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GUARANTEED AND ROBUST A POSTERIORI ERROR ESTIMATION BASED ON FLUX RECONSTRUCTION FOR DISCONTINUOUS GALERKIN METHODS*

ALEXANDRE ERN[†], ANNETTE F. STEPHANSEN^{†‡}, AND MARTIN VOHRALÍK[§]

Abstract. We propose and study a posteriori error estimates for convection–diffusion–reaction problems approximated by discontinuous Galerkin methods. Firstly, we show that accurate $\mathbf{H}(\text{div})$ -conforming diffusive and convective fluxes can be reconstructed for such methods, even on meshes with hanging nodes. We next present abstract a posteriori error estimates for potential- and flux-nonconforming approximations, which are sharp since they hold for arbitrary conforming reconstructions of the discrete solution and its fluxes. In particular the classical elementwise residual can be transformed in most cases into a higher-order “data oscillation” term. We then derive fully computable a posteriori error estimates. In the pure diffusion case, certain parts of the estimates are robust with respect to diffusion inhomogeneities whenever diffusion-dependent weighted averages are used in the discontinuous Galerkin method. The other parts are so under some additional assumption. In the general case including singularly perturbed regimes with dominant convection or reaction, the local efficiency of the estimates in the energy norm depends on cutoff functions of the local Péclet and Damköhler numbers. The estimates become fully robust if the energy norm is augmented by a dual seminorm of the skew-symmetric part of the differential operator and a certain mesh-dependent seminorm of the jumps. Finally, numerical experiments are presented to illustrate the theoretical results.

Key words. convection–diffusion–reaction equation, inhomogeneous and anisotropic diffusion, dominant convection, dominant reaction, discontinuous Galerkin finite element method, a posteriori error estimates, nonmatching meshes, robustness

AMS subject classifications. 65N15, 65N30, 76S05

1. Introduction. We consider the convection–diffusion–reaction problem

$$-\nabla \cdot (\mathbf{K} \nabla u) + \beta \cdot \nabla u + \mu u = f \quad \text{in } \Omega, \tag{1.1a}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.1b}$$

where $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a polyhedral domain, \mathbf{K} the diffusion tensor, β the velocity field, μ the reaction coefficient, and f the source term. Our intention is to study a posteriori error estimates for the approximation of (1.1a)–(1.1b) by interior-penalty discontinuous Galerkin (DG) methods with the twofold objective of deriving estimates without undetermined constants and analyzing carefully the robustness of the estimates in several practically important situations such as diffusion inhomogeneities and anisotropies and singularly perturbed regimes due to dominant convection or reaction. We are mainly interested in the case where the error is measured in the natural energy (semi)norm associated with the symmetric part of the differential operator in (1.1a). In the case of dominant convection, however, a stronger norm including an additional contribution from the skew-symmetric part of the operator (containing the dual seminorm of the convective derivative) and a certain mesh-dependent seminorm of the jumps will be considered as well.

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For the pure diffusion problem ((1.1a)–(1.1b) with $\beta = \mu = 0$), residual-based a posteriori energy (semi)norm error estimates for DG methods can be traced back to [6, 19]. Although the estimates derived therein are both reliable (that is, they yield an upper bound on the difference between the exact and approximate solution) and locally efficient (that is, they give local lower bounds for the error as well), they feature various undetermined constants. As such, they can be used to drive mesh adaptation procedures, but not to deliver certified error upper bounds. This shortcoming has been remedied recently upon introducing estimators based either on equilibrated fluxes [3] (for the first-order symmetric interior-penalty DG scheme in the case $d = 2$ and $\mathbf{K} = Id$) or, in parallel with the present work, on a comparison of the local diffusive flux $-\mathbf{K}\nabla_h u_h$ of the approximate DG solution with a reconstructed $\mathbf{H}(\text{div})$ -conforming flux [21, 12]. We also mention [24] where numerical experiments for similar estimators are presented. Error estimates for continuous finite element methods using reconstructed $\mathbf{H}(\text{div})$ -conforming fluxes can be traced back to the seminal work of Prager and Synge [27], while more recent developments include [22, 23, 13].

The first focus of the present work is to investigate two important issues in the pure diffusion case which have not been addressed previously. Firstly, previous work was restricted to meshes without hanging nodes. Since the use of nonmatching meshes (that is, meshes with hanging nodes) constitutes one of the advantages of DG methods, we derive and analyze in this work accurate $\mathbf{H}(\text{div})$ -conforming flux reconstructions for DG methods on nonmatching meshes. Secondly, previous work did not address the important issue of robustness of the derived estimates, that is, the sharpest possible dependency of the upper and lower error bounds with respect to diffusion inhomogeneities and anisotropies. Therefore, we shall tackle this issue here by elaborating on recent work [18, 14] where the use of diffusivity-dependent weighted averages was advocated to formulate the consistency terms in DG methods. We shall prove that using such weights allows for full robustness with respect to inhomogeneities for certain estimators. The other are then robust under the usual assumption of “monotonicity around vertices” distribution of the inhomogeneities.

Our second aim is to investigate a posteriori error estimates in the case of singularly perturbed regimes with dominant convection or reaction. With the exception of the residual-based estimates derived in [17, 31], this appears to be a novel topic for DG methods. In this work we derive two guaranteed estimators based on flux reconstruction and we consider the case of nonmatching meshes. The first one applies to the error still measured in the energy (semi)norm associated with the differential operator and yields constants in the lower error bounds that depend on certain cut-off functions of the local Péclet and Damköhler numbers similarly to the estimates derived by Verfürth for stabilized conforming finite elements [34] and in [17] for DG methods. Our second estimator, which is fully robust, features an additional term in the error norm involving the skew-symmetric part of the differential operator as proposed again by Verfürth for stabilized conforming finite elements [35] and a certain mesh-dependent seminorm of the jumps specific to DG methods.

The starting point of our results is an extension of the abstract framework of [38] to approximations u_h which are such that both u_h and $-\mathbf{K}\nabla_h u_h$ are nonconforming, as in DG methods. This extended framework is sharp in both the energy and augmented (semi)norms since it is established for arbitrary conforming reconstructions of the discrete solution itself and of its fluxes. In many cases, it also allows to transform the classical elementwise residual into a higher-order term, sometimes considered sep-

arately and called “data oscillation.” We thus fully take advantage of the degrees of freedom within each element available in DG methods. The derived estimators yield local lower bounds for the classical residual-based estimators, require no convexity of Ω , no additional regularity of the weak solution, no saturation assumption, and do not use the Helmholtz decomposition. We only consider homogeneous Dirichlet boundary conditions for the sake of simplicity; extensions to inhomogeneous Dirichlet and Neumann boundary conditions are possible. Furthermore, the present analysis can be extended to meshes consisting of general elements, simplices are only considered for simplicity.

This paper is organized as follows. We introduce the setting in Section 2. We then describe and analyze the $\mathbf{H}(\text{div})$ -conforming flux reconstructions for DG methods on nonmatching meshes in Section 3. We address the energy (semi)norm a posteriori error estimates and their robustness for pure diffusion problems in Section 4. Then, we extend the analysis to convection–diffusion–reaction problems including singularly perturbed regimes with dominant convection or reaction, in Section 5 for the energy (semi)norm and in Section 6 for the augmented norm. Finally, some numerical experiments illustrating the theoretical analysis are presented in Section 7.

2. The setting. This section introduces the main notation, the basic assumptions, and the continuous and discrete problems.

2.1. Notation. Let $\{\mathcal{T}_h\}_{h>0}$ be a family of simplicial triangulations of the domain Ω . A generic element in \mathcal{T}_h is denoted by T , h_T stands for its diameter, $|T|$ for its measure, and \mathbf{n}_T for its unit outward normal. We suppose that the meshes cover $\overline{\Omega}$ exactly. Meshes can possess “hanging nodes”, as depicted in Figure 2.1. Meshes without hanging nodes are referred to as “matching”. We will be using the so-called broken Sobolev space

$$H^s(\mathcal{T}_h) := \{v \in L^2(\Omega); v|_T \in H^s(T) \quad \forall T \in \mathcal{T}_h\},$$

along with its DG approximation space

$$V^k(\mathcal{T}_h) := \{v_h \in L^2(\Omega); v_h|_T \in \mathbb{P}_k(T) \quad \forall T \in \mathcal{T}_h\},$$

where $\mathbb{P}_k(T)$, $k \geq 1$, is the set of polynomials of degree less than or equal to k on an element T . The space $V^0(\mathcal{T}_h)$ spanned by piecewise constant functions will be considered as well. For all $l \geq 0$, Π_l denotes the L^2 -orthogonal projection onto $V^l(\mathcal{T}_h)$. The L^2 -scalar product and its associated norm on a region $R \subset \Omega$ are indicated by the subscript $0, R$; shall R coincide with Ω , this subscript will be dropped. For $s \geq 1$, a norm (seminorm) with the subscript s, R stands for the usual norm (seminorm) in $H^s(R)$. Finally, ∇_h denotes the broken gradient operator, that is, for $v \in H^1(\mathcal{T}_h)$, $\nabla_h v \in [L^2(\Omega)]^d$ and for all $T \in \mathcal{T}_h$, $(\nabla_h v)|_T = \nabla(v|_T)$.

We say that F is an interior face of a given mesh if it has positive $(d-1)$ -dimensional measure and if there are distinct $T^-(F)$ and $T^+(F)$ in \mathcal{T}_h such that $F = \partial T^-(F) \cap \partial T^+(F)$ and we define \mathbf{n}_F as the unit normal vector to F pointing from $T^-(F)$ towards $T^+(F)$, cf. Figure 2.1. Similarly, we say that F is a boundary face of the mesh if it has positive $(d-1)$ -dimensional measure and there is $T(F) \in \mathcal{T}_h$ such that $F = \partial T(F) \cap \partial\Omega$ and we define \mathbf{n}_F as the unit outward normal to $\partial\Omega$ (the arbitrariness in the orientation of \mathbf{n}_F is irrelevant in the sequel). All the interior (resp., boundary) faces of the mesh are collected into the set $\mathcal{F}_h^{\text{int}}$ (resp., $\mathcal{F}_h^{\text{ext}}$) and we define $\mathcal{F}_h := \mathcal{F}_h^{\text{int}} \cup \mathcal{F}_h^{\text{ext}}$. For $F \in \mathcal{F}_h$ and $l \geq 0$, $\Pi_{l,F}$ denotes the L^2 -orthogonal

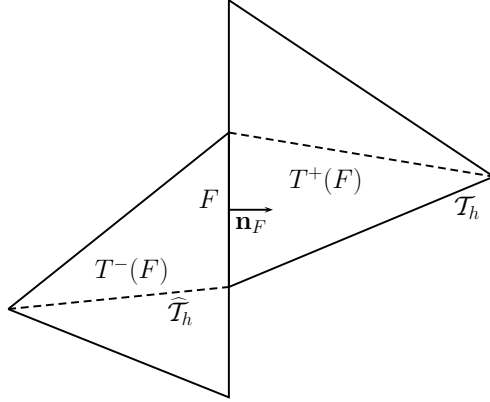


FIG. 2.1. Notation for nonmatching meshes

projection onto $\mathbb{P}_l(F)$. For a function v that is double-valued on a face $F \in \mathcal{F}_h^{\text{int}}$, its jump and arithmetic average on F are defined as

$$[[v]]_F := v|_{T^-(F)} - v|_{T^+(F)}, \quad \{\{v\}\}_F := \frac{1}{2}(v|_{T^-(F)} + v|_{T^+(F)}). \quad (2.1)$$

We set $[[v]]_F := v|_F$ and $\{\{v\}\}_F := \frac{1}{2}v|_F$ on boundary faces. The subscript F in the above jumps and averages is omitted if there is no ambiguity. Furthermore, weighted averages are defined on an interior face $F \in \mathcal{F}_h^{\text{int}}$ as

$$\{\{v\}\}_\omega := \omega_{T^-(F),F} v|_{T^-(F)} + \omega_{T^+(F),F} v|_{T^+(F)}, \quad (2.2)$$

where the nonnegative weights satisfy $\omega_{T^-(F),F} + \omega_{T^+(F),F} = 1$. On boundary faces, we set $\{\{v\}\}_\omega := v|_F$ and $\omega_{T(F),F} := 1$. Finally, for all $T \in \mathcal{T}_h$ and for all $F \subset \partial T$, we set $\bar{\omega}_{T,F} := 1 - \omega_{T,F}$.

It is convenient to define the following sets: For all $T \in \mathcal{T}_h$,

$$\begin{aligned} \mathcal{F}_T &= \{F \in \mathcal{F}_h; F \subset \partial T\}, & \mathfrak{F}_T &= \{F \in \mathcal{F}_h; F \cap \partial T \neq \emptyset\}, \\ \mathcal{T}_T &= \{T' \in \mathcal{T}_h; \mathcal{F}_T \cap \mathcal{F}_{T'} \neq \emptyset\}, & \mathfrak{T}_T &= \{T' \in \mathcal{T}_h; T \cap T' \neq \emptyset\}, \end{aligned}$$

and for all $F \in \mathcal{F}_h$,

$$\mathcal{T}_F = \{T \in \mathcal{T}_h; F \in \mathcal{F}_T\}, \quad \mathfrak{T}_F = \{T \in \mathcal{T}_h; F \cap \partial T \neq \emptyset\}.$$

For each \mathcal{T}_h , there exists a matching simplicial submesh $\widehat{\mathcal{T}}_h$ of \mathcal{T}_h such that $\widehat{\mathcal{T}}_h = \mathcal{T}_h$ if \mathcal{T}_h is itself matching. If \mathcal{T}_h is nonmatching, $\widehat{\mathcal{T}}_h$ can lead to a subdivision of the faces in \mathcal{F}_h . For all $l \geq 0$, $\widehat{\Pi}_l$ denotes the L^2 -orthogonal projection onto $V^l(\widehat{\mathcal{T}}_h)$. Interior and boundary faces of $\widehat{\mathcal{T}}_h$ are collected, respectively, in the sets $\widehat{\mathcal{F}}_h^{\text{int}}$ and $\widehat{\mathcal{F}}_h^{\text{ext}}$ and we define $\widehat{\mathcal{F}}_h := \widehat{\mathcal{F}}_h^{\text{int}} \cup \widehat{\mathcal{F}}_h^{\text{ext}}$ and $\widehat{\Pi}_{l,F}$ for $F \in \widehat{\mathcal{F}}_h$ as above. For all $F \in \widehat{\mathcal{F}}_h$ such that there is $F' \in \mathcal{F}_h$ with $F \subset F'$, \mathbf{n}_F is chosen with the same orientation as that of $\mathbf{n}_{F'}$. We also define for all $T \in \widehat{\mathcal{T}}_h$, the sets $\widehat{\mathcal{F}}_T$ and $\widehat{\mathfrak{F}}_T$ as above. Finally, for all $T \in \mathcal{T}_h$, we consider the refinement of T by $\widehat{\mathcal{T}}_h$, namely

$$\mathfrak{R}_T = \{T' \in \widehat{\mathcal{T}}_h; T' \subset T\}.$$

Clearly, $\mathfrak{R}_T = \{T\}$ if \mathcal{T}_h is matching.

2.2. Assumptions. We assume that $\mathbf{K} \in [L^\infty(\Omega)]^{d \times d}$ is a symmetric, uniformly positive definite, and piecewise constant tensor and for all $T \in \mathcal{T}_h$, we denote by $c_{\mathbf{K},T}$ and $C_{\mathbf{K},T}$, respectively, its minimum and maximum eigenvalue on T . We also assume that $\boldsymbol{\beta} \in [L^\infty(\Omega)]^d$ with $\nabla \cdot \boldsymbol{\beta} \in L^\infty(\Omega)$, $\mu \in L^\infty(\Omega)$ and $\mu - \frac{1}{2} \nabla \cdot \boldsymbol{\beta} \geq 0$ a.e. in Ω . For all $T \in \mathcal{T}_h$, $c_{\boldsymbol{\beta},\mu,T}$ indicates the (essential) minimum value of $\mu - \frac{1}{2} \nabla \cdot \boldsymbol{\beta}$ on T ; we suppose that if $c_{\boldsymbol{\beta},\mu,T} = 0$, then $\|\mu\|_{\infty,T} = \|\frac{1}{2} \nabla \cdot \boldsymbol{\beta}\|_{\infty,T} = 0$. Finally, we assume $f \in L^2(\Omega)$. The above assumptions are sufficient for the existence and uniqueness of both continuous and discrete solutions and for our a posteriori error estimates, as well as for the global efficiency of the abstract estimates; for the proofs of the local efficiency of the computable estimates, however, we shall tighten them. For such proofs, we shall also assume the following on the meshes:

- (A1) $\{\widehat{\mathcal{T}}_h\}_{h>0}$ is shape-regular in the sense that there exists a constant $\kappa_{\widehat{\mathcal{T}}} > 0$ such that $\min_{T \in \widehat{\mathcal{T}}_h} |T|/h_T^d \geq \kappa_{\widehat{\mathcal{T}}}$ for all $h > 0$.
- (A2) There exists a constant $\nu_{\mathcal{T}} > 0$ such that $\min_{T' \in \mathfrak{R}_T} h_{T'}/h_T \geq \nu_{\mathcal{T}}$ for all $T \in \mathcal{T}_h$ and all $h > 0$.

Observe that the above assumptions imply the shape-regularity of $\{\mathcal{T}_h\}_{h>0}$.

2.3. The continuous problem. For all $u, v \in H^1(\mathcal{T}_h)$, we define the bilinear form

$$\mathcal{B}(u, v) := (\mathbf{K} \nabla_h u, \nabla_h v) + (\boldsymbol{\beta} \cdot \nabla_h u, v) + (\mu u, v), \quad (2.3)$$

and the corresponding energy (semi)norm

$$\|v\|^2 := \sum_{T \in \mathcal{T}_h} \|v\|_T^2, \quad \|v\|_T^2 := \|\mathbf{K}^{\frac{1}{2}} \nabla v\|_{0,T}^2 + \|(\mu - \frac{1}{2} \nabla \cdot \boldsymbol{\beta})^{\frac{1}{2}} v\|_{0,T}^2. \quad (2.4)$$

We remark that $\|\cdot\|$ is always a norm on $H_0^1(\Omega)$, whereas it is a norm on $H^1(\mathcal{T}_h)$ only if $c_{\boldsymbol{\beta},\mu,T} > 0$ for all $T \in \mathcal{T}_h$.

The weak formulation of (1.1a)–(1.1b) consists of finding $u \in H_0^1(\Omega)$ such that

$$\mathcal{B}(u, v) = (f, v) \quad \forall v \in H_0^1(\Omega). \quad (2.5)$$

The assumptions of the previous section, the Green theorem, and the Cauchy–Schwarz inequality imply that $\mathcal{B}(v, v) = \|v\|^2$ for all $v \in H_0^1(\Omega)$ and that for all $u, v \in H^1(\mathcal{T}_h)$,

$$\mathcal{B}(u, v) \leq \max \left\{ 1, \max_{T \in \mathcal{T}_h} \left\{ \frac{\|\mu\|_{\infty,T}}{c_{\boldsymbol{\beta},\mu,T}} \right\} \right\} \|u\| \|v\| + \max_{T \in \mathcal{T}_h} \left\{ \frac{\|\boldsymbol{\beta}\|_{\infty,T}}{c_{\mathbf{K},T}^{1/2}} \right\} \|u\| \|v\|. \quad (2.6)$$

Hence, the problem (2.5) admits a unique solution.

