpha, \beta)_{ij}$  for each  $(A_i, A_j) \in E \cup E(f')$ . Consequently there is no cycle of positive length in  $G^{f'}$  and  $\delta_{0(n+1)}^{f'} \leq \delta_{0(n+1)}^{f\alpha\beta}$ .  $\square$



Theorem 1 shows that, to solve  $(\text{RCPSP}_{xf})$ , we can restrict the search space to pairs of activity sets  $(\alpha, \beta)$  without considering explicitly amounts  $q_{ij}$ . In what follows, we refer to a feasible insertion position as an insertion position  $(\alpha, \beta)$  such that there is no positive length cycle in  $G^{f\alpha\beta}$  and we identify the feasible insertion positions with the solutions to  $(\text{RCAIP}_{xf})$ .

#### 4 Feasibility conditions under a makespan upper bound

In this section we establish feasibility conditions of a given insertion position  $(\alpha, \beta)$ .

The search variant of  $(\text{RCAIP}_{xf})$  is considered. It is denoted by  $(\text{FRCAIP}_{xfv})$ . The problem is to find a feasible solution to  $(\text{RCAIP}_{xf})$  with a makespan not greater than  $v$ . This can be achieved by searching for a feasible solution to  $(\text{RCAIP}_{xf})$  after adding an arc from  $A_{n+1}$  to  $A_0$  valued by  $\max(-v, l_{(n+1)0})$ . Let  $E(f, v) = E(f) \cup \{(A_{n+1}, A_0)\}$  and  $l(f, v)$  denote the so modified set of arcs and values. For a given insertion position  $(\alpha, \beta)$ , let  $l(f, v, \alpha, \beta)$  denote the corresponding arc values. With these definitions,  $(\text{FRCAIP}_{xfv})$  amounts to searching for an insertion position  $(\alpha, \beta)$  such that there is no positive length cycle in  $G^{fv\alpha\beta} = G(V, E \cup E(f, v) \cup E(\alpha, \beta), l(f, v, \alpha, \beta))$ .

As a preliminary remark, if there is a positive length cycle in  $G^{fv}$ , there is obviously no feasible solution to  $(\text{FRCAIP}_{xfv})$ . Again, this can be determined in  $O(n^3)$  by the Floyd-Warshall algorithm. However, according to the last remark of section 2, if  $f$  represents a feasible solution to  $(P_x)$ , there is a positive length cycle in  $G^{fv}$  if and only if  $v < \delta_{0(n+1)}^f$ . In what follows, we assume  $v \geq \delta_{0(n+1)}^f$  and we consider  $(\delta_{ij}^{fv})_{A_i, A_j \in V}$  defined as the distance matrix corresponding to the all-pairs longest path lengths in  $G^{fv}$ .

The remaining of the section is devoted to the evaluation of the feasibility of an insertion position  $(\alpha, \beta)$  (without generating graph  $G^{fv\alpha\beta}$  nor computing matrix  $(\delta_{ij}^{fv\alpha\beta})$ ). For this purpose, we define the following three categories of elementary cycles in  $G^{fv\alpha\beta}$  involving inserted activity  $A_x$  :

- An elementary cycle of type  $\mathcal{C}_1(v, \alpha, \beta)$  involves an arc  $(A_a, A_x)$  with  $A_a \in \alpha$  and  $l(f, v, \alpha, \beta)_{ax} = p_a$  and no arc  $(A_x, A_b)$  with  $A_b \in \beta$ . Note that in  $G^{fv\alpha\beta}$ , a longest path from  $A_x$  to  $A_a$  not traversing any node of  $\beta$  is also a longest path from  $A_x$  to  $A_a$  in  $G^{fv}$  of length  $\delta_{xa}^{fv}$  (since the path cannot traverse  $A_x$  again). Therefore  $L_1(v, \alpha, \beta) = \max_{A_a \in \alpha} (p_a + \delta_{xa}^{fv})$  is the length of the cycles of  $\mathcal{C}_1(v, \alpha, \beta)$  having the largest length.
- An elementary cycle of type  $\mathcal{C}_2(v, \alpha, \beta)$  includes an arc  $(A_x, A_b)$  with  $A_b \in \beta$  and  $l(f, v, \alpha, \beta)_{xb} = p_x$  and no arc  $(A_a, A_x)$  with  $A_a \in \alpha$ . Using the same arguments as the ones use in the previous item, the length of the longest

path from  $A_b$  to  $A_x$  in  $G^{fv\alpha\beta}$  is equal to  $\delta_{bx}^{fv}$ . Consequently,  $L_2(v, \alpha, \beta) = p_x + \max_{A_b \in \beta}(\delta_{bx}^{fv})$  denotes the length of the cycle in  $\mathcal{C}_2(v, \alpha, \beta)$  having the largest length.

- A cycle of type  $\mathcal{C}_3(v, \alpha, \beta)$  involves two arcs  $(A_a, A_x), (A_x, A_b)$  with  $l(f, v, \alpha, \beta)_{ax} = p_a$  and  $l(f, v, \alpha, \beta)_{xb} = p_x$ . Again, the longest path in  $G^{fv\alpha\beta}$  passes only through arcs existing in  $G(V, E \cup E(f, v), l(v, f))$  and thus has length  $\delta_{ba}^{fv}$ . Hence, let  $L_3(v, \alpha, \beta) = p_x + \max_{(A_a, A_b) \in \alpha \times \beta}(p_a + \delta_{ba}^{fv})$  be the length of the cycle of  $\mathcal{C}_3(v, \alpha, \beta)$  having the largest length.

In what follows,  $\mathcal{C}_q(v, \alpha, \beta)$  and  $L_q(v, \alpha, \beta)$  ( $q = 1, 2, 3$ ) will be simply denoted  $\mathcal{C}_q$  and  $L_q$  when there is no ambiguity. Figure 4 illustrates the three considered cycle types.

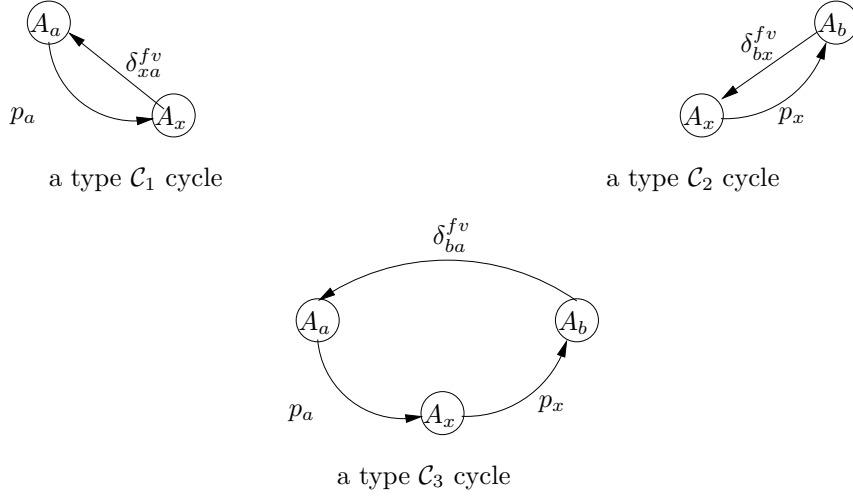


Fig. 4. The three cycle types generated by the insertion

**Theorem 2** An insertion position  $(\alpha, \beta)$  is feasible for  $(FRCAIP_{xfv})$  if and only if  $\max(L_1, L_2, L_3) \leq 0$ .

**Proof** This amounts to show that there is no cycle of positive length in  $G^{fv\alpha\beta}$  if and only if  $\max(L_1, L_2, L_3) \leq 0$ .

( $\Rightarrow$  part) If there is no positive length cycle in  $G^{fv\alpha\beta}$ , since each  $L_i$  is not larger than an actual cycle length, then  $L_1, L_2$  and  $L_3$  are non positive.

( $\Leftarrow$  part) We know that in a valuated digraph there is no positive length cycle if and only if there is no elementary positive length cycle. Thus, it is sufficient to show that if  $\max(L_1, L_2, L_3) \leq 0$ , there is no elementary positive length cycle in  $G^{fv\alpha\beta}$ . Among the possible elementary cycles, we may restrict to the ones traversing at least one arc  $(A_a, A_x)$  with  $A_a \in \alpha$  and  $l(f, v, \alpha, \beta)_{ax} = p_a$  (type (i) arc) or one arc  $(A_x, A_b)$  with  $A_b \in \beta$  and  $l(f, v, \alpha, \beta)_{xb} = p_x$  (type (ii) arc). Indeed, any cycle traversing none of these arcs has a length equal to the length of a cycle in  $G^{fv}$  which is by hypothesis non-negative. Furthermore, at

most one type (i) arc and at most one type (ii) arc may be traversed by any elementary cycle.

Suppose now that  $\mathcal{C}$  denotes an elementary cycle of  $G^{fv\alpha\beta}$  with the largest length but not belonging to any category among  $\mathcal{C}_1$ ,  $\mathcal{C}_2$ , nor  $\mathcal{C}_3$ . According to the presence of type (i) and (ii) arcs there are three possibilities. If  $\mathcal{C}$  traverses exactly one type (i) arc and no type (ii) arcs, the path in  $\mathcal{C}$  from  $A_x$  to  $A_a$  is not larger than  $\delta_{xa}^{fv}$ . Indeed, by construction, for any activities  $A_i, A_j, A_k$  we have  $\delta_{ij}^{fv} \geq \delta_{ik}^{fv} + \delta_{kj}^{fv}$ . Consequently, there exists in  $\mathcal{C}_1$  a cycle not shorter than  $\mathcal{C}$ . For the same reason, if  $\mathcal{C}$  traverses exactly one type (ii) arc and no type (i) arcs, the path in  $\mathcal{C}$  from  $A_b$  to  $A_x$  is not larger than  $\delta_{bx}^{fv}$  and there exists in  $\mathcal{C}_2$  a cycle not shorter than  $\mathcal{C}$ . Last, in the case  $\mathcal{C}$  traverses exactly one type (i) arc and one type (ii) arc, the path in  $\mathcal{C}$  from  $A_b$  to  $A_a$  is not larger than  $\delta_{ba}^{fv}$  and there exists in  $\mathcal{C}_3$  a cycle not shorter than  $\mathcal{C}$ . It follows that there exists an elementary longest length cycle that belongs either to  $\mathcal{C}_1$ ,  $\mathcal{C}_2$  or  $\mathcal{C}_3$ .  $\square$

## 5 Computational complexity of the resource-constrained activity insertion problem with minimum and maximum time lags

In this section, we prove that  $(\text{RCAIP}_{xf})$  is NP-hard. We show that the decision problem  $(\text{DRCAIP}_{xfv})$ , stating whether there exists a feasible solution to  $(\text{FRCAIP}_{xfv})$  or not, is NP-complete by reduction from the independent set problem. We recall that, given a graph  $G(\mathcal{V}, \mathcal{E})$ , where  $\mathcal{V}$  is a set of vertices and  $\mathcal{E}$  is a set of edges, and an integer  $k$ , the independent set problem lies in finding a subset of vertices  $\mathcal{W} \subseteq \mathcal{V}$  of cardinality  $k$  such that  $\forall x, y \in \mathcal{W}$ ,  $(x, y) \notin \mathcal{E}$  (i.e. the vertices of  $\mathcal{W}$  are pairwise independent). Such a problem is known to be NP-hard. Let  $p$  denotes the number of vertices:  $p = |\mathcal{V}|$ .

