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Improved energy norm a posteriori error estimation based on flux reconstruction for discontinuous Galerkin methods*

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Abstract

We propose and study a new approach to residual a posteriori error estimation in the discontinuous Galerkin finite element method. The main idea, which consists of constructing an $\mathbf{H}(\text{div})$ -conforming Raviart–Thomas flux on the basis of the conservative discontinuous Galerkin side fluxes, is first exposed for a pure diffusion second-order elliptic problem. In this case, the classical elementwise residual can be transformed into a higher-order term (sometimes considered separately and called “data oscillation term”), thus fully taking advantage of the spectral degrees of freedom within each element available in the discontinuous Galerkin method. Moreover, the classical estimator based on normal gradient jumps is simultaneously replaced by a comparison of the original and reconstructed diffusive fluxes. Finally, our error bound consists of one last estimator which measures the nonconformity of the actual discrete solution by comparing it to its so-called Oswald interpolate. In the second part of the paper, we extend our results to convection–diffusion–reaction problems, where we introduce an additional convective flux reconstruction. Our estimators are based on an abstract upper bound which is sharp since it is established for arbitrary conforming reconstructions of the discrete solution itself and of its diffusive and convective fluxes. They yield a guaranteed upper bound since all constants are evaluated, are locally efficient, represent local lower bounds of the classical residual estimators, and numerical examples presented at the end of the paper confirm their accuracy and robustness. Incidentally, the $\mathbf{H}(\text{div})$ -conforming Raviart–Thomas diffusive and convective flux reconstructions are of independent interest.

Keywords: convection–diffusion–reaction equation, inhomogeneous and anisotropic diffusion, convection dominance, discontinuous Galerkin finite element method, a posteriori error estimates

AMS: 65N15, 65N30, 76S05

1 Introduction

Let us consider a convection–diffusion–reaction problem

$$-\nabla \cdot (\mathbf{K} \nabla u) + \beta \cdot \nabla u + \mu u = f \quad \text{in } \Omega, \quad (1.1a)$$

$$u = 0 \quad \text{on } \partial\Omega, \quad (1.1b)$$

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where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a polygonal (polyhedral) domain, \mathbf{K} is a diffusion tensor, $\boldsymbol{\beta}$ is a velocity field, μ is a reaction function, and f is a source term. Our intention is to study a posteriori energy norm error estimates for the approximation of (1.1a)–(1.1b) by interior-penalty discontinuous Galerkin methods with the twofold objective to derive estimates without undetermined constants and to analyze carefully the robustness of the estimates in several practically important situations, *e.g.*, diffusion inhomogeneities, dominant convection, or dominant reaction.

For the pure diffusion problem ((1.1a)–(1.1b) with $\boldsymbol{\beta} = \mu = 0$), a posteriori error estimates have now been presented in the literature for all major numerical methods. In particular, for the discontinuous Galerkin (DG) one, residual-based energy norm error estimators can be found in the work of Karakashian and Pascal [16], Becker *et al.* [6], and Houston *et al.* [15]. New results, closer in spirit to the present approach since they avoid undetermined constants, appeared in the finalization phase of this paper; they include the works of Ainsworth [3], Kim [17, 18], Lazarov *et al.* [21], Cochez-Dhondt and Nicaise [9], and Ern and Stephansen [13].

Although the residual-based energy norm error estimates in [16, 6, 15] are proved to be both reliable (yield an upper bound on the difference between the exact and approximate solution) and locally efficient (give local lower bounds for the error as well), there is, in our opinion, still room for improvement. First of all, in all these estimates, various undetermined constants appear. As such, the derived estimators should rather be called error indicators, since they are fully usable for the usual practice of identifying the parts of the computational domain with insufficient precision, but not for the actual control over the error. Hence, the first motivation for our work was to remedy this inconvenience. Secondly, in all these references, the residual estimator in an element T is given by $c_{\mathbf{K}}h_T\|R(u_h)\|_{0,T}$, where $R(u_h) := f + \nabla \cdot (\mathbf{K}\nabla_h u_h)$ is the elementwise residue, h_T is the element diameter, and the constant $c_{\mathbf{K}}$ depends only on \mathbf{K} (the modifications of [15] do not influence the basic ideas of what follows). In particular, for piecewise constant \mathbf{K} and a DG scheme employing first-order polynomials, this reduces to $c_{\mathbf{K}}h_T\|f\|_{0,T}$. We believe that this is not an optimal estimator. In contrast to this situation, the a posteriori error estimates for mixed finite element or finite volume methods recently derived in [30, 31, 32] lead to residual estimators of the form $c_{\mathbf{K}}h_T\|f - \Pi_k(f)\|_{0,T}$, where Π_k is the L^2 -orthogonal projection onto piecewise polynomials of degree k ($k = 0$ for finite volumes and it is the scalar unknown polynomial degree for mixed finite elements), which is obviously of one order better for $k = 0$ as soon as f possess an $H^1(T)$ regularity. This result is based on the elementwise conservativity of these methods. Hence, a second motivation for our work was to extend this result to DG methods as well, since these methods are likewise locally conservative. A first result in this direction can be found in [13] where the fact that piecewise constant functions are contained in the DG finite element space is exploited to improve the classical residual estimator to $c_{\mathbf{K}}h_T\|R(u_h) - \Pi_0(R(u_h))\|_{0,T}$, which reduces to $c_{\mathbf{K}}h_T\|f - \Pi_0(f)\|_{0,T}$ if first-order polynomials are used. Finally, it is quite usual in the a posteriori error estimation literature to encounter a residual estimator in each element of the form $c_{\mathbf{K}}h_T\|R(u_h)\|_{0,T}$ and a separate “data oscillation term” $c_{\mathbf{K}}h_T\|f - \Pi_k(f)\|_{0,T}$. In our approach, these two terms are merged into a single residual estimator of the form $c_{\mathbf{K}}h_T\|f - \Pi_k(f)\|_{0,T}$.