REMARK 2.1 (Notation). *If $c_{\boldsymbol{\beta},\mu,T} = 0$, the term $\|\mu\|_{\infty,T}/c_{\boldsymbol{\beta},\mu,T}$ in estimate (2.6) should be evaluated as zero, since in this case we assume $\|\mu\|_{\infty,T} = 0$. To simplify the notation, we will systematically use the convention $0/0 = 0$.*

2.4. The discontinuous Galerkin method. The interior-penalty DG methods considered in this paper are associated with the bilinear form

$$\begin{aligned} \mathcal{B}_h(u, v) &:= (\mathbf{K} \nabla_h u, \nabla_h v) + ((\mu - \nabla \cdot \boldsymbol{\beta})u, v) - (u, \boldsymbol{\beta} \cdot \nabla_h v) \\ &\quad - \sum_{F \in \mathcal{F}_h} \{(\mathbf{n}_F \cdot \{\{\mathbf{K} \nabla_h u\}\}_\omega, [v])_{0,F} + \theta(\mathbf{n}_F \cdot \{\{\mathbf{K} \nabla_h v\}\}_\omega, [u])_{0,F}\} \\ &\quad + \sum_{F \in \mathcal{F}_h} \{(\gamma_F [u], [v])_{0,F} + (\boldsymbol{\beta} \cdot \mathbf{n}_F \{u\}, [v])_{0,F}\}. \end{aligned} \quad (2.7)$$

The discrete problem consists of finding $u_h \in V^k(\mathcal{T}_h)$ such that

$$\mathcal{B}_h(u_h, v_h) = (f, v_h) \quad \forall v_h \in V^k(\mathcal{T}_h). \quad (2.8)$$

Taking in (2.7) the weights on interior faces equal to $1/2$ and letting $\theta \in \{-1, 0, 1\}$ leads to the well-known Nonsymmetric, Incomplete, or Symmetric Interior-Penalty DG methods. The stabilization parameter γ_F takes the general form

$$\gamma_F := \alpha_F \gamma_{\mathbf{K}, F} h_F^{-1} + \gamma_{\beta, F} \quad \forall F \in \mathcal{F}_h, \quad (2.9)$$

where α_F is a (user-dependent) positive parameter, $\gamma_{\mathbf{K}, F}$ a positive scalar-valued function depending on \mathbf{K} , and $\gamma_{\beta, F}$ a nonnegative scalar-valued function depending on β and vanishing if $\beta = 0$; we suppose here that $\gamma_{\beta, F} = \frac{1}{2} |\beta \cdot \mathbf{n}_F|$, which amounts to so-called upwinding. As usual with interior-penalty methods, the parameters α_F must be taken large enough to ensure the coercivity of the discrete bilinear form \mathcal{B}_h on $V^k(\mathcal{T}_h)$ whenever $\theta \neq -1$. Some additional assumptions on the weights and the penalty coefficient $\gamma_{\mathbf{K}, F}$ will be introduced later to ensure the robustness of our estimates with respect to diffusion inhomogeneities; see Section 4.3 below.

3. $\mathbf{H}(\text{div}, \Omega)$ -conforming flux reconstructions on nonmatching meshes.

The approximate DG diffusive flux $-\mathbf{K}\nabla_h u_h$ is nonconforming since it does not belong to the space $\mathbf{H}(\text{div}, \Omega)$ as the exact diffusive flux $-\mathbf{K}\nabla u$ does. One may then be interested in a $\mathbf{H}(\text{div}, \Omega)$ -conforming postprocessing of the approximate diffusive flux. Previous works in this direction on matching meshes include [4, 16, 21]. We generalize in this section the approach of [16, 21] to nonmatching meshes and also introduce a convective flux reconstruction for convection–diffusion–reaction problems. Our purpose is to use the postprocessed fluxes in the context of a posteriori error estimation. These postprocessed fluxes are, however, of independent interest, *e.g.*, to obtain an appropriate velocity field for a contaminant transport model after a DG flow simulation.

The postprocessed fluxes will belong to the Raviart–Thomas–Nédélec spaces of vector functions on the mesh $\widehat{\mathcal{T}}_h$ (recall that $\widehat{\mathcal{T}}_h = \mathcal{T}_h$ if \mathcal{T}_h is matching),

$$\mathbf{RTN}^l(\widehat{\mathcal{T}}_h) = \{ \mathbf{v}_h \in \mathbf{H}(\text{div}, \Omega); \mathbf{v}_h|_T \in \mathbf{RTN}_T^l \quad \forall T \in \widehat{\mathcal{T}}_h \},$$

where $l \in \{k-1, k\}$ (recall that k is the polynomial degree used for the DG approximation) and $\mathbf{RTN}_T^l = \mathbb{P}_l^d(T) + \mathbf{xP}_l(T)$. In particular, $\mathbf{v}_h \in \mathbf{RTN}^l(\widehat{\mathcal{T}}_h)$ is such that $\nabla \cdot \mathbf{v}_h \in \mathbb{P}_l(T)$ for all $T \in \widehat{\mathcal{T}}_h$, $\mathbf{v}_h \cdot \mathbf{n}_F \in \mathbb{P}_l(F)$ for all $F \in \widehat{\mathcal{F}}_T$ and all $T \in \widehat{\mathcal{T}}_h$, and such that its normal trace is continuous, *cf.* [28, 25, 8, 30].

3.1. Diffusive flux reconstructions. We present here two ways to reconstruct a $\mathbf{H}(\text{div}, \Omega)$ -conforming diffusive flux on nonmatching meshes.

3.1.1. Reconstruction by direct prescription. In this section the reconstructed diffusive flux is directly prescribed by the values of the DG approximate solution u_h . Using the specification of the degrees of freedom of functions in \mathbf{RTN}_T^l , the $\mathbf{H}(\text{div}, \Omega)$ -conforming diffusive flux $\mathbf{t}_h \in \mathbf{RTN}^l(\widehat{\mathcal{T}}_h)$ is prescribed locally on all $T \in \widehat{\mathcal{T}}_h$ as follows: For all $F \in \widehat{\mathcal{F}}_T$ and all $q_h \in \mathbb{P}_l(F)$,

$$(\mathbf{t}_h \cdot \mathbf{n}_F, q_h)_{0, F} = (-\mathbf{n}_F \cdot \{ \mathbf{K} \nabla_h u_h \}_\omega + \alpha_F \gamma_{\mathbf{K}, F} h_F^{-1} \llbracket u_h \rrbracket, q_h)_{0, F}, \quad (3.1)$$

and for all $\mathbf{r}_h \in \mathbb{P}_{l-1}^d(T)$,

$$(\mathbf{t}_h, \mathbf{r}_h)_{0, T} = -(\mathbf{K} \nabla u_h, \mathbf{r}_h)_{0, T} + \theta \sum_{F \in \widehat{\mathcal{F}}_T} \omega_{T, F} (\mathbf{n}_F \cdot \mathbf{K} \mathbf{r}_h, \llbracket u_h \rrbracket)_{0, F}. \quad (3.2)$$

Observe that the quantities prescribing the moments of $\mathbf{t}_h \cdot \mathbf{n}_F$ are uniquely defined for each face $F \in \widehat{\mathcal{F}}_h$, whence the continuity of the normal trace of \mathbf{t}_h . Observe also that α_F , $\gamma_{\mathbf{K},F}$ and $\omega_{T,F}$ need only be evaluated on the faces of \mathcal{T}_h (where they are actually defined) since $\llbracket u_h \rrbracket = 0$ and $\{\{\mathbf{K}\nabla_h u_h\}\}_\omega = \mathbf{K}\nabla u_h$ on the remaining faces of $\widehat{\mathcal{T}}_h$. The above construction leads to the following crucial lemma:

LEMMA 3.1 (Reconstructed diffusion residual). *There holds*

$$(\nabla \cdot \mathbf{t}_h, \xi_h)_{0,T} = (f, \xi_h)_{0,T} \quad \forall T \in \mathcal{T}_h, \forall \xi_h \in \mathbb{P}_l(T). \quad (3.3)$$

For matching meshes ($\widehat{\mathcal{T}}_h = \mathcal{T}_h$), this yields

$$\nabla \cdot \mathbf{t}_h|_T = \Pi_l f|_T \quad \forall T \in \mathcal{T}_h. \quad (3.4)$$

Proof. Let $T \in \mathcal{T}_h$ and let $\xi_h \in \mathbb{P}_l(T)$. Owing to the Green theorem,

$$\begin{aligned} (\nabla \cdot \mathbf{t}_h, \xi_h)_{0,T} &= \sum_{T' \in \mathfrak{R}_T} (\nabla \cdot \mathbf{t}_h, \xi_h)_{0,T'} \\ &= \sum_{T' \in \mathfrak{R}_T} -(\mathbf{t}_h, \nabla \xi_h)_{0,T'} + \sum_{T' \in \mathfrak{R}_T} \sum_{F \in \widehat{\mathcal{F}}_{T'}} (\mathbf{t}_h \cdot \mathbf{n}_{T'}, \xi_h)_{0,F} := T_1 + T_2. \end{aligned}$$

Since $\nabla \xi_h|_{T'} \in \mathbb{P}_{l-1}(T')^d$ for all $T' \in \mathfrak{R}_T$ and $\llbracket u_h \rrbracket = 0$ on those faces $F \in \widehat{\mathcal{F}}_{T'}$ that lie in the interior of T , using (3.2) yields

$$T_1 = (\mathbf{K}\nabla u_h, \nabla \xi_h)_{0,T} - \theta \sum_{F \in \mathcal{F}_T} \omega_{T,F} (\mathbf{n}_F \cdot \mathbf{K}\nabla \xi_h, \llbracket u_h \rrbracket)_{0,F}.$$

Furthermore, owing to the continuity of ξ_h and that of the normal component of \mathbf{t}_h in the interior of T and the fact that $\xi_h|_F \in \mathbb{P}_l(F)$ for all $F \in \widehat{\mathcal{F}}_{T'}$ and all $T' \in \mathfrak{R}_T$,

$$T_2 = \sum_{F \in \mathcal{F}_T} (-\mathbf{n}_F \cdot \{\{\mathbf{K}\nabla_h u_h\}\}_\omega + \alpha_F \gamma_{\mathbf{K},F} h_F^{-1} \llbracket u_h \rrbracket, \mathbf{n}_T \cdot \mathbf{n}_F \xi_h)_{0,F}.$$

Letting 1_T be the characteristic function of T and using (2.7)–(2.8) yields

$$T_1 + T_2 = B_h(u_h, \xi_h 1_T) = (f, \xi_h)_{0,T},$$

whence (3.3) is inferred. Finally, (3.4) results from (3.3) and the fact that for matching meshes, $\nabla \cdot \mathbf{t}_h|_T \in \mathbb{P}_l(T)$. \square

3.1.2. Reconstruction by solving local linear systems. The reconstruction of the previous section is very cheap but we only have (3.3), that is, a local conservation property on each $T \in \mathcal{T}_h$, but not on each $T \in \widehat{\mathcal{T}}_h$. Moreover, (3.4) does not hold if \mathcal{T}_h is nonmatching. For the price of a local linear system solution, the reconstruction of this section improves on these two points in the pure diffusion case.

For a given $T \in \mathcal{T}_h$, let

$$\begin{aligned} \mathbf{RTN}_{N,\text{DG}}^l(\mathfrak{R}_T) &= \{\mathbf{v}_h \in \mathbf{H}(\text{div}, T); \mathbf{v}_h|_{T'} \in \mathbf{RTN}_{T'}^l, \forall T' \in \mathfrak{R}_T, \\ &\quad (\mathbf{v}_h \cdot \mathbf{n}_F, q_h)_{0,F} = (\phi_F(u_h), q_h)_{0,F} \forall F \in \mathcal{F}_T, \forall q_h \in \mathbb{P}_l(F)\}, \end{aligned}$$

where $\phi_F(u_h) = -\mathbf{n}_F \cdot \{\{\mathbf{K}\nabla_h u_h\}\}_\omega + \alpha_F \gamma_{\mathbf{K},F} h_F^{-1} \llbracket u_h \rrbracket$. Let also $\mathbf{RTN}_{N,0}^l(\mathfrak{R}_T)$ be defined as above, but with the normal flux condition $(\mathbf{v}_h \cdot \mathbf{n}_F, q_h)_{0,F} = 0$. We then define $\mathbf{t}_h \in \mathbf{RTN}^l(\widehat{\mathcal{T}}_h)$ by solving on each $T \in \mathcal{T}_h$, the following minimization problem:

$$\mathbf{t}_h|_T = \arg \inf_{\mathbf{v}_h \in \mathbf{RTN}_{N,\text{DG}}^l(\mathfrak{R}_T), \nabla \cdot \mathbf{v}_h = \widehat{\Pi}_l f} \|\mathbf{K}^{\frac{1}{2}} \nabla u_h + \mathbf{K}^{-\frac{1}{2}} \mathbf{v}_h\|_{0,T}. \quad (3.5)$$

Observe that indeed $\mathbf{t}_h \in \mathbf{RTN}^l(\widehat{\mathcal{T}}_h)$ and that the local conservation property $\nabla \cdot \mathbf{t}_h = \widehat{\Pi}_l f$ is directly enforced. Moreover, the minimizing functional that has been chosen is motivated by the fact that the quantity $\|\mathbf{K}^{\frac{1}{2}} \nabla u_h + \mathbf{K}^{-\frac{1}{2}} \mathbf{t}_h\|_{0,T}$ is used in the a posteriori error estimates; see (4.5) below. Finally, we notice that the minimizing set is not empty since owing to (2.7)–(2.8),

$$(\widehat{\Pi}_l f, 1)_{0,T} = (f, 1)_{0,T} = \sum_{F \in \mathcal{F}_T} \mathbf{n}_T \cdot \mathbf{n}_F (\phi_F(u_h), 1)_{0,F} = \sum_{F \in \mathcal{F}_T} (\mathbf{v}_h \cdot \mathbf{n}_T, 1)_{0,F},$$

for all $\mathbf{v}_h \in \mathbf{RTN}_{N,\text{DG}}^l(\mathfrak{R}_T)$. Letting $\mathbb{P}_l^*(\mathfrak{R}_T)$ be spanned by piecewise l -th order polynomials on \mathfrak{R}_T with zero mean on T , it is easy to show that (3.5) is equivalent to finding $\mathbf{t}_h \in \mathbf{RTN}_{N,\text{DG}}^l(\mathfrak{R}_T)$ and $p_h \in \mathbb{P}_l^*(\mathfrak{R}_T)$ such that

$$\begin{aligned} (\mathbf{K}^{-1} \mathbf{t}_h + \nabla u_h, \mathbf{v}_h)_{0,T} - (p_h, \nabla \cdot \mathbf{v}_h)_{0,T} &= 0 & \forall \mathbf{v}_h \in \mathbf{RTN}_{N,0}^l(\mathfrak{R}_T), \\ (\nabla \cdot \mathbf{t}_h, \phi_h)_{0,T} &= (f, \phi_h)_{0,T} & \forall \phi_h \in \mathbb{P}_l^*(\mathfrak{R}_T), \end{aligned}$$

and that this problem is well-posed, cf. [8, 30, 40]. Letting $\mathbf{t}'_h := \mathbf{t}_h + \mathbf{K}\nabla_h u_h$, (\mathbf{t}'_h, p_h) corresponds to the mixed finite element approximation to the local problem

$$\begin{aligned} -\nabla \cdot (\mathbf{K}\nabla p) &= f + \nabla \cdot (\mathbf{K}\nabla u_h) & \text{in } T, \\ -\mathbf{K}\nabla p \cdot \mathbf{n}_T &= (\mathbf{n}_T \cdot \mathbf{n}_F) \phi_F(u_h) + \mathbf{K}\nabla u_h|_T \cdot \mathbf{n}_T & \text{on all } F \in \mathcal{F}_T, \\ (p, 1)_T &= 0. \end{aligned}$$

Finally, we notice that if \mathcal{T}_h is matching, the reconstructions by direct prescription and by solving local linear systems yield the same result if $l \in \{0, 1\}$.

3.2. Convective flux reconstruction. We present here a reconstruction of the convective flux by direct prescription and a reconstruction of the convective–diffusive flux by solving local linear systems. Only the former will be used in the context of a posteriori error estimates, but the latter can be of independent interest.

3.2.1. Reconstruction by direct prescription. As for the diffusive flux, the reconstruction \mathbf{q}_h of the approximate DG convective flux βu_h belongs to $\mathbf{RTN}^l(\widehat{\mathcal{T}}_h)$ with $l \in \{k-1, k\}$ and is prescribed locally on all $T \in \widehat{\mathcal{T}}_h$ as follows: For all $F \in \widehat{\mathcal{F}}_T$ and all $q_h \in \mathbb{P}_l(F)$,

$$(\mathbf{q}_h \cdot \mathbf{n}_F, q_h)_{0,F} = (\beta \cdot \mathbf{n}_F \{\{u_h\}\} + \gamma_{\beta,F} \llbracket u_h \rrbracket, q_h)_{0,F}, \quad (3.6)$$

and for all $\mathbf{r}_h \in \mathbb{P}_{l-1}^d(T)$,

$$(\mathbf{q}_h, \mathbf{r}_h)_{0,T} = (u_h, \beta \cdot \mathbf{r}_h)_{0,T}. \quad (3.7)$$

Note in particular that the quantities prescribing the moments of $\mathbf{q}_h \cdot \mathbf{n}_F$ are uniquely defined for each face $F \in \widehat{\mathcal{F}}_h$, whence the continuity of the normal trace of \mathbf{q}_h . The above construction leads to the following generalization of Lemma 3.1 with \mathbf{t}_h still

defined by (3.1)–(3.2). The proof, which is similar to that of Lemma 3.1, is skipped for brevity.

LEMMA 3.2 (Reconstructed convection residual). *There holds*

$$(\nabla \cdot \mathbf{t}_h + \nabla \cdot \mathbf{q}_h + (\mu - \nabla \cdot \boldsymbol{\beta})u_h, \xi_h)_{0,T} = (f, \xi_h)_{0,T} \quad \forall T \in \mathcal{T}_h, \forall \xi_h \in \mathbb{P}_l(T). \quad (3.8)$$

Moreover, for matching meshes,

$$(\nabla \cdot \mathbf{t}_h + \nabla \cdot \mathbf{q}_h + \Pi_l((\mu - \nabla \cdot \boldsymbol{\beta})u_h))|_T = \Pi_l f|_T \quad \forall T \in \mathcal{T}_h, \quad (3.9)$$

and, in particular, when μ and $\nabla \cdot \boldsymbol{\beta}$ are elementwise constant and $l = k$,

$$(\nabla \cdot \mathbf{t}_h + \nabla \cdot \mathbf{q}_h + (\mu - \nabla \cdot \boldsymbol{\beta})u_h)|_T = \Pi_k f|_T \quad \forall T \in \mathcal{T}_h. \quad (3.10)$$

3.2.2. Reconstruction by solving local linear systems. Proceeding as in Section 3.1.2, the reconstructed convective–diffusive flux $\mathbf{y}_h \in \mathbf{RTN}^l(\widehat{\mathcal{T}}_h)$ is defined on each $T \in \mathcal{T}_h$ by solving the following minimization problem:

$$\mathbf{y}_h|_T = \arg \inf_{\mathbf{v}_h \in \mathbf{RTN}_{N,\text{DG}}^l(\mathfrak{R}_T), \nabla \cdot \mathbf{v}_h = \widehat{\Pi}_l f_0} \|\mathbf{K}^{\frac{1}{2}} \nabla u_h - \mathbf{K}^{-\frac{1}{2}} \boldsymbol{\beta} u_h + \mathbf{K}^{-\frac{1}{2}} \mathbf{v}_h\|_{0,T}, \quad (3.11)$$

where the space $\mathbf{RTN}_{N,\text{DG}}^l(\mathfrak{R}_T)$ is defined as before, but with the normal flux data $\phi_F(u_h) = -\mathbf{n}_F \cdot \{\{\mathbf{K} \nabla_h u_h\}\}_\omega + \boldsymbol{\beta} \cdot \mathbf{n}_F \{u_h\} + \gamma_F \llbracket u_h \rrbracket$, and where $f_0 = f - (\mu - \nabla \cdot \boldsymbol{\beta})u_h$. Again, using (2.7)–(2.8), it is readily verified that the minimizing set is not empty. Moreover, (3.11) is equivalent to finding $\mathbf{y}_h \in \mathbf{RTN}_{N,\text{DG}}^l(\mathfrak{R}_T)$ and $p_h \in \mathbb{P}_l^*(\mathfrak{R}_T)$ such that

$$\begin{aligned} (\mathbf{K}^{-1} \mathbf{y}_h + \nabla u_h - \mathbf{K}^{-1} \boldsymbol{\beta} u_h, \mathbf{v}_h)_{0,T} - (p_h, \nabla \cdot \mathbf{v}_h)_{0,T} &= 0 \quad \forall \mathbf{v}_h \in \mathbf{RTN}_{N,0}^l(\mathfrak{R}_T), \\ (\nabla \cdot \mathbf{y}_h, \phi_h)_{0,T} &= (f - (\mu - \nabla \cdot \boldsymbol{\beta})u_h, \phi_h)_{0,T} \quad \forall \phi_h \in \mathbb{P}_l^*(\mathfrak{R}_T). \end{aligned}$$

Letting $\mathbf{y}'_h := \mathbf{y}_h + \mathbf{K} \nabla u_h - \boldsymbol{\beta} u_h$, (\mathbf{y}'_h, p_h) corresponds to the mixed finite element approximation to the local problem

$$\begin{aligned} -\nabla \cdot (\mathbf{K} \nabla p) &= f + \nabla \cdot (\mathbf{K} \nabla u_h) - \boldsymbol{\beta} \cdot \nabla u_h - \mu u_h && \text{in } T, \\ -\mathbf{K} \nabla p \cdot \mathbf{n}_T &= (\mathbf{n}_T \cdot \mathbf{n}_F) \phi_F(u_h) + (\mathbf{K} \nabla u_h - \boldsymbol{\beta} u_h)|_T \cdot \mathbf{n}_T && \text{on all } F \in \mathcal{F}_T, \\ (p, 1)_T &= 0. \end{aligned}$$

4. Guaranteed and robust energy norm a posteriori error estimates in the pure diffusion case. This section deals with energy norm a posteriori error estimates in the pure diffusion case. The main results are Theorems 4.1, 4.6, and 4.9.

