



HAL
open science

\mathcal{T} -class algorithms for pseudocontractions and κ -strict pseudocontractions in Hilbert spaces

Jean-Philippe Chancelier

► **To cite this version:**

Jean-Philippe Chancelier. \mathcal{T} -class algorithms for pseudocontractions and κ -strict pseudocontractions in Hilbert spaces. 2007. hal-00193621

HAL Id: hal-00193621

<https://hal.science/hal-00193621>

Preprint submitted on 4 Dec 2007

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

\mathcal{T} -class algorithms for pseudocontractions and κ -strict pseudocontractions in Hilbert spaces

Jean-Philippe Chancelier

December 4, 2007

Abstract

In this paper we study iterative algorithms for finding a common element of the set of fixed points of κ -strict pseudocontractions or finding a solution of a variational inequality problem for a monotone, Lipschitz continuous mapping. The last problem being related to finding fixed points of pseudocontractions. These algorithms were already studied in [1] and [9] but our aim here is to provide the links between these know algorithms and the general framework of \mathcal{T} -class algorithms studied in [3].

1 Introduction

Let C be a closed convex subset of a Hilbert space \mathcal{H} and P_C be the metric projection from \mathcal{H} onto C . A mapping $Q : C \mapsto C$ is said to be a *strict pseudocontraction* if there exists a constant $0 \leq \kappa < 1$ such that :

$$\|Qx - Qy\|^2 \leq \|x - y\|^2 + \kappa\|(I - Q)x - (I - Q)y\|^2, \quad (1)$$

for all $x, y \in C$. A mapping Q for which (1) holds is also called a κ -strict pseudocontraction. As pointed out in [1] iterative methods for finding a common element of the set of fixed points of strict pseudocontractions are far less developed than iterative methods for nonexpansive mappings ($\kappa = 0$) [2, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15]. We will, in section 2 of this article, consider the algorithm 1 studied in [1] and we will show that this algorithm can be viewed as a \mathcal{T} -class algorithm as defined and studied in [3].

Section 3 is devoted to the case $\kappa = 1$ for which previous algorithm cannot be used. A mapping A for which (1) holds with $\kappa = 1$ is called *pseudocontractive*. We will see that *pseudocontractive* mappings are related to monotone Lipschitz continuous mappings. A mapping $A : C \mapsto \mathcal{H}$ is called *monotone* if

$$\langle Au - Av, u - v \rangle \geq 0 \quad \text{for all } (u, v) \in C^2.$$

A is called k -Lipschitz continuous if there exists a positive real number k such that

$$\|Au - Av\| \leq k\|u - v\| \quad \text{for all } (u, v) \in C^2.$$

Let the mapping $A : C \mapsto \mathcal{H}$ be monotone and Lipschitz continuous. The variational inequality problem is to find a $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0 \quad \text{for all } v \in C.$$

The set of solutions of the variational inequality problem is denoted by $VI(C, A)$.

Assume that a mapping $Q : C \mapsto C$ is pseudocontractive and k -Lipschitz-continuous then the mapping $A = I - Q$ is monotone and $(k + 1)$ -Lipschitz-continuous and moreover $Fix(Q) = VI(C, A)$ [9, Theorem 4.5] where $Fix(Q)$ is the set of fixed points of Q , that is

$$Fix(Q) \stackrel{\text{def}}{=} \{x \in C : Qx = x\} \quad (2)$$

Thus, to cover the case $\kappa = 1$, algorithms which aims at computing $P_{VI(C,A)}x$ for a monotone and k -Lipschitz-continuous mapping A are investigated. We will, in section 3 mainly use results from [9] to prove that the general algorithm that they use can be rephrased in a slightly extended \mathcal{T} -class algorithm framework.

2 \mathcal{T} -class iterative algorithm for a sequence of κ -strict pseudocontractions

Let $(Q_n)_{n \geq 0}$ be a sequence of κ -strict pseudocontractions, $\kappa \in [0, 1)$ and $(\alpha_n)_{n \geq 0}$ a sequence of real numbers chosen so that $\alpha_n \in (\kappa, 1)$. We consider as in [1] the following algorithm :

Algorithm 1 *Given $x_0 \in C$, we consider the sequence $(x_n)_{n \geq 0}$ generated by the following algorithm :*

$$\begin{aligned} y_n &= \alpha_n x_n + (1 - \alpha_n) Q_n x_n, \\ C_n &\stackrel{\text{def}}{=} \left\{ z \in C \mid \|y_n - z\|^2 \leq \|x_n - z\|^2 - (1 - \alpha_n)(\alpha_n - \kappa) \|x_n - Q_n x_n\|^2 \right\}, \\ D_n &\stackrel{\text{def}}{=} \{z \in C \mid \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{(C_n \cap D_n)} x_0. \end{aligned}$$

We will show that this algorithm belong to the \mathcal{T} -class algorithms as defined in [3] and deduce its strong convergence to $P_F x_0$ when $F \neq \emptyset$ and where $F \stackrel{\text{def}}{=} \bigcap_{n \geq 0} Fix(Q_n)$.