One obtains $c_{\mathbf{K}}h_T\|R(u_h)\|_{0,T}$ as the residual term when the elliptic operator is applied directly to the discrete solution u_h , after an integration by parts has been performed. Since the diffusive flux $-\mathbf{K}\nabla_h u_h$ of the approximate DG solution u_h is not in $\mathbf{H}(\text{div}, \Omega)$, there also appears a so-called mass balance estimator, typically of the form $c_{\mathbf{K}}h_F^{1/2}\|\mathbf{n}_F^t[\mathbf{K}\nabla_h u_h]\|_{0,F}$ for each face F , where h_F is the diameter of F and where $[\cdot]$ is the jump operator given by equation (2.1) below. By such a direct approach, one in some sense ignores the local conservativity imbedded in DG schemes. The basic idea of our approach is to first introduce an $\mathbf{H}(\text{div}, \Omega)$ -conforming reconstruction of the diffusive flux \mathbf{t}_h . By suitably choosing \mathbf{t}_h in Raviart–Thomas spaces, used extensively in the mixed finite element method, *cf.* [7, 27], the mass balance estimator is replaced by a comparison of the

original and reconstructed diffusive fluxes of the form $\|\mathbf{K}^{1/2}\nabla u_h + \mathbf{K}^{-1/2}\mathbf{t}_h\|_{0,T}$. We next prove that this new estimator represents a lower bound for the original mass balance estimator plus a part of the classical nonconformity estimator (see below), which, together with the results of the previous paragraph, closes the improvement circle. Lastly, this locally computable estimate is only one possible realization of the general estimator $\inf_{\mathbf{t}\in\mathbf{H}(\text{div},\Omega)} \|\mathbf{K}^{1/2}\nabla_h u_h + \mathbf{K}^{-1/2}\mathbf{t}\|_{0,\Omega}$ that we show to be optimally efficient.

The last typical DG residual estimator measures the nonconformity in the approximate solution u_h and usually takes the form $c_{\mathbf{K}}h_F^{-1/2}\|[[u_h]]\|_{0,F}$ for each face F . However, it appears unnecessary at the estimation stage to go up to this form. The term $\|\mathbf{K}^{1/2}\nabla(u_h - \mathcal{I}_{\text{Os}}(u_h))\|_{0,T}$, with $\mathcal{I}_{\text{Os}}(u_h)$ the Oswald $H_0^1(\Omega)$ -conforming interpolate of the original nonconforming u_h , is the usual starting point, it is a lower bound for the above one, and presents the additional advantages that it does not feature any undetermined interpolation constant and that it yields a direct (and correct) dependence on \mathbf{K} . Again, the completely general form for this estimator is $\inf_{s\in H_0^1(\Omega)} \|\mathbf{K}^{1/2}\nabla_h(u_h - s)\|_{0,\Omega}$.

Estimators based on comparisons with reconstructed $\mathbf{H}(\text{div})$ -conforming fluxes in the continuous finite element method can be traced back to the ideas of Prager and Synge [24] and include, *e.g.*, the works of Ladevèze [19], Ladevèze and Leguillon [20], and Destuynder and Métivet [10]. The estimates [3, 17, 18, 21, 9] for DG discretizations of pure diffusion problems develop this way. In particular, Ainsworth [3] gives a fully computable estimate for the symmetric interior-penalty DG scheme in the case $d = 2, k = 1$, and $\mathbf{K} = Id$ (actually, the reconstructed flux \mathbf{t}_h is not directly computed). Kim in [17] uses an $\mathbf{H}(\text{div})$ -conforming flux reconstruction and gives an estimate for the original unknown and this reconstruction for $d = 2$. Next, in [18], he presents a result similar to that of Cochez-Dhondt and Nicaise [9] and to the one given here for the pure diffusion case. Finally, Lazarov *et al.* [21] present essentially numerical experiments for yet a similar estimator.