4.1. Abstract framework. The following theorem provides the abstract framework for a posteriori error estimates in the pure diffusion case.

THEOREM 4.1 (Abstract a posteriori error estimate and its efficiency in the pure diffusion case). *Let $\boldsymbol{\beta} = \mu = 0$, let u be the unique solution of (2.5), and let $u_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then,*

$$\begin{aligned} \| \|u - u_h\| \|^2 &\leq \inf_{s \in H_0^1(\Omega)} \| \|u_h - s\| \|^2 \\ &\quad + \inf_{\mathbf{t} \in \mathbf{H}(\text{div}, \Omega)} \sup_{\varphi \in H_0^1(\Omega), \|\varphi\|=1} ((f - \nabla \cdot \mathbf{t}, \varphi) - (\mathbf{K} \nabla_h u_h + \mathbf{t}, \nabla \varphi))^2 \\ &\leq 2 \| \|u - u_h\| \|^2. \end{aligned} \quad (4.1)$$

Proof. It follows from [20, Lemma 4.4] that

$$\|u - u_h\|^2 \leq \inf_{s \in H_0^1(\Omega)} \|u_h - s\|^2 + \sup_{\varphi \in H_0^1(\Omega), \|\varphi\|=1} \mathcal{B}(u - u_h, \varphi)^2. \quad (4.2)$$

Owing to (2.5), $\mathcal{B}(u, \varphi) = (f, \varphi)$, whence it is inferred employing the Green theorem that for arbitrary $\mathbf{t} \in \mathbf{H}(\text{div}, \Omega)$,

$$\begin{aligned} \mathcal{B}(u - u_h, \varphi) &= (f, \varphi) - (\mathbf{K}\nabla_h u_h, \nabla\varphi) = (f, \varphi) - (\mathbf{K}\nabla_h u_h + \mathbf{t}, \nabla\varphi) + (\mathbf{t}, \nabla\varphi) \\ &= (f - \nabla \cdot \mathbf{t}, \varphi) - (\mathbf{K}\nabla_h u_h + \mathbf{t}, \nabla\varphi). \end{aligned}$$

Combining the above results yields the upper error bound. For the lower error bound, it suffices to put $s = u$, $\mathbf{t} = -\mathbf{K}\nabla u$, and to use the Cauchy–Schwarz inequality and the fact that $\|\varphi\| = 1$. \square

REMARK 4.2 (Form of the abstract estimate of Theorem 4.1). *It has been already noted, e.g., in [1, 2, 38, 20] that the first term of (4.2) evaluates the so-called “nonconforming error” in the scalar unknown u_h , by measuring its distance to the space $H_0^1(\Omega)$. The second term of (4.2) is sometimes referred to as the “conforming error”, cf. [2, 20]. Relation (4.1) actually shows that this second term evaluates the nonconformity in the flux $-\mathbf{K}\nabla_h u_h$, by measuring its distance to the space $\mathbf{H}(\text{div}, \Omega)$, along with the residual, which is expressed using the auxiliary vector field \mathbf{t} which has the appropriate regularity.*

REMARK 4.3 (Global effectivity index). *The abstract a posteriori error estimate of Theorem 4.1 is quasi-exact in the sense that the effectivity index, i.e., the ratio of the estimated to the actual error, is equal to $\sqrt{2}$. A variant of this result, where the effectivity index is improved to 1, is discussed in Remark 4.16 below.*

REMARK 4.4 (Robustness with respect to data, polynomial degree, and meshes). *The abstract a posteriori error estimate of Theorem 4.1 is fully robust with respect to \mathbf{K} without any assumption on its distribution, with respect to f (no polynomial form needed), with respect to the polynomial degree k , and with respect to the mesh, which can be nonmatching, anisotropic, and not even shape-regular, in the sense that the effectivity index does not depend on these quantities.*

REMARK 4.5 (A first computable a posteriori error estimate). *Owing to the Cauchy–Schwarz inequality, the Friedrichs inequality $\|\varphi\|^2 \leq C_{F,\Omega} \ell_\Omega^2 \|\nabla\varphi\|^2$ (where ℓ_Ω is a length scale associated with Ω), the definition (2.4) of the energy norm, and the fact that $\|\varphi\| = 1$, it is inferred from (4.1) that*

$$\|u - u_h\|^2 \leq \|u_h - s\|^2 + \left(\frac{C_{F,\Omega}^{1/2} \ell_\Omega}{\min_{T \in \mathcal{T}_h} c_{\mathbf{K},T}^{1/2}} \|f - \nabla \cdot \mathbf{t}\| + \|\mathbf{K}^{\frac{1}{2}} \nabla_h u_h + \mathbf{K}^{-\frac{1}{2}} \mathbf{t}\| \right)^2 \quad (4.3)$$

for any $s \in H_0^1(\Omega)$ and any $\mathbf{t} \in \mathbf{H}(\text{div}, \Omega)$. This is an estimate similar to that proposed by Lazarov et al. [24] for isotropic and homogeneous diffusion ($\mathbf{K} = \text{Id}$). As promoted in [24], this estimate is scheme-independent. However, scheme-independence also entails that some information specific to the properties of the scheme can be missed. As we will see later, this information can actually be used to improve the residual. Another disadvantage of the above estimate is that the dependence on the diffusion tensor \mathbf{K} is very unfavorable in the presence of strong inhomogeneities.

4.2. Locally computable a posteriori error estimate. To derive a computable estimate from Theorem 4.1, particular $s \in H_0^1(\Omega)$ and $\mathbf{t} \in \mathbf{H}(\text{div}, \Omega)$ need to be chosen. As far as \mathbf{t} is concerned, we promote the two options discussed in Section 3. Concerning s , the Oswald interpolate of u_h was already considered in, *e.g.*, [1, 19, 15, 39] for matching grids. For nonmatching grids, we define $\mathcal{I}_{\text{Os}} : V^k(\mathcal{T}_h) \rightarrow V^k(\widehat{\mathcal{T}}_h) \cap H_0^1(\Omega)$ following [39] as follows: For a function $v_h \in V^k(\mathcal{T}_h)$, observing that $V^k(\mathcal{T}_h) \subset V^k(\widehat{\mathcal{T}}_h)$, $\mathcal{I}_{\text{Os}}(v_h)$ is constructed using the usual Oswald interpolate defined on $V^k(\widehat{\mathcal{T}}_h)$, that is, the values of $\mathcal{I}_{\text{Os}}(v_h)$ are prescribed at suitable (Lagrange) nodes of the simplices of $\widehat{\mathcal{T}}_h$. At the nodes located inside Ω , the average of the values of v_h at this node is used,

$$\mathcal{I}_{\text{Os}}(v_h)(V) = \frac{1}{\#(\widehat{\mathcal{T}}_V)} \sum_{T \in \widehat{\mathcal{T}}_V} v_h|_T(V),$$

where $\widehat{\mathcal{T}}_V$ is the set of those $T \in \widehat{\mathcal{T}}_h$ to which the node V belongs and where for any set S , $\#(S)$ denotes its cardinality. Note that $\mathcal{I}_{\text{Os}}(v_h)(V) = v_h(V)$ at those nodes V lying in the interior of some $T \in \mathcal{T}_h$. At boundary nodes, the value of $\mathcal{I}_{\text{Os}}(v_h)$ is set to zero.

We are now ready to state our a posteriori error estimators. For all $T \in \mathcal{T}_h$, we define the *nonconformity estimator* $\eta_{\text{NC},T}$, the *diffusive flux estimator* $\eta_{\text{DF},T}$, and the *residual estimator* $\eta_{\text{R},T}$ as

$$\eta_{\text{NC},T} := \| \|u_h - \mathcal{I}_{\text{Os}}(u_h)\| \|_T, \quad (4.4)$$

$$\eta_{\text{DF},T} := \| \mathbf{K}^{\frac{1}{2}} \nabla u_h + \mathbf{K}^{-\frac{1}{2}} \mathbf{t}_h \|_{0,T}, \quad (4.5)$$

$$\eta_{\text{R},T} := m_T \| f - \nabla \cdot \mathbf{t}_h \|_{0,T}, \quad (4.6)$$

$$\eta_{\text{R},T} := \left\{ \sum_{T' \in \mathfrak{R}_T} m_{T'}^2 \| f - \widehat{\Pi}_l f \|_{0,T'}^2 \right\}^{1/2}, \quad (4.7)$$

where (4.6) is used if \mathbf{t}_h is prescribed by (3.1)–(3.2), whereas (4.7) is used if \mathbf{t}_h solves the minimization problem (3.5). Recall that in both cases, $\mathbf{t}_h \in \mathbf{RTN}^l(\widehat{\mathcal{T}}_h)$ with $l \in \{k-1, k\}$. Furthermore, for a generic element T of \mathcal{T}_h or of $\widehat{\mathcal{T}}_h$, we have set

$$m_T^2 := C_P c_{\mathbf{K},T}^{-1} h_T^2,$$

where C_P is the constant from the Poincaré inequality

$$\| \varphi - \Pi_0 \varphi \|_{0,T}^2 \leq C_P h_T^2 \| \nabla \varphi \|_{0,T}^2 \quad \forall \varphi \in H^1(T), \quad (4.8)$$

with $\Pi_0 \varphi$ changed into $\widehat{\Pi}_0 \varphi$ if $T \in \widehat{\mathcal{T}}_h$. The constant C_P can be evaluated as $1/\pi^2$ owing to the convexity of simplices [26, 5]. The main result of this section is then:

THEOREM 4.6 (Locally computable a posteriori error estimate in the pure diffusion case). *Let $\beta = \mu = 0$, let u be the unique solution of (2.5), and let u_h be its DG approximation solving (2.8). Then,*

$$\| \|u - u_h\| \| \leq \left\{ \sum_{T \in \mathcal{T}_h} \{ \eta_{\text{NC},T}^2 + (\eta_{\text{R},T} + \eta_{\text{DF},T})^2 \} \right\}^{1/2}.$$

Proof. Put $s = \mathcal{I}_{\text{Os}}(u_h)$ and $\mathbf{t} = \mathbf{t}_h$ in Theorem 4.1. If \mathbf{t}_h is defined by (3.1)–(3.2), then for each $T \in \mathcal{T}_h$,

$$|(f - \nabla \cdot \mathbf{t}_h, \varphi)_{0,T}| = |(f - \nabla \cdot \mathbf{t}_h, \varphi - \Pi_0 \varphi)_{0,T}| \leq \eta_{\mathbf{R},T} \|\varphi\|_T, \quad (4.9)$$

using Lemma 3.1, the Poincaré inequality (4.8), the definition (2.4) of the energy norm, and the Cauchy–Schwarz inequality. If \mathbf{t}_h solves (3.5), proceeding similarly yields

$$|(f - \nabla \cdot \mathbf{t}_h, \varphi)_{0,T}| \leq \sum_{T' \in \mathfrak{R}_T} |(f - \widehat{\Pi}_l f, \varphi - \widehat{\Pi}_0 \varphi)_{0,T'}| \leq \eta_{\mathbf{R},T} \|\varphi\|_T.$$

Next, $|(\mathbf{K} \nabla_h u_h + \mathbf{t}_h, \nabla \varphi)_{0,T}| \leq \eta_{\text{DF},T} \|\varphi\|_T$ is immediate. It now suffices to use the Cauchy–Schwarz inequality and to notice that $\|\varphi\| = 1$ to complete the proof. \square

REMARK 4.7 (Properties of the estimate of Theorem 4.6). *The following properties of the estimate of Theorem 4.6 can be mentioned:*

- It yields a guaranteed upper bound, i.e., features no undetermined constant.
- The residual estimator $\eta_{\mathbf{R},T}$ coincides with the classical (properly weighted) “data oscillation term” for matching meshes, whence it represents a major improvement of the classical residual estimator of the form $c_{\mathbf{K}} h_T \|f + \nabla \cdot (\mathbf{K} \nabla_h u_h)\|_{0,T}$. In particular, for piecewise constant \mathbf{K} and $k = 1$, this latter estimator reduces to $c_{\mathbf{K}} h_T \|f\|_{0,T}$, whereas $\eta_{\mathbf{R},T}$ is, for piecewise smooth f , one to two orders better. In practice, although it represents a higher-order term, $\eta_{\mathbf{R},T}$ should not be neglected since it can be significant on coarse grids or for strongly inhomogeneous or anisotropic \mathbf{K} .
- The Poincaré constant $C_{\mathbf{P}}$ is independent of the shape-regularity of the mesh and of the polynomial degree of u_h . Hence, the present estimate is, in contrast to [19, 6], valid also on anisotropic meshes and uniformly with respect to k .
- No assumption on the polynomial form of f is needed at this stage.
- If $\eta_{\mathbf{R},T}$ is evaluated by (4.6), the present estimate is valid for any $\mathbf{t}_h \in \mathbf{H}(\text{div}, \Omega)$ such that $(\nabla \cdot \mathbf{t}_h, 1)_{0,T} = (f, 1)_{0,T}$ for all $T \in \mathcal{T}_h$, which is a local (conservation) property, in contrast to the global Galerkin orthogonality used traditionally for conforming finite element methods.

REMARK 4.8 (Alternative form of Theorem 4.6). *For nonmatching meshes, letting for all $T \in \mathcal{T}_h$, $\mathbf{t}_T^{\text{pre}} \in \mathbf{RTN}_{\mathbf{N},\text{DG}}^l(\mathfrak{R}_T)$ be defined by (3.1)–(3.2) and $\mathbf{t}_T^{\text{sol}} \in \mathbf{RTN}_{\mathbf{N},\text{DG}}^l(\mathfrak{R}_T)$ solve (3.5), the assertion of Theorem 4.6 can be improved to*

$$\| \|u - u_h\| \| \leq \left\{ \sum_{T \in \mathcal{T}_h} \left\{ \eta_{\mathbf{NC},T}^2 + \min_{\mathbf{t}_T \in \{\mathbf{t}_T^{\text{pre}}, \mathbf{t}_T^{\text{sol}}\}} (\eta_{\mathbf{R},T}(\mathbf{t}_T) + \eta_{\text{DF},T}(\mathbf{t}_T))^2 \right\} \right\}^{1/2}.$$

4.3. Local efficiency of the locally computable estimate. In this section, we prove the local efficiency of the a posteriori error estimate derived in Theorem 4.6, or more precisely, of the bound stated in Remark 4.8. Hence, it is sufficient to examine the local efficiency of $\eta_{\mathbf{R},T}$ and $\eta_{\text{DF},T}$ evaluated with \mathbf{t}_h defined by (3.1)–(3.2).

To achieve robustness with respect to diffusion inhomogeneities, we restrict here the class of interior-penalty DG schemes by setting

$$\gamma_{\mathbf{K},F} := \frac{\delta_{\mathbf{K},F+} \delta_{\mathbf{K},F-}}{\delta_{\mathbf{K},F+} + \delta_{\mathbf{K},F-}}, \quad \delta_{\mathbf{K},F\mp} := \mathbf{n}_F \cdot \mathbf{K}|_{T\mp(F)} \mathbf{n}_F, \quad (4.10)$$

for all $F = \partial T^-(F) \cap \partial T^+(F) \in \mathcal{F}_h^{\text{int}}$ and $\gamma_{\mathbf{K},F} = \mathbf{n}_F \cdot \mathbf{K}|_{T(F)} \mathbf{n}_F$ for all $F = \partial T(F) \cap \partial \Omega \in \mathcal{F}_h^{\text{ext}}$, while the weights are chosen such that for all $F \in \mathcal{F}_h^{\text{int}}$,

$$\omega_{T^-(F),F} := \frac{\delta_{\mathbf{K},F+}}{\delta_{\mathbf{K},F+} + \delta_{\mathbf{K},F-}}, \quad \omega_{T^+(F),F} := \frac{\delta_{\mathbf{K},F-}}{\delta_{\mathbf{K},F+} + \delta_{\mathbf{K},F-}}, \quad (4.11)$$

as recently proposed in [18]. This choice implies that for all $F \in \mathcal{F}_h$,

$$\frac{1}{2} \min_{T \in \mathcal{T}_F} c_{\mathbf{K},T} \leq \gamma_{\mathbf{K},F} \leq \min_{T \in \mathcal{T}_F} C_{\mathbf{K},T}, \quad (4.12)$$

and for all $F \in \mathcal{F}_h^{\text{int}}$, $T \in \mathcal{T}_F$, and T' such that $F = \partial T \cap \partial T'$,

$$(\mathbf{n}_F \cdot \mathbf{K}|_{T} \mathbf{n}_F) \omega_{T,F} \leq \gamma_{\mathbf{K},F}, \quad C_{\mathbf{K},T}^{-1/2} c_{\mathbf{K},T'}^{1/2} \bar{\omega}_{T,F} \leq \frac{1}{2}. \quad (4.13)$$

We are now ready to state the main result of this section.