**Theorem 3** *The decision variant of the resource-constrained activity insertion problem is NP-complete.*

**Proof** The decision variant of the insertion problem amounts to find an answer to the following question. Does there exist a feasible insertion position for  $(\text{FRCAIP}_{xfv})$ ? The problem is obviously in NP since (i) for any tentative solution  $(\alpha, \beta)$  checking whether constraints (9) are satisfied can be made in  $O(n^2m)$  time since  $|\alpha| \leq n$  and  $|\beta| \leq n$  and (ii) checking whether there is no type  $\mathcal{C}_1$ ,  $\mathcal{C}_2$  and  $\mathcal{C}_3$  cycles take  $O(n)$ ,  $O(n)$  and  $O(n^2)$  time, respectively. Matrix  $(\delta_{ij}^{fv})$  can be computed in  $O(n^3)$  by the Floyd-Warshall algorithm.

In what follows, we show how to associate to any independent set problem an instance of the insertion problem with  $n = 2p + 1$  non-dummy activities and

a single resource of availability  $B_1 = p$  (the resource index will be omitted in the rest of the demonstration).

The non-dummy activities are partitioned into three subsets  $\Theta = \{A_1, \dots, A_p\}$ ,  $\Omega = \{A_{p+1}, \dots, A_{2p}\}$  and  $A_n = A_{2p+1}$ ,  $A_n$  being the activity to insert. All activities but  $A_n$  are such that  $b_i = p_i = 1, \forall A_i \in \Theta \cup \Omega$ . For  $A_n$  we have  $p_n = 3$  and we set  $b_n$  to  $k$ . The resource flow  $f$  is such that

$$\begin{aligned} f_{0i} &= 1 \quad \forall A_i \in \Theta \\ f_{i(p+i)} &= 1 \quad \forall A_i \in \Theta \\ f_{j(n+1)} &= 1 \quad \forall A_j \in \Omega. \end{aligned}$$

All other resource flows are zero. The precedence constraints  $E$  are such that there is an arc  $(A_0, A_i)$  with  $l_{0i} = 0$ , for each  $A_i \in \Theta$ . There is an arc  $(A_j, A_{n+1})$  with  $l_{j(n+1)} = 1$  for each  $j \in \Omega$ . There is an arc  $(A_i, A_{p+i})$  with  $l_{i(p+i)} = 2$  for each  $A_i \in \Theta$ . There is an arc  $(A_{n+1}, A_0)$  such that  $l_{(n+1)0} = -5$  (i.e. the makespan has to be not greater than 5). For the inserted task  $A_n$ , there is an arc  $(A_0, A_n)$  and an arc  $(A_n, A_{n+1})$  with  $l_{0n} = 0$  and  $l_{n(n+1)} = 3$ , respectively.

The set of edges  $\mathcal{E}$  of the graph of the independent set problem is arbitrarily orientated. Let  $G(\mathcal{V}, \mathcal{U})$  denotes the graph so obtained. Now we create in  $E$  an arc  $(A_{p+j}, A_i)$  with  $l_{(p+j)i} = -3$  for each  $(i, j) \in \mathcal{U}$  to represent the edges in  $\mathcal{E}$ . Figure 5 shows how the insertion problem instance is built from an example graph of six nodes. Below each arc the time lag is indicated and the transferred resource flow is displayed between braces.

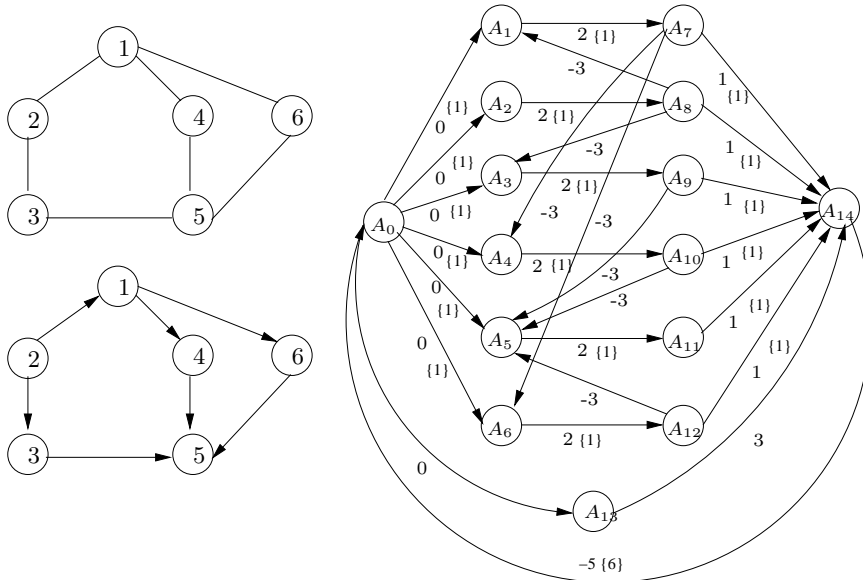


Fig. 5. Reduction from independent set

It can be easily checked that the minimal distance matrix  $\delta^{fv}$  for the so-defined insertion problem instance (with  $v = 5$ ) is such that for each pair  $A_i, A_j$  with  $i \in \Theta$  and  $j \in \Omega$ , we have  $\delta_{ji}^{fv} = -3$  if  $(j - p, i) \in \mathcal{U}$  and  $\delta_{ji}^{fv} = -4$  otherwise.

Below we show that there exists an insertion position for  $A_n$  in  $G^{fv}$  if and only if there exists an independent set of cardinality  $k$  in  $G(\mathcal{V}, \mathcal{E})$ . First, one can observe that if  $W$  is an independent set of cardinality  $k$ , then there is a feasible insertion position  $(\alpha, \beta)$ . Indeed, if  $\alpha = \{A_i | i \in W\}$  and  $\beta = \{A_{p+i} | i \in W\}$  then, because the flow from  $A_i \in \Theta$  to  $A_{i+p}$  is equal to 1, we have  $\sum_{A_i \in \alpha, A_j \in \beta} f_{ij} = k$ . Consequently  $(\alpha, \beta)$  verifies condition (9). Moreover, there is no maximum time lags involving the inserted activity  $A_n$  and, consequently, no possible positive length cycles of types  $\mathcal{C}_1$  nor  $\mathcal{C}_2$ . For any pair of nodes  $x, y \in W$  we have no edge  $(x, y)$  in  $\mathcal{E}$ , therefore we have no arc  $(x + p, y)$  in  $\mathcal{U}$ . As a result,  $\forall A_j \in \Omega, \forall A_i \in \Theta$ , we have  $\delta_{ji}^{fv} = -4$ . The largest cycle of type  $\mathcal{C}_3$  has a length equal to  $L_3 = \max_{(A_a, A_b) \in \alpha \times \beta} (p_a + p_x + \delta_{ba})$ . Since  $p_a = 1$  and  $p_x = 3$ , we have  $L_3 = 0$ . It follows that the insertion problem is feasible.

Now let us show that if the insertion is feasible, there is an independent set of cardinality  $k$ . Let  $(\alpha, \beta)$  denotes the feasible insertion position. Obviously, the insertion position such that  $\alpha = \{A_0\}$  and  $\beta = \Theta \cup \Omega \cup A_{n+1}$  is not feasible since it will increase the project duration to 6, whereas it is bounded by 5. The same remark holds for the insertion position such that  $\alpha = A_0 \cup \Theta \cup \Omega$  and  $\beta = \{A_{n+1}\}$ . Therefore,  $A_n$  can only be located between some activities of  $\Theta$  and some activities of  $\Omega$  (i.e.  $\alpha \subset \Theta$  and  $\beta \subset \Omega$ ). For a feasible  $(\alpha, \beta)$  insertion position, since no positive length cycles have been generated, we have  $\delta_{ba}^{fv} = -4 \forall A_a \in \alpha, \forall A_b \in \beta$ . Subsequently there is no arc  $(b - p, a)$  in  $\mathcal{U}, \forall A_a \in \alpha, \forall A_b \in \beta$  and no edge  $(x, y)$  in  $E, \forall a \in \alpha$ . Furthermore, since the insertion is feasible, we have  $|\alpha| = k$  therefore  $\alpha$  is an independent set of cardinality  $k$ . Figure 6 illustrates the correspondence between an independent set (bold nodes) and an insertion position.  $\square$

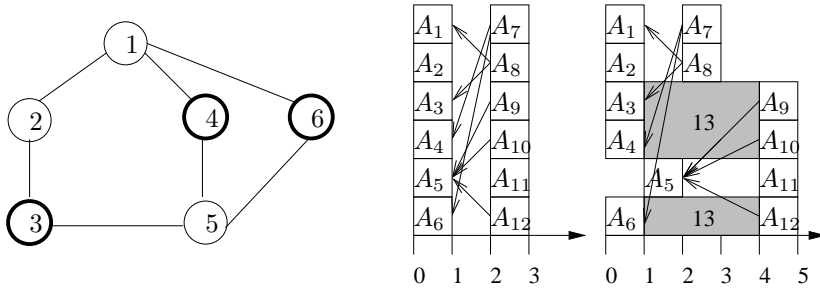


Fig. 6. A feasible independent set and its corresponding feasible insertion position

While for the standard RCPSP, Artigues *et al.* proposed a polynomial algorithm to solve the insertion problem [1,2], the introduction of minimum and maximum time lags clearly makes the problem intractable despite of its apparent simplicity. This illustrates the deep modifications of the problem structure

brought by the maximum time lags. To confirm this, we exhibit in the following section a polynomial algorithm, able to solve the insertion problem when only minimum time lags are considered and when activity durations are strictly positive.

## 6 A polynomial algorithm for the resource-constrained activity insertion problem with minimum time lags

We suppose in this section that all time lags are non-negative and that  $p_i \geq 1$ ,  $\forall A_i \in \mathcal{A}$ . We still assume that  $f$  represents a feasible solution to  $(P_x)$  of makespan  $\delta_{0(n+1)}^f$  corresponding to the length of the longest path between 0 and  $n+1$  in  $G^f$ .

We already know that the insertion problem is polynomial for the standard RCPSP as stated in [2]. The standard RCPSP corresponds to the case where all minimum time lags are such that  $l_{ij} = p_i$ . It follows that in the case where all minimum time lags are such that  $l_{ij} \geq p_i$ , we can trivially show that the problem is polynomially solvable by transforming the RCPSP instance with minimum time lags into a standard RCPSP instance as follows. For each  $(A_i, A_j) \in E$ , create an activity  $A_{ij}$  of duration  $l_{ij} - p_i$  and zero resource demand and augment  $E$  with the precedence constraints  $(A_i, A_{ij})$ ,  $(A_{ij}, A_j)$ . Then, set all arc values to the duration of the origin activity. Since the initial flow is trivially valid for the so-generated instance, the polynomial insertion algorithm for the standard RCPSP can be applied. Thus, the problem remains open only in the case there is at least one minimum time lag such that  $0 \leq l_{ij} \leq p_i$ .

We show that in this case the insertion problem  $(RCAIP_{xf})$  is polynomially solvable. We first prove that for any  $v \geq \delta_{0(n+1)}^f$ , the following lemma holds.