The setting of the present paper includes a large class of interior-penalty DG schemes. We treat the complete convection–diffusion–reaction case and present an abstract framework, established for arbitrary conforming reconstructions of the discrete solution itself and of its diffusive flux and convective fluxes, and show that this framework is optimal. Our estimates are given in the natural energy seminorm for the DG approximate solution u_h , which is the energy norm for the flux $-\mathbf{K}\nabla_h u_h$. We then prove rigorously the local efficiency of the derived estimators, this time in a norm including a term with jumps. We also pay a special attention to the case of inhomogeneous and anisotropic diffusion tensor \mathbf{K} ; it turns out that some fully robust results with respect to diffusion inhomogeneities can be obtained for our new diffusive flux estimator for a certain class of DG schemes such as those introduced by Ern *et al.* [14]. These schemes use diffusivity-dependent weighted averages to formulate the consistency terms and the harmonic average of normal diffusivity to penalize jumps at interfaces. Next, our error estimates, as well as the upper and global lower bounds within the abstract framework, do not require the mesh to be shape-regular and the data can be as general functions as possible (the usual requirement of shape-regularity and of polynomial data, or, equivalently, the introduction of higher-order oscillation terms, is only needed for the local efficiency proofs). Also, no saturation assumption, no convexity of Ω , no additional regularity of the weak solution of (1.1a)–(1.1b), and no Helmholtz decomposition are needed in our setting. Finally, we have only considered the homogeneous Dirichlet boundary condition for the sake of simplicity; extensions to inhomogeneous Dirichlet and Neumann boundary conditions are possible using the concepts of, *e.g.*, [9, 17, 30]. A similar remark applies also for nonmatching meshes, *cf.*, *e.g.*, [30], while constructing the Oswald interpolate as well as the conforming diffusive and convective flux reconstructions on a matching refinement of the given nonmatching grid.

The paper is organized as follows: we first introduce the schemes, notation, assumptions, and the continuous problems in Section 2. We then present the details for the pure diffusion

problem. First, in Section 3, we state both the abstract (containing the above-discussed infimum over continuous spaces) and locally computable (using particular conforming scalar and diffusive flux reconstructions) forms of our a posteriori error estimates. In Section 4, we then show that our abstract framework gives a quasi-optimal global efficiency of $\sqrt{2}$ and that the locally computable estimate is optimal up to inhomogeneities and anisotropies. An abstract a posteriori error estimate for the reconstructed diffusive flux itself, which allows to improve the global efficiency to the optimal constant 1, is then given in Section 5. In Sections 6 and 7, we then extend the results of the pure diffusion case to the full convection–diffusion–reaction one. While the abstract upper bound stays quasi-optimal with global efficiency of 2, the presented choice of discrete reconstructions leads only to semi-robust estimates in this case, with local efficiency depending on local variations in the coefficients and on the local Péclet number. Finally, numerical experiments of Section 8 confirm the accuracy and robustness of our estimators.

2 Notation, assumptions, and continuous and discrete problems

2.1 Notation

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of triangulations of the domain Ω , consisting of simplices (triangles if $d = 2$, tetrahedra if $d = 3$). A generic element in \mathcal{T}_h is denoted by T , h_T stands for the diameter of T , and \mathbf{n}_T for its outward unit normal. We suppose that \mathcal{T}_h is matching in the sense that it contains no “hanging nodes”, *i.e.*, such that if $T, T' \in \mathcal{T}_h$, $T \neq T'$, then $T \cap T'$ is either an empty set or their common face, edge, or vertex. For the local efficiency proofs of our estimators, we will later need the assumption that \mathcal{T}_h is shape-regular in the sense that there exists a constant $\kappa_{\mathcal{T}} > 0$ such that $\min_{T \in \mathcal{T}_h} |T|/h_T^d \geq \kappa_{\mathcal{T}}$ for all $h > 0$. We will be using the “broken Sobolev space” $H^s(\mathcal{T}_h)$,

$$H^s(\mathcal{T}_h) := \{v \in L^2(\Omega); v|_T \in H^s(T) \quad \forall T \in \mathcal{T}_h\},$$

along with its DG approximation space

$$V_h^k := \{v_h \in L^2(\Omega); v_h|_T \in \mathbb{P}_k(T) \quad \forall T \in \mathcal{T}_h\},$$

where $\mathbb{P}_k(T)$ is the set of polynomials of degree less than or equal to k on an element T , $k \geq 1$. The L^2 -scalar product and its associated norm on a region $R \subset \Omega$ are indicated by the subscript 0, R ; shall R coincide with Ω , this subscript will be dropped off. For $s \geq 1$, a norm (seminorm) with the subscript s, R designates the usual norm (seminorm) in $H^s(R)$. Finally, we use the symbol $\nabla_h v_h$ in order to denote the piecewise gradient of $v \in H^1(\mathcal{T}_h)$, that is, $\nabla_h v \in [L^2(\Omega)]^d$ and for all $T \in \mathcal{T}_h$, $(\nabla_h v)|_T = \nabla(v|_T)$.