THEOREM 4.9 (Local efficiency of the estimate of Theorem 4.6). *Let $\beta = \mu = 0$, let $\{\mathcal{T}_h\}_{h>0}$ satisfy Assumptions (A1)–(A2), let f be a piecewise polynomial of degree m , let u be the unique solution of (2.5), and let u_h be its DG approximation solving (2.8) with penalty parameters $\gamma_{\mathbf{K},F}$ and weights $\omega_{T,F}$ defined by (4.10)–(4.11). Let us finally set $c_{\mathbf{K},\mathfrak{T}_T} := \min_{T' \in \mathfrak{T}_T} c_{\mathbf{K},T'}$, and*

$$\|v\|_{*,\mathcal{F}}^2 := \sum_{F \in \mathcal{F}} \|\gamma_F^{1/2} \llbracket v \rrbracket\|_{0,F}^2 \quad v \in H^1(\mathcal{T}_h), \quad (4.14)$$

where we will take either $\mathcal{F} = \mathcal{F}_h$, $\mathcal{F} = \mathcal{F}_T$, or $\mathcal{F} = \mathfrak{T}_T$. Let $T \in \mathcal{T}_h$ and let $\eta_{\text{NC},T}$, $\eta_{\text{DF},T}$ and $\eta_{\text{R},T}$ be defined by (4.4)–(4.6) with \mathbf{t}_h defined by (3.1)–(3.2). Then,

$$\eta_{\text{R},T} \leq C \frac{C_{\mathbf{K},T}^{1/2}}{c_{\mathbf{K},T}^{1/2}} (\|u - u_h\|_T + \eta_{\text{DF},T}), \quad (4.15)$$

$$\eta_{\text{NC},T} \leq \tilde{C} \frac{C_{\mathbf{K},T}^{1/2}}{c_{\mathbf{K},\mathfrak{T}_T}^{1/2}} \|u - u_h\|_{*,\mathfrak{T}_T}, \quad (4.16)$$

$$\eta_{\text{DF},T} \leq \tilde{C} \frac{C_{\mathbf{K},T}^{1/2}}{c_{\mathbf{K},T}^{1/2}} \left(\|u - u_h\|_{*,\mathcal{F}_T} + \sum_{T' \in \mathcal{T}_T} \frac{C_{\mathbf{K},T'}^{1/2}}{c_{\mathbf{K},T'}^{1/2}} \|u - u_h\|_{T'} \right), \quad (4.17)$$

where the constant C only depends on the space dimension d , the polynomial degree k , the shape-regularity parameters $\kappa_{\hat{\tau}}$ and ι_T , and the polynomial degree m of f , and \tilde{C} in addition depends on the DG parameters α_F and θ .

Proof. Let C and \tilde{C} denote generic constants depending on the parameters as in the statement of the theorem. Let $T \in \mathcal{T}_h$. We proceed in three steps.

(1) Bound on $\eta_{\text{R},T}$. The estimate

$$\|f - \nabla \cdot \mathbf{t}_h\|_{0,T} \leq CC_{\mathbf{K},T}^{1/2} h_T^{-1} \|\mathbf{K}^{\frac{1}{2}} \nabla u + \mathbf{K}^{-\frac{1}{2}} \mathbf{t}_h\|_{0,T}$$

is classically derived using element bubble functions [33]; (4.15) then results from the triangle inequality and the definition of m_T .

(2) Bound on $\eta_{\text{NC},T}$. For $T' \in \hat{\mathcal{T}}_h$ and $v_h \in V^k(\hat{\mathcal{T}}_h)$, recall the estimate

$$\|\nabla(v_h - \mathcal{I}_{\text{Os}}(v_h))\|_{0,T'}^2 \leq C \sum_{F \in \hat{\mathfrak{T}}_{T'}} h_F^{-1} \|\llbracket v_h \rrbracket\|_{0,F}^2,$$

proved in [19, Theorem 2.2] (see also [9]). Using this estimate, (4.12), and the fact that $\llbracket u_h \rrbracket = 0$ on those faces that lie in the interior of T , it is easy to see that

$$\eta_{\text{NC},T}^2 \leq C_{\mathbf{K},T} \sum_{T' \in \mathfrak{R}_T} \|\nabla(u_h - \mathcal{I}_{\text{Os}}(u_h))\|_{0,T'}^2 \leq \tilde{C} \frac{C_{\mathbf{K},T}}{c_{\mathbf{K},\mathfrak{R}_T}} \sum_{F \in \mathfrak{F}_T} \alpha_F \gamma_{\mathbf{K},F} h_F^{-1} \|\llbracket u_h \rrbracket\|_{0,F}^2.$$

Since $\llbracket u_h - u \rrbracket = \llbracket u_h \rrbracket$, this yields (4.16).

(3) Bound on $\eta_{\text{DF},T}$. Set $\mathbf{v}_h := (\mathbf{K}\nabla u_h + \mathbf{t}_h)|_T \in \mathbf{RTN}^l(\mathfrak{R}_T)$. Let $T' \in \mathfrak{R}_T$. Standard arguments show that

$$\|\mathbf{v}_h\|_{0,T'}^2 \leq C \left\{ \sum_{F \in \hat{\mathcal{F}}_{T'}} h_F \|\mathbf{v}_h \cdot \mathbf{n}_F\|_{0,F}^2 + \left(\sup_{\mathbf{r}_h \in \mathbb{P}_{l-1}^d(T')} \frac{(\mathbf{v}_h, \mathbf{r}_h)_{0,T'}}{\|\mathbf{r}_h\|_{0,T'}} \right)^2 \right\}.$$

Owing to (3.2), (4.13), the Cauchy–Schwarz inequality, and an inverse inequality,

$$(\mathbf{v}_h, \mathbf{r}_h)_{0,T'} = \theta \sum_{F \in \hat{\mathcal{F}}_{T'}} \omega_{T',F} (\mathbf{n}_F \cdot \mathbf{K} \mathbf{r}_h, \llbracket u_h \rrbracket)_{0,F} \leq \tilde{C} \|\mathbf{r}_h\|_{0,T'} \sum_{F \in \hat{\mathcal{F}}_{T'}} \gamma_{\mathbf{K},F} h_F^{-1/2} \|\llbracket u_h \rrbracket\|_{0,F},$$

and the last sum can be restricted to those faces located in ∂T . Furthermore, owing to (3.1), it is inferred that for $F \in \hat{\mathcal{F}}_{T'}$, $\mathbf{v}_h \cdot \mathbf{n}_F = 0$ if $F \not\subset \partial T$, whereas if $F \subset \partial T$,

$$\mathbf{v}_h \cdot \mathbf{n}_F = \bar{\omega}_{T,F} \mathbf{n}_F \cdot \llbracket \mathbf{K} \nabla_h u_h \rrbracket + \alpha_F \gamma_{\mathbf{K},F} h_F^{-1} \hat{\Pi}_{l,F}(\llbracket u_h \rrbracket).$$

As a result,

$$\begin{aligned} \eta_{\text{DF},T}^2 &\leq c_{\mathbf{K},T}^{-1} \|\mathbf{v}_h\|_{0,T}^2 = c_{\mathbf{K},T}^{-1} \sum_{T' \in \mathfrak{R}_T} \|\mathbf{v}\|_{0,T'}^2 \\ &\leq \tilde{C} c_{\mathbf{K},T}^{-1} \left\{ \sum_{F \in \mathcal{F}_T} h_F \|\bar{\omega}_{T,F} \mathbf{n}_F \cdot \llbracket \mathbf{K} \nabla_h u_h \rrbracket\|_{0,F}^2 + \gamma_{\mathbf{K},F}^2 h_F^{-1} \|\llbracket u_h \rrbracket\|_{0,F}^2 \right\}. \end{aligned}$$

Estimating the first term as in [17, Proposition 3.2] and using (4.12) for the second yields (4.17). \square

REMARK 4.10 (Local efficiency norm and global efficiency with respect to the energy norm). *The local efficiency result of Theorem 4.9 is stated for the DG energy norm, which is the energy norm augmented by the $\|\cdot\|_{*,\mathcal{F}_h}$ -jump seminorm. Owing to the result of Ainsworth [3, Theorem 3], the global efficiency of $\eta_{\text{NC},T}$ and $\eta_{\text{DF},T}$ in the energy norm $\|\cdot\|$ follows from (4.16)–(4.17) for sufficiently large parameters α_F in the case $d = 2$, $k = 1$, $\mathbf{K} = \text{Id}$, $\theta = 1$, and matching meshes.*

REMARK 4.11 (Efficiency of the residual estimator). *On matching meshes, the residual estimator $\eta_{\text{R},T}$ coincides with the usual “data oscillation term” and is of higher order if f is locally smooth.*

REMARK 4.12 (Robustness with respect to inhomogeneities and anisotropies). *Owing to (4.15) and (4.17), the estimators $\eta_{\text{R},T}$ and $\eta_{\text{DF},T}$ are fully robust with respect to diffusion inhomogeneities. This is an important property in practical applications, e.g., when dealing with underground flows. The properties (4.12)–(4.13) play a crucial role in this respect. Furthermore, under the assumption of “monotonicity around vertices” for the distribution of inhomogeneities introduced by Bernardi and Verfürth [7, Hypothesis 2.7], also the nonconformity estimator $\eta_{\text{NC},T}$ evaluated with the weighted Oswald interpolate proposed by Ainsworth [2] can be shown to be robust*

with respect to inhomogeneities. Finally, full robustness with respect to anisotropies is not achieved, but, at least, the local efficiency estimates only depend on local (elementwise) anisotropies.

REMARK 4.13 (Arbitrary number of hanging nodes). *It can be observed that Theorem 4.9 still holds if Assumption (A2) is lifted, thus allowing for an arbitrary number of hanging nodes, provided the faces of the original mesh \mathcal{T}_h are not subdivided by the faces of $\hat{\mathcal{T}}_h$.*

REMARK 4.14 (Generalization to other DG schemes). *Making appropriate changes in the proof of Theorem 4.9, all the presented results (up to the robustness with respect to inhomogeneities) extend to all the DG schemes included in the setting (2.7)–(2.8), even if the properties (4.12)–(4.13) are not satisfied.*

REMARK 4.15 (Lower bound for the classical estimator). *We observe that the nonconformity estimator $\eta_{\text{NC},T}$ represents a lower bound for the classical nonconformity estimator $\{\sum_{F \in \hat{\mathcal{F}}_T} h_F^{-1} \|\llbracket u_h \rrbracket\|_{0,F}^2\}^{1/2}$. Similarly, it follows from the proof of Theorem 4.9 that the diffusive flux estimator $\eta_{\text{DF},T}$ represents a lower bound for the classical gradient jump estimator $\{\sum_{F \in \mathcal{F}_T} h_F \|\mathbf{n}_F \cdot \llbracket \mathbf{K} \nabla_h u_h \rrbracket\|_{0,F}^2\}^{1/2}$ plus again the classical nonconformity estimator.*

REMARK 4.16 (A posteriori error estimates for the reconstructed flux). *Using the ideas of [40, Corollary 6.5], it is readily seen that an a posteriori error estimate for the postprocessed flux \mathbf{t}_h can be derived in the form*

$$\|\mathbf{K}^{-\frac{1}{2}} \mathbf{t}_h + \mathbf{K}^{\frac{1}{2}} \nabla u\|^2 \leq \inf_{s \in H_0^1(\Omega)} \|\mathbf{K}^{-\frac{1}{2}} \mathbf{t}_h + \mathbf{K}^{\frac{1}{2}} \nabla s\|^2 + \eta_{\mathbf{R}}^2 \leq \|\mathbf{K}^{-\frac{1}{2}} \mathbf{t}_h + \mathbf{K}^{\frac{1}{2}} \nabla u\|^2 + \eta_{\mathbf{R}}^2,$$

with $\eta_{\mathbf{R}}^2 = \sum_{T \in \mathcal{T}_h} \eta_{\mathbf{R},T}^2$. Whenever the residual estimators are of higher order, this yields the optimal effectivity index of 1. For practical purposes, a first choice for s is $s = \mathcal{I}_{\text{Os}}(u_h)$. More precise choices are suggested and studied in [40] for mixed finite elements.

5. General case: guaranteed and semi-robust a posteriori error estimates in the energy norm. We extend in this section the analysis of Section 4 to the general case (1.1a)–(1.1b). We are still interested in estimating the energy norm of the error. To achieve semi-robustness, we show that the estimators can be scaled by suitable cutoff functions of the local Péclet and Damköhler numbers (or variations thereof) which, on each $T \in \mathcal{T}_h$, can be defined as $h_T \|\boldsymbol{\beta}\|_{\infty, T} c_{\mathbf{K}, T}^{-1}$ and $h_T^2 c_{\boldsymbol{\beta}, \mu, T} c_{\mathbf{K}, T}^{-1}$, respectively. The main results of this section are Theorems 5.1, 5.5, and 5.10.

5.1. Abstract a posteriori error estimate and its efficiency. The abstract form of our a posteriori error estimate now takes the following form:

THEOREM 5.1 (Abstract energy norm a posteriori error estimate and its efficiency in the general case). *Let u be the unique solution of (2.5) and let $u_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then,*

$$\begin{aligned} \| \|u - u_h\| \| &\leq \inf_{s \in H_0^1(\Omega)} \left\{ \| \|u_h - s\| \| \right. \\ &\quad + \inf_{\mathbf{t} \in \mathbf{H}(\text{div}, \Omega)} \sup_{\varphi \in H_0^1(\Omega), \| \varphi \| = 1} \left\{ (f - \nabla \cdot \mathbf{t} - \boldsymbol{\beta} \cdot \nabla_h s - \mu s, \varphi) \right. \\ &\quad \left. \left. - (\mathbf{K} \nabla_h u_h + \mathbf{t}, \nabla \varphi) + \left(\left(\mu - \frac{1}{2} \nabla \cdot \boldsymbol{\beta} \right) (s - u_h), \varphi \right) \right\} \right\} \\ &\leq 2 \| \|u - u_h\| \|. \end{aligned} \quad (5.1)$$

Proof. It has been proved in [38, Lemma 7.1] (see also [17, Lemma 3.1]) that

$$\begin{aligned} \|u - u_h\| \leq & \inf_{s \in H_0^1(\Omega)} \left\{ \|u_h - s\| + \sup_{\varphi \in H_0^1(\Omega), \|\varphi\|=1} \left\{ \mathcal{B}(u - u_h, \varphi) \right. \right. \\ & \left. \left. + (\boldsymbol{\beta} \cdot \nabla_h(u_h - s) + \frac{1}{2}(\nabla \cdot \boldsymbol{\beta})(u_h - s), \varphi) \right\} \right\}. \end{aligned} \quad (5.2)$$

It suffices to use (2.5) therein, to introduce an arbitrary $\mathbf{t} \in \mathbf{H}(\text{div}, \Omega)$, and to employ the Green theorem to infer

$$\begin{aligned} \mathcal{B}(u - u_h, \varphi) + (\boldsymbol{\beta} \cdot \nabla_h(u_h - s) + \frac{1}{2}(\nabla \cdot \boldsymbol{\beta})(u_h - s), \varphi) &= (f - \nabla \cdot \mathbf{t} - \boldsymbol{\beta} \cdot \nabla_h u_h - \mu u_h, \varphi) \\ &- (\mathbf{K} \nabla_h u_h + \mathbf{t}, \nabla \varphi) + (\boldsymbol{\beta} \cdot \nabla_h(u_h - s) + \frac{1}{2}(\nabla \cdot \boldsymbol{\beta})(u_h - s), \varphi), \end{aligned}$$

yielding the upper error bound. For the lower error bound, put $s = u$ and $\mathbf{t} = -\mathbf{K} \nabla u$ and use the Cauchy–Schwarz inequality and the fact that $\|\varphi\| = 1$. \square

REMARK 5.2 (Comparison with Theorem 4.1). *In comparison with Theorem 4.1, Theorem 5.1 is applicable to the general convection–diffusion–reaction case. In particular, there is an additional contribution from the skew-symmetric part of the bilinear form \mathcal{B} , which can be evaluated using an arbitrary $s \in H_0^1(\Omega)$. The price for this generality is that Theorem 5.1 yields a triangular-like inequality instead of a Pythagorean-like inequality.*

REMARK 5.3 (A computable a posteriori error estimate). *As in the pure diffusion case (cf. Remark 4.5), using the Friedrichs inequality $\|\varphi\|^2 \leq C_{F,\Omega} \ell_\Omega^2 \|\nabla \varphi\|^2$ or the inequality $\|\varphi\| \leq \{\min_{T \in \mathcal{T}_h} c_{\boldsymbol{\beta}, \mu, T}^{1/2}\}^{-1} \|(\mu - \frac{1}{2} \nabla \cdot \boldsymbol{\beta})^{1/2} \varphi\|$, the Cauchy–Schwarz inequality, the definition (2.4) of the energy norm, and the fact that $\|\varphi\| = 1$, it is inferred from (5.1) that*

$$\begin{aligned} \|u - u_h\| \leq & \|u_h - s\| + \left(\|\mathbf{K}^{\frac{1}{2}} \nabla_h u_h + \mathbf{K}^{-\frac{1}{2}} \mathbf{t}\|^2 + \|(\mu - \frac{1}{2} \nabla \cdot \boldsymbol{\beta})^{\frac{1}{2}}(u_h - s)\|^2 \right)^{1/2} \\ & + \min \left\{ \frac{C_{F,\Omega}^{1/2} \ell_\Omega}{\min_{T \in \mathcal{T}_h} c_{\mathbf{K}, T}^{1/2}}, \frac{1}{\min_{T \in \mathcal{T}_h} c_{\boldsymbol{\beta}, \mu, T}^{1/2}} \right\} \|f - \nabla \cdot \mathbf{t} - \boldsymbol{\beta} \cdot \nabla_h s - \mu s\|, \end{aligned}$$

for any $s \in H_0^1(\Omega)$ and any $\mathbf{t} \in \mathbf{H}(\text{div}, \Omega)$. This is a fully computable and scheme-independent estimate, but all the points made in Remark 4.5 apply here also.

To derive a locally computable estimate in the next section, we would like to consider the same $H_0^1(\Omega)$ -conforming scalar function s and the same $\mathbf{H}(\text{div}, \Omega)$ -conforming diffusive flux \mathbf{t} as in Sections 4.2 and 3, respectively. With this choice, however, the residual $f - \nabla \cdot \mathbf{t} - \boldsymbol{\beta} \cdot \nabla_h s - \mu s$ does not satisfy an orthogonality property such as (3.3) and is not even necessarily of zero mean elementwise, a property that is necessary to obtain a computable estimate with the proper scaling as in (4.9). This motivates the following form of Theorem 5.1, where \mathbf{q} is a $\mathbf{H}(\text{div}, \Omega)$ -conforming convective flux

reconstruction. The proof, which is straightforward, is skipped for brevity.

THEOREM 5.4 (Another form of Theorem 5.1). *There holds*

$$\begin{aligned}
 \| \|u - u_h\| \| &\leq \inf_{s \in H_0^1(\Omega)} \left\{ \| \|u_h - s\| \| \right. \\
 &\quad + \inf_{\mathbf{q} \in \mathbf{H}(\operatorname{div}, \Omega)} \inf_{\mathbf{t} \in \mathbf{H}(\operatorname{div}, \Omega)} \sup_{\varphi \in H_0^1(\Omega), \| \varphi \| = 1} \left\{ (f - \nabla \cdot \mathbf{t} - \nabla \cdot \mathbf{q} - (\mu - \nabla \cdot \boldsymbol{\beta})u_h, \varphi) \right. \\
 &\quad \left. \left. - (\mathbf{K} \nabla_h u_h + \mathbf{t}, \nabla \varphi) + (\nabla \cdot \mathbf{q} - \nabla \cdot (\boldsymbol{\beta}s), \varphi) - \left(\frac{1}{2} (\nabla \cdot \boldsymbol{\beta})(u_h - s), \varphi \right) \right\} \right\} \\
 &\leq 2 \| \|u - u_h\| \|. \tag{5.3}
 \end{aligned}$$

5.2. Locally computable a posteriori error estimate. To keep things simple, we begin by treating the case of matching meshes; the modifications for nonmatching meshes are described in Remark 5.9 below. To formulate the locally computable a posteriori error estimates, we will need the constants appearing in the following approximation results: For all $\varphi \in H_0^1(\Omega)$,

$$\| \varphi - \Pi_0 \varphi \|_{0,T} \leq m_T \| \varphi \|_T \quad \forall T \in \mathcal{T}_h, \tag{5.4}$$

$$\| \varphi - \Pi_0 \varphi |_T \|_{0,F} \leq C_{t,T,F}^{1/2} \tilde{m}_T^{1/2} \| \varphi \|_T \quad \forall T \in \mathcal{T}_h, \forall F \in \mathcal{F}_T, \tag{5.5}$$

$$\| [\Pi_0 \varphi] \|_{0,F} \leq m_F \sum_{T \in \mathcal{T}_F} \| \varphi \|_T \quad \forall F \in \mathcal{F}_h, \tag{5.6}$$

with the cutoff functions

$$m_T^2 := \min \{ C_P h_T^2 c_{\mathbf{K},T}^{-1}, c_{\boldsymbol{\beta},\mu,T}^{-1} \}, \tag{5.7}$$

$$\tilde{m}_T := \min \{ (C_P + C_P^{1/2}) h_T c_{\mathbf{K},T}^{-1}, h_T^{-1} c_{\boldsymbol{\beta},\mu,T}^{-1} + c_{\boldsymbol{\beta},\mu,T}^{-1/2} c_{\mathbf{K},T}^{-1/2} / 2 \}, \tag{5.8}$$

$$m_F^2 := \min \left\{ \max_{T \in \mathcal{T}_F} \left\{ C_{F,T,F} \frac{|F| h_T^2}{|T| c_{\mathbf{K},T}} \right\}, \max_{T \in \mathcal{T}_F} \left\{ \frac{|F|}{|T| c_{\boldsymbol{\beta},\mu,T}} \right\} \right\}, \tag{5.9}$$

where $|F|$ denotes the measure of F . Recall that C_P is the constant from the Poincaré inequality (4.8). In addition, $C_{t,T,F}$ and $C_{F,T,F}$ are respectively the constants from the following trace and generalized Friedrichs inequalities:

$$\| \varphi \|_{0,F}^2 \leq C_{t,T,F} (h_T^{-1} \| \varphi \|_T^2 + \| \varphi \|_T \| \nabla \varphi \|_T), \tag{5.10}$$

$$\| \varphi - \Pi_0 \varphi \|_{0,T}^2 \leq C_{F,T,F} h_T^2 \| \nabla \varphi \|_{0,T}^2, \tag{5.11}$$

valid for all $T \in \mathcal{T}_h$, $\varphi \in H^1(T)$, and $F \in \mathcal{F}_T$. It follows from [32, p. 73] that $C_{t,T,F} = |F| h_T / |T|$ for a simplex T and its face F ; see also [10]. Furthermore, it follows from [37, Lemma 4.1] that $C_{F,T,F} = 3d$ for a simplex T and its face F . The estimate (5.4) is readily inferred from the Poincaré inequality (4.8) and the fact that $\| \varphi - \Pi_0 \varphi \|_{0,T} \leq \| \varphi \|_{0,T}$. The estimate (5.5) is established in [11]. Finally, the estimate (5.6) is proved in [39, Lemma 4.5].