For $(x, y) \in \mathcal{H}^2$ define the mappings H as follows :

$$H(x, y) \stackrel{\text{def}}{=} \{z \in \mathcal{H} \mid \langle z - y, x - y \rangle \leq 0\} \quad (3)$$

and denote by $Q(x, y, z)$ the projection of x onto $H(x, y) \cap H(y, z)$. Note that $H(x, x) = \mathcal{H}$ and for $x \neq y$, $H(x, y)$ is a closed affine half space onto which y is the projection of x .

Lemma 1 *The sequence generated by Algorithm 1 coincide with the sequence given by $x_{n+1} = Q(x_0, x_n, T_n x_n)$ with :*

$$T_n(x) \stackrel{\text{def}}{=} \frac{x + R_n y}{2} + \frac{1}{2} \left(\frac{\kappa - \alpha_n}{1 - \alpha_n} \right) (x - R_n y), \text{ and } R_n(x) \stackrel{\text{def}}{=} \alpha_n x + (1 - \alpha_n) Q_n(x). \quad (4)$$

Moreover, we have :

$$2T_n - I = \kappa I + (1 - \kappa) Q_n x. \quad (5)$$

Proof :Let $\kappa \in [0, 1)$, $\alpha \in (\kappa, 1)$, $y \stackrel{\text{def}}{=} \alpha x + (1 - \alpha) Qx$ for a κ -strict pseudo-contractions Q and define $\Gamma(x, y)$ as follows :

$$\Gamma(x, y) \stackrel{\text{def}}{=} \left\{ z \in \mathcal{H} \mid \|y - z\|^2 \leq \|x - z\|^2 - (1 - \alpha)(\alpha - \kappa) \|x - Qx\|^2 \right\}. \quad (6)$$

We first prove that $\Gamma(x, y) = H(x, Tx)$ where T is defined by equation (4).

$$\begin{aligned} & \|y - z\|^2 - \|x - z\|^2 \leq -(1 - \alpha)(\alpha - \kappa) \|x - Qx\|^2 \\ \Leftrightarrow & \langle y - z, y - z \rangle - \|x - z\|^2 \leq -(1 - \alpha)(\alpha - \kappa) \|x - Qx\|^2 \\ \Leftrightarrow & \langle y - x, y - z \rangle + \langle x - z, y - z \rangle - \|x - z\|^2 \leq -(1 - \alpha)(\alpha - \kappa) \|x - Qx\|^2 \\ \Leftrightarrow & \langle y - x, y - z \rangle + \langle x - z, y - x \rangle \leq -(1 - \alpha)(\alpha - \kappa) \|x - Qx\|^2 \\ \Leftrightarrow & \langle y - x, y - z \rangle + \langle x - z, y - x \rangle \leq (\alpha - \kappa) \langle y - x, x - Qx \rangle \\ \Leftrightarrow & \langle y - x, y + x - 2z + (\kappa - \alpha)(x - Qx) \rangle \leq 0 \\ \Leftrightarrow & \left\langle y - x, y + x - 2z + \left(\frac{\kappa - \alpha}{1 - \alpha} \right) (x - y) \right\rangle \leq 0 \end{aligned}$$

which gives :

$$\left\langle z - \frac{x + y}{2} - \frac{1}{2} \left(\frac{\kappa - \alpha}{1 - \alpha} \right) (x - y), x - y \right\rangle \leq 0$$

and since we have $x - Tx = (1/2)(1 - \frac{\kappa - \alpha}{1 - \alpha})(x - y)$ with $(1 - \frac{\kappa - \alpha}{1 - \alpha}) > 0$ this is equivalent to $\langle z - Tx, x - Tx \rangle \leq 0$. For $y_n = \alpha_n x_n + (1 - \alpha_n) Q_n x_n$, we thus obtain that $C_n = \Gamma(x_n, y_n) = H(x_n, T_n x_n)$ and since by definition of H we have $D_n = H(x_0, x_n)$ the result follows. The last statement of the lemma (5) is obtained by simple rewrite from equation (4) \square

We prove now that T_n for all $n \in \mathbb{N}$ belongs to the \mathcal{T} class of mappings.