We say that F is an interior face of the mesh if there are $T^-(F)$ and $T^+(F)$ in \mathcal{T}_h such that $F = T^-(F) \cap T^+(F)$ and we let \mathbf{n}_F be the unit normal vector to F pointing from $T^-(F)$ towards $T^+(F)$. Similarly, we say that F is a boundary face of the mesh if there is $T(F) \in \mathcal{T}_h$ such that $F = T(F) \cap \partial\Omega$ and we let \mathbf{n}_F coincide with the outward normal to $\partial\Omega$. All the interior (resp., boundary) faces of the mesh are collected into the set $\mathcal{F}_h^{\text{int}}$ (resp., $\mathcal{F}_h^{\text{ext}}$) and we let $\mathcal{F}_h = \mathcal{F}_h^{\text{int}} \cup \mathcal{F}_h^{\text{ext}}$; \mathcal{F}_T is then the set of faces of a given $T \in \mathcal{T}_h$ and $\tilde{\mathcal{F}}_T$ is the set of such faces that share at least a vertex with T . Similarly, \mathcal{T}_T is the set including the simplex T and its neighbors and $\tilde{\mathcal{T}}_T$ contains all $T' \in \mathcal{T}_h$ that share at least a vertex with T (including T itself). Henceforth, we shall often deal with functions that are double-valued on $\mathcal{F}_h^{\text{int}}$ and single-valued on $\mathcal{F}_h^{\text{ext}}$. This is the case, for instance, of functions in V_h^k . On interior faces, when the two branches of the function in question, say v , are associated with restrictions to the neighboring elements $T^\mp(F)$, these branches are denoted by v^\mp and the jump of v across F is defined as

$$[[v]]_F := v^- - v^+. \quad (2.1)$$

On an interior face $F \in \mathcal{F}_h^{\text{int}}$, we define the standard (arithmetic) average as $\{v\}_F := \frac{1}{2}(v^- + v^+)$; the subscript F in the above jumps and averages is omitted if there is no ambiguity. For convenience, we set $\llbracket v \rrbracket_F := v|_F$ and $\{v\}_F := \frac{1}{2}v|_F$ on boundary faces. Finally, the weighted average of a two-valued function on an interior face $F \in \mathcal{F}_h^{\text{int}}$ is defined as

$$\{v\}_\omega := \omega_{T^-(F),F}v^- + \omega_{T^+(F),F}v^+, \quad (2.2)$$

where the nonnegative weights have to satisfy $\omega_{T^-(F),F} + \omega_{T^+(F),F} = 1$. On boundary faces, we set $\{v\}_\omega := v$ and $\omega_{T(F),F} := 1$. Finally, for all $T \in \mathcal{T}_h$ and $F \in \mathcal{F}_T$, we let $\bar{\omega}_{T,F} := 1 - \omega_{T,F}$.

2.2 Assumptions

We suppose in this paper that $\mathbf{K} \in [L^\infty(\Omega)]^{d \times d}$ is a symmetric, uniformly positive definite, and piecewise constant tensor and we denote by $c_{\mathbf{K},T}$ and $C_{\mathbf{K},T}$, respectively, its minimum and maximum eigenvalue on $T \in \mathcal{T}_h$. Next, $\boldsymbol{\beta} \in \mathbf{H}(\text{div}, \Omega) \cap [L^\infty(\Omega)]^d$, $\mu \in L^\infty(\Omega)$, and $\mu - \frac{1}{2}\nabla \cdot \boldsymbol{\beta} \geq 0$ are supposed and we use $c_{\boldsymbol{\beta},\mu,T}$ to indicate the (essential) minimum value of $\mu - \frac{1}{2}\nabla \cdot \boldsymbol{\beta}$ on T ; we also suppose that if $c_{\boldsymbol{\beta},\mu,T} = 0$, then $\|\mu\|_{\infty,T} = \|\frac{1}{2}\nabla \cdot \boldsymbol{\beta}\|_{\infty,T} = 0$. Finally, $f \in L^2(\Omega)$ is supposed. These assumptions will be sufficient for the existence and uniqueness of both continuous and discrete problems and for our a posteriori error estimates, as well as for the global efficiency of the abstract estimates; for the present proof of the local efficiency of the locally computable estimates, however, we shall later tighten them.