We are now ready to formulate our a posteriori error estimators. Let $T \in \mathcal{T}_h$. While we keep the definition (4.4) of the nonconformity estimator $\eta_{\text{NC},T}$, the *residual estimator* $\eta_{\text{R},T}$ and the *diffusive flux estimator* $\eta_{\text{DF},T}$ are now defined as

$$\eta_{\text{R},T} := m_T \| f - \nabla \cdot \mathbf{t}_h - \nabla \cdot \mathbf{q}_h - (\mu - \nabla \cdot \boldsymbol{\beta})u_h \|_{0,T}, \tag{5.12}$$

$$\eta_{\text{DF},T} := \min \left\{ \eta_{\text{DF},T}^{(1)}, \eta_{\text{DF},T}^{(2)} \right\}, \tag{5.13}$$

where

$$\eta_{\text{DF},T}^{(1)} := \|\mathbf{K}^{\frac{1}{2}}\nabla u_h + \mathbf{K}^{-\frac{1}{2}}\mathbf{t}_h\|_{0,T}, \quad (5.14)$$

$$\begin{aligned} \eta_{\text{DF},T}^{(2)} &:= m_T \|(Id - \Pi_0)(\nabla \cdot (\mathbf{K}\nabla u_h + \mathbf{t}_h))\|_{0,T} \\ &\quad + \tilde{m}_T^{1/2} \sum_{F \in \mathcal{F}_T} C_{t,T,F}^{1/2} \|(\mathbf{K}\nabla u_h + \mathbf{t}_h) \cdot \mathbf{n}_F\|_{0,F}, \end{aligned} \quad (5.15)$$

with \mathbf{t}_h defined by (3.1)–(3.2). Finally, we define the two *convection estimators* $\eta_{\text{C},1,T}$ and $\eta_{\text{C},2,T}$ and the *upwinding estimator* $\eta_{\text{U},T}$ as

$$\eta_{\text{C},1,T} := m_T \|(Id - \Pi_0)(\nabla \cdot (\mathbf{q}_h - \beta s_h))\|_{0,T}, \quad (5.16)$$

$$\eta_{\text{C},2,T} := c_{\beta,\mu,T}^{-1/2} \left\| \frac{1}{2} (\nabla \cdot \beta)(u_h - s_h) \right\|_{0,T}, \quad (5.17)$$

$$\eta_{\text{U},T} := \sum_{F \in \mathcal{F}_T} m_F \|\Pi_{0,F}((\mathbf{q}_h - \beta s_h) \cdot \mathbf{n}_F)\|_{0,F}, \quad (5.18)$$

with $s_h = \mathcal{I}_{\text{Os}}(u_h)$ and \mathbf{q}_h defined by (3.6)–(3.7). We can now state the main result of this section.

THEOREM 5.5 (Locally computable energy norm a posteriori error estimate in the general case). *Let u be the unique solution of (2.5) and let u_h be its DG approximation solving (2.8). Then,*

$$\| \|u - u_h\| \| \leq \eta,$$

where

$$\eta := \left\{ \sum_{T \in \mathcal{T}_h} \eta_{\text{NC},T}^2 \right\}^{1/2} + \left\{ \sum_{T \in \mathcal{T}_h} (\eta_{\text{R},T} + \eta_{\text{DF},T} + \eta_{\text{C},1,T} + \eta_{\text{C},2,T} + \eta_{\text{U},T})^2 \right\}^{1/2}.$$

Proof. We start by putting $s = s_h = \mathcal{I}_{\text{Os}}(u_h)$, $\mathbf{t} = \mathbf{t}_h$, and $\mathbf{q} = \mathbf{q}_h$ in the abstract estimate of Theorem 5.4. We next write

$$\begin{aligned} &(f - \nabla \cdot \mathbf{t}_h - \nabla \cdot \mathbf{q}_h - (\mu - \nabla \cdot \beta)u_h, \varphi) - (\mathbf{K}\nabla_h u_h + \mathbf{t}_h, \nabla \varphi) + (\nabla \cdot \mathbf{q}_h - \nabla \cdot (\beta s_h), \varphi) \\ &- \left(\frac{1}{2} (\nabla \cdot \beta)(u_h - s_h), \varphi \right) = \sum_{T \in \mathcal{T}_h} \left\{ (f - \nabla \cdot \mathbf{t}_h - \nabla \cdot \mathbf{q}_h - (\mu - \nabla \cdot \beta)u_h, \varphi - \Pi_0 \varphi)_{0,T} \right. \\ &- (\mathbf{K}\nabla u_h + \mathbf{t}_h, \nabla \varphi)_{0,T} - \left(\frac{1}{2} (\nabla \cdot \beta)(u_h - s_h), \varphi \right)_{0,T} + (\nabla \cdot (\mathbf{q}_h - \beta s_h), \varphi - \Pi_0 \varphi)_{0,T} \\ &\left. + \sum_{F \in \mathcal{F}_T} ((\mathbf{q}_h - \beta s_h) \cdot \mathbf{n}_T, \Pi_0 \varphi)_{0,F} \right\}, \end{aligned} \quad (5.19)$$

using Lemma 3.2 in the first term and subtracting $(\nabla \cdot (\mathbf{q}_h - \beta s_h), \Pi_0 \varphi)_{0,T}$ and adding the same quantity rewritten using the Green theorem in the last two terms. Next, in these last two terms, we replace $\nabla \cdot (\mathbf{q}_h - \beta s_h)$ by $(Id - \Pi_0)(\nabla \cdot (\mathbf{q}_h - \beta s_h))$ and $(\mathbf{q}_h - \beta s_h) \cdot \mathbf{n}_T$ by $\Pi_{0,F}((\mathbf{q}_h - \beta s_h) \cdot \mathbf{n}_T)$. Furthermore, following [11], there are two ways to bound the term $-(\mathbf{K}\nabla u_h + \mathbf{t}_h, \nabla \varphi)_{0,T}$. Either one simply uses

$$-(\mathbf{K}\nabla u_h + \mathbf{t}_h, \nabla \varphi)_{0,T} \leq \eta_{\text{DF},T}^{(1)} \|\varphi\|_T,$$

or one notices using (5.4) and (5.5) that

$$\begin{aligned} -(\mathbf{K}\nabla u_h + \mathbf{t}_h, \nabla\varphi)_{0,T} &= -(\mathbf{K}\nabla u_h + \mathbf{t}_h, \nabla(\varphi - \Pi_0\varphi))_{0,T} \\ &= (\nabla \cdot (\mathbf{K}\nabla u_h + \mathbf{t}_h), \varphi - \Pi_0\varphi)_{0,T} \\ &\quad - \sum_{F \in \mathcal{F}_T} ((\mathbf{K}\nabla u_h + \mathbf{t}_h) \cdot \mathbf{n}_T, \varphi - \Pi_0\varphi)_{0,F} \leq \eta_{\text{DF},T}^{(2)} \|\varphi\|_T. \end{aligned}$$

Finally, using (5.6) and the continuity of the normal component of $(\mathbf{q}_h - \beta s_h)$ for the last term in (5.19), it is inferred that

$$\sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} (\Pi_{0,F}((\mathbf{q}_h - \beta s_h) \cdot \mathbf{n}_T), \Pi_0\varphi)_{0,F} \leq \sum_{T \in \mathcal{T}_h} \eta_{\text{U},T} \|\varphi\|_T.$$

Collecting the above bounds leads to

$$\begin{aligned} &(f - \nabla \cdot \mathbf{t}_h - \nabla \cdot \mathbf{q}_h - (\mu - \nabla \cdot \beta)u_h, \varphi) - (\mathbf{K}\nabla_h u_h + \mathbf{t}_h, \nabla\varphi) + (\nabla \cdot \mathbf{q}_h - \nabla \cdot (\beta s_h), \varphi) \\ &- \left(\frac{1}{2}(\nabla \cdot \beta)(u_h - s_h), \varphi\right) \leq \sum_{T \in \mathcal{T}_h} \left(\eta_{\text{R},T} + \eta_{\text{DF},T} + \eta_{\text{C},1,T} + \eta_{\text{C},2,T} + \eta_{\text{U},T}\right) \|\varphi\|_T, \end{aligned}$$

whence the conclusion is straightforward. \square

REMARK 5.6 (Properties of the estimate of Theorem 5.5). *As in the pure diffusion case, the estimate of Theorem 5.5 yields a guaranteed upper bound, the residual represents a higher-order term in the case of matching meshes owing to Lemma 3.2, the estimate is valid uniformly with respect to the polynomial degree k , no polynomial data form is needed for f , and, finally, the estimate is valid for any $\mathbf{t}_h, \mathbf{q}_h \in \mathbf{H}(\text{div}, \Omega)$ such that $(\nabla \cdot \mathbf{t}_h + \nabla \cdot \mathbf{q}_h + (\mu - \nabla \cdot \beta)u_h, 1)_{0,T} = (f, 1)_{0,T}$ for all $T \in \mathcal{T}_h$.*

REMARK 5.7 (Mean values in $\eta_{\text{C},1,T}$ and $\eta_{\text{U},T}$). *As observed in [39, Remark 4.1], subtracting or using mean values in the estimators $\eta_{\text{C},1,T}$ and $\eta_{\text{U},T}$ can only lower these quantities, with noteworthy improvements in some situations.*

REMARK 5.8 (Form of $\eta_{\text{DF},T}$). *The idea of defining the diffusive flux estimator $\eta_{\text{DF},T}$ as a minimum between two quantities has been proposed in [11]. The purpose is to obtain in singularly perturbed regimes resulting from dominant convection or reaction appropriate cutoff functions in the expression for $\eta_{\text{DF},T}^{(2)}$. This way of proceeding is coherent with the recent observation made by Verfürth [36] that the diffusive flux estimator $\eta_{\text{DF},T}^{(1)}$ alone cannot be shown to be robust.*

REMARK 5.9 (Nonmatching meshes). *In the case of nonmatching meshes, the approximation results (5.4)–(5.6) are employed on $\widehat{\mathcal{T}}_h$ and the cutoff functions m_T, \widetilde{m}_T and m_F as well as the constants $C_{\text{t},T,F}$ and $C_{\text{F},T,F}$ are redefined accordingly for all $T \in \widehat{\mathcal{T}}_h$ and $F \in \widehat{\mathcal{F}}_T$. Then, for all $T \in \mathcal{T}_h$, the estimators $\eta_{\text{NC},T}$, $\eta_{\text{R},T}$, $\eta_{\text{DF},T}^{(1)}$ and $\eta_{\text{C},2,T}$ are kept unchanged. To see how the estimators $\eta_{\text{C},1,T}$ and $\eta_{\text{U},T}$ are modified, consider the last but one term in the right-hand side of (5.19) and introduce $\widehat{\Pi}_0\varphi$ instead of $\Pi_0\varphi$. Then, using $(\cdot, \cdot)_{0,T} = \sum_{T' \in \mathfrak{R}_T} (\cdot, \cdot)_{0,T'}$, it is easily verified that $(\nabla \cdot (\mathbf{q}_h - \beta s_h), \varphi - \widehat{\Pi}_0\varphi)_{0,T} \leq \eta_{\text{C},1,T} \|\varphi\|_T$ with*

$$\eta_{\text{C},1,T} := \left\{ \sum_{T' \in \mathfrak{R}_T} m_{T'}^2 \|(Id - \widehat{\Pi}_0)(\nabla \cdot (\mathbf{q}_h - \beta s_h))\|_{0,T'}^2 \right\}^{1/2}, \quad (5.20)$$

and that

$$\eta_{\text{U},T} := \left\{ \sum_{T' \in \mathfrak{R}_T} \left(\sum_{F \in \widehat{\mathcal{F}}_{T'}, F \cap T \neq \emptyset} m_F \|\widehat{\Pi}_{0,F}((\mathbf{q}_h - \beta s_h) \cdot \mathbf{n}_F)\|_{0,F} \right)^2 \right\}^{1/2}, \quad (5.21)$$

since for $F \cap \partial T = \emptyset$, $(\mathbf{q}_h - \boldsymbol{\beta}s_h) \cdot \mathbf{n}_F = 0$. Proceeding similarly yields

$$\eta_{\text{DF},T}^{(2)} := \left\{ \sum_{T' \in \mathfrak{R}_T} \left(m_{T'} \|(Id - \widehat{\Pi}_0)(\nabla \cdot (\mathbf{K} \nabla u_h + \mathbf{t}_h))\|_{0,T'} \right. \right. \quad (5.22)$$

$$\left. \left. + \widetilde{m}_{T'}^{1/2} \sum_{F \in \widehat{\mathcal{F}}_{T'}, F \subset \partial T} C_{\mathbf{t},T',F}^{1/2} \|(\mathbf{K} \nabla u_h + \mathbf{t}_h) \cdot \mathbf{n}_F\|_{0,F} \right)^2 \right\}^{1/2},$$

since for $F \not\subset \partial T$, $(\mathbf{K} \nabla u_h + \mathbf{t}_h) \cdot \mathbf{n}_F = 0$.

5.3. Local efficiency of the locally computable estimate. It will be convenient to state the local efficiency result using the following residual-based a posteriori error estimators: For all $T \in \mathcal{T}_h$,

$$\rho_{1,T} := m_T \|f + \nabla \cdot (\mathbf{K} \nabla_h u_h) - \boldsymbol{\beta} \cdot \nabla_h u_h - \mu u_h\|_{0,T}, \quad (5.23)$$

$$\rho_{2,T} := m_T^{1/2} c_{\mathbf{K},T}^{-1/4} \sum_{F \in \mathcal{F}_T} \bar{\omega}_{T,F} \|\mathbf{n}_F \cdot [\mathbf{K} \nabla u_h]\|_{0,F}. \quad (5.24)$$

The following semi-robust bounds are proved in [17, Propositions 3.3 and 3.4] under the assumption that f , $\boldsymbol{\beta}$, and μ are piecewise polynomials of degree m and that the penalty parameters $\gamma_{\mathbf{K},F}$ and weights $\omega_{T,F}$ satisfy (4.12)–(4.13):

$$\rho_{1,T} \leq C m_T (C_{\mathbf{K},T}^{1/2} h_T^{-1} + \min(\alpha_{1,T}, \alpha_{2,T})) \|u - u_h\|_T,$$

$$\rho_{2,T} \leq C \frac{C_{\mathbf{K},T}^{1/2}}{c_{\mathbf{K},T}^{1/2}} m_T^{1/2} c_{\mathbf{K},T}^{1/4} \sum_{T' \in \mathcal{T}_T} m_{T'}^{-1/2} c_{\mathbf{K},T'}^{-1/4} \left(\frac{C_{\mathbf{K},T'}^{1/2}}{c_{\mathbf{K},T'}^{1/2}} + m_{T'} \alpha_{1,T'} \right) \|u - u_h\|_{T'},$$

with $\alpha_{1,T} := \|\mu\|_{\infty,T} c_{\boldsymbol{\beta},\mu,T}^{-1/2} + \|\boldsymbol{\beta}\|_{\infty,T} c_{\mathbf{K},T}^{-1/2}$ and $\alpha_{2,T} := c_{\boldsymbol{\beta},\mu,T}^{-1/2} (\|\mu - \nabla \cdot \boldsymbol{\beta}\|_{\infty,T} + \|\boldsymbol{\beta}\|_{\infty,T} h_T^{-1})$. The constant C only depends on d , k , m , $\kappa_{\widehat{\mathcal{T}}}$, and ι_T . We can now state the main result of this section. We treat the case of nonmatching meshes.