**Lemma 1**  $\forall A_i, A_j \in V^2$ ,  $\delta_{ij}^{fv} = \max(\delta_{ij}^f, \delta_{0j}^f + \delta_{i(n+1)}^f - v)$

**Proof** Since  $\delta_{ij}^{fv}$  is the length of the longest path in  $G^{fv}$ , let  $\mathcal{P}$  denotes such a longest path. If  $\mathcal{P}$  traverses arc  $(A_{n+1}, A_0)$  (valuated by  $-v$  since we have no maximum time lags), by definition of the longest paths we have  $\delta_{ij}^{fv} = \delta_{0j}^{fv} + \delta_{i(n+1)}^{fv} - v$ . Since  $G^{fv}$  is obtained from  $G^f$  by adding arc  $(A_{n+1}, A_0)$ , we have  $\delta_{0i}^{fv} = \delta_{0i}^f$  and  $\delta_{j(n+1)}^{fv} = \delta_{j(n+1)}^f$ . If  $\mathcal{P}$  does not traverse arc  $(A_{n+1}, A_0)$  it has for the same reason length  $\delta_{ij}^f$ .  $\square$

With this property, we subdivide each cycle type  $\mathcal{C}_q(v, \alpha, \beta)$  in  $G^{fv\alpha\beta}$  into two

subtypes, one being independent of  $v$ , as follows:

- $\mathcal{C}_1^1(v, \alpha, \beta)$  is the subset of elementary cycles  $\mathcal{C}_1(v, \alpha, \beta)$  (traversing an arc  $(A_a, A_x)$  with  $A_a \in \alpha$ ) such that  $\delta_{xa}^{fv} = \delta_{0a}^f + \delta_{x(n+1)}^f - v$ . Consequently, the longest cycle of this type has a length equal to  $L_1^1(v, \alpha, \beta) = \max_{A_a \in \alpha} (\delta_{0a}^f + p_a) + \delta_{x(n+1)}^f - v$ .
- $\mathcal{C}_1^2(\alpha, \beta)$  is the subset of elementary cycles  $\mathcal{C}_1(v, \alpha, \beta)$  such that  $\delta_{xa}^{fv} = \delta_{xa}^f$ . The longest cycle of this type is of length  $L_1^2(\alpha, \beta) = \max_{A_a \in \alpha} (\delta_{xa}^f + p_a)$ . Note we have  $L_1^2(\alpha, \beta) = -\infty$  if  $\forall A_a \in \alpha, \delta_{xa}^f = -\infty$ .
- $\mathcal{C}_2^1(v, \alpha, \beta)$  is the subset of elementary cycles  $\mathcal{C}_2(v, \alpha, \beta)$  (traversing an arc  $(A_x, A_b)$  with  $A_b \in \beta$ ) such that  $\delta_{bx}^{fv} = \delta_{0x}^f + \delta_{b(n+1)}^f - v$ . The longest cycle of this type is of length  $L_2^1(v, \alpha, \beta) = \delta_{0x}^f + p_x + \max_{A_b \in \beta} (\delta_{b(n+1)}^f) - v$ .
- $\mathcal{C}_2^2(\alpha, \beta)$  is the subset of elementary cycles  $\mathcal{C}_2(v, \alpha, \beta)$  such that  $\delta_{bx}^{fv} = \delta_{bx}^f$ . The longest cycle of this type is of length  $L_2^2(\alpha, \beta) = \max_{A_b \in \beta} (\delta_{bx}^f) + p_x$ . We have  $L_2^2(\alpha, \beta) = -\infty$  if  $\forall A_b \in \beta, \delta_{bx}^f = -\infty$ .
- $\mathcal{C}_3^1(v, \alpha, \beta)$  is the subset of elementary cycles  $\mathcal{C}_3(v, \alpha, \beta)$  (traversing an arc  $(A_a, A_x)$  with  $A_a \in \alpha$  and an arc  $(A_x, A_b)$  with  $A_b \in \beta$ ) such that  $\delta_{ba}^{fv} = \delta_{0a}^f + \delta_{b(n+1)}^f - v$ . The longest cycle of this type is of length  $L_3^1(v, \alpha, \beta) = \max_{A_a \in \alpha} (\delta_{0a}^f + p_a) + p_x + \max_{A_b \in \beta} (\delta_{b(n+1)}^f) - v$ .
- $\mathcal{C}_3^2(\alpha, \beta)$  is the subset of elementary cycles  $\mathcal{C}_3(v, \alpha, \beta)$  such that  $\delta_{ba}^{fv} = \delta_{ba}^f$ . The longest cycle of this type is of length  $L_3^2(\alpha, \beta) = \max_{A_a \in \alpha, A_b \in \beta} (\delta_{ba}^f + p_a) + p_x$ . We have  $L_3^2(\alpha, \beta) = -\infty$  if  $\delta_{ba}^f = -\infty$ .

Figure 7 illustrates the 6 types of relevant cycles.

We consider now the insertion problem  $(\text{RCAIP}_{xf})$  instead of its search variant. An insertion position  $(\alpha, \beta)$  is feasible for  $(\text{RCAIP}_{xf})$  if there exists an arbitrary large value  $M \geq \delta_{0(n+1)}^f$  such that  $(\text{FRCAIP}_{xfM})$  is feasible.  $(\text{RCAIP}_{xf})$  amounts to finding the smallest  $M \geq \delta_{0(n+1)}^f$  such that there exists a feasible insertion position for  $(\text{FRCAIP}_{xfM})$ . Remarking that  $L_1^2, L_2^2$  and  $L_3^2$  do not depend on  $v$ , the following theorem can be stated:

**Theorem 4** *An insertion position  $(\alpha, \beta)$  is feasible for  $(\text{RCAIP}_{xf})$  if and only if it satisfies constraints (9) and  $\max(L_1^1(\alpha, \beta), L_2^1(\alpha, \beta), L_3^1(\alpha, \beta)) \leq 0$ .*

**Proof** Searching a feasible solution to  $(\text{RCAIP}_{xf})$  amounts to searching a feasible solution to  $(\text{FRCAIP}_{xf\infty})$  where  $v = \infty$ . It follows that for any  $(\alpha, \beta)$ ,  $L_1^1(\alpha, \beta) = L_2^1(\alpha, \beta) = L_3^1(\alpha, \beta) = -\infty$ . It follows that there is no positive length cycle in  $G^{f\alpha\beta}$  if and only if the considered property holds.  $\square$

Thus, the feasibility of an insertion position only depends on the cycle subtype  $\mathcal{C}_x^2$ .

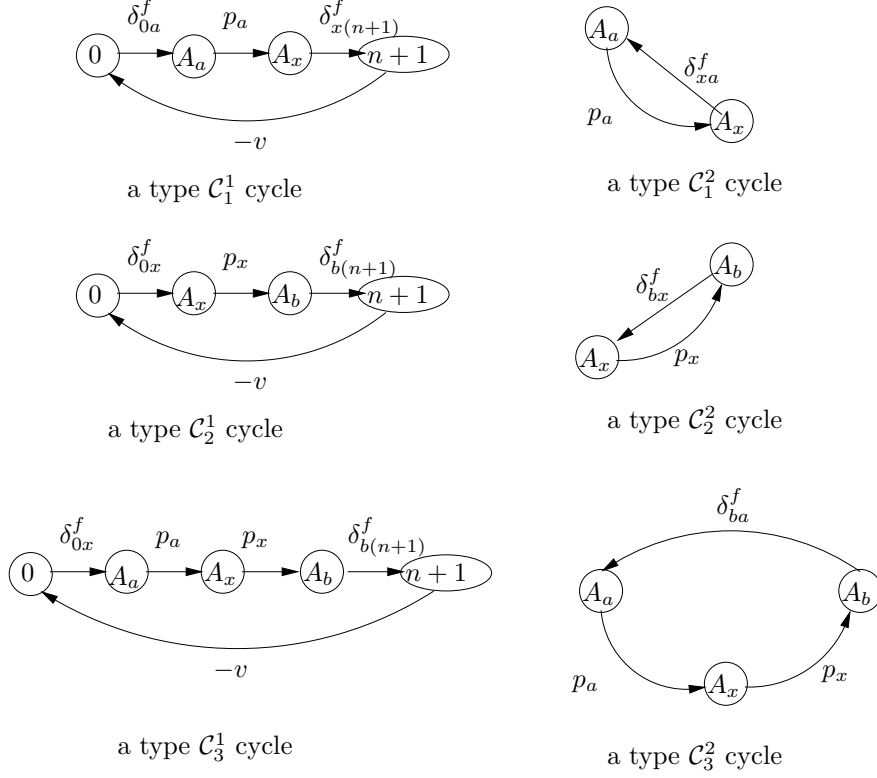


Fig. 7. The six types of cycles generated by the insertion (minimum time lags only)

**Theorem 5** *If  $(\alpha, \beta)$  is a feasible insertion position for  $(RCAIP_{xf})$ , we have*

$$\delta_{0(n+1)}^{f\alpha\beta} = \max\left(\delta_{0(n+1)}^f, L_1^1(0, \alpha, \beta), L_2^1(0, \alpha, \beta), L_3^1(0, \alpha, \beta)\right) \quad (10)$$

**Proof** Since  $(\alpha, \beta)$  is feasible for  $(RCAIP_{xf})$ , we have  $L_1(v, \alpha, \beta) = L_1^1(v, \alpha, \beta)$ ,  $L_2(v, \alpha, \beta) = L_2^1(v, \alpha, \beta)$  and  $L_3(v, \alpha, \beta) = L_3^1(v, \alpha, \beta)$ ,  $\forall v \geq \delta_{0(n+1)}^f$ . The makespan of the solution represented by  $(\alpha, \beta)$  is either equal to  $\delta_{0(n+1)}^f$  or to the smallest value  $v$  such that  $\max(L_1^1(v, \alpha, \beta), L_2^1(v, \alpha, \beta), L_3^1(v, \alpha, \beta)) \leq 0$  which yields the expression.  $\square$

The makespan of a feasible insertion position depends on the cycle subtype  $\mathcal{C}_x^1$ . Below, we derive a necessary and sufficient condition for the existence of a feasible insertion position and we define a feasible insertion position in the case the condition is verified.

Let  $\gamma_0 = \{A_i \in \mathcal{A} \mid \delta_{ix}^f = 0 \text{ and } \delta_{xi}^f = 0\}$ . The constraint  $\delta_{ix}^f = 0$  and  $\delta_{xi}^f = 0$  is called a synchronization constraint and  $\gamma_0$  is the set of non-dummy activities that are synchronized with  $A_x$ . Note that  $A_{n+1}$  cannot be synchronized with  $A_x$  since  $l_{x(n+1)} = p_x$ .  $A_0$  can only be synchronized with  $A_x$  if  $l_{0x} = 0$ . By convention, in the special case where  $l_{0x} = l_{x0} = 0$ , dummy activity  $A_0$  does not belong to  $\gamma_0$ . Consequently, for any activity  $A_i \in \gamma_0$  we cannot have  $A_i \in \alpha$



(otherwise a cycle of length  $p_a$  would be issued) nor  $A_i \in \beta$  (otherwise a cycle of length  $p_x$  would be issued). Furthermore, in any feasible schedule respecting the precedence constraints, the synchronized activities are constrained to start exactly at the same time. Consequently, there exists a solution to the insertion problem if the following inequalities hold.

$$\sum_{A_i \in \gamma_0} b_{ik} + b_{xk} \leq B_k \quad \forall R_k \in \mathcal{R} \quad (11)$$

Let  $(\alpha_0, \beta_0)$  the insertion position such that  $\alpha_0 = \{A_i \in V \setminus \gamma_0 \mid \delta_{ix}^f \geq 0\}$  and  $\beta_0 = V \setminus (\gamma_0 \cup \alpha_0)$ .