2.3 The continuous problem

We define the bilinear form \mathcal{B} by

$$\mathcal{B}(u, v) := \sum_{T \in \mathcal{T}_h} \{(\mathbf{K}\nabla u, \nabla v)_{0,T} + (\boldsymbol{\beta} \cdot \nabla u, v)_{0,T} + (\mu u, v)_{0,T}\} \quad u, v \in H^1(\mathcal{T}_h) \quad (2.3)$$

and the corresponding energy (semi)norm by

$$\|v\|^2 := \sum_{T \in \mathcal{T}_h} \|v\|_T^2, \quad \|v\|_T^2 := \|\mathbf{K}^{\frac{1}{2}}\nabla v\|_{0,T}^2 + \|(\mu - \frac{1}{2}\nabla \cdot \boldsymbol{\beta})^{\frac{1}{2}}v\|_{0,T}^2 \quad v \in H^1(\mathcal{T}_h). \quad (2.4)$$

We remark that $\|\cdot\|$ is always a norm on $H_0^1(\Omega)$, whereas it is a norm on $H^1(\mathcal{T}_h)$ only when $c_{\boldsymbol{\beta},\mu,T} > 0$ for all $T \in \mathcal{T}_h$.

The weak formulation of the problem (1.1a)–(1.1b) is then to find $u \in H_0^1(\Omega)$ such that

$$\mathcal{B}(u, v) = (f, v) \quad \forall v \in H_0^1(\Omega). \quad (2.5)$$

The assumptions of the previous section, the Green theorem, and the Cauchy–Schwarz inequality imply that

$$\mathcal{B}(v, v) = \|v\|^2 \quad \forall v \in H_0^1(\Omega), \quad (2.6)$$

$$\begin{aligned} \mathcal{B}(u, v) &\leq \max \left\{ 1, \max_{T \in \mathcal{T}_h} \left\{ \frac{\|\mu\|_{\infty,T}}{c_{\boldsymbol{\beta},\mu,T}} \right\} \right\} \|u\| \|v\| \\ &\quad + \max_{T \in \mathcal{T}_h} \left\{ \frac{\|\boldsymbol{\beta}\|_{\infty,T}}{c_{\mathbf{K},T}^{1/2}} \right\} \|u\| \|v\| \quad \forall u, v \in H^1(\mathcal{T}_h). \end{aligned} \quad (2.7)$$

Hence, problem (2.5) admits a unique solution.

Remark 2.1 (Notation). *If $c_{\boldsymbol{\beta},\mu,T} = 0$, then the term $\|\mu\|_{\infty,T}/c_{\boldsymbol{\beta},\mu,T}$ in estimate (2.7) should be evaluated as zero, since in this case we assume $\|\mu\|_{\infty,T} = 0$. To simplify the notation, we will systematically use the convention $0/0 = 0$ throughout the text.*

2.4 The discontinuous Galerkin method

The interior-penalty DG methods considered in this paper are associated with the bilinear form

$$\begin{aligned} \mathcal{B}_h(u, v) &:= (\mathbf{K}\nabla_h u, \nabla_h v) + ((\mu - \nabla \cdot \boldsymbol{\beta})u, v) - (u, \boldsymbol{\beta} \cdot \nabla_h v) \\ &\quad - \sum_{F \in \mathcal{F}_h} \{(\mathbf{n}_F^t \{ \mathbf{K}\nabla_h u \}_\omega, \llbracket v \rrbracket)_{0,F} + \theta(\mathbf{n}_F^t \{ \mathbf{K}\nabla_h v \}_\omega, \llbracket u \rrbracket)_{0,F}\} \\ &\quad + \sum_{F \in \mathcal{F}_h} \{(\gamma_F \llbracket u \rrbracket, \llbracket v \rrbracket)_{0,F} + (\boldsymbol{\beta} \cdot \mathbf{n}_F \{u\}, \llbracket v \rrbracket)_{0,F}\}. \end{aligned} \quad (2.8)$$

The discrete problem now consists of finding $u_h \in V_h^k$ such that

$$\mathcal{B}_h(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h^k. \quad (2.9)$$

Taking in (2.8) the weights on interior faces equal to 1/2 and letting $\theta = 0$, $\theta = -1$, or $\theta = 1$ leads to the well-known Incomplete, Nonsymmetric, or Symmetric Interior-Penalty discontinuous Galerkin methods. The stabilization parameter γ_F takes the general form

$$\gamma_F := \alpha_F \frac{\gamma_{\mathbf{K},F}}{h_F} + \gamma_{\boldsymbol{\beta},F} \quad \forall F \in \mathcal{F}_h, \quad (2.10)$$

where α_F is a (user-dependent) positive parameter, $\gamma_{\mathbf{K},F}$ a positive scalar-valued function depending on \mathbf{K} , and $\gamma_{\boldsymbol{\beta},F}$ a nonnegative scalar-valued function depending on $\boldsymbol{\beta}$ and vanishing if $\boldsymbol{\beta} = 0$ (the usual choice is $\gamma_{\boldsymbol{\beta},F} = \frac{1}{2}|\boldsymbol{\beta} \cdot \mathbf{n}_F|$, which amounts to so-called upwinding). As usual with interior-penalty methods, the parameters α_F must be taken large enough to ensure the coercivity of the discrete bilinear form \mathcal{B}_h on V_h^k whenever $\theta \neq -1$. Some additional assumptions on the weights and the penalty coefficient $\gamma_{\mathbf{K},F}$ will be introduced later in order to ensure the robustness of our estimates with respect to diffusion inhomogeneities; see Theorems 4.4 and 7.2 below. The recently derived weighted interior penalty DG method of [14] satisfies these assumptions.