THEOREM 5.10 (Local efficiency of the energy norm estimate of Theorem 5.5). *Let $\{\mathcal{T}_h\}_{h>0}$ satisfy Assumptions (A1)–(A2). Assume for simplicity that $\nabla \cdot (\mathbf{q}_h - \boldsymbol{\beta}s_h) \in \mathbb{P}_l(T)$ and $\nabla \cdot (\mathbf{q}_h - \boldsymbol{\beta}u_h) \in \mathbb{P}_l(T)$ for all $T \in \widehat{\mathcal{T}}_h$. Let u be the unique solution of (2.5) and let u_h be its DG approximation solving (2.8) with penalty parameters $\gamma_{\mathbf{K},F}$ and weights $\omega_{T,F}$ defined by (4.10)–(4.11). Assume for simplicity that $\gamma_{\boldsymbol{\beta},F}$ is facewise constant. For all $T \in \mathcal{T}_h$, recalling that $c_{\mathbf{K},\mathfrak{T}_T} := \min_{T' \in \mathfrak{T}_T} c_{\mathbf{K},T'}$ and setting $c_{\mathbf{K},\mathcal{T}_T} := \min_{T' \in \mathcal{T}_T} c_{\mathbf{K},T'}$, $c_{\boldsymbol{\beta},\mathfrak{F}_T} := \min_{F \in \mathfrak{F}_T} \gamma_{\boldsymbol{\beta},F}$ and $c_{\boldsymbol{\beta},\mathcal{F}_T} := \min_{F \in \mathcal{F}_T} \gamma_{\boldsymbol{\beta},F}$, we introduce the cutoff functions*

$$\chi_{\mathfrak{T}_T} := \min(h_T c_{\mathbf{K},\mathfrak{T}_T}^{-1/2}, h_T^{1/2} c_{\boldsymbol{\beta},\mathfrak{F}_T}^{-1/2}), \quad \chi_{\mathcal{T}_T} := \min(h_T c_{\mathbf{K},\mathcal{T}_T}^{-1/2}, h_T^{1/2} c_{\boldsymbol{\beta},\mathcal{F}_T}^{-1/2}), \quad (5.25)$$

as well as $m_{\mathcal{T}_T} = \min(h_T c_{\mathbf{K},\mathcal{T}_T}^{-1/2}, c_{\boldsymbol{\beta},\mu,\mathcal{T}_T}^{-1/2})$ where $c_{\boldsymbol{\beta},\mu,\mathcal{T}_T} := \min_{T' \in \mathcal{T}_T} c_{\boldsymbol{\beta},\mu,T'}$. Let $\eta_{\text{NC},T}$ be defined by (4.4), $\eta_{\text{R},T}$, $\eta_{\text{DF},T}$, $\eta_{\text{C},1,T}$, $\eta_{\text{C},2,T}$ and $\eta_{\text{U},T}$ by (5.12)–(5.18) (or as in Remark 5.9 for nonmatching meshes) with \mathbf{t}_h defined by (3.1)–(3.2), \mathbf{q}_h by (3.6)–(3.7)

and $s_h = \mathcal{I}_{\text{Os}}(u_h)$. Then,

$$\begin{aligned} \eta_{\text{NC},T} &\leq C \left(\frac{C_{\mathbf{K},T}^{1/2}}{c_{\mathbf{K},\mathfrak{T}_T}^{1/2}} + \left\| \mu - \frac{1}{2} \nabla \cdot \boldsymbol{\beta} \right\|_{\infty,T}^{1/2} \chi_{\mathfrak{T}_T} \right) \| \|u - u_h\|_{*,\mathfrak{F}_T}, \\ \eta_{\text{C},2,T} &\leq C \left\| \frac{1}{2} \nabla \cdot \boldsymbol{\beta} \right\|_{\infty,T} c_{\boldsymbol{\beta},\mu,T}^{-1/2} \chi_{\mathfrak{T}_T} \| \|u - u_h\|_{*,\mathfrak{F}_T}, \\ \eta_{\text{U},T} &\leq C m_{\mathcal{T}_T} h_T^{-1} \|\boldsymbol{\beta}\|_{\infty,T} \chi_{\mathfrak{T}_T} \| \|u - u_h\|_{*,\mathfrak{F}_T}, \\ \eta_{\text{C},1,T} &\leq C m_T h_T^{-1} \|\boldsymbol{\beta}\|_{\infty,T} \chi_{\mathfrak{T}_T} \| \|u - u_h\|_{*,\mathfrak{F}_T}, \\ \eta_{\text{R},T} &\leq \rho_{1,T} + C_{\zeta_T} \rho_{2,T} + C \left(\zeta_T^2 \frac{C_{\mathbf{K},T}^{1/2}}{c_{\mathbf{K},T}^{1/2}} + m_T h_T^{-1} \|\boldsymbol{\beta}\|_{\infty,T} \chi_{\mathcal{T}_T} \right) \| \|u - u_h\|_{*,\mathcal{F}_T}, \\ \eta_{\text{DF},T} &\leq C \rho_{2,T} + C_{\zeta_T} \frac{C_{\mathbf{K},T}^{1/2}}{c_{\mathbf{K},T}^{1/2}} \| \|u - u_h\|_{*,\mathcal{F}_T}, \end{aligned}$$

where $\zeta_T := m_T^{1/2} h_T^{-1/2} c_{\mathbf{K},T}^{1/4} \leq C_P^{1/4}$ by construction. Here, the constant C only depends on the space dimension d , the polynomial degree k of u_h , the shape-regularity parameters $\kappa_{\widehat{\mathcal{T}}}$ and ι_T , and the DG parameters α_F and θ .

Proof. Let C denote a generic constant depending on the parameters as in the statement of the theorem. Let $T \in \mathcal{T}_h$. The proof is decomposed in two parts.

(1) Bounds on the estimators involving s_h . First, it follows from (2.9) and the fact that $\llbracket u - u_h \rrbracket = -\llbracket u_h \rrbracket$ that

$$\sum_{F \in \widehat{\mathfrak{F}}_T} \|\llbracket u_h \rrbracket\|_{0,F} \leq C h_T^{-1/2} \chi_{\mathfrak{T}_T} \| \|u - u_h\|_{*,\mathfrak{F}_T}, \quad (5.26)$$

with $\chi_{\mathfrak{T}_T}$ defined by (5.25). Furthermore, for $T' \in \widehat{\mathcal{T}}_h$ and $v_h \in V^k(\widehat{\mathcal{T}}_h)$, it is well-known (see, e.g., [9, Lemma 3.2]) that

$$\|v_h - \mathcal{I}_{\text{Os}}(v_h)\|_{0,T'} \leq C \sum_{F \in \widehat{\mathfrak{F}}_{T'}} h_F^{1/2} \|\llbracket v_h \rrbracket\|_{0,F},$$

so that summing over $T' \in \mathfrak{R}_T$, observing that $\llbracket u_h \rrbracket = 0$ on those faces that lie in the interior of T and using (5.26) yields

$$\|u_h - s_h\|_{0,T} \leq C \chi_{\mathfrak{T}_T} \| \|u - u_h\|_{*,\mathfrak{F}_T}. \quad (5.27)$$

The bound on $\eta_{\text{NC},T}$ follows from (5.27) for the L^2 -part of the $\|\cdot\|$ -norm and by proceeding as in the proof of Theorem 4.9 for the diffusive part. Moreover, the bound on $\eta_{\text{C},2,T}$ is readily inferred from (5.27). Next, we observe that for all $T' \in \mathfrak{R}_T$ and all $F \in \widehat{\mathcal{F}}_{T'}$, $m_F \leq C h_T^{-1/2} m_{\mathcal{T}_T}$, and that owing to (3.6) and the fact that $\|\Pi_{0,F} g\|_{0,F} \leq \|g\|_{0,F}$ for all $g \in L^2(F)$,

$$\begin{aligned} \|\Pi_{0,F}((\mathbf{q}_h - \boldsymbol{\beta} s_h) \cdot \mathbf{n}_F)\|_{0,F} &= \|\Pi_{0,F}(\boldsymbol{\beta} \cdot \mathbf{n}_F \llbracket u_h \rrbracket + \gamma_{\boldsymbol{\beta},F} \llbracket u_h \rrbracket - \boldsymbol{\beta} \cdot \mathbf{n}_F s_h)\|_{0,F} \\ &\leq \|\boldsymbol{\beta} \cdot \mathbf{n}_F \llbracket u_h \rrbracket + \gamma_{\boldsymbol{\beta},F} \llbracket u_h \rrbracket - \boldsymbol{\beta} \cdot \mathbf{n}_F s_h\|_{0,F} \\ &\leq C \|\boldsymbol{\beta}\|_{\infty,T} \sum_{F' \in \widehat{\mathfrak{F}}_{T'}} \|\llbracket u_h \rrbracket\|_{0,F'}, \end{aligned}$$

since $\|u_h - s_h\|_{0,F} \leq C \sum_{F' \in \widehat{\mathfrak{F}}_{T'}} \|\llbracket u_h \rrbracket\|_{0,F'}$ for the Oswald interpolate. Hence, using (5.21), (5.26), and the fact that $\llbracket u_h \rrbracket = 0$ on those faces that lie in the interior of

T , the bound on $\eta_{U,T}$ is inferred. Finally, to prove the bound on $\eta_{C,1,T}$, we observe that for all $T' \in \mathfrak{R}_T$,

$$\|(Id - \widehat{\Pi}_0)(\nabla \cdot (\mathbf{q}_h - \beta s_h))\|_{0,T'} \leq \|\nabla \cdot (\mathbf{q}_h - \beta s_h)\|_{0,T'} = \sup_{\xi \in \mathbb{P}_l(T')} \frac{(\nabla \cdot (\mathbf{q}_h - \beta s_h), \xi)_{0,T'}}{\|\xi\|_{0,T'}},$$

using the assumption that $\nabla \cdot (\mathbf{q}_h - \beta s_h) \in \mathbb{P}_l(T')$. Using the Green theorem and the definitions (3.6)–(3.7) of \mathbf{q}_h yields

$$(\nabla \cdot (\mathbf{q}_h - \beta s_h), \xi)_{0,T'} = -(u_h - s_h, \beta \cdot \nabla \xi)_{0,T'} + \sum_{F \in \widehat{\mathcal{F}}_{T'}} ((\mathbf{q}_h - \beta s_h) \cdot \mathbf{n}_{T'}, \xi)_{0,F}.$$

Using (5.20), the Cauchy–Schwarz inequality, the above bounds for $\|u_h - s_h\|_{0,T'}$ and $\|\widehat{\Pi}_{l,F}((\mathbf{q}_h - \beta s_h) \cdot \mathbf{n}_F)\|_{0,F}$, inverse inequalities to estimate $\|\nabla \xi\|_{0,T'}$ and $\|\xi\|_{0,F}$, and the fact that $m_{T'} \leq Cm_T$ for all $T' \in \mathfrak{R}_T$, the bound on $\eta_{C,1,T}$ is inferred.

(2) Bounds on $\eta_{R,T}$ and $\eta_{DF,T}$. Using the triangle inequality yields

$$\eta_{R,T} \leq \rho_{1,T} + m_T \|\nabla \cdot (\mathbf{K} \nabla u_h + \mathbf{t}_h)\|_{0,T} + m_T \|\nabla \cdot (\mathbf{q}_h - \beta u_h)\|_{0,T},$$

with $\rho_{1,T}$ defined by (5.23). To bound the last two terms in the right-hand side, we proceed as we did above for $\nabla \cdot (\mathbf{q}_h - \beta s_h)$. Since $\nabla \cdot (\mathbf{q}_h - \beta u_h) \in \mathbb{P}_l(T')$ for all $T' \in \mathfrak{R}_T$, it is easy to see that

$$m_T \|\nabla \cdot (\mathbf{q}_h - \beta u_h)\|_{0,T} \leq Cm_T h_T^{-1} \|\beta\|_{\infty, T \chi_{T_T}} \|u - u_h\|_{*, \mathcal{F}_T}.$$

Similarly, for all $T' \in \mathfrak{R}_T$,

$$\sup_{\xi \in \mathbb{P}_l(T')} \frac{(\mathbf{K} \nabla u_h + \mathbf{t}_h, \nabla \xi)_{0,T'}}{\|\xi\|_{0,T'}} \leq C \sum_{F \in \widehat{\mathcal{F}}_{T'}} \gamma_{\mathbf{K},F} h_F^{-3/2} \|[[u_h]]\|_{0,F},$$

and for all $F \in \widehat{\mathcal{F}}_{T'}$,

$$\|(\mathbf{K} \nabla u_h + \mathbf{t}_h) \cdot \mathbf{n}_F\|_{0,F} \leq C(\bar{\omega}_{T,F} \|\mathbf{n}_F \cdot [[\mathbf{K} \nabla u_h]]\|_{0,F} + \gamma_{\mathbf{K},F} h_F^{-1} \|[[u_h]]\|_{0,F}). \quad (5.28)$$

Hence,

$$\|\nabla \cdot (\mathbf{K} \nabla u_h + \mathbf{t}_h)\|_{0,T'} \leq C \sum_{F \in \widehat{\mathcal{F}}_{T'}} (\gamma_{\mathbf{K},F} h_F^{-3/2} \|[[u_h]]\|_{0,F} + h_F^{-1/2} \bar{\omega}_{T,F} \|\mathbf{n}_F \cdot [[\mathbf{K} \nabla u_h]]\|_{0,F}),$$

where only the faces $F \subset \partial T$ need to be considered. As a result,

$$m_T \|\nabla \cdot (\mathbf{K} \nabla u_h + \mathbf{t}_h)\|_{0,T} \leq C \left(\varsigma_T^2 \frac{C_{\mathbf{K},T}^{1/2}}{c_{\mathbf{K},T}^{1/2}} \|u - u_h\|_{*, \mathcal{F}_T} + \varsigma_T \rho_{2,T} \right),$$

with $\rho_{2,T}$ defined by (5.24), whence the bound on $\eta_{R,T}$ is inferred. Finally, since $\eta_{DF,T} \leq \eta_{DF,T}^{(2)}$ owing to (5.13), it suffices to bound $\eta_{DF,T}^{(2)}$. The volume term in (5.22) can be bounded as above since $m_{T'} \leq Cm_T$ for all $T' \in \mathfrak{R}_T$ and $\|(Id - \widehat{\Pi}_0)g\|_{0,T'} \leq \|g\|_{0,T'}$ for all $g \in L^2(T')$. For the face term, we use (5.28) along with the estimate $\tilde{m}_{T'} \leq Cm_T c_{\mathbf{K},T}^{-1/2}$ for all $T' \in \mathfrak{R}_T$ proven in [11]. \square

REMARK 5.11 (Comments on the results of Theorem 5.10). *In the DG energy norm, the a posteriori error estimate of Theorem 5.5 is semi-robust in the sense*

that the bounds on the estimators involve cutoff functions of the local Péclet and Damköhler numbers in various forms. This result is of the same quality as those achieved in [34, 38, 39, 17]. Moreover, as $h \rightarrow 0$, the estimators $\eta_{C,1,T}$, $\eta_{C,2,T}$, and $\eta_{U,T}$ will loose influence, whereas $\eta_{NC,T}$ and $\eta_{DF,T}$ will become optimally efficient. Numerical experiments suggest that $\eta_{DF,T}^{(1)}$ is often well-behaved.

REMARK 5.12 (Efficiency of the residual estimator). *On matching meshes with $l = k$ and μ and $\nabla \cdot \beta$ piecewise constant, Lemma 3.2 implies that $\eta_{R,T}$ coincides with the usual “data oscillation term” which is of higher order if f is piecewise smooth.*

REMARK 5.13 (Efficiency of $\eta_{U,T}$ in comparison with finite volumes or mixed finite elements). *In finite volume or mixed finite element schemes, upwinding can likewise be employed in the convection-dominated regime. However, it seems that no efficiency of the corresponding upwinding estimator can be proved for these schemes, cf. [38, 39], in contrast to the situation with DG methods.*

REMARK 5.14 (Efficiency of $\eta_{C,1,T}$ and $\eta_{U,T}$). *We remark that the improvements in $\eta_{C,1,T}$ and $\eta_{U,T}$ described in Remark 5.7 were not taken into account in the proof of Theorem 5.10. Hence, the actual efficiency of these estimators may still be better.*

6. General case: guaranteed and robust a posteriori error estimates in the augmented norm. In this section we adapt to the DG setting the idea of Verfürth [35] on augmenting the energy norm by a dual norm of the convective derivative. For a related work on matching meshes, we refer to Schötzau and Zhu [31]. The main results of this section are Theorems 6.1, 6.2, and 6.6. For simplicity, the analysis is presented in the case of matching meshes. Modifications for nonmatching meshes are described in Remark 6.5.

Let us first define for $u, v \in H^1(\mathcal{T}_h)$,

$$\begin{aligned} \mathcal{B}_S(u, v) &:= (\mathbf{K} \nabla_h u, \nabla_h v) + ((\mu - \frac{1}{2} \nabla \cdot \beta) u, v), \\ \mathcal{B}_A(u, v) &:= (\beta \cdot \nabla_h u + \frac{1}{2} (\nabla \cdot \beta) u, v), \\ \mathcal{B}_D(u, v) &:= - \sum_{F \in \mathcal{F}_h} (\beta \cdot \mathbf{n}_F \llbracket u \rrbracket, \{\{\Pi_0 v\}\}_{0,F}). \end{aligned}$$

Observe that \mathcal{B}_A is skew-symmetric on $H_0^1(\Omega)$ and that

$$\mathcal{B}(u, v) = \mathcal{B}_S(u, v) + \mathcal{B}_A(u, v) \quad \forall u, v \in H^1(\mathcal{T}_h) \quad (6.1)$$

and recall that $\mathcal{B}_S(v, v) = \|v\|^2$ for all $v \in H^1(\mathcal{T}_h)$. The so-called augmented norm that we will be using for error control is defined as

$$\|v\|_{\oplus} := \|v\| + \sup_{\varphi \in H_0^1(\Omega), \|\varphi\|=1} \{\mathcal{B}_A(v, \varphi) + \mathcal{B}_D(v, \varphi)\} \quad v \in H^1(\mathcal{T}_h). \quad (6.2)$$

Whenever $\|\nabla \cdot \beta\|_{\infty, T}$ is controlled by $c_{\beta, \mu, T}$ for all $T \in \mathcal{T}_h$, the zero-order contribution in \mathcal{B}_A can be discarded in the definition of the augmented norm, recovering the dual norm introduced by Verfürth [35] for conforming finite elements. The additional contribution from \mathcal{B}_D in the augmented norm is specific to the DG setting and has been introduced in the present work to sharpen the global efficiency result; see Remark 6.3 below.

6.1. Abstract a posteriori error estimate and its efficiency. The abstract form of our a posteriori error estimate in the $\|\cdot\|_{\oplus}$ -norm takes the following form:

THEOREM 6.1 (Abstract $\|\cdot\|_{\oplus}$ -norm a posteriori error estimate and its efficiency). *Let u be the unique solution of (2.5) and let $u_h \in H^1(\mathcal{T}_h)$ be arbitrary.*

Then,

$$\begin{aligned}
& \| \|u - u_h\| \|_{\oplus} \\
& \leq 2 \inf_{s \in H_0^1(\Omega)} \left\{ \| \|u_h - s\| \| + \inf_{\mathbf{t} \in \mathbf{H}(\operatorname{div}, \Omega)} \sup_{\varphi \in H_0^1(\Omega), \| \varphi \| = 1} \{ (f - \nabla \cdot \mathbf{t} - \boldsymbol{\beta} \cdot \nabla_h s - \mu s, \varphi) \right. \\
& \quad \left. - (\mathbf{K} \nabla_h u_h + \mathbf{t}, \nabla \varphi) + ((\mu - \frac{1}{2} \nabla \cdot \boldsymbol{\beta})(s - u_h), \varphi) \} \right\} + \sup_{\varphi \in H_0^1(\Omega), \| \varphi \| = 1} \\
& \quad \left\{ \inf_{\mathbf{t} \in \mathbf{H}(\operatorname{div}, \Omega)} \{ (f - \nabla \cdot \mathbf{t} - \boldsymbol{\beta} \cdot \nabla_h u_h - \mu u_h, \varphi) - (\mathbf{K} \nabla_h u_h + \mathbf{t}, \nabla \varphi) \} - \mathcal{B}_D(u_h, \varphi) \right\} \\
& \leq 5 \| \|u - u_h\| \|_{\oplus}. \tag{6.3}
\end{aligned}$$

Proof. Using the definition (6.2) of the $\| \cdot \|_{\oplus}$ -norm and that of the $\| \cdot \|$ -norm, (6.1), the Cauchy–Schwarz inequality, and the fact that $\mathcal{B}_D(u - u_h, \cdot) = -\mathcal{B}_D(u_h, \cdot)$, it is inferred that

$$\| \|u - u_h\| \|_{\oplus} \leq 2 \| \|u - u_h\| \| + \sup_{\varphi \in H_0^1(\Omega), \| \varphi \| = 1} \{ \mathcal{B}(u - u_h, \varphi) - \mathcal{B}_D(u_h, \varphi) \}.$$

For the first term, we simply use (5.1). For the second term, we use (2.5), add and subtract $(\mathbf{t}, \nabla \varphi)$ for an arbitrary $\mathbf{t} \in \mathbf{H}(\operatorname{div}, \Omega)$, and employ the Green theorem. This yields the upper error bound. For the lower error bound, it suffices to use (5.1) for the first term and the fact that

$$\mathcal{B}(u - u_h, \varphi) - \mathcal{B}_D(u_h, \varphi) = \mathcal{B}_S(u - u_h, \varphi) + (\mathcal{B}_A + \mathcal{B}_D)(u - u_h, \varphi) \leq \| \|u - u_h\| \|_{\oplus} \| \varphi \|,$$

for the second one. \square

6.2. Locally computable a posteriori error estimate. To derive a locally computable error upper bound from Theorem 6.1, we define the *modified convection estimator* $\tilde{\eta}_{C,1,T}$ and the *modified upwinding estimator* $\tilde{\eta}_{U,T}$ as

$$\tilde{\eta}_{C,1,T} := m_T \| (Id - \Pi_0)(\nabla \cdot (\mathbf{q}_h - \boldsymbol{\beta} u_h)) \|_{0,T}, \tag{6.4}$$

$$\tilde{\eta}_{U,T} := \sum_{F \in \mathcal{F}_T} m_F \| \Pi_{0,F}(\gamma_{\boldsymbol{\beta},F} \llbracket u_h \rrbracket) \|_{0,F}, \tag{6.5}$$

where \mathbf{q}_h is still defined by (3.6)–(3.7). Let η , $\eta_{R,T}$, and $\eta_{DF,T}$ be as defined in Section 5.2. We can now state our main result.