Definition 2 $\mathcal{T} \stackrel{\text{def}}{=} \{T : \mathcal{H} \mapsto \mathcal{H} \mid \text{dom}T = \mathcal{H} \text{ and } (\forall x \in \mathcal{H}) \text{Fix}(T) \subset H(x, Tx)\}$

Lemma 3 *for all $n \in \mathbb{N}$ and T_n defined by equation (4) we have $T_n \in \mathcal{T}$.*

Proof :Using Lemma 1 we have $2T_n - I = \kappa I + (1 - \kappa) Q_n$. If we can prove that when Q is a κ -strict pseudocontraction the mapping $\kappa I + (1 - \kappa) Q$ is

quasi-nonexpansive then the result will follow from [3, Proposition 2.3 (v)]. For $(x, y) \in \mathcal{H}^2$ we have :

$$\begin{aligned}
\|\kappa x + (1 - \kappa)Qx - y - (1 - \kappa)y\|^2 &= \|\kappa(x - y) + (1 - \kappa)(Qx - Qy)\|^2 \\
&= \kappa\|x - y\|^2 + (1 - \kappa)\|Qx - Qy\|^2 - \kappa(1 - \kappa)\|x - y - (Qx - Qy)\|^2 \\
&= \kappa\|x - y\|^2 + (1 - \kappa)\|Qx - Qy\|^2 - \kappa(1 - \kappa)\|x - y - (Qx - Qy)\|^2 \\
&\leq \kappa\|x - y\|^2 + (1 - \kappa)\left(\|Qx - Qy\|^2 - \kappa\|(I - Q)x - (I - Q)y\|^2\right) \\
&\leq \kappa\|x - y\|^2 + (1 - \kappa)\|x - y\|^2 = \|x - y\|^2
\end{aligned}$$

Thus the mapping $\kappa I + (1 - \kappa)Q$ is nonexpansive and thus also quasi-nonexpansive. \square

Definition 4 [3] *A sequence $(T_n)_{n \geq 0}$ such that $T_n \in \mathcal{T}$ is coherent if for every bounded sequence $\{z_n\}_{n \geq 0} \in \mathcal{H}$ there holds :*

$$\left\{ \begin{array}{l} \sum_{n \geq 0} \|z_{n+1} - z_n\|^2 < \infty \\ \sum_{n \geq 0} \|z_n - T_n z_n\|^2 < \infty \end{array} \right. \Rightarrow \mathcal{M}(z_n)_{n \geq 0} \subset \bigcap_{n \geq 0} \text{Fix}(T_n) \quad (7)$$

where $\mathcal{M}(z_n)_{n \geq 0}$ is the set of weak cluster points of the sequence $(z_n)_{n \geq 0}$.

Lemma 5 *Let $(Q_n)_{n \geq 0}$ be a sequence of κ -strict pseudocontraction such that $\text{Fix}(Q_n) = F$ which does not depends on n and for each subsequence $\sigma(n)$ we can find a sub-sequence $\mu(n)$ such that $Q_{\mu(n)} \rightarrow Q$ with $\text{Fix}(Q) = F$ and Q is a κ -strict pseudocontraction. Then, the sequence $(T_n)_{n \geq 0}$ given by (4) is coherent.*

Proof : Suppose that $(z_n)_{n \geq 0}$ is a bounded sequence such that the left hand side of (7) is satisfied. Using (5) we have $\|z_n - T_n z_n\| = (1 - \kappa)/2 \|z_n - Q_n z_n\|$ and $\text{Fix}(T_n) = \text{Fix}(Q_n)$. Thus, verifying the coherence of $(T_n)_{n \geq 0}$ or the coherence of $(Q_n)_{n \geq 0}$ is equivalent. Consider now $u \in \mathcal{M}(z_n)_{n \geq 0}$, by hypothesis $\|z_n - Q_n z_n\| \rightarrow 0$. Let $\sigma(n)$ a subsequence such that $z_{\sigma(n)} \rightharpoonup u$, we extract a subsequence $\mu(n)$ such that $Q_{\mu(n)} \rightarrow Q$ and we thus obtain that $z_{\mu(n)} \rightharpoonup u$ and $\|z_{\mu(n)} - Q z_{\mu(n)}\| \rightarrow 0$. Now, if Q is a κ -strict pseudocontraction, using [1, Proposition 2.6] we have that $I - Q$ is demi-closed and thus $u \in \text{Fix}(Q) = F$. \square