3 Improved energy norm a posteriori error estimates in the pure diffusion case

We present in this section our a posteriori estimates on the error between the weak solution u and the DG approximate solution u_h in the pure diffusion case. Note that at this stage, neither additional assumptions on the data (in particular, f need not be a polynomial) nor the shape-regularity of the mesh are required.

3.1 Abstract framework

The following lemma gives the basic abstract framework for our a posteriori error estimates in the pure diffusion case. It follows from [17, Lemma 4.4] and it is analogous to, but simpler than, the Helmholtz decomposition of, *e.g.*, [3, Theorem 1]; a similar but more general result, applicable also in the convection–diffusion–reaction case (and used in Section 6 below) is given in [31, Lemma 7.1].

Lemma 3.1 (Abstract framework in the pure diffusion case). *Let $\boldsymbol{\beta} = \mu = 0$ and let $u \in H_0^1(\Omega)$ and $u_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then*

$$\| \|u - u_h\| \|^2 \leq \inf_{s \in H_0^1(\Omega)} \| \|u_h - s\| \|^2 + \sup_{\varphi \in H_0^1(\Omega), \| \varphi \| = 1} \mathcal{B}(u - u_h, \varphi)^2. \quad (3.1)$$

Proof. Following [17, Lemma 4.4], let $\psi \in H_0^1(\Omega)$ be such that

$$\mathcal{B}(\psi, v) = \mathcal{B}(u_h, v) \quad \forall v \in H_0^1(\Omega).$$

Then

$$\| \|u - u_h\| \|^2 = \| \|u_h - \psi\| \|^2 + \mathcal{B} \left(u - u_h, \frac{u - \psi}{\| \|u - \psi\| \|} \right)^2,$$

whence the conclusion is straightforward. \square

3.2 Abstract a posteriori error estimate

We next give here an abstract form of our a posteriori error estimate.

Theorem 3.2 (Abstract a posteriori error estimate in the pure diffusion case). *Let $\beta = \mu = 0$, let u be the unique solution of (2.5), and let $u_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then*

$$\begin{aligned} \| \|u - u_h\| \|^2 &\leq \inf_{s \in H_0^1(\Omega)} \| \|u_h - s\| \|^2 \\ &+ \inf_{\mathbf{t} \in \mathbf{H}(\text{div}, \Omega)} \sup_{\varphi \in H_0^1(\Omega), \| \varphi \| = 1} ((f - \nabla \cdot \mathbf{t}, \varphi) - (\mathbf{K} \nabla_h u_h + \mathbf{t}, \nabla \varphi))^2. \end{aligned} \quad (3.2)$$

Proof. By (2.5), we immediately have $\mathcal{B}(u, \varphi) = (f, \varphi)$. Using this we obtain, for an arbitrary $\mathbf{t} \in \mathbf{H}(\text{div}, \Omega)$ and employing the Green theorem,

$$\begin{aligned} \mathcal{B}(u - u_h, \varphi) &= (f, \varphi) - (\mathbf{K} \nabla_h u_h, \nabla \varphi) = (f, \varphi) - (\mathbf{K} \nabla_h u_h + \mathbf{t}, \nabla \varphi) + (\mathbf{t}, \nabla \varphi) \\ &= (f - \nabla \cdot \mathbf{t}, \varphi) - (\mathbf{K} \nabla_h u_h + \mathbf{t}, \nabla \varphi). \end{aligned}$$

\square

Remark 3.3 (Form of the abstract estimate of Theorem 3.2). *It has been already noted in, e.g., [1, 11, 2, 31, 17] that the first term of (3.1) evaluates the “nonconforming error” in the scalar unknown u_h . The second term of (3.1) is called in [2, 17] the “conforming error”. Relation (3.2) actually shows that this second term is related to the residual and to the nonconformity in the flux $-\mathbf{K} \nabla_h u_h$.*