THEOREM 6.2 (Locally computable $\| \cdot \|_{\oplus}$ -norm a posteriori error estimate). *Let u be the unique solution of (2.5) and let u_h be its DG approximation solving (2.8). Then,*

$$\| \|u - u_h\| \|_{\oplus} \leq \tilde{\eta} := 2\eta + \left\{ \sum_{T \in \mathcal{T}_h} (\eta_{R,T} + \eta_{DF,T} + \tilde{\eta}_{C,1,T} + \tilde{\eta}_{U,T})^2 \right\}^{1/2}. \tag{6.6}$$

Proof. We start from the abstract estimate of Theorem 6.1, where we put $s = s_h = \mathcal{I}_{Os}(u_h)$ and $\mathbf{t} = \mathbf{t}_h$ defined by (3.1)–(3.2). As the first term is bounded by 2η owing to Theorem 5.5, we only bound the second one. Proceeding as in the proof of

Theorem 5.5 leads to

$$\begin{aligned}
 & (f - \nabla \cdot \mathbf{t}_h - \boldsymbol{\beta} \cdot \nabla_h u_h - \mu u_h, \varphi) - (\mathbf{K} \nabla_h u_h + \mathbf{t}_h, \nabla \varphi) - \mathcal{B}_D(u_h, \varphi) \\
 &= \sum_{T \in \mathcal{T}_h} \left\{ (f - \nabla \cdot \mathbf{t}_h - \nabla \cdot \mathbf{q}_h - (\mu - \nabla \cdot \boldsymbol{\beta}) u_h, \varphi - \Pi_0 \varphi)_{0,T} - (\mathbf{K} \nabla_h u_h + \mathbf{t}_h, \nabla \varphi)_{0,T} \right. \\
 & \quad \left. + (\nabla \cdot (\mathbf{q}_h - \boldsymbol{\beta} u_h), \varphi - \Pi_0 \varphi)_{0,T} + \sum_{F \in \mathcal{F}_T} ((\mathbf{q}_h - \boldsymbol{\beta} u_h) \cdot \mathbf{n}_T, \Pi_0 \varphi)_{0,F} \right\} - \mathcal{B}_D(u_h, \varphi) \\
 & \leq \sum_{T \in \mathcal{T}_h} (\eta_{R,T} + \eta_{DF,T} + \tilde{\eta}_{C,1,T} + \tilde{\eta}_{U,T}) \|\varphi\|_T.
 \end{aligned}$$

For the last two terms, letting $\mathbf{y}_h = \mathbf{q}_h - \boldsymbol{\beta} u_h$, we have used the relation

$$\begin{aligned}
 \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} (\mathbf{y}_h \cdot \mathbf{n}_T, \Pi_0 \varphi)_{0,F} &= \sum_{F \in \mathcal{F}_h} (\mathbf{n}_F \cdot \llbracket \mathbf{y}_h \rrbracket, \{\{\Pi_0 \varphi\}\})_{0,F} + (\mathbf{n}_F \cdot \llbracket \mathbf{y}_h \rrbracket, \llbracket \Pi_0 \varphi \rrbracket)_{0,F} \\
 &= \mathcal{B}_D(u_h, \varphi) + \sum_{F \in \mathcal{F}_h} (\Pi_{0,F}(\gamma_{\boldsymbol{\beta},F} \llbracket u_h \rrbracket), \llbracket \Pi_0 \varphi \rrbracket)_{0,F},
 \end{aligned}$$

and the right-hand side is estimated using (5.6), leading to the $\tilde{\eta}_{U,T}$ estimator. \square

REMARK 6.3 (Role of \mathcal{B}_D in the augmented norm). *Adding the bilinear form \mathcal{B}_D to the augmented norm plays an important role in that it eliminates the term $\mathcal{B}_D(u_h, \varphi)$ from the above expression.*

REMARK 6.4 (Practical relevance of Theorem 6.2). *We observe that the estimator $\tilde{\eta}$ is fully computable. This is, however, not the case of the $\|\cdot\|_{\oplus}$ -norm, which, as usual, can only be approximated. Moreover, the estimator $\tilde{\eta}$ has basically the same structure as the estimator η derived in Theorem 5.5, so that for practical purposes, η appears to be sufficient. The motivation for Theorem 6.2 is mainly theoretical, since it paves the road for a fully robust a posteriori error estimate, as discussed below.*

REMARK 6.5 (Nonmatching meshes). *In the case of nonmatching meshes, the bilinear form \mathcal{B}_D is modified as*

$$\mathcal{B}_D(u, v) := - \sum_{F \in \hat{\mathcal{F}}'_h} (\boldsymbol{\beta} \cdot \mathbf{n}_F \llbracket u \rrbracket, \{\{\hat{\Pi}_0 v\}\})_{0,F},$$

where $\hat{\mathcal{F}}'_h = \{F \in \hat{\mathcal{F}}_h; \exists T \in \mathcal{T}_h, F \subset \partial T\}$, while the estimators $\tilde{\eta}_{C,1,T}$ and $\tilde{\eta}_{U,T}$ are modified similarly to the estimators $\eta_{C,1,T}$ and $\eta_{U,T}$ in Remark 5.9; see (5.20) and (5.21). Then, the proof of Theorem 6.2 proceeds as above by considering $\hat{\Pi}_0 \varphi$ instead of $\Pi_0 \varphi$.

6.3. Global efficiency of the locally computable estimate. We show here that the $\|\cdot\|_{\oplus}$ -norm a posteriori error estimate of Theorem 6.2 is globally efficient and fully robust.

THEOREM 6.6 (Global efficiency of the $\|\cdot\|_{\oplus}$ -norm estimate of Theorem 6.2). *Along with the assumptions of Theorem 5.10, assume that f , $\boldsymbol{\beta}$, and μ are piecewise polynomials of degree m . For all $v \in H^1(\mathcal{T}_h)$, define*

$$\begin{aligned}
 \|v\|_{\#, \mathcal{F}_h}^2 &:= \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathfrak{F}_T} \frac{1}{\#(\mathfrak{F}_F)} \left\{ \frac{c_{\mathbf{K},T}}{c_{\mathbf{K},\mathfrak{T}_T}} \alpha_F \gamma_{\mathbf{K},F} h_F^{-1} \|v\|_{0,F}^2 + c_{\boldsymbol{\beta},\mu,T} h_F \|v\|_{0,F}^2 \right. \\
 & \quad \left. + m_{\mathcal{T}_T}^2 \|\boldsymbol{\beta}\|_{\infty, \mathcal{T}_T}^2 h_F^{-1} \|v\|_{0, \mathcal{F}_F \cap \mathfrak{F}_T}^2 \right\}, \tag{6.7}
 \end{aligned}$$

where $m_{\mathcal{T}_T}$ is defined in Theorem 5.10 and \mathcal{F}_F collects the faces of the one or two elements in \mathcal{T}_F . Then,

$$\tilde{\eta} \leq \tilde{C}(\|u - u_h\|_{\oplus} + \|u - u_h\|_{\#, \mathcal{F}_h}),$$

where the constant \tilde{C} depends on the same parameters as the constant C in Theorem 5.10 and in addition on the polynomial degree m of f , β , and μ , the ratios $C_{\mathbf{K},T}/c_{\mathbf{K},T}$ and $(\|\mu\|_{\infty,T} + \|\frac{1}{2}\nabla \cdot \beta\|_{\infty,T})/c_{\beta,\mu,T}$ for all $T \in \mathcal{T}_h$ and the ratios $c_{\beta,\mu,T}/c_{\beta,\mu,T'}$ for all $T, T' \in \mathcal{T}_h$ sharing a face.

Proof. Proceeding as in the proof of Theorem 5.10 and using similar bounds on the estimators $\tilde{\eta}_{C,1,T}$ and $\tilde{\eta}_{U,T}$ of Theorem 6.2, it is inferred that

$$\tilde{\eta} \leq \tilde{C} \left\{ \sum_{T \in \mathcal{T}_h} (\rho_{1,T}^2 + \rho_{2,T}^2) \right\}^{1/2} + \tilde{C} \|u_h\|_{\#, \mathcal{F}_h},$$

where $\rho_{1,T}$ and $\rho_{2,T}$ are defined by (5.23)–(5.24) (recall that these quantities represent residual-based a posteriori error estimators). Since $\|u_h\|_{\#, \mathcal{F}_h} = \|u - u_h\|_{\#, \mathcal{F}_h}$, it remains to bound the contributions from the residuals $\rho_{1,T}$ and $\rho_{2,T}$. For simplicity, we present the proof for matching meshes; it can be extended to nonmatching meshes by proceeding as before. For all $T \in \mathcal{T}_h$, let ψ_T be the element bubble function introduced by Verfürth in [34, §3], $R_T := (f + \nabla \cdot (\mathbf{K} \nabla u_h) - \beta \cdot \nabla u_h - \mu u_h)|_T$ and $\Psi_T := \psi_T R_T$. Observe that

$$\sum_{T \in \mathcal{T}_h} \rho_{1,T}^2 \leq \tilde{C} \sum_{T \in \mathcal{T}_h} m_T^2 (\mathcal{B}_S(u - u_h, \Psi_T) + (\mathcal{B}_A + \mathcal{B}_D)(u - u_h, \Psi_T) - \mathcal{B}_D(u - u_h, \Psi_T)).$$

Since $m_T \|\Psi_T\|_T \leq \tilde{C} \|R_T\|_{0,T}$ with a constant \tilde{C} depending on the local ratios $C_{\mathbf{K},T}/c_{\mathbf{K},T}$ and $(\|\mu\|_{\infty,T} + \|\frac{1}{2}\nabla \cdot \beta\|_{\infty,T})/c_{\beta,\mu,T}$, it is easy to see that the first two terms in the above right-hand side are bounded by $\|u - u_h\|_{\oplus} \{\sum_{T \in \mathcal{T}_h} \rho_{1,T}^2\}^{1/2}$. Concerning the last term, we use an inverse inequality to infer

$$\sum_{T \in \mathcal{T}_h} m_T^2 \mathcal{B}_D(u - u_h, \Psi_T) \leq \tilde{C} \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} m_T \|R_T\|_{0,T} m_T \|\beta\|_{\infty,T} h_F^{-1/2} \|[u_h]\|_{0,F},$$

which can be bounded by $\|u - u_h\|_{\#, \mathcal{F}_h} \{\sum_{T \in \mathcal{T}_h} \rho_{1,T}^2\}^{1/2}$. Consider now $\rho_{2,T}$. For all $F \in \mathcal{F}_h$, let ψ_F be the face bubble function introduced by Verfürth in [34, §3] (see also [17]), $R_F := \mathbf{n}_F \cdot [\mathbf{K} \nabla_h u_h]$, and let Ψ_F be the lifting of $\psi_F R_F$ to \mathcal{T}_F . Observe that

$$\begin{aligned} \sum_{T \in \mathcal{T}_h} \rho_{2,T}^2 &\leq \tilde{C} \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} m_T c_{\mathbf{K},T}^{-1/2} \bar{\omega}_{T,F}^2 \left\{ -\mathcal{B}_S(u - u_h, \Psi_F) - (\mathcal{B}_A + \mathcal{B}_D)(u - u_h, \Psi_F) \right. \\ &\quad \left. + \mathcal{B}_D(u - u_h, \Psi_F) + \sum_{T' \in \mathcal{T}_F} (R_{T'}, \Psi_F)_{0,T'} \right\} := T_1 + T_2 + T_3 + T_4. \end{aligned}$$

We first consider T_1 and observe that (up to a multiplicative constant \tilde{C})

$$\begin{aligned} |T_1| &\leq \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} m_T c_{\mathbf{K},T}^{-1/2} \bar{\omega}_{T,F}^2 \sum_{T' \in \mathcal{T}_F} \|u - u_h\|_{T'} \|\Psi_F\|_{T'} \\ &\leq \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} m_T^{1/2} c_{\mathbf{K},T}^{-1/4} \bar{\omega}_{T,F} \|R_F\|_{0,F} \sum_{T' \in \mathcal{T}_F} (m_T^{1/2} c_{\mathbf{K},T}^{-1/4} \bar{\omega}_{T,F} m_{T'}^{-1/2} c_{\mathbf{K},T'}^{1/4}) \|u - u_h\|_{T'} \\ &\leq \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} m_T^{1/2} c_{\mathbf{K},T}^{-1/4} \bar{\omega}_{T,F} \|R_F\|_{0,F} \sum_{T' \in \mathcal{T}_F} \|u - u_h\|_{T'}, \end{aligned}$$

since $\|\Psi_F\|_{T'} \leq \tilde{C} m_{T'}^{-1/2} c_{\mathbf{K},T'}^{1/4} \|R_F\|_{0,F}$ and since, owing to (4.13),

$$m_T^{1/2} c_{\mathbf{K},T}^{-1/4} \bar{\omega}_{T,F} m_{T'}^{-1/2} c_{\mathbf{K},T'}^{1/4} \leq m_T^{1/2} \bar{\omega}_{T,F}^{1/2} m_{T'}^{-1/2} \leq \tilde{C}, \quad (6.8)$$

with \tilde{C} depending on the ratios $c_{\beta,\mu,T}/c_{\beta,\mu,T'}$. The bound on T_2 is similar (details are skipped for brevity) leading to $|T_1| + |T_2| \leq \tilde{C} \|u - u_h\|_{\oplus} \{\sum_{T \in \mathcal{T}_h} \rho_{2,T}^2\}^{1/2}$. We next consider T_3 and observe that (up to a multiplicative constant \tilde{C})

$$\begin{aligned} |T_3| &\leq \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} m_T c_{\mathbf{K},T}^{-1/2} \bar{\omega}_{T,F}^2 \|\beta\|_{\infty, \mathcal{T}_T} \sum_{F' \in \mathcal{F}_F} \|[u_h]\|_{0,F'} \|\{\Pi_0 \Psi_F\}\|_{0,F'} \\ &\leq \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} m_T^{1/2} c_{\mathbf{K},T}^{-1/4} \bar{\omega}_{T,F} \|R_F\|_{0,F} m_{T'} \|\beta\|_{\infty, \mathcal{T}_T} \sum_{F' \in \mathcal{F}_F} h_{F'}^{-1/2} \|[u_h]\|_{0,F'}, \end{aligned}$$

where we have used the inverse inequality $\|\{\Pi_0 \Psi_F\}\|_{0,F'} \leq \tilde{C} h_{F'}^{-1/2} \|\Psi_F\|_{0, \mathcal{T}_{F'} \cap \mathcal{T}_F}$, the fact that $\|\Psi_F\|_{0,T'} \leq \tilde{C} m_{T'}^{1/2} c_{\mathbf{K},T'}^{1/4} \|R_F\|_{0,F}$, and the bound (6.8). This yields $|T_3| \leq \tilde{C} \|u - u_h\|_{\#, \mathcal{F}_h} \{\sum_{T \in \mathcal{T}_h} \rho_{2,T}^2\}^{1/2}$. Finally, we proceed similarly to bound T_4 to obtain $|T_4| \leq \tilde{C} \{\sum_{T \in \mathcal{T}_h} \rho_{1,T}^2\}^{1/2} \{\sum_{T \in \mathcal{T}_h} \rho_{2,T}^2\}^{1/2}$. Using the previous estimate for $\{\sum_{T \in \mathcal{T}_h} \rho_{1,T}^2\}^{1/2}$ completes the proof. \square

REMARK 6.7 (Comparison with the results of [31]). *The result of Theorem 6.6 is in its form comparable with that reported in [31]. One essential difference is, however, that our discrete jump seminorm $\|\cdot\|_{\#, \mathcal{F}_h}$ contains the cutoff factors $m_{T'}$ in front of $\|\beta\|_{\infty, \mathcal{T}_T} h_F^{-1/2} \|[v]\|_{0, \mathcal{F}_F \cap \mathcal{F}_T}$, which can considerably reduce the size of this term.*

REMARK 6.8 (Robustness with respect to reaction- and convection-dominance). *Theorem 6.2 yields a guaranteed and fully robust estimate provided the error is measured in the $(\|\cdot\|_{\oplus} + \|\cdot\|_{\#, \mathcal{F}_h})$ -norm, since*

$$\|u - u_h\|_{\oplus} + \|u - u_h\|_{\#, \mathcal{F}_h} \leq \tilde{\eta} + \|u_h\|_{\#, \mathcal{F}_h} \leq \tilde{C} (\|u - u_h\|_{\oplus} + \|u - u_h\|_{\#, \mathcal{F}_h}).$$

However, the discrete seminorm $\|\cdot\|_{\#, \mathcal{F}_h}$ is not fully satisfactory since it does not appear in the natural DG stability norm. In particular, a priori error estimates including this new seminorm have not been established. Moreover, the $\|\cdot\|_{\#, \mathcal{F}_h}$ -seminorm is not easily localizable with respect to data.

7. Numerical experiments. We present in this section the results of several numerical experiments.

7.1. Pure diffusion. We have examined three test cases, all posed on the domain $\Omega = \{-1 < x, y < 1\}$ with Dirichlet boundary conditions. The diffusion tensor is isotropic (but heterogeneous in test cases 2 and 3) and can thus be identified with a scalar diffusion coefficient κ . The weighted interior-penalty DG scheme proposed in [18] is used with polynomial degree $k = 1$ and penalty parameter and weights defined by (4.10)–(4.11). The diffusive flux \mathbf{t}_h was reconstructed using (3.1)–(3.2) for both $l = 0$ or $l = 1$. Next, the piecewise affine Oswald interpolate $\mathcal{I}_{\text{Os}}(u_h)$ of the discrete solution u_h was used. In the subsequent tables, sequences of uniformly refined, structured or unstructured meshes are considered to evaluate the convergence rates and N indicates the number of mesh elements. For simplicity, only matching meshes are considered. Columns labeled “eff” report the global effectivity index, that is the ratio of the a posteriori error estimate to the actual error, both quantities being computed over all mesh elements. We employ the following notation for the error estimators: $\eta_{\text{NC}} := \{\sum_{T \in \mathcal{T}_h} \eta_{\text{NC},T}^2\}^{1/2}$, $\eta_{\text{R}} := \{\sum_{T \in \mathcal{T}_h} \eta_{\text{R},T}^2\}^{1/2}$, and so on.

| N | $\ u - u_h\ $ | η_{NC} | $l = 0$ | | | $l = 1$ | | |
|-------|---------------|--------------------|-------------------|--------------------|------|-------------------|--------------------|------|
| | | | η_{R} | η_{DF} | eff. | η_{R} | η_{DF} | eff. |
| 128 | 3.28e-1 | 1.89e-1 | 7.23e-2 | 3.38e-1 | 1.2 | 5.50e-3 | 4.32e-1 | 1.4 |
| 512 | 1.62e-1 | 9.72e-2 | 1.82e-2 | 1.69e-1 | 1.2 | 6.90e-4 | 2.22e-1 | 1.5 |
| 2048 | 8.04e-2 | 4.89e-2 | 4.54e-3 | 8.39e-2 | 1.2 | 8.64e-5 | 1.12e-1 | 1.5 |
| 8192 | 4.01e-2 | 2.45e-2 | 1.14e-3 | 4.18e-2 | 1.2 | 1.08e-5 | 5.64e-2 | 1.5 |
| order | 1.0 | 1.0 | 2.0 | 1.0 | - | 3.0 | 1.0 | - |

TABLE 7.1

Convergence rates of error estimators for test case 1, structured meshes

| N | $\ u - u_h\ $ | η_{NC} | $l = 0$ | | | $l = 1$ | | |
|-------|---------------|--------------------|-------------------|--------------------|------|-------------------|--------------------|------|
| | | | η_{R} | η_{DF} | eff. | η_{R} | η_{DF} | eff. |
| 112 | 3.16e-1 | 1.25e-1 | 7.01e-2 | 3.60e-1 | 1.2 | 5.13e-3 | 3.58e-1 | 1.2 |
| 448 | 1.58e-1 | 6.85e-2 | 1.76e-2 | 1.82e-1 | 1.2 | 6.90e-4 | 2.22e-1 | 1.5 |
| 1792 | 7.88e-2 | 3.53e-2 | 4.40e-3 | 9.10e-2 | 1.2 | 8.05e-5 | 9.43e-2 | 1.3 |
| 7168 | 3.93e-2 | 1.77e-2 | 1.10e-3 | 4.55e-2 | 1.2 | 1.01e-5 | 4.76e-2 | 1.3 |
| order | 1.1 | 1.1 | 2.1 | 1.1 | - | 3.2 | 1.1 | - |

TABLE 7.2

Convergence rates of error estimators for test case 1, unstructured meshes

For test case 1 the exact solution is $u(x, y) = \cos(0.5\pi x) \cos(0.5\pi y)$ and κ is equal to unity. The purpose is to assess the convergence rate of all the estimators in the case of a smooth solution. Tables 7.1 and 7.2 report the results obtained on structured and unstructured meshes, respectively. As expected, the residual estimator η_{R} converges to order $(l + 2)$, *i.e.*, super-converges with respect to the nonconformity estimator and to the diffusive flux estimator, the latter always dominating the former by a factor between 2 and 3. For $l = 0$, the effectivity index is equal to 1.2. For $l = 1$, the effectivity index is equal to 1.5 on structured meshes and to 1.3 on unstructured meshes, which confirms the sharpness of the estimate for $l = 1$ also. The effectivity index for $l = 1$ is slightly larger than for $l = 0$, showing that for the present test case, the lowest-order diffusive flux reconstruction yields the sharpest results (a different conclusion is reached in the two following test cases).