Remark 6 *Given an integer $N \geq 1$, let, for each $1 \leq i \leq N$, $S_i : C \mapsto C$ be a κ_i -strict pseudocontraction for some $0 \leq \kappa_i < 1$. Let $\kappa \stackrel{\text{def}}{=} \max\{\kappa_i : 1 \leq i \leq N\}$. Assume the common fixed point set $F \stackrel{\text{def}}{=} \bigcap_{i=1}^N \text{Fix}(S_i)$ of $\{S_i\}$ is nonempty. Assume also for each n , $\{\lambda_{n,i}\}_{i=1,\dots,N}$ is a finite sequence of positive numbers such that $\sum_{i=1}^N \lambda_{n,i} = 1$ and $\inf_n \lambda_{n,i} > 0$ for all $1 \leq i \leq N$. Let the mapping $Q_n : C \mapsto C$ be defined by :*

$$Q_n x \stackrel{\text{def}}{=} \sum_{i=1}^N \lambda_{n,i} S_i x. \quad (8)$$

Then using [1], for all $n \in \mathbb{N}$, Q_n is a κ -strict pseudocontraction and $\text{Fix}(Q_n) = F$. Moreover for each subsequence $\lambda_{i,(\sigma_n)}$ we can extract a subsequence $\lambda_{i,\mu(n)}$ and $(\bar{\lambda}_i)_{1 \leq i \leq N} \in (0, 1)^N$ such that $\lambda_{i,\mu(n)} \rightarrow \bar{\lambda}_i$ for all $1 \leq i \leq N$. We thus have $Q_{\mu(n)} \rightarrow \sum_i \bar{\lambda}_i S_i$ and using previous lemma the sequence $(T_n)_{n \geq 0}$ is coherent.

Given $T_n \in \mathcal{T}$ we can also consider [3] the following algorithm :

Algorithm 2 Given $\epsilon \in (0, 1]$ and $x_0 \in C$ we consider the sequence given by the iterations $x_{n+1} = x_n + (2 - \epsilon)(T_n x_n - x_n)$.

Gathering previous result the strong convergence of Algorithm 1 to $P_F x_0$ and the weak convergence of Algorithm 2 is obtained by [3, Theorem 4.2] that we recall now :

Theorem 7 [3, Theorem 4.2] Suppose that $(T_n)_{n \geq 0}$ is coherent. Then
(i) if $F \neq \emptyset$, then every orbit of Algorithm 2 converges weakly to a point in F
(ii) For an arbitrary orbit of Algorithm 1, exactly one of the following alternatives holds :

- (a) $F \neq \emptyset$ and $x_n \rightarrow_n P_F x_0$.
- (b) $F = \emptyset$ and $x_n \rightarrow_n +\infty$.
- (c) $F = \emptyset$ and the algorithm terminates.

Remark 8 Note that using previous theorem and Remark 6 we obtain an other proof of [1, Theorem 5.1]. In fact the proofs are very similar but we just hilitate here the role played by \mathcal{T} -class sequences.

3 \mathcal{T} -class iterative algorithm for a sequence of pseudo contractions

Let F be a closed convex of \mathcal{H} we define \mathcal{U}_F as follows :

$$\mathcal{U}_F \stackrel{\text{def}}{=} \{T : \mathcal{H} \mapsto \mathcal{H} \mid \text{dom}T = \mathcal{H} \quad \text{and} \quad (\forall x \in \mathcal{H}) F \subset H(x, Tx)\} . \quad (9)$$

Of course we have $T \in \mathcal{T} \Leftrightarrow T \in \mathcal{U}_{F_{ix}(T)}$.

A mapping $Q : \mathcal{H} \mapsto \mathcal{H}$ is said F -quasi-nonexpansive if

$$\forall (x, y) \in \mathcal{H} \times F \quad \|Qx - y\| \leq \|x - y\| \quad (10)$$

and we can characterize elements of \mathcal{U}_F using the following easy lemma :

Lemma 9 $2T - I$ is F -quasi-nonexpansive is equivalent to $T \in \mathcal{U}_F$.

Proof :The proof follows from the equality [3, (2.6)] :

$$(\forall (x, y) \in \mathcal{H}^2) \quad 4 \langle y - Tx, x - Tx \rangle = \|(2T - I)x - y\|^2 - \|x - y\|^2 . \quad (11)$$

□

Definition 10 A sequence $\{T_n\}_{n \geq 0} \subset \mathcal{U}_F$ is F -coherent if for every bounded sequence $\{z_n\}_{n \geq 0} \in \mathcal{H}$ there holds :

$$\begin{cases} \sum_{n \geq 0} \|z_{n+1} - z_n\|^2 < \infty \\ \sum_{n \geq 0} \|z_n - T_n z_n\|^2 < \infty \end{cases} \Rightarrow \mathcal{M}(z_n)_{n \geq 0} \subset F \quad (12)$$

We propose now the following extension of [3, Theorem 4.2] for the two algorithms 2 and 3.