**Lemma 2** ( $RCAIP_{xf}$ ) *is feasible and  $(\alpha_0, \beta_0)$  is a feasible insertion position if and only if  $\gamma_0$  satisfies constraints (11).*

**Proof** First, we remark that the definition of  $(\alpha_0, \beta_0)$  implies that  $A_0 \in \alpha_0$  and  $A_{n+1} \in \beta_0$  and that  $\alpha_0$ ,  $\beta_0$  and  $\gamma_0$  define a partition of  $V$ . Consider a resource  $R_k \in \mathcal{R}$ . Partitioning the incoming flow of task  $A_{n+1} \in \beta_0$  according to  $\alpha_0$ ,  $\beta_0$  and  $\gamma_0$ , we get

$$B_k = \sum_{A_i \in \gamma_0} f_{i(n+1)k} + \sum_{A_i \in \alpha_0} f_{i(n+1)k} + \sum_{A_i \in \beta_0} f_{i(n+1)k}. \quad (12)$$

The same decomposition applied to the other activities of  $\beta_0$  yields:

$$\forall A_j \in \beta_0 \setminus \{A_{n+1}\}, b_{jk} = \sum_{A_i \in \gamma_0} f_{ijk} + \sum_{A_i \in \alpha_0} f_{ijk} + \sum_{A_i \in \beta_0} f_{ijk} \quad (13)$$

Summing all equations (12) and (13) it comes:

$$\sum_{A_j \in \beta_0 \setminus \{A_{n+1}\}} b_{jk} + B_k = \sum_{A_i \in \gamma_0, A_j \in \beta_0} f_{ijk} + \sum_{A_i \in \alpha_0, A_j \in \beta_0} f_{ijk} + \sum_{A_i, A_j \in \beta_0} f_{ijk} \quad (14)$$

Consider  $A_i \in \gamma_0$ . We have  $\delta_{ix}^f = \delta_{xi}^f = 0$ . Suppose there exists  $A_j \in \alpha_0$  and  $R_k \in \mathcal{R}$  such that  $f_{ijk} > 0$ . Then  $\delta_{ij}^f \geq p_i$ . Since  $A_j \in \alpha_0$  we have  $\delta_{jx}^f \geq 0$  which implies  $\delta_{ix}^f \geq p_i$  and yields a positive length cycle with  $\delta_{xi}^f = 0$ . It follows that  $\forall A_i \in \gamma_0, \forall A_j \in \alpha_0$  and  $\forall R_k \in \mathcal{R}$ , we have  $f_{ijk} = 0$ . For any pair of distinct activities  $A_i, A_j \in \gamma_0$ , we have also  $f_{ijk} = 0, \forall R_k \in \mathcal{R}$  since otherwise a cycle of length  $p_i$  would exist. It follows that each activity  $A_i \in \gamma_0$  sends flow units only to activities of  $\beta_0$ , which yields  $\forall R_k \in \mathcal{R}, \sum_{A_i \in \gamma_0, A_j \in \beta_0} f_{ijk} = \sum_{A_i \in \gamma_0} b_{ik}$ .

Consider  $A_i \in \beta_0, A_j \in \alpha_0 \cup \gamma_0$  and  $R_k \in \mathcal{R}$  such that  $f_{ijk} > 0$ . It follows that  $\delta_{ij}^f \geq p_i$  and, in addition, we have  $\delta_{jx}^f \geq 0$  by definition of  $\alpha_0$  and  $\gamma_0$ .

By transitivity, it comes  $\delta_{ix}^f \geq p_i$  which implies that  $A_i \in \alpha_0$ , a contradiction. Consequently each activity  $A_i \in \beta_0$  sends flow units only to other activities of  $\beta_0$ , and we have  $\sum_{A_i, A_j \in \beta_0} f_{ijk} = \sum_{A_i \in \beta_0 \setminus \{A_{n+1}\}} b_{ik}$ ,  $\forall A_i \in \beta_0$ . Inserting these results into (14), we get  $\sum_{A_i \in \alpha_0, A_j \in \beta_0} f_{ijk} = B_k - \sum_{A_i \in \gamma_0} b_{ik}$ . It follows that  $(\alpha_0, \beta_0)$  can be a feasible insertion position only if  $\gamma_0$  satisfies constraints (11).

It remains to show that if  $\gamma_0$  satisfies constraints (11),  $(\alpha_0, \beta_0)$  is a feasible insertion position for  $(\text{RCAIP}_{x,f})$ . From the definition of  $\alpha_0$ , there is no path in  $G^f$  from any activity  $A_a \neq A_0 \in \alpha_0$  and  $x$  which yields  $\delta_{xa}^f = -\infty$  for these activities. For the special case of  $A_0$  we have  $\delta_{x0} \leq 0$ . It follows that  $L_1^2(\alpha_0, \beta_0) \leq 0$ . From the definition of  $\beta_0$ , we have  $\delta_{bx}^f = -\infty$ ,  $\forall A_b \in \beta$ , which yields  $L_2^2(\alpha_0, \beta_0) = -\infty$ . Last, suppose there exists  $A_a \in \alpha_0$  and  $A_b \in \beta_0$  such that  $\delta_{ba}^f \geq 0$ . Since  $\delta_{ax}^f \geq 0$ , it comes  $\delta_{bx}^f \geq 0$  by transitivity of the longest path lengths and we have  $A_b \in \alpha_0$  or  $A_b \in \gamma_0$ , a contradiction. It follows that  $\forall A_a \in \alpha_0$  and  $\forall A_b \in \beta_0$  we have  $\delta_{ba}^f = -\infty$  which implies  $L_3^2 = -\infty$ .  $\square$

According to Lemma 2, the necessary and sufficient condition (11) for the existence of a feasible insertion position can be checked in  $O(nm)$ . Feasible insertion position  $(\alpha_0, \beta_0)$  can be obtained in  $O(|E \cup E(f)|)$  since checking if  $\delta_{ix}^f \geq 0$  can be done by a simple depth-first search algorithm in  $G^f$ .

To illustrate the feasibility condition, let us consider a RCPSP example with 10 activities and two resources such that  $B_1 = 7$  and  $B_2 = 4$ . The processing times and the resource demands are indicated on Table 2. Minimum time lags are displayed on Figure 8. We assume that all activities, except Activity  $A_5$  that has to be inserted, are already scheduled as shown on the Gantt diagram of Figure 10. The flow network  $f$  corresponding to this partial schedule is displayed on Figure 9 and gives a makespan  $C_{\max} = 11$ .

$i$	0	1	2	3	4	5	6	7	8	9	10	11
$p_i$	0	6	1	1	2	5	3	5	3	2	4	0
$(b_{i1}, b_{i2})$	(7,4)	(2,1)	(1,0)	(3,1)	(2,0)	(1,2)	(2,1)	(2,0)	(0,1)	(1,2)	(1,1)	(7,4)

Table 2  
Processing times and resource demands

The values of  $\delta_{0i}^f + p_i$  and  $\delta_{i(n+1)}^f$ , for all activities  $A_i$ , needed to check the feasibility or compute the makespan of any insertion position can be computed using the Bellmann-Ford's algorithm. There are indicated on Table 3.

Sets  $\gamma_0$ ,  $\alpha_0$  and  $\beta_0$  are defined as follows.  $\gamma_0 = \{6\}$  since  $A_6$  is the only activity which is synchronized with activity 5 ( $S_5 = S_6$ ). Then we find  $\alpha_0 = \{A_0, A_2, A_3, A_4\}$  and  $\beta_0 = \{A_1, A_7, A_8, A_9, A_{10}, A_{11}\}$  (see the cut denoted  $\rho = 0$  on Figure 9). This insertion position  $(\alpha_0, \beta_0)$  is resource-feasible since,  $\forall k$ ,  $\sum_{(A_i, A_j) \in \alpha_0 \times \beta_0} f_{ijk} \geq b_{5k}$ .

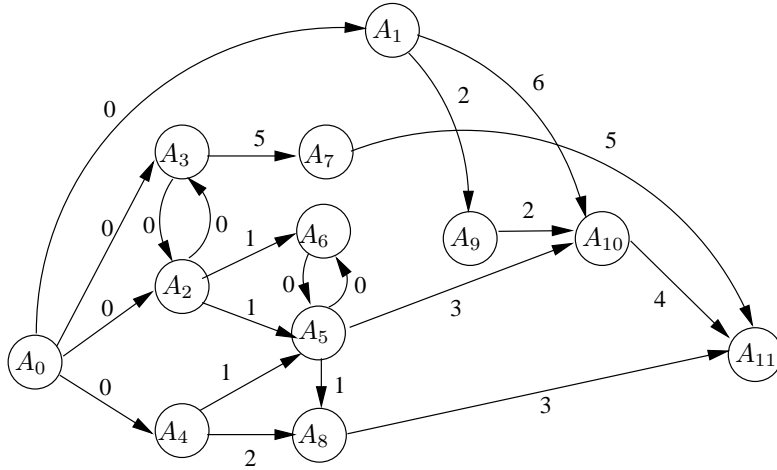


Fig. 8. Example of a RCPSP with minimum time lags

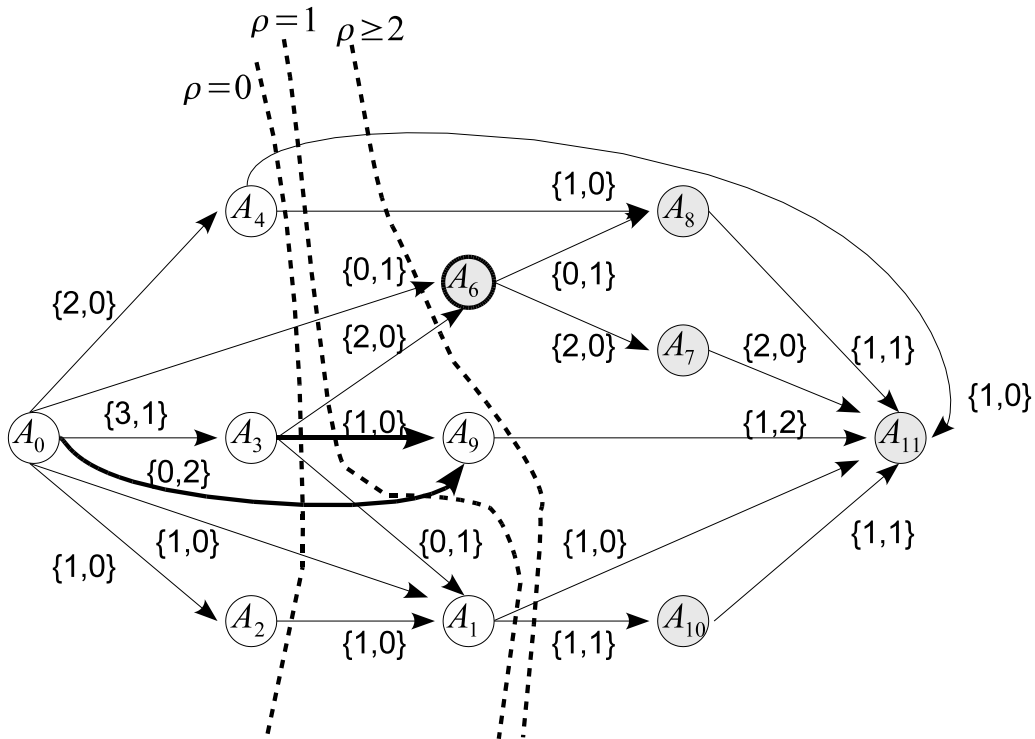


Fig. 9. Resource flow network

$i$	0	1	2	3	4	5	6	7	8	9	10	11
$\delta_{0i}^f + p_i$	0	7	1	1	2	4	4	10	7	5	11	11
$\delta_{i(n+1)}^f$	11	10	11	11	9	8	8	5	3	6	4	0

Table 3

Values of  $\delta_{0i}^f + p_i$  and  $\delta_{i(n+1)}^f$

In the remaining of the section, we present an algorithm to find an optimal insertion position. We prove its optimality and provide its time complexity.