Before moving to the following test cases, it is useful to compare the present error estimators to those previously available in the literature. We focus here on the classical error estimator for the pure diffusion case which consists of four terms: the nonconformity estimator (evaluated using the Oswald interpolate as in this paper) and three additional terms, namely the residual estimator η_{R}^* , the diffusive flux (mass balance) estimator η_{DF}^* , and the jump estimator η_{J}^* defined as follows:

$$\begin{aligned}
(\eta_{\text{R}}^*)^2 &= \sum_{T \in \mathcal{T}_h} m_T^2 \|f + \nabla \cdot (\mathbf{K} \nabla u_h)\|_{0,T}^2, & (\eta_{\text{J}}^*)^2 &= \sum_{T \in \mathcal{T}_h} C_T \gamma_{\mathbf{K}, F} h_T^{-1} \|[u_h]\|_{0,\partial T}^2, \\
(\eta_{\text{DF}}^*)^2 &= \sum_{T \in \mathcal{T}_h} C_T h_T c_{\mathbf{K}, T}^{-1} \|\mathbf{n}_F \cdot [\mathbf{K} \nabla_h u_h]\|_{0,\partial T \setminus \partial \Omega}^2,
\end{aligned}$$

where $C_T = 3dh_T |\partial T|/|T|$. Results are presented in Table 7.3. In particular, the 5th column, which displays the effectivity index for the classical estimator, shows that the

| N | η_R^* | η_{DF}^* | η_J^* | eff.* | eff. [17] |
|-------|------------|---------------|------------|-------|-----------|
| 112 | 1.74 | 1.38 | 3.57e-1 | 7.5 | 5.2 |
| 448 | 8.73e-1 | 5.86e-1 | 2.03e-1 | 7.2 | 4.9 |
| 1792 | 4.37e-1 | 2.75e-1 | 1.07e-1 | 7.1 | 4.7 |
| 7168 | 2.19e-1 | 1.31e-1 | 5.42e-2 | 7.1 | 4.5 |
| order | 1.1 | 1.1 | 1.0 | - | - |

TABLE 7.3

Comparison with other error estimators for test case 1, unstructured meshes

error is overestimated by a factor of 7. It should be observed that the main source of overestimation is the residual estimator η_R^* , which we have transformed into a superconvergent term in the present work. The diffusive flux estimator η_{DF}^* is also observed to be about three-times larger than the estimators η_{DF} evaluated using the present reconstructed flux \mathbf{t}_h either with $l = 0$ or with $l = 1$. For completeness, the last column of Table 7.3 proposes a comparison with the recent results of [17] where the mean value is subtracted from the residue within each mesh element, while the diffusive flux estimator and the nonconformity estimator are considered together as one unique term. We see that the estimate becomes sharper, though it still overestimates the error by a factor of 4.5, mainly because the reconstructed flux \mathbf{t}_h is not used.

Test cases 2 and 3, which were proposed in [29], include diffusion inhomogeneities. The domain Ω is split along the Cartesian axes into four subregions Ω_i . The subregion $\{x > 0, y > 0\} \cap \Omega$ is indicated by Ω_1 and the subsequent numbering is done in a counterclockwise manner. The diffusion coefficient is equal to κ_i in subregion Ω_i where κ_i is a constant. Taking the forcing term equal to zero, the analytical solution with corresponding nonhomogeneous Dirichlet boundary conditions in polar coordinates is $u(r, \phi)|_{\Omega_i} = r^\alpha (a_i \sin(\alpha\phi) + b_i \cos(\alpha\phi))$, where the subscript i refers to the corresponding subregion. Owing to the singularity at the origin, the expected convergence rate of the DG approximation is 2α in the L^2 -norm and α in the energy norm. For test case 2, we take $\kappa_1 = \kappa_3 = 5$ and $\kappa_2 = \kappa_4 = 1$, $\alpha = 0.53544095$ and

$$\begin{aligned} (a_1, a_2, a_3, a_4) &= (0.44721360, -0.74535599, -0.94411759, -2.40170264), \\ (b_1, b_2, b_3, b_4) &= (1.00000000, 2.33333333, 0.55555556, -0.48148148). \end{aligned}$$

For test case 3, we take $\kappa_1 = \kappa_3 = 100$ and $\kappa_2 = \kappa_4 = 1$, $\alpha = 0.12690207$ and

$$\begin{aligned} (a_1, a_2, a_3, a_4) &= (0.10000000, -9.60396040, -0.48035487, 7.70156488), \\ (b_1, b_2, b_3, b_4) &= (1.00000000, 2.96039604, -0.88275659, -6.45646175). \end{aligned}$$

The results for test case 2 are shown in Tables 7.4 and 7.5 for structured and unstructured meshes, respectively. Since the forcing term is zero, the residual estimator is also equal to zero, and has not been reported. The interpolation error on nonhomogeneous Dirichlet boundary conditions is not reported either. We observe that the expected convergence rate of order α is obtained for both the nonconformity estimator η_{NC} and for the diffusive flux estimator η_{DF} . Both estimators take comparable values. The effectivity index is 1.9 for $l = 0$ and 1.8 for $l = 1$; hence, for this test case, employing $l = 1$ for the reconstruction leads to a slightly sharper estimator.

The results for test case 3 are shown in Tables 7.6 and 7.7 for structured and unstructured meshes, respectively. The order of convergence of the error estimators

| N | $\ u - u_h\ $ | η_{NC} | $l = 0$ | | $l = 1$ | |
|-------|---------------|--------------------|--------------------|------|--------------------|------|
| | | | η_{DF} | eff. | η_{DF} | eff. |
| 128 | 6.61e-01 | 9.60e-1 | 8.02e-1 | 1.9 | 6.54e-1 | 1.8 |
| 512 | 4.58e-01 | 6.68e-1 | 5.63e-1 | 1.9 | 4.63e-1 | 1.8 |
| 2048 | 3.17e-01 | 4.62e-1 | 3.92e-1 | 1.9 | 3.23e-1 | 1.8 |
| 8192 | 2.19e-01 | 3.20e-1 | 2.72e-1 | 1.9 | 2.25e-1 | 1.8 |
| order | 0.53 | 0.53 | 0.53 | - | 0.53 | - |

TABLE 7.4

Convergence rates of error estimators for test case 2, structured meshes

| N | $\ u - u_h\ $ | η_{NC} | $l = 0$ | | $l = 1$ | |
|-------|---------------|--------------------|--------------------|------|--------------------|------|
| | | | η_{DF} | eff. | η_{DF} | eff. |
| 112 | 6.11e-01 | 8.70e-1 | 7.43e-1 | 1.9 | 6.00e-1 | 1.7 |
| 448 | 4.28e-01 | 6.09e-1 | 5.35e-1 | 1.9 | 4.32e-1 | 1.7 |
| 1792 | 2.97e-01 | 4.23e-1 | 3.74e-1 | 1.9 | 3.05e-1 | 1.8 |
| 7168 | 2.01e-01 | 2.92e-1 | 2.60e-1 | 1.9 | 2.12e-1 | 1.8 |
| order | 0.53 | 0.53 | 0.53 | - | 0.52 | - |

TABLE 7.5

Convergence rates of error estimators for test case 2, unstructured meshes

is close to α , and the error is overestimated by a factor of approximately 3.8. This is because the nonconformity error estimator now dominates over the diffusive flux estimator. Hence, although the diffusive flux estimator is lower for $l = 1$ than for $l = 0$, this difference is scarcely reflected in the effectivity index. Finally, we notice that the fact that the diffusion coefficient is not monotone around the singularity precludes the use of weighted variants of the Oswald interpolate, *cf.* [2]. On the other hand, one can employ a piecewise quadratic Oswald interpolate as in [38, 39].

7.2. Convection–diffusion–reaction. We consider the domain $\Omega = \{0 < x, y < 1\}$, the reaction coefficient $\mu = 1$, the convection field $\beta = (1, 0)^t$, and an isotropic homogeneous diffusion tensor represented by a diffusion coefficient κ . We run tests with $\kappa = 10^{-2}$ (test case 4) and $\kappa = 10^{-4}$ (test case 5). The source term f is such that the exact solution with homogeneous Dirichlet boundary conditions is

$$u = \frac{1}{2}x(x-1)y(y-1)(1 - \tanh(10 - 20x)).$$

For brevity, only results for uniformly refined structured meshes are presented.

Tables 7.8 and 7.9 report the results for $\kappa = 10^{-2}$. Table 7.8 focuses on the global effectivity index when both the diffusive and convective fluxes are reconstructed using $l = 0$ or $l = 1$. Both choices yield comparable results with effectivity indices ranging between 8 and 14 approximately. A more detailed comparison can be found in Table 7.9. The residual estimator η_{R} super-converges and converges faster for $l = 1$ than for $l = 0$. The classical residual estimator η_{R}^* evaluated using solely the discrete solution is also reported; it takes, as expected, larger values. The diffusive flux estimator η_{DF} yields the smallest contribution among the different terms in the error estimate. The upwinding estimator η_{U} is dominant, along with the first convection estimator $\eta_{\text{C},1}$ for $l = 1$, while this latter estimator vanishes for $l = 0$ since in this

| N | $\ u - u_h\ $ | η_{NC} | $l = 0$ | | $l = 1$ | |
|-------|---------------|--------------------|--------------------|------|--------------------|------|
| | | | η_{DF} | eff. | η_{DF} | eff. |
| 128 | 3.49 | 12.4 | 2.68 | 3.6 | 2.02 | 3.6 |
| 512 | 3.29 | 11.9 | 2.57 | 3.7 | 1.95 | 3.6 |
| 2048 | 3.09 | 11.3 | 2.45 | 3.7 | 1.86 | 3.7 |
| 8192 | 2.88 | 10.7 | 2.32 | 3.8 | 1.76 | 3.8 |
| order | 0.10 | 0.08 | 0.08 | - | 0.08 | - |

TABLE 7.6

Convergence rates of error estimators for test case 3, structured meshes

| N | $\ u - u_h\ $ | η_{NC} | $l = 0$ | | $l = 1$ | |
|-------|---------------|--------------------|--------------------|------|--------------------|------|
| | | | η_{DF} | eff. | η_{DF} | eff. |
| 112 | 3.27 | 11.8 | 2.39 | 3.7 | 1.89 | 3.7 |
| 448 | 3.11 | 11.3 | 2.33 | 3.7 | 1.84 | 3.7 |
| 1792 | 2.93 | 10.8 | 2.23 | 3.8 | 1.77 | 3.7 |
| 7168 | 2.75 | 10.3 | 2.12 | 3.8 | 1.68 | 3.8 |
| order | 0.09 | 0.08 | 0.08 | - | 0.07 | - |

TABLE 7.7

Convergence rates of error estimators for test case 3, unstructured meshes

case, $\nabla \cdot (\mathbf{q}_h - \beta \mathcal{I}_{\text{Os}}(u_h))$ is by construction piecewise constant. Finally, the second convection estimator $\eta_{\text{C},2}$ vanishes identically because β is divergence-free. All in all, there is little gain when going from $l = 0$ to $l = 1$.

Tables 7.10 and 7.11 report the results for $\kappa = 10^{-4}$. Table 7.10 focuses on the global effectivity index for the $l = 0$ and $l = 1$ flux reconstructions. Again, both choices yield similar results, and the effectivity indices are roughly ten-times larger than those observed for $\kappa = 10^{-2}$, in agreement with the cut-off coefficients employed in front of the estimators. A more detailed comparison can be found in Table 7.11. As for test case 4, the residual estimator η_{R} converges faster for $l = 1$ than for $l = 0$, but this gain is compensated by the first convection estimator $\eta_{\text{C},1}$. The diffusive flux estimator η_{DF} yields the smallest contribution, while the upwinding estimator η_{U} dominates the overall estimate.

7.3. Adaptive meshes. We conclude this section by an example on how the error estimator with $l = 0$ can be used to adapt the mesh. Test case 2 is considered. The adaptive mesh refinement procedure flags 5% of the mesh elements yielding the largest error indicators. Results are reported in Table 7.12. The effectivity index fluctuates between 1.7 and 2 and decreases as finer meshes are constructed. Comparing with Table 7.5, we observe that the energy norm of the error on an adapted mesh with 494 elements is comparable to that obtained on a uniformly refined mesh with 7168 elements. Finally, Figure 7.1 presents two meshes obtained within the adaptive refinement procedure, one with 342 elements and one with 494 elements. We see that the adaptive refinement correctly aims at capturing the singularity at the origin.

| N | $\ u - u_h\ $ | η_{NC} | eff. $l = 0$ | eff. $l = 1$ |
|-------|---------------|--------------------|--------------|--------------|
| 128 | 8.12e-3 | 3.62e-3 | 13.8 | 14.4 |
| 512 | 4.01e-3 | 1.84e-3 | 11.1 | 10.9 |
| 2048 | 1.89e-3 | 8.84e-4 | 8.10 | 7.75 |
| order | 1.1 | 1.1 | - | - |

TABLE 7.8
Error and effectivity indices for test case 4 ($\kappa = 10^{-2}$)

| N | $l = 0$ | | | | $l = 1$ | | | |
|-------|---------------------|-------------------|--------------------|-------------------|-------------------|--------------------|-------------------|---------------------|
| | η_{R}^* | η_{R} | η_{DF} | η_{U} | η_{R} | η_{DF} | η_{U} | $\eta_{\text{C},1}$ |
| 128 | 4.91e-2 | 3.94e-2 | 8.82e-3 | 6.35e-2 | 1.12e-2 | 8.73e-3 | 6.35e-2 | 3.28e-2 |
| 512 | 1.44e-2 | 9.86e-3 | 4.93e-3 | 2.87e-2 | 1.66e-3 | 4.73e-3 | 2.87e-2 | 7.69e-3 |
| 2048 | 4.63e-3 | 2.42e-3 | 2.51e-3 | 9.77e-3 | 3.19e-4 | 2.37e-3 | 9.77e-3 | 1.53e-3 |
| order | 1.6 | 2.0 | 1.0 | 1.6 | 2.4 | 1.0 | 1.6 | 2.3 |

TABLE 7.9
Convergence of error estimators for test case 4 ($\kappa = 10^{-2}$)

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| N | $\ u - u_h\ $ | η_{NC} | eff. $l = 0$ | eff. $l = 1$ |
|-------|---------------|--------------------|--------------|--------------|
| 128 | 1.72e-3 | 2.73e-3 | 80 | 89 |
| 512 | 5.68e-4 | 6.74e-4 | 124 | 128 |
| 2048 | 2.14e-4 | 1.66e-4 | 145 | 152 |
| 8192 | 1.00e-4 | 6.78e-5 | 126 | 127 |
| order | 1.1 | 1.3 | - | - |

TABLE 7.10
 Error and effectivity indices for test case 5 ($\kappa = 10^{-4}$)

| N | η_{R}^* | $l = 0$ | | | $l = 1$ | | | |
|-------|---------------------|-------------------|--------------------|-------------------|-------------------|--------------------|-------------------|---------------------|
| | | η_{R} | η_{DF} | η_{U} | η_{R} | η_{DF} | η_{U} | $\eta_{\text{C},1}$ |
| 128 | 7.77e-2 | 6.84e-2 | 1.06e-3 | 6.98e-2 | 1.92e-2 | 1.03e-3 | 6.98e-2 | 6.55e-2 |
| 512 | 3.90e-2 | 3.41e-2 | 6.20e-4 | 3.60e-2 | 3.44e-3 | 5.71e-4 | 3.60e-2 | 3.38e-2 |
| 2048 | 1.87e-2 | 1.63e-2 | 3.23e-4 | 1.47e-2 | 2.01e-3 | 2.86e-4 | 1.60e-2 | 1.46e-2 |
| 8192 | 6.69e-3 | 5.80e-3 | 1.60e-4 | 6.70e-3 | 3.66e-4 | 1.45e-4 | 6.70e-3 | 5.68e-3 |
| order | 1.5 | 1.5 | 1.0 | 1.1 | 2.5 | 1.0 | 1.1 | 1.5 |

TABLE 7.11
 Convergence of error estimators for test case 5 ($\kappa = 10^{-4}$)

| N | $\ u - u_h\ $ | η_{NC} | η_{DF} | eff. |
|-----|---------------|--------------------|--------------------|------|
| 112 | 6.11e-1 | 8.70e-1 | 7.43e-1 | 1.87 |
| 148 | 4.58e-1 | 6.17e-1 | 5.78e-1 | 1.84 |
| 204 | 3.59e-1 | 5.59e-1 | 4.63e-1 | 2.02 |
| 264 | 2.96e-1 | 4.21e-1 | 3.76e-1 | 1.91 |
| 342 | 2.50e-1 | 3.05e-1 | 3.23e-1 | 1.78 |
| 494 | 2.10e-1 | 2.20e-1 | 2.78e-1 | 1.68 |

TABLE 7.12
 Error as a function of mesh elements on adaptive meshes

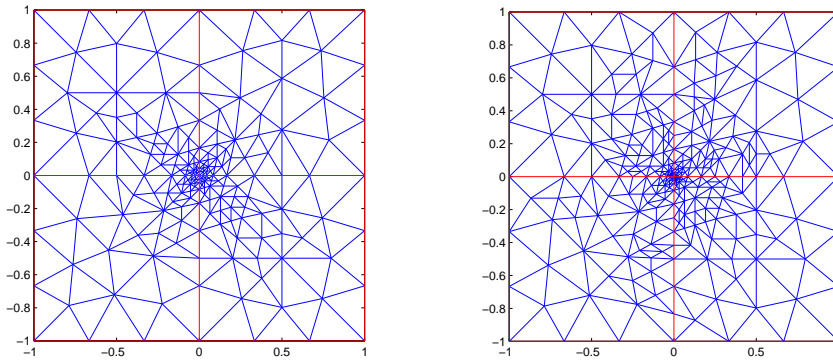


FIG. 7.1. Two meshes successively refined using the error estimator with $l = 0$ reconstruction: 342 elements (left) and 494 elements (right)

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