Algorithm 3 Given $x_0 \in C$ we consider the sequence given by the iterations

$$x_{n+1} = Q(x_0, x_n, T_n x_n)$$

Theorem 11 Suppose that $(T_n)_{n \geq 0}$ is F -coherent for a closed convex F . Then (i) if $F \neq \emptyset$, then every orbit of Algorithm 2 converges weakly to a point in F (ii) For an arbitrary orbit of Algorithm 3, exactly one of the following alternatives holds :

- (a) $F \neq \emptyset$ and $x_n \rightarrow_n P_F x_0$.
- (b) $F = \emptyset$ and $x_n \rightarrow_n +\infty$.
- (c) $F = \emptyset$ and the algorithm terminates.

Proof :The result is very similar to [3, Theorem 2.9] and a careful reading of the proof and remarks in [3, 4] leads to the conclusion that it remains true as stated here. \square

We give now a typical application of this theorem.

Definition 12 For $A : C \mapsto C$ a monotone and k -Lipschitz mapping, let $T_\lambda : \mathcal{H} \times \mathcal{H} \mapsto \mathcal{H}$ the mapping defined by $T_\lambda(x, y) \stackrel{\text{def}}{=} P_C(x - \lambda Ay)$. We also define $T_\lambda^{(1)} x \stackrel{\text{def}}{=} T_\lambda(x, x)$ and $T_\lambda^{(2)} x \stackrel{\text{def}}{=} T_\lambda(x, T_\lambda(x, x)) = T_\lambda(x, T_\lambda^{(1)} x)$.

We assume that $\lambda k \in [a, b] \subset (0, 1)$ and consider $(\lambda_n)_{n \geq 0}$ a sequence of real numbers such that $\lambda_n k \in [a, b]$. To simplify the notations we will use $T_n^{(1)}$ (resp. $T_n^{(2)}$) for denoting $T_{\lambda_n}^{(1)}$ (resp. $T_{\lambda_n}^{(2)}$).

Let $F \stackrel{\text{def}}{=} VI(C, A)$, It is known that F is closed convex and that we have $Fix T_\lambda^{(1)} = F$. It is easy to see that $F \subset Fix(T_\lambda^{(2)})$ but the inclusion may be strict and thus we do not expect the mapping $T_\lambda^{(2)}$ to be quasi-nonexpansive. Following inequalities contained in the proof of [9, Theorem 3.1] we obtain F -quasi-nonexpansive property as exposed now.

Lemma 13 $T_\lambda^{(2)}$ is F -quasi-nonexpansive where $F \stackrel{\text{def}}{=} VI(C, A)$ or using Lemma 9 $(T_\lambda^{(2)} + I)/2 \in \mathcal{U}_F$.

Proof : Let $y = T_\lambda^{(1)}(x)$ and $u \in VI(C, A)$. We use the fact that for all $x \in \mathcal{H}$ and $y \in C$ $P_C x$ can be characterized as follows :

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \quad (13)$$

and since A is a monotone mapping following the steps of the proof of [9, Theorem 3.1] that we reproduce here we obtain :

$$\begin{aligned} \|T_\lambda^{(2)}(x) - u\|^2 &\leq \|x - \lambda Ay - u\|^2 - \|x - \lambda Ay - T_\lambda^{(2)}(x)\|^2 \\ &= \|x - u\|^2 - \|x - T_\lambda^{(2)}(x)\|^2 + 2\lambda \langle Ay, u - T_\lambda^{(2)}(x) \rangle \\ &= \|x - u\|^2 - \|x - T_\lambda^{(2)}(x)\|^2 \\ &\quad + 2\lambda(\langle Ay - Au, u - y \rangle + \langle Au, u - y \rangle + \langle Ay, y - T_\lambda^{(2)}(x) \rangle) \\ &\leq \|x - u\|^2 - \|x - T_\lambda^{(2)}(x)\|^2 + 2\lambda \langle Ay, y - T_\lambda^{(2)}(x) \rangle \\ &= \|x - u\|^2 - \|x - y\|^2 - 2 \langle x - y, y - T_\lambda^{(2)}(x) \rangle - \|y - T_\lambda^{(2)}(x)\|^2 \\ &\quad + 2\lambda \langle Ay, y - T_\lambda^{(2)}(x) \rangle \\ &= \|x - u\|^2 - \|x - y\|^2 - \|y - T_\lambda^{(2)}(x)\|^2 \\ &\quad + 2 \langle x - \lambda Ay - y, T_\lambda^{(2)}(x) - y \rangle. \end{aligned}$$