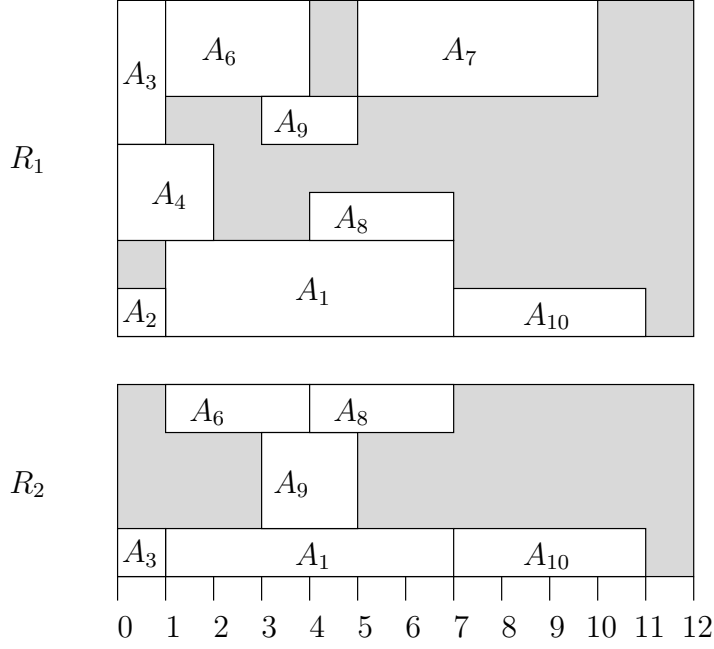


Fig. 10. Partial solution not including Activity  $A_5$

Let  $(\alpha, \beta)$  denotes a feasible insertion position for  $(\text{RCAIP}_{xf})$ . Let  $\mu$  denotes the subset of  $\alpha$  such that  $\mu = \{A_i \in \alpha \mid \delta_{0i}^f + p_i = \max_{A_a \in \alpha} (\delta_{0a}^f + p_a)\}$  and  $\nu$  denotes the subset of  $\beta$  such that  $\nu = \{A_i \in \beta \mid \delta_{i(n+1)}^f = \max_{A_b \in \beta} (\delta_{b(n+1)}^f)\}$ . Due to the expression of  $\delta_{0(n+1)}^{f\alpha\beta}$  given by equation (10), there is no other feasible insertion position  $(\alpha', \beta')$  such that  $C_{\max}(\alpha', \beta') < C_{\max}(\alpha, \beta)$ ,  $\mu \subseteq \alpha'$  and  $\nu \subseteq \beta'$ . Indeed  $\mu$  denotes the set of activities of  $\alpha$  yielding the longest cycle in  $\mathcal{C}_1^1(\alpha, \beta)$  given by expression  $L_1^1(\alpha, \beta)$ .  $\nu$  denotes the set of activities of  $\beta$  yielding the longest cycle in  $\mathcal{C}_2^1(\alpha, \beta)$  given by expression  $L_2^1(\alpha, \beta)$ .  $(\mu, \nu)$  denote the subset of  $(\alpha, \beta)$  yielding the longest cycle in  $\mathcal{C}_3^1(\alpha, \beta)$  given by expression  $L_3^1(\alpha, \beta)$ . Thus, to have an insertion position  $(\alpha', \beta')$  of makespan lower than  $\delta_{0(n+1)}^{f\alpha\beta}$  we must have either  $\mu \cap \alpha' = \emptyset$  or  $\nu \cap \beta' = \emptyset$ , or both conditions.

Again, let us consider the insertion position given  $\alpha_0 = \{A_0, A_2, A_3, A_4\}$  and  $\beta_0 = \{A_1, A_7, A_8, A_9, A_{10}, A_{11}\}$ . The insertion position yields a makespan

$$\delta_{0(n+1)}^{f\alpha\beta} = \max \left( \delta_{0(n+1)}^f, L_1^1(0, \alpha_0, \beta_0), L_2^1(0, \alpha_0, \beta_0), L_3^1(0, \alpha_0, \beta_0) \right) = 15.$$

In Table 3, we may check that  $\mu = \{A_4\}$  and  $\nu = \{A_1\}$ , which means that  $(\alpha_0, \beta_0)$  is an optimal insertion position among all insertion positions such that  $A_4$  is a resource predecessor and  $A_1$  is a resource successor.

Based on these dominance rules, in the case the problem is feasible, we can define a series of  $\Lambda \geq 1$  insertion positions  $(\alpha_\rho, \beta_\rho)_{0 \leq \rho \leq \Lambda-1}$  such that

$$(\alpha_\rho, \beta_\rho) = (\alpha_{\rho-1} \cup \nu'_{\rho-1}, \beta_{\rho-1} \setminus \nu_{\rho-1}) \quad \rho = 1, \dots, \Lambda - 1 \quad (15)$$

where

$$\nu_\rho = \{A_i \in \beta_\rho \mid \delta_{i(n+1)}^f = \max_{A_b \in \beta_\rho} (\delta_{b(n+1)}^f)\} \quad \rho = 0, \dots, \Lambda - 1, \quad (16)$$

$$\nu'_\rho = \{A_i \in \nu_\rho \mid \delta_{xi}^f = -\infty\} \quad \rho = 0, \dots, \Lambda - 1, \quad (17)$$

and

$$\Lambda = \min\{\rho \geq 1, \exists R_k \in \mathcal{R}, \sum_{(A_i, A_j) \in (\alpha_{\rho-1} \cup \nu'_{\rho-1}) \times (\beta_{\rho-1} \setminus \nu_{\rho-1})} f_{ijk} < b_{ik}\} \quad (18)$$

Namely, starting with  $\rho = 1$ , we define insertion position  $(\alpha_\rho, \beta_\rho)$  from  $(\alpha_{\rho-1}, \beta_{\rho-1})$  by removing from the set of possible successors  $\beta_{\rho-1}$  the set of activities  $\nu_{\rho-1}$  involved in the longest cycles in  $\mathcal{C}_2^1(\alpha_{\rho-1}, \beta_{\rho-1}) \cup \mathcal{C}_3^1(\alpha_{\rho-1}, \beta_{\rho-1})$  and adding to the set of possible successors  $\alpha_{\rho-1}$  a subset of activities  $\nu'_{\rho-1} \subseteq \nu_{\rho-1}$ . This subset contains by definition activities  $A_i \in \nu_{\rho-1}$  such that there is no path in  $G^f$  from  $A_x$  to  $A_i$ . Indeed, the activities belonging to  $\nu_{\rho-1}$  but not to  $\nu'_{\rho-1}$  can not be resource predecessors of  $A_x$  without causing the creation of a positive length cycle. The recursion is stopped before considering  $(\alpha_\Lambda, \beta_\Lambda)$  which would violate resource feasibility condition (9) as stated by the definition of  $\Lambda$  (18).

In figure 9, cases  $\rho = 0$ ,  $\rho = 1$  and  $\rho = 2$  are illustrated by cuts in the resource-flow network.  $(\alpha_1, \beta_1)$  is generated from  $(\alpha_0, \beta_0)$  by removing  $\nu_0 = \{A_1\}$  from  $\beta_0$  and by inserting  $\nu'_0 = \nu_0$  into  $\alpha_0$ .  $(\alpha_2, \beta_2)$  is generated from  $(\alpha_1, \beta_1)$  by removing  $\nu_1 = \{A_9\}$  from  $\beta_1$  and by inserting  $\nu'_1 = \nu_1$  into  $\alpha_1$ . We will illustrate the complete generation process at the end of the paper.

For each so-defined insertion position  $(\alpha_\rho, \beta_\rho)$ , we define a sub-series of  $\lambda_\rho \geq 1$  insertion positions  $(\alpha_\rho^r, \beta_\rho^r)_{r=0, \dots, \lambda_\rho-1}$  defined as follows.

$$\alpha_\rho^r = \alpha_\rho \quad r = 0 \quad (19)$$

$$\alpha_\rho^r = \alpha_\rho^{r-1} \setminus \mu_\rho^{r-1} \quad r = 1, \dots, \lambda_\rho - 1 \quad (20)$$

with

$$\mu_\rho^r = \{A_i \in \alpha_\rho^r \mid \delta_{0i}^f + p_i = \max_{A_a \in \alpha_\rho^r} (\delta_{0a}^f + p_a)\} \quad r = 0, \dots, \lambda_\rho - 1 \quad (21)$$

and

$$\lambda_\rho = \min\{r \geq 1 \mid \exists R_k \in \mathcal{R}, \sum_{(A_i, A_j) \in \alpha_\rho^{r-1} \setminus \mu_\rho^{r-1} \times \beta_\rho} f_{ijk} < b_{ik}\}. \quad (22)$$

Starting with  $(\alpha_\rho^0, \beta_\rho) = (\alpha_\rho, \beta_\rho)$ , we define insertion position  $(\alpha_\rho^r, \beta_\rho)$  from  $(\alpha_\rho^{r-1}, \beta_\rho)$  by removing from the set of possible predecessors  $\alpha_\rho^{r-1}$  the set of activities  $\mu_\rho^{r-1}$  involved in the longest cycles in  $\mathcal{C}_1^1(\alpha_\rho^{r-1}, \beta_\rho) \cup \mathcal{C}_3^1(\alpha_\rho^{r-1}, \beta_\rho)$ . The recursion is stopped before considering  $(\alpha_\rho^{\lambda_\rho}, \beta_\rho)$  which would violate resource feasibility condition (9) as stated by the definition of  $\lambda_\rho$  (22).

In the illustrative example, we have  $(\alpha_0^0, \beta_0) = (\alpha_0, \beta_0) = (\{A_0, A_2, A_3, A_4\}, \{A_1, A_7, A_8, A_9, A_{10}, A_{11}\})$  and  $\mu_0 = \{A_4\}$ . Consequently  $\alpha_0^1$  is generated by removing  $A_4$  from  $\alpha_0^0$ . The generation process of the sub-series of insertion positions is fully illustrated at the end of the paper.

We now focus on the demonstration that the considered insertion positions are feasible and, in addition, dominant for the RCAIP.

**Lemma 3** *If RCAIP<sub>xf</sub> is feasible, each insertion position  $(\alpha_\rho^r, \beta_\rho)$ ,  $\rho = 0, \dots, \Lambda - 1$ ,  $r = 0, \dots, \lambda_\rho - 1$ , is feasible and  $\sum_{\rho=0}^{\Lambda-1} \lambda_\rho \leq n^2$ .*

**Proof** If (RCAIP<sub>xf</sub>) has a solution, then  $(\alpha_0, \beta_0)$  is feasible as stated through Lemma 2.

We first show with a recursion argument that if  $(\alpha_\rho, \beta_\rho)$  is feasible with  $\rho \geq 0$  then so is  $(\alpha_{\rho+1}, \beta_{\rho+1})$  for  $\rho+1 < \Lambda$ . Feasibility is checked by examining  $\mathcal{C}_1^2$ ,  $\mathcal{C}_2^2$  and  $\mathcal{C}_2^3$  cycles, as stated by Theorem 4. Suppose a positive length type  $\mathcal{C}_1^2$  cycle is generated by adding  $\nu'_\rho$  to  $\alpha_\rho$ . Since  $(\alpha_\rho, \beta_\rho)$  is feasible we have  $\delta_{xa}^f + p_a \leq 0$ ,  $\forall A_a \in \alpha_\rho$ . To obtain a positive length type  $\mathcal{C}_1^2$  cycle we must have an activity  $A_a \in \nu'$  such that  $\delta_{xa}^f + p_a > 0$ . This yields a contradiction with the definition of  $\nu'$  stating that for each  $A_a \in \nu'$ ,  $\delta_{xa}^f = -\infty$ . No positive length type  $\mathcal{C}_2^2$  cycle can be generated since  $(\alpha_\rho, \beta_\rho)$  is feasible and  $\beta_{\rho+1} \subset \beta_\rho$ . Suppose now that a positive length type  $\mathcal{C}_2^3$  cycle is generated. Since  $(\alpha_\rho, \beta_\rho)$  is feasible we have  $\forall A_a \in \alpha_\rho$ ,  $\forall A_b \in \beta_\rho$ ,  $\delta_{ba}^f + p_a + p_x \leq 0$ . The only way to generate a type  $\mathcal{C}_2^3$  cycle is to find  $A_b \in \beta \setminus \nu$  and  $A_a \in \nu'$  such that  $\delta_{ba}^f + p_a + p_x > 0$ . Suppose that for such two activities we have  $\delta_{ba}^f \geq 0$ . In this case we have  $\delta_{b(n+1)}^f \geq \delta_{ba}^f + \delta_{a(n+1)}^f \geq \delta_{a(n+1)}^f$  which implies that  $A_b \in \nu$ , a contradiction. Consequently,  $\forall A_b \in \beta \setminus \nu$  and  $\forall A_a \in \nu'$ ,  $\delta_{ba}^f = -\infty$  and no type  $\mathcal{C}_2^3$  cycle can be generated. The only feasibility condition that can be violated is the resource condition (9), which is impossible by definition of  $\Lambda$  (18).