Remark 3.4 (A first computable a posteriori error estimate). *We remark that using the Cauchy–Schwarz inequality, the Friedrichs inequality $\| \varphi \|^2 \leq C_{F, \Omega} h_\Omega^2 \| \nabla \varphi \|^2$, the definition (2.4) of the energy seminorm, and the fact that $\| \varphi \| = 1$, it follows readily from (3.2) that*

$$\| \|u - u_h\| \|^2 \leq \| \|u_h - s\| \|^2 + \left(\frac{C_{F, \Omega}^{1/2} h_\Omega}{\min_{T \in \mathcal{T}_h} C_{\mathbf{K}, T}^{1/2}} \| f - \nabla \cdot \mathbf{t} \| + \| \mathbf{K}^{1/2} \nabla_h u_h + \mathbf{K}^{-1/2} \mathbf{t}_h \| \right)^2$$

for any $s \in H_0^1(\Omega)$ and any $\mathbf{t} \in \mathbf{H}(\text{div}, \Omega)$. This is an estimate similar to that proposed by Lazarov et al. [21]. As promoted in [21], this estimate is scheme-independent. On the other hand, being scheme-independent means that we are not using all the information that we have once the computation has been finished. As we will see later, this information can be used to improve the residual. Another disadvantage of the above estimate is that the dependence on the diffusion tensor \mathbf{K} is very unfavorable in the presence of strong inhomogeneities; this point was however not addressed in [21].

Although Theorem 3.2 gives a framework for a quasi-optimal a posteriori error estimate (see Section 4.1 below), such an estimate is not practically computable. To this purpose, we have to choose a particular $s \in H_0^1(\Omega)$ and $\mathbf{t} \in \mathbf{H}(\text{div}, \Omega)$. We devote the two following sections to this point.

3.3 Oswald interpolation operator

The Oswald interpolate of u_h was already used as a suitable $s \in H_0^1(\Omega)$ in a posteriori error estimation in nonconforming or DG methods, *cf.* [1, 16, 11]. It has been analyzed in detail in [16, 8]. The Oswald interpolation operator $\mathcal{I}_{\text{Os}} : V_h^k \rightarrow V_h^k \cap H_0^1(\Omega)$ is defined as follows: given a function $v_h \in V_h^k$, the value of $\mathcal{I}_{\text{Os}}(v_h)$ is prescribed at suitable (*e.g.*, Lagrangian) vertices of the simplices of \mathcal{T}_h . At the vertices located inside Ω , the average of the values of v_h is specified by

$$\mathcal{I}_{\text{Os}}(v_h)(V) = \frac{1}{|\mathcal{T}_V|} \sum_{T \in \mathcal{T}_V} v_h|_T(V),$$

where \mathcal{T}_V is the set of $T \in \mathcal{T}_h$ that contain the vertex V , while at boundary vertices, the value of $\mathcal{I}_{\text{Os}}(v_h)$ is set to zero. The following results have been proved in [8, Lemma 3.2 and Remark 3.2] and [16, Theorem 2.2]:

Lemma 3.5 (Oswald interpolation operator). *Let \mathcal{T}_h be shape-regular, let $v_h \in V_h^k$, and let $\mathcal{I}_{\text{Os}}(v_h)$ be constructed as above. Then,*

$$\begin{aligned} \|v_h - \mathcal{I}_{\text{Os}}(v_h)\|_{0,T}^2 &\leq C \sum_{F \in \tilde{\mathcal{F}}_T} h_F \|[[v_h]]\|_{0,F}^2, \\ \|\nabla(v_h - \mathcal{I}_{\text{Os}}(v_h))\|_{0,T}^2 &\leq C \sum_{F \in \tilde{\mathcal{F}}_T} h_F^{-1} \|[[v_h]]\|_{0,F}^2, \end{aligned}$$

where the constants C only depend on the space dimension d , the maximal polynomial degree k , and the shape regularity parameter $\kappa_{\mathcal{T}}$.

3.4 Diffusive flux reconstruction

A choice of suitable $\mathbf{t} \in \mathbf{H}(\text{div}, \Omega)$ in Theorem 3.2 is a more delicate question. Remark in particular that $\mathbf{t} \in \mathbf{H}(\text{div}, \Omega)$ is a necessary condition but some result on the divergence of \mathbf{t} will also be necessary in view of the abstract a posteriori error estimate. A previous work on $\mathbf{H}(\text{div}, \Omega)$ flux postprocessing in DG methods includes the paper of Bastian and Rivière [4], but we shall choose here the postprocessing recently derived in [12] or in [18].

To this purpose, we will need the Raviart–Thomas–Nédélec spaces of vector functions (*cf.* [25, 22, 7, 27])

$$\begin{aligned} \mathbf{RTN}_T^l &= \mathbb{P}_l^d(T) + \mathbf{x}\mathbb{P}_l(T), \\ \mathbf{RTN}_h^l &= \{\mathbf{v}_h \in \mathbf{H}(\text{div}, \Omega); \mathbf{v}_h|_T \in \mathbf{RTN}_T^l \quad \forall T \in \mathcal{T}_h\}. \end{aligned}$$

In particular, $\mathbf{v}_h \in \mathbf{RTN}_h^l$ is such that $\nabla \cdot \mathbf{v}_h \in \mathbb{P}_l(T)$ for all $T \in \mathcal{T}_h$, $\mathbf{v}_h \cdot \mathbf{n}_F \in \mathbb{P}_l(F)$ for all $F \in \mathcal{F}_T$ and all $T \in \mathcal{T}_h$, and such that its normal trace is continuous.