Further, since $y = P_C(x - \lambda Ax)$ and A is k -Lipschitz-continuous, we have

$$\begin{aligned} \langle x - \lambda Ay - y, T_\lambda^{(2)}(x) - y \rangle &= \langle x - \lambda Ax - y, T_\lambda^{(2)}(x) - y \rangle \\ &+ \langle \lambda Ax - \lambda Ay, T_\lambda^{(2)}(x) - y \rangle \leq \langle \lambda Ax - \lambda Ay, T_\lambda^{(2)}(x) - y \rangle \\ &\leq \lambda k \|x - y\| \|T_\lambda^{(2)}(x) - y\|. \end{aligned}$$

So, we have ;

$$\begin{aligned} \|T_\lambda^{(2)}(x) - u\|^2 &\leq \|x - u\|^2 - \|x - y\|^2 - \|y - T_\lambda^{(2)}(x)\|^2 + 2\lambda k \|x - y\| \|T_\lambda^{(2)}(x) - y\| \\ &\leq \|x - u\|^2 + (\lambda^2 k^2 - 1) \max \left(\|x - y\|^2, \|T_\lambda^{(2)}(x) - y\|^2 \right) \quad (14) \\ &\leq \|x - u\|^2. \end{aligned}$$

□

Corollary 14 *If we consider $R \stackrel{\text{def}}{=} \alpha I + (1 - \alpha)S$ where S is a non-expansive mapping and define $\tilde{F} = \text{Fix}(S) \cap VI(C, A)$ then we obtain immediately that $R \circ T_\lambda^{(2)}$ is a \tilde{F} -quasi-nonexpansive mapping.*

Proof :Let $u \in \tilde{F}$ then $u = Ru$ and we have $\|R \circ T_\lambda^{(2)} - u\| \leq \|T_\lambda^{(2)} - u\|$ and the previous lemma ends the proof. \square

Lemma 15 *The sequence $Q_n = 1/2(T_n^{(2)} + I)$ is F -coherent.*

Proof :Let $(y_n)_{n \geq 0}$ a bounded sequence satisfying the left hand side of equation (12) and $\varphi \in \mathcal{M}(y_n)_{n \geq 0}$. We can find a subsequence $y_{\sigma(n)}$ which converges weakly to φ . For simplicity, we use the notation y_n for the subsequence and since it satisfies the left hand side of equation (12) we have $\|y_n - Q_n y_n\| \rightarrow 0$. By definition of Q_n we also have $\|y_n - T_n^{(2)} y_n\| \rightarrow 0$ and thus $T_n^{(2)} y_n \rightarrow u$ From equation (14) we obtain :

$$\|T_\lambda^{(2)} x - u\|^2 \leq \|x - u\|^2 + (\lambda^2 k^2 - 1) \max \left(\|x - T_\lambda^{(1)} x\|^2, \|T_\lambda^{(2)} x - T_\lambda^{(1)} x\|^2 \right)$$

Thus :

$$\begin{aligned} \max \left(\|x - T_\lambda^{(1)} x\|^2, \|T_\lambda^{(2)} x - T_\lambda^{(1)} x\|^2 \right) &\leq \frac{1}{1 - \lambda^2 k^2} \left(\|x - u\|^2 - \|T_\lambda^{(2)} x - u\|^2 \right) \\ &\leq K \left(\|x - u\| + \|T_\lambda^{(2)} x - u\| \right) \|x - T_\lambda^{(2)} x\| \end{aligned} \quad (15)$$

Using Lemma 13, the sequence $T_n^{(2)} y_n$ is bounded and we thus have from the previous inequality $\|y_n - T_n^{(1)} y_n\| \rightarrow 0$ and $\|T_n^{(2)} y_n - T_n^{(1)} y_n\| \rightarrow 0$.