Second, we show that given that  $(\alpha_\rho, \beta_\rho)$  is feasible, all considered  $(\alpha_\rho^r, \beta_\rho)$  are also feasible for  $r \leq \lambda_\rho - 1$ . This is obviously verified for  $r = 0$ , since  $\alpha_\rho^r = \alpha_\rho$ . Suppose now that  $(\alpha_\rho^r, \beta_\rho)$  is feasible. Removing  $\mu_\rho^r$  from  $\alpha_\rho^r$  cannot increase the length of any cycle. Consequently  $(\alpha, \beta_\rho)$  with  $\alpha \subseteq \alpha_\rho^r$  is feasible if and only if there is a sufficient amount of flow sent from  $\alpha$  to  $\beta_0$ , which is precisely the condition ensured by the definition of  $\lambda_\rho$  (22).

We now provide a bound on the number of considered insertion positions. For each insertion position  $(\alpha_\rho, \beta_\rho)$ , there is a finite number of sub-positions obtained by removing set  $\mu_\rho^r$ . Since  $\mu_\rho^r$  includes at least one activity there can be no more than  $n$  insertion positions  $(\alpha_\rho^r, \beta_\rho)$  for a fixed  $\rho$  (i.e.  $\lambda_\rho \leq n$ ). Also, set  $\nu_\rho$  contains at least one activity, so the repeated removal of  $\nu_\rho$  from  $\beta_\rho$  yields at most  $n$  insertion positions  $(\alpha_\rho, \beta_\rho)$  (i.e.  $\Lambda \leq n$ ). Consequently there are no more than  $n^2$  considered insertion positions.  $\square$

We now present a property of series  $(\alpha_\rho, \beta_\rho)_{\rho=0, \dots, \Lambda-1}$  that will allow us to show the series of insertion positions  $(\alpha_\rho^r, \beta_\rho)_{\rho=0, \dots, \Lambda-1, r=0, \dots, \lambda_\rho-1}$  is dominant, i.e. that it includes an optimal insertion position.

**Lemma 4** *Each insertion position  $(\alpha_\rho, \beta_\rho)$ ,  $\rho = 0, \dots, \Lambda - 1$ , is such that among insertion positions  $(\alpha, \beta)$  verifying  $\beta \subseteq \beta_\rho$  and  $\beta \cap \nu_\rho \neq \emptyset$  the insertion positions that verifies in addition  $\alpha \subseteq \alpha_\rho$  are dominant.*

### Proof

As a preliminary remark, given an insertion position  $(\alpha_\rho, \beta_\rho)$  and any other insertion position  $(\alpha, \beta)$  we have  $\alpha \subset (\alpha_\rho \cup \beta_\rho)$ . Indeed, any activity  $A_i$  not belonging to  $\alpha_\rho \cup \beta_\rho$  belongs either to  $\gamma_0$  or to a set  $\beta_{\rho'}$  with  $\rho' < \rho$  and to its subset  $\nu_{\rho'} \setminus \nu'_{\rho'}$ . In the first case, we have a path of length 0 from  $A_i$  to  $A_x$  and so  $A_i$  cannot be a resource predecessor of  $A_x$ . In the second case, the activity  $A_i$  is such that  $\delta_{xi}^f > 0$  (so it is not added to  $\alpha_{\rho'+1}$ ) and so it cannot be neither a resource predecessor of  $A_x$ . It follows that if  $(\alpha, \beta)$  is a feasible insertion position,  $\alpha \subseteq \alpha_\rho$  if and only if  $\alpha \cap \beta_\rho = \emptyset$ .

Thus, we partition the set of insertion positions verifying  $\beta \subseteq \beta_\rho$  and  $\beta \cap \nu_\rho \neq \emptyset$  into two subsets defined by  $\alpha \cap \beta_\rho = \emptyset$  (i) for the first one and  $\alpha \cap \beta_\rho \neq \emptyset$  (ii) for the second one.

Consider the (ii) sub-case, i.e. an improving insertion position  $(\alpha, \beta)$  such that  $\beta \cap \nu_\rho \neq \emptyset$  and  $\alpha \cap \beta_\rho \neq \emptyset$ .

Let  $\Omega = (\beta_\rho \cap \alpha) \cup \{A_i \in \beta_\rho \mid \exists A_j \in \beta_\rho \cap \alpha, \delta_{ijk}^f \geq 0\}$ .  $\Omega$  includes the activities of  $\alpha \cap \beta_\rho$  and all their ancestors in  $\beta_\rho$ . Note  $\Omega$  is not empty since  $\alpha \cap \beta_\rho \neq \emptyset$ . Let  $\Omega^- = \{A_i \in V \setminus \Omega \mid \exists A_j \in \Omega \text{ and } R_k \in \mathcal{R}, f_{ijk} > 0\}$  denotes the set of activities outside  $\Omega$  that send a non zero amount of flow to activities of  $\Omega$  and  $\Omega^+ = \{A_i \in V \setminus \Omega \mid \exists A_j \in \Omega \text{ and } R_k \in \mathcal{R}, f_{jik} > 0\}$  denotes the set of activities outside  $\Omega$  that receive a non zero amount of flow from the activities of  $\Omega$ .

We now prove by contradiction that  $\Omega^- \subseteq \alpha_\rho$ . Suppose this is not the case and there is an activity  $A_i \in \Omega^-$ ,  $A_i \notin \alpha_\rho$ . We have either  $A_i \in \beta_\rho$ ,  $A_i \in \gamma_0$  or  $A_i \in \beta_{\rho'}$  with  $\rho' < \rho$  and to its subset  $\nu_{\rho'} \setminus \nu'_{\rho'}$ . If  $A_i \in \beta_\rho$ , by definition

of  $\Omega$  and  $\Omega'$ ,  $A_i \in \Omega$  and  $A_i \notin \Omega'$ , a contradiction. If  $A_i \in \gamma_0$ , by definition of  $\gamma_0$  there is a path from  $A_x$  to  $A_i$ . Since, by definition of  $\omega'$  we have also a path from  $A_i$  to an activity  $A_j \in \alpha \cap \beta_\rho$  inducing a path from  $A_x$  to  $A_j$ , a contradiction with  $A_j \in \alpha$ . If  $A_i \in \beta_{\rho'}$  with  $\rho' < \rho$  and to its subset  $\nu_{\rho'} \setminus \nu_\rho$ ,  $A_i$  is such that  $\delta_{xi}^f > 0$  (so it is not added to  $\alpha_{\rho'+1}$ ) which yields the same contradiction as for the preceding case. It follows that  $\Omega^- \subseteq \alpha_\rho$ .

Consider the activity sets  $\alpha' = (\alpha \cap \alpha_\rho) \cup \Omega^-$  and  $\beta' = \beta_\rho$ . Since  $\Omega^- \subseteq \alpha_\rho$ , we have  $\alpha' \subseteq \alpha_\rho$  and  $\alpha' \cap \beta_\rho = \emptyset$ . If  $(\alpha', \beta')$  is a valid insertion position, then it belongs to type (i).

We now prove that, if  $(\alpha, \beta)$  is a feasible insertion position,  $(\alpha', \beta')$  is a feasible insertion position. Recall that  $\alpha$  is partitioned into two subsets  $\alpha \cap \alpha_\rho$  and  $\alpha \cap \beta_\rho$  which yields:

$$\sum_{A_i \in \alpha, A_j \in \beta} f_{ijk} = \sum_{A_i \in \alpha \cap \alpha_\rho, A_j \in \beta} f_{ijk} + \sum_{A_i \in \alpha \cap \beta_\rho, A_j \in \beta} f_{ijk} \quad (23)$$

We have  $\Omega^+ \subseteq \beta_\rho$  since there is a path from any activity of  $\alpha \cap \beta_\rho$  to any activity of  $\Omega^+$ . Since  $\alpha \cap \beta_\rho \subseteq \Omega$  we have  $\sum_{A_i \in \alpha \cap \beta_\rho, A_j \in \beta} f_{ijk} \leq \sum_{A_i \in \Omega, A_j \in \beta} f_{ijk}$ . Since in addition  $\Omega^+$  includes all activities receiving flow from  $\Omega$  this yields  $\sum_{A_i \in \Omega, A_j \in \beta} f_{ijk} \leq \sum_{A_i \in \Omega, A_j \in \Omega^+} f_{ijk}$ . Inserting this result into (23) we obtain:

$$\sum_{A_i \in \alpha, A_j \in \beta} f_{ijk} \leq \sum_{A_i \in \alpha \cap \alpha_\rho, A_j \in \beta} f_{ijk} + \sum_{A_i \in \Omega, A_j \in \Omega^+} f_{ijk} \quad (24)$$

The flow conservation implies  $\sum_{\forall A_i \in \Omega^-, A_j \in \Omega} f_{ijk} = \sum_{\forall A_i \in \Omega, A_j \in \Omega^+} f_{ijk}$  which immediately yields

$$\sum_{A_i \in \alpha, A_j \in \beta} f_{ijk} \leq \sum_{A_i \in \alpha \cap \alpha_\rho, A_j \in \beta} f_{ijk} + \sum_{\forall A_i \in \Omega^-, A_j \in \Omega} f_{ijk} \quad (25)$$

On the other hand we have  $\alpha \cap \alpha_\rho \subseteq \alpha'$  and  $\Omega^- \subseteq \alpha'$ . We obtain:

$$\sum_{A_i \in \alpha, A_j \in \beta} f_{ijk} \leq \sum_{A_i \in \alpha', A_j \in \beta} f_{ijk} + \sum_{\forall A_i \in \alpha', A_j \in \Omega} f_{ijk} \quad (26)$$

We remark that  $\beta \cap \Omega = \emptyset$  since otherwise there would be a path from an activity of  $\beta$  to an activity of  $\alpha \cap \beta_\rho$ , which contradicts the feasibility of  $(\alpha, \beta)$ . Furthermore  $\beta \subseteq \beta_\rho$  and  $\Omega \subseteq \beta_\rho$ .  $\beta$  and  $\Omega$  are disjoint subsets of  $\beta_\rho$  which yields

$$\sum_{A_i \in \alpha, A_j \in \beta} f_{ijk} \leq \sum_{A_i \in \alpha', A_j \in \beta_\rho} f_{ijk} \quad (27)$$



Equation 27 show that  $(\alpha', \beta_\rho)$  is an insertion position. Furthermore, since  $\alpha' \subseteq \alpha_\rho$  and  $(\alpha_\rho, \beta_\rho)$  is feasible,  $(\alpha', \beta_\rho)$  is feasible.