Using next lemma (Lemma 17) we therefore obtain that for $(v, w) \in G(T)$:

$$\langle v - \varphi, w \rangle = \lim_{n \rightarrow \infty} \langle v - T_n^{(2)} y_n, w \rangle \geq 0.$$

Thus we obtain that $\langle v - \varphi, w \rangle \geq 0$ which gives $\varphi \in T^{-1}(0)$ since T is maximal monotone and then $\varphi \in F = VI(C, A)$. Thus Q_n is F -coherent. \square

Corollary 16 *Let $(R_n)_{n \geq 0}$ a sequence of nonexpansive mappings such that for each subsequence $\sigma(n)$ it is possible to extract a subsequence $\mu(n)$ and find R_μ such that $R_{\mu(n)} y_n \rightarrow_{n \rightarrow \infty} R_\mu y_n$ for every bounded sequence $(y_n)_{n \geq 0}$ with Fix $R_\mu = \mathcal{S}$ a fixed set such that $\mathcal{S} \cap \mathcal{S} \neq \emptyset$. Then, we also have that $Q_n = 1/2((R_n \circ T_n^{(2)}) + I)$ is $F \cap \mathcal{S}$ -coherent.*

Proof :Let $u \in \mathcal{S} \cap \mathcal{S}$, since R_n is nonexpansive we have : $\|R_n \circ T_\lambda^{(2)} - u\| \leq \|T_\lambda^{(2)} - u\|$, Thus equation (15) can be replaced by :

$$\|R_n \circ T_\lambda^{(2)} x - u\|^2 \leq \|x - u\|^2 + (\lambda^2 k^2 - 1) \max \left(\|x - T_\lambda^{(1)} x\|^2, \|T_\lambda^{(2)} x - T_\lambda^{(1)} x\|^2 \right)$$

proceeding as in previous lemma we obtain that for $(y_n)_{n \geq 0}$ a bounded sequence satisfying the left hand side of equation (12) for the sequence of mapping $R_n \circ T_n^{(2)}$ we also have up to subsequences that $\|y_n - T_n^{(1)} y_n\| \rightarrow 0$ and

$\|T_n^{(2)}y_n - T_n^{(1)}y_n\| \rightarrow 0$ and thus also $\|y_n - T_n^{(2)}y_n\| \rightarrow 0$. Thus, as before, if φ is a weak limit of $(y_n)_{n \geq 0}$ we have $\varphi \in F$. Moreover, we have :

$$\begin{aligned} \|T_n^{(2)}y_n - R_\mu\nu\| &\leq \|T_n^{(2)}y_n - y_n\| + \|y_n - R_n \circ T_n^{(2)}y_n\| \\ &\quad + \|R_n \circ T_n^{(2)}y_n - R_\mu \circ T_n^{(2)}y_n\| + \|T_n^{(2)}y_n - \nu\| \end{aligned} \quad (16)$$

Thus

$$\liminf_{n \rightarrow \infty} \|T_n^{(2)}y_n - R_\mu\nu\| \leq \liminf_{n \rightarrow \infty} \|T_n^{(2)}y_n - \nu\|$$

which by Opial's condition is only possible if $R_\mu\nu = \nu$. We conclude that $\nu \in F \cap \mathcal{S}$ which ends the proof. \square

Lemma 17 [9] *Let $T : \mathcal{H} \mapsto H$ the mapping defined by $Tv \stackrel{\text{def}}{=} Av + N_Cv$ when $v \in C$ and $Tv = 0$ when $v \notin C$ where N_C is the normal cone to C at $v \in C$. Let $G(T)$ be the graph of T and $(v, w) \in G(T)$. Then for $x \in C$ we have the following inequality :*

$$\left\langle v - T_\lambda^{(2)}x, w \right\rangle \geq \left\langle v - T_\lambda^{(2)}x, AT_\lambda^{(2)}x - AT_\lambda^{(1)}x \right\rangle - \left\langle v - T_\lambda^{(2)}x, \frac{T_\lambda^{(2)}x - x}{\lambda} \right\rangle$$

Proof : The proof of this inequality is given in [9], we reproduce it for the sake of completeness. The mapping T is maximal monotone, and $0 \in Tv$ if and only if $v \in VI(C, A)$. Let $(v, w) \in G(T)$. Then, we have $w \in Tv = Av + N_Cv$ and hence $w - Av \in N_Cv$. So, we have $\langle v - t, w - Av \rangle \geq 0$ for all $t \in C$. On the other hand, from $T_\lambda^{(2)}(x) = P_C(x - \lambda AT_\lambda^{(1)}(x))$ and $v \in C$ we have $\langle x - \lambda Ay - T_\lambda^{(2)}(x), T_\lambda^{(2)}(x) - v \rangle \geq 0$ and hence $\langle v - T_\lambda^{(2)}(x), T_\lambda^{(2)}(x) - x\lambda + AT_\lambda^{(1)}x \rangle \geq 0$. From $\langle v - t, w - Av \rangle \geq 0$ for all $t \in C$ and $T_\lambda^{(2)}(x) \in C$, we have

$$\begin{aligned} \left\langle v - T_\lambda^{(2)}x, w \right\rangle &\geq \left\langle v - T_\lambda^{(2)}x, Av \right\rangle \\ &\geq \left\langle v - T_\lambda^{(2)}x, Av \right\rangle - \left\langle v - T_\lambda^{(2)}x, \frac{T_\lambda^{(2)}x - x}{\lambda} + AT_\lambda^{(1)}x \right\rangle \\ &= \left\langle v - T_\lambda^{(2)}x, Av - AT_\lambda^{(2)}x \right\rangle + \left\langle v - T_\lambda^{(2)}x, AT_\lambda^{(2)}x - AT_\lambda^{(1)}x \right\rangle \\ &\quad - \left\| v - T_\lambda^{(2)}x, \frac{T_\lambda^{(2)}x - x}{\lambda} \right\| \\ &\geq \left\langle v - T_\lambda^{(2)}x, AT_\lambda^{(2)}x - AT_\lambda^{(1)}x \right\rangle - \left\langle v - T_\lambda^{(2)}x, \frac{T_\lambda^{(2)}x - x}{\lambda} \right\rangle \end{aligned}$$

\square

We end this section by gathering previous results in a main theorem. The proof is immediate by applying Theorem 11. The first statement is a new result. The second statement when applied to the sequence $R_n = \alpha_n Id + (1 - \alpha_n)S$ with $\alpha_n \in [0, c)$ and $c < 1$ gives the same result as [9, Theorem 3.1].

Theorem 18 *Let $(R_n)_{n \geq 0}$ a sequence of nonexpansive mappings satisfying the hypothesis of Corollary 16 and $(T_n^{(2)})_{n \geq 0}$ the sequence of mappings defined on Definition 12. Then, every orbit of Algorithm 2 applied to the sequence of mappings $R_n \circ T_n^{(2)}$ converges weakly to a point in F and the sequence generated by Algorithm 1 converges strongly to $P_F x_0$.*

References

- [1] G. L. Acedo, H.-K. Xu, Iterative methods for strict pseudo-contractions in hilbert spaces, *Nonlinear Analysis* 67 (2007) 2258–2271.
- [2] H. Bauschke, The approximation of fixed points of compositions of nonexpansive mappings in hilbert space, *J. Math. Anal. Appl.* 202 (1996) 150–159.
- [3] H. H. Bauschke, P. L. Combettes, A weak-to-strong convergence principle for fejér-monotone methods in hilbert spaces, *Mathematics of Operations Research* 26 (2) (2001) 248–264.
- [4] P. Combettes, S. Histoaga, Equilibrium programming in hilbert spaces, *Journal of Nonlinear and Convex Analysis* 6 (1) (2005) 117–136.
- [5] K. Goebel, W. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Studies in Advanced Mathematics, Cambridge University Press ed., 1990.
- [6] B. Halpern, Fixed points of nonexpanding maps, *Bull. Amer. Math. Soc.* 73 (1967) 957–961.
- [7] T. Kim, H. Xu, Strong convergence of modified mann iterations for asymptotically nonexpansive mappings and semigroups, *Nonlinear Anal.* 64 (2006) 1140–1152.
- [8] P. Lions, Approximation de points fixes de contractions, *C. R. Acad. Sci. Série A–B Paris* 284 (1977) 1357–1359.
- [9] N. Nadezhkina, W. Takahashi, strong convergence theorem by a hybrid method for nonexpansive mappings and lipschitz-continuous monotone mappings, *siam j. optim* 16 (4) (2006) 1230–1241.
- [10] K. Nakajo, W. Takahashi, Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, *J. Math. Anal. Appl.* 279 (2003) 372–379.

- [11] S. Reich, Weak convergence theorems for nonexpansive mappings in banach spaces, *J. Math. Anal. Appl.* 67 (1979) 274–276.
- [12] N. Shioji, W. Takahashi, Strong convergence of approximated sequences for nonexpansive mappings in banach spaces, *Proc. Amer. Math. Soc.* 125 (1997) 3641–3645.
- [13] R. Wittmann, Approximation of fixed points of nonexpansive mappings, *Arch. Math.* 58 (1992) 486–491.
- [14] H. Xu, Iterative algorithms for nonlinear operators, *J. London Math. Soc.* 66 (2002) 240–256.
- [15] H. Xu, Strong convergence of an iterative method for nonexpansive mappings and accretive operators, *J. Math. Anal. Appl.* 314 (2006) 631–643.