We finally show that  $(\alpha', \beta_\rho)$  cannot be worse than  $(\alpha, \beta)$ . Since by hypothesis we have  $\nu_\rho \cap \beta \neq \emptyset$ , it follows that  $\max_{A_b \in \beta} \delta_{b(n+1)}^f = \max_{A_b \in \beta_\rho} \delta_{b(n+1)}^f$ . Since for each activity of  $A_i \in \Omega^-$  we have a path of length at least  $p_i$  to an activity of  $\alpha \cap \beta_\rho$ . We have  $\max_{A_a \in \alpha \cap \beta_\rho} (\delta_{0a}^f + p_a) \geq \max_{A_a \in \Omega^-} (\delta_{0a}^f + p_a)$ . Since  $\alpha' = \alpha \setminus (\alpha \cap \beta_\rho) \cup \Omega^-$ , it comes  $\delta_{0(n+1)}^{f\alpha\beta} \geq \delta_{0(n+1)}^{f\alpha'\beta_0}$ . Consequently, we have only to consider improving insertion positions of the (i) category; i.e. such that  $\alpha \subseteq \alpha_\rho$ .  $\square$

This property can be simply illustrated by considering the insertion position  $(\alpha_0, \beta_0)$  from the illustrative example. We may try to find an insertion position  $(\alpha, \beta)$  improving  $(\alpha_0, \beta_0)$  while keeping  $A_1$ , which belongs to  $\nu_0$ , as a possible resource successor, i.e. in  $\beta$ . Now, Lemma 4 indicates that, under this condition, it is not useful to consider as possible resource predecessor any activity of  $\beta_0$ . For instance, given that  $A_1$  is a resource successor, it is better to select  $A_3 \in \alpha_0$  as a resource predecessor than to select  $A_9 \in \beta_0$  (while the latter choice would be feasible).

**Theorem 6** *If (RCAIP<sub>xf</sub>) is feasible, the insertion position  $(\alpha, \beta)$  of the series  $(\alpha_\rho^r, \beta_\rho)_{\rho=0, \dots, \Lambda-1, r=0, \dots, \lambda_\rho-1}$  having the minimal makespan  $\delta_{0(n+1)}^{f\alpha\beta}$  is optimal.*

## Proof

We show that the set of evaluated insertion positions includes an optimal one. Let  $(\alpha_\rho^*, \beta_\rho^*)$  denote the insertion position of minimal makespan among  $(\alpha_{\rho'}^r, \beta_{\rho'})_{\rho'=0, \dots, \rho, r=0, \dots, \lambda_{\rho'}}$ .

We now establish the following property by recursion:  $(\alpha_\rho^*, \beta_\rho^*)$  dominates any insertion position  $(\alpha, \beta)$  such that  $\beta \cap \nu_\rho \neq \emptyset$ .

We first show the property holds for  $\rho = 0$ . Consider an insertion position  $(\alpha, \beta)$  such that  $\beta \cap \nu_0 \neq \emptyset$ . we have  $\beta \subseteq \beta_0$  since no activity of  $\alpha_0 \cup \lambda_0$  can be a resource successor of  $A_x$ . In this case, Lemma 4 states that insertion positions such that  $\alpha \subseteq \alpha_0$  are dominant. Since  $\beta \cap \nu_0 \neq \emptyset$ , the only way to have an insertion position better than  $(\alpha_0, \beta_0)$  is to remove  $\mu_0^0$  from  $\alpha_0$ , then  $\mu_0^1$ , etc. This yields series  $(\alpha_0^r, \beta_0)_r$  until the amount of flow sent from  $\alpha_0^r$  to  $\beta_0$  would become insufficient, which is reached for  $r = \lambda_0 - 1$ .

Let us assume the invariant is verified for  $\tau = 0, \dots, \rho - 1$  with  $\rho \geq 1$ . To find an insertion position  $(\alpha, \beta)$  better than  $(\alpha_\rho^*, \beta_\rho^*)$  with  $\nu_\rho \cap \beta \neq \emptyset$ , we consider the following possible disjoint possibilities for  $\beta$ : (a)  $\beta \cap \alpha_\rho \neq \emptyset$  and

(b)  $\beta \cap \alpha_\rho = \emptyset$ . Consider first case (a) and let  $A_i \in \beta \cap \alpha_\rho$ . Note we cannot have  $A_i \in \alpha_0$  since  $(\alpha, \beta)$  is feasible and no activity of  $\alpha_0$  can be a resource successor of  $A_x$ . Consequently, there exists  $0 \leq \tau < \rho$  with  $A_i \in \nu_\tau$  and, by recursion,  $(\alpha, \beta)$  is dominated by the insertion position  $(\alpha_\tau^*, \beta_\tau^*)$  already found at step  $\tau$ . So, we have only to consider an improving insertion position  $(\alpha, \beta)$  such that  $\beta \subseteq \beta_\rho$ . In this case, Lemma 4 states that insertion positions such that  $\alpha \subseteq \alpha_\rho$  are dominant. Since  $\beta \cap \nu_\rho \neq \emptyset$ , the only way to have an insertion position better than  $(\alpha_\rho, \beta_\rho)$  is to remove  $\mu_\rho^0$  from  $\alpha_\rho$ , then  $\mu_\rho^1$ , etc. This yields series  $(\alpha_\rho^r, \beta_\rho)_r$  until the amount of flow sent from  $\alpha_\rho^r$  to  $\beta_\rho$  would become insufficient, which is reached for  $r = \lambda_\rho - 1$ .

Thus, we have established the property that the set of considered insertion position dominates any insertion position  $(\alpha, \beta)$  such that there exists  $\rho \in \{0, \dots, \Lambda - 1\}$ ,  $\beta \cap \nu_\rho \neq \emptyset$ . Consider  $\beta_0$  the set of possible resource successors for activity  $A_x$ . We have  $\beta_0 = \nu_0 \cup \nu_1 \cup \dots \cup \nu_{\Lambda-1} \cup \beta_\Lambda$ , where  $\beta_\Lambda$  is the set of activities that remains as possible resource successors but without enough flow from  $\alpha_\Lambda$  to  $\beta_\Lambda$ . It follows that to get a sufficient amount of flow, we must transfer at least one activity from  $\alpha_\Lambda$  to  $\beta_\Lambda$ . Such activity must belong to one set  $\nu_\tau$  with  $0 \leq \tau < \Lambda$  and is consequently dominated. Since any possible resource successor  $A_i$  belong to  $\beta_0$  any insertion position  $(\alpha, \beta)$  with  $A_i \in \beta$  is dominated by an insertion position of series  $(\alpha_\rho^r, \beta_\rho)_{\rho=0, \dots, \Lambda-1, r=0, \dots, \lambda_\rho-1}$   $\square$

Algorithm 1 generates all the so-defined dominant insertion positions and stores the one of minimal makespan through steps 1-1. Note that steps 1-1 correspond to the generation of the sub-series of insertion positions  $(\alpha_\rho^r, \beta_\rho)$ . A variable named  $\alpha$  is used to avoid using indices  $r$  and  $\rho$ . The remaining steps 1-1 actually update flow  $f$  (actually inserting activity  $A_x$ ) from the insertion position  $(\alpha^*, \beta^*)$  following the principles presented in section 3. The algorithm can be implemented in  $O(n^2m)$  time. Note that the preliminary computations of values  $\delta_{0i}^f + p_i$  and  $\delta_{i(n+1)}^f$ , for all activities  $A_i$ , can be done through the Bellman-Ford algorithm in  $O(n|E \cup E(f)|)$ .

Finally we give the complete behaviour of the algorithm on the illustrative example. For  $\rho = 0$ , starting with  $(\alpha_0, \beta_0)$ , Table 4 displays the various insertion positions that are explored assuming that  $\beta_0$  is kept unchanged, as well as their evaluation. The insertion position  $(\alpha_0, \beta_0)$  is feasible and leads to

$$\delta_{0(n+1)}^{f\alpha\beta} = \max\left(\delta_{0(n+1)}^f, L_1^1(0, \alpha_0, \beta_0), L_2^1(0, \alpha_0, \beta_0), L_3^1(0, \alpha_0, \beta_0)\right) = 15.$$

This value can be decreased to 14 by removing  $A_4$  from  $\alpha$ . At the end of this phase, since only  $A_0$  still belongs to  $\alpha$ , the algorithm considers the case  $\rho = 1$ , by transferring the activities belonging to  $\nu'_0 = \nu_0 = \{A_1\}$  from  $\beta_0$  to  $\alpha_0$ .

Case  $\rho = 1$  is described on Table 5. We know that  $\alpha_1 = \{A_0, A_1, A_2, A_3, A_4\}$  and  $\beta_1 = \{A_7, A_8, A_9, A_{10}, A_{11}\}$ . For this case, three insertion positions can be

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Algorithm 1. OPTINSERT( $x, f$ ) insert  $A_x$  into flow  $f$  minimizing the makespan

- 1:  $\gamma_0 \leftarrow \{A_i \in \mathcal{A} \mid \delta_{ix}^f = 0 \text{ and } \delta_{xi}^f = 0\}$ ;
- 2: **if**  $\sum_{A_i \in \gamma_0} b_{ik} + b_{xk} \leq B_k \quad \forall R_k \in \mathcal{R}$  **then**
- 3:    $\alpha_0 \leftarrow \{A_i \in V \setminus \gamma_0 \mid \delta_{ix}^f \geq 0\}$ ;  $\beta_0 \leftarrow \{V \setminus (\gamma_0 \cup \alpha_0)\}$ ;
- 4:    $(\alpha^*, \beta^*) \leftarrow (\alpha_0, \beta_0)$ ;  $\rho \leftarrow 0$ ;
- 5:   **repeat**
- 6:      $\alpha \leftarrow \alpha_\rho$ ;
- 7:     **while**  $\sum_{(A_i, A_j) \in \alpha \times \beta_\rho} f_{ijk} \geq b_{xk} \quad \forall R_k \in \mathcal{R}$  **do**
- 8:       **if**  $\delta_{0(n+1)}^{f\alpha\beta_\rho} < \delta_{0(n+1)}^{f\alpha^*\beta^*}$  **then**
- 9:          $(\alpha^*, \beta^*) \leftarrow (\alpha, \beta_\rho)$ ;
- 10:       **end if**
- 11:        $\mu \leftarrow \{A_i \in \alpha \mid \delta_{0i}^f + p_i = \max_{A_a \in \alpha} (\delta_{0a}^f + p_a)\}$ ;
- 12:        $\alpha \leftarrow \alpha \setminus \mu$ ;
- 13:     **end while**
- 14:      $\nu_\rho \leftarrow \{A_i \in \beta_\rho \mid \delta_{i(n+1)}^f = \max_{A_b \in \beta_\rho} (\delta_{b(n+1)}^f)\}$ ;
- 15:      $\nu'_\rho \leftarrow \{A_i \in \nu_\rho \mid \delta_{xi}^f = -\infty\}$ ;
- 16:      $\alpha_{\rho+1} \leftarrow \alpha_\rho \cup \nu'_\rho$ ;  $\beta_{\rho+1} \leftarrow \beta_\rho \setminus \nu_\rho$ ;  $\rho \leftarrow \rho + 1$ ;
- 17:   **until**  $\exists R_k \in \mathcal{R}, \sum_{(A_i, A_j) \in \alpha_\rho \times \beta_\rho} f_{ijk} < b_{ik}$ ;
- 18:   **for**  $R_k \in \mathcal{R}$  **do**
- 19:      $z_k \leftarrow b_{xk}$ ;
- 20:     **for**  $A_i \in \alpha^*$  **do**
- 21:       **for**  $A_j \in \beta^*$  **do**
- 22:          $\phi = \min(f_{ijk}, z_k)$ ;  $z_k \leftarrow z_k - \phi$ ;
- 23:          $f_{ijk} \leftarrow f_{ijk} - \phi$ ,  $f_{ixk} \leftarrow f_{ixk} + \phi$ ;  $f_{xjk} \leftarrow f_{xjk} + \phi$ ;
- 24:       **end for**
- 25:     **end for**
- 26:   **end for**
- 27: **else**
- 28:   There is no solution to (RCAIP $_x f$ );
- 29: **end if**

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$\alpha$	$\sum_{(A_i, A_j) \in \alpha \times \beta_\rho} f_{ijk}$	$L_1^1(0, \alpha, \beta_\rho)$	$L_2^1(0, \alpha, \beta_\rho)$	$L_3^1(0, \alpha, \beta_\rho)$	$\mu$
$\{A_0, A_2, A_3, A_4\}$	(5,3)	10	14	15	$\{A_4\}$
$\{A_0, A_2, A_3\}$	(3,3)	9	14	14	$\{A_2, A_3\}$
$\{A_0\}$	(1,3)	8	14	13	$\emptyset$

Table 4

$\rho = 0$ ,  $\alpha_0 = \{A_0, A_2, A_3, A_4\}$  and  $\beta_0 = \{A_1, A_7, A_8, A_9, A_{10}, A_{11}\}$

considered. The best one (according to the greatest  $L_i^1(0, \alpha, \beta)$  values) is found for  $\alpha = \{A_0, A_2, A_3\}$ . It allows for inserting  $A_5$  in the partial schedule while keeping the  $C_{\max}$  value unchanged. Note that at this point, since an insertion

position that keeps the project duration unchanged has been found, it is not necessary any more to explore further insertion positions. Anyway, in order to explain all the features of the algorithm, we detail the other stages.

$\alpha$	$\sum_{(A_i, A_j) \in \alpha \times \beta_\rho} f_{ijk}$	$L_1^1(0, \alpha, \beta_\rho)$	$L_2^1(0, \alpha, \beta_\rho)$	$L_3^1(0, \alpha, \beta_\rho)$	$\mu$
$\{A_0, A_1, A_2, A_3, A_4\}$	(5,3)	15	10	16	$\{A_1\}$
$\{A_0, A_2, A_3, A_4\}$	(3,2)	10	10	11	$\{A_4\}$
$\{A_0, A_2, A_3\}$	(1,2)	9	10	10	$\{A_2, A_3\}$
$\{A_0\}$	(0,2)	-	-	-	-

Table 5

$\rho = 1$ ,  $\alpha_1 = \{A_0, A_1, A_2, A_3, A_4\}$  and  $\beta_1 = \{A_7, A_8, A_9, A_{10}, A_{11}\}$

Case  $\rho = 2$  is considered next: the activities belonging to  $\nu'_1 = \nu_1 = \{A_9\}$  from  $\beta_1$  to  $\alpha_1$ . It is described on Table 6. We have  $\alpha_2 = \{A_0, A_1, A_2, A_3, A_4, A_9\}$  and  $\beta_2 = \{A_7, A_8, A_{10}, A_{11}\}$ . For this case, two insertion positions are considered but, as one can see on the table, they are both worse than the best known one. At this step, it is not any longer possible to transfer any activity from  $\beta$  to  $\alpha$  because there is a path from  $A_5$  to each activity  $A_7, A_8, A_{10}$  and  $A_{11}$  remaining in  $\beta$  ( $\nu'_\rho = \emptyset \forall \rho \geq 2$ ). For this reason, these activities have been tinted gray on the resource network of Figure 9. Therefore, these activities can only be removed from  $\beta$  until the remaining flow becomes lower than the resource demand. According to the  $\delta_{i(n+1)}$  values,  $A_7$  is removed first (see Table 7), then  $A_{10}$  (see Table 8) and lastly  $A_8$  (see Table 9). As one can see on Tables 7-9, no insertion positions are found that is better than the best one already known at this point (i.e.  $\alpha^* = \{A_0, A_2, A_3\}$  and  $\beta^* = \{A_7, A_8, A_9, A_{10}, A_{11}\}$ ). Figure 11 displays the obtained schedule after  $A_5$  insertion. Note that only the resource flows coming from  $A_0$  and  $A_3$  are used (whereas  $A_2$  also belongs to  $\alpha^*$ ) since  $f_{2jk} = 0 \forall j \in \beta^*$ .

$\alpha$	$\sum_{(A_i, A_j) \in \alpha \times \beta_\rho} f_{ijk}$	$L_1^1(0, \alpha, \beta_\rho)$	$L_2^1(0, \alpha, \beta_\rho)$	$L_3^1(0, \alpha, \beta_\rho)$	$\mu$
$\{A_0, A_1, A_2, A_3, A_4, A_9\}$	(5,3)	15	9	15	$\{A_1\}$
$\{A_0, A_2, A_3, A_4, A_9\}$	(3,2)	13	9	13	$\{A_9\}$
$\{A_0, A_2, A_3, A_4\}$	(1,0)	-	-	-	-

Table 6

$\rho = 2$ ,  $\alpha_2 = \{A_0, A_1, A_2, A_3, A_4, A_9\}$  and  $\beta_2 = \{A_7, A_8, A_{10}, A_{11}\}$

$\alpha$	$\sum_{(A_i, A_j) \in \alpha \times \beta_\rho} f_{ijk}$	$L_1^1(0, \alpha, \beta_\rho)$	$L_2^1(0, \alpha, \beta_\rho)$	$L_3^1(0, \alpha, \beta_\rho)$	$\mu$
$\{A_0, A_1, A_2, A_3, A_4, A_9\}$	(5,3)	15	9	14	$\{A_1\}$
$\{A_0, A_2, A_3, A_4, A_9\}$	(3,2)	13	8	12	$\{A_9\}$
$\{A_0, A_2, A_3, A_4\}$	(1,0)	-	-	-	-

Table 7

$\rho = 3$ ,  $\alpha_3 = \{A_0, A_1, A_2, A_3, A_4, A_9\}$  and  $\beta_3 = \{A_8, A_{10}, A_{11}\}$

$\alpha$	$\sum_{(A_i, A_j) \in \alpha \times \beta_\rho} f_{ijk}$	$L_1^1(0, \alpha, \beta_\rho)$	$L_2^1(0, \alpha, \beta_\rho)$	$L_3^1(0, \alpha, \beta_\rho)$	$\mu$
$\{A_0, A_1, A_2, A_3, A_4, A_9\}$	(4,2)	15	7	13	$\{A_1\}$
$\{A_0, A_2, A_3, A_4, A_9\}$	(3,2)	13	7	11	$\{A_9\}$
$\{A_0, A_2, A_3, A_4\}$	(1,0)	-	-	-	-

Table 8

$\rho = 4$ ,  $\alpha_4 = \{0, 1, 2, 3, 4, 9\}$  and  $\beta_4 = \{8, 11\}$

$\alpha$	$\sum_{(A_i, A_j) \in \alpha \times \beta_\rho} f_{ijk}$	$L_1^1(0, \alpha, \beta_\rho)$	$L_2^1(0, \alpha, \beta_\rho)$	$L_3^1(0, \alpha, \beta_\rho)$	$\mu$
$\{A_0, A_1, A_2, A_3, A_4, A_9\}$	(3,2)	15	4	10	$\{A_1\}$
$\{A_0, A_2, A_3, A_4, A_9\}$	(2,2)	13	4	8	$\{A_9\}$
$\{A_0, A_2, A_3, A_4\}$	(1,0)	-	-	-	-

Table 9

$\rho = 5$ ,  $\alpha_5 = \{A_0, A_1, A_2, A_3, A_4, A_9\}$  and  $\beta_5 = \{A_{11}\}$

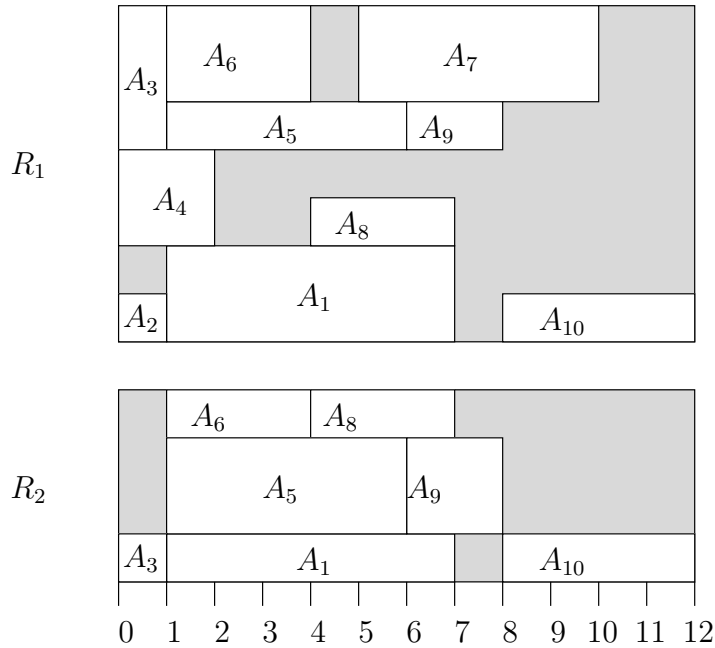


Fig. 11. Complete solution after  $A_5$  insertion

## 7 Conclusion

In this paper, we considered the problem of inserting a single activity  $A_x$  inside an existing partial schedule while preserving its structure in presence of minimum and maximum time lags between activities. The objective is to minimize the project duration increase. We stated that the problem consists of selecting the best ordered pair of sets  $(\alpha^*, \beta^*)$  inside the resource flow network associated with the existing partial schedule, so that both the flow taken from  $\alpha^*$  and sent back to  $\beta^*$  satisfies the resource demand of  $A_x$  and the insertion does not cause any temporal inconsistency. We have shown, given a pair  $(\alpha, \beta)$ , how the latter constraint can easily be checked by considering three categories of elementary cycles.

Whereas the insertion problem is polynomially solvable for the standard RCPSP (where only precedence constraints are taken into account), we showed that the introduction of minimum and maximum time lags makes the problem NP-hard. Nevertheless, when only minimum time lags are considered and when activity durations are strictly positive, the problem turns back polynomially solvable and we proposed an algorithm to solve it.

Once again, we observed that dealing with maximum time lag constraints is difficult. It was already known that finding a resource-feasible schedule that respects both the minimum and maximum time lags is NP-hard. We demonstrated here that, though apparently easier, even the problem of inserting a single activity in an already existing resource- and time-feasible schedule is NP-hard. Therefore, it might appear relevant in some situations to relax maximum time lags, intending to find a schedule which minimizes the maximum time lag violation (a maximum makespan value being preliminary set). Indeed, in many schedule environments, maximum time lags correspond to user preferences and hence it makes sense to produce a schedule that does not fully respect them. In this case, one can also define priorities in order to define a hierarchy among maximum time lags. In this case, a possible objective function would be the minimization of the weighted sum of the time lag violations.

Another way of tackling insertion problems in presence of maximum time lags, would be to define another structure of the partial schedule to turn the RCAIP into a polynomially solvable problem. This remains a critical issue for designing efficient local search methods for the RCPSP/max based on individual activity reinsertions.

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