Using the specification of the degrees of freedom of functions in \mathbf{RTN}_T^l given in the above citations, our $\mathbf{H}(\text{div}, \Omega)$ -conforming diffusive flux reconstruction \mathbf{t}_h will belong to \mathbf{RTN}_h^l with $l = k$ or $l = k - 1$ and we prescribe it locally on all $T \in \mathcal{T}_h$ as follows:

$$(\mathbf{t}_h \cdot \mathbf{n}_F, q_h)_{0,F} = \left(-\mathbf{n}_F^t \{ \mathbf{K} \nabla_h u_h \}_\omega + \alpha_F \frac{\gamma_{\mathbf{K},F}}{h_F} [[u_h]], q_h \right)_{0,F} \quad \forall q_h \in \mathbb{P}_l(F), \forall F \in \mathcal{F}_T, \quad (3.3)$$

$$(\mathbf{t}_h, \mathbf{r}_h)_{0,T} = -(\mathbf{K} \nabla_h u_h, \mathbf{r}_h)_{0,T} + \theta \sum_{F \in \mathcal{F}_T} \omega_{T,F} (\mathbf{n}_F^t \mathbf{K} \mathbf{r}_h, [[u_h]])_{0,F} \quad \forall \mathbf{r}_h \in \mathbb{P}_{l-1}^d(T). \quad (3.4)$$

Note in particular that the quantities prescribing the moments of $\mathbf{t}_h \cdot \mathbf{n}_F$ are uniquely defined for each face $F \in \mathcal{F}_h$, whence the continuity of the normal trace of \mathbf{t}_h . By this construction, we have the following crucial lemma:

Lemma 3.6 (Reconstructed diffusion residual). *There holds*

$$(\nabla \cdot \mathbf{t}_h, \xi_h)_{0,T} = (f, \xi_h)_{0,T} \quad \forall T \in \mathcal{T}_h, \forall \xi_h \in \mathbb{P}_l(T), \quad (3.5)$$

which yields, using that $\nabla \cdot \mathbf{t}_h|_T \in \mathbb{P}_l(T)$,

$$\nabla \cdot \mathbf{t}_h|_T = \Pi_l(f)|_T \quad \forall T \in \mathcal{T}_h.$$

Proof. Let $\xi_h \in \mathbb{P}_l(T)$ be arbitrary. We then have, using the Green theorem, the fact that $\xi_h|_F \in \mathbb{P}_l(F)$ for all $F \in \mathcal{F}_T$, $\nabla \xi_h \in \mathbb{P}_{l-1}^d(T)$, the definition (3.3)–(3.4) of \mathbf{t}_h , and putting $v_h = \xi_h$ on T and $v_h = 0$ otherwise,

$$\begin{aligned} (\nabla \cdot \mathbf{t}_h, \xi_h)_{0,T} &= -(\mathbf{t}_h, \nabla \xi_h)_{0,T} + \sum_{F \in \mathcal{F}_T} (\mathbf{t}_h \cdot \mathbf{n}_T, \xi_h)_{0,F} = (\mathbf{K} \nabla_h u_h, \nabla_h v_h)_{0,T} \\ &\quad - \sum_{F \in \mathcal{F}_T} \left\{ \theta(\mathbf{n}_F^t \{ \mathbf{K} \nabla_h v_h \}_\omega, \llbracket u_h \rrbracket)_{0,F} + \left(\mathbf{n}_F^t \{ \mathbf{K} \nabla_h u_h \}_\omega - \alpha_F \frac{\gamma_{\mathbf{K},F}}{h_F} \llbracket u_h \rrbracket, \llbracket v_h \rrbracket \right)_{0,F} \right\} \\ &= \mathcal{B}_h(u_h, v_h) = (f, v_h) = (f, \xi_h)_{0,T}, \end{aligned}$$

employing finally the definition of the DG bilinear form (2.8) and that of the DG approximate solution (2.9). \square

3.5 Locally computable a posteriori error estimate

With the results of the three previous sections, we are now ready to state our practical locally computable a posteriori error estimate for DG methods.

Let us define the *nonconformity estimator* $\eta_{\text{NC},T}$ by

$$\eta_{\text{NC},T} := \| \|u_h - \mathcal{I}_{\text{Os}}(u_h)\| \|_T, \quad (3.6)$$

and the *diffusive flux estimator* $\eta_{\text{DF},T}$ by

$$\eta_{\text{DF},T} := \| \mathbf{K}^{\frac{1}{2}} \nabla u_h + \mathbf{K}^{-\frac{1}{2}} \mathbf{t}_h \|_{0,T}, \quad (3.7)$$

where $\mathbf{t}_h \in \mathbf{RTN}_h^l$ is given by (3.3)–(3.4). Finally, let us put

$$m_{T,\mathbf{K}}^2 := C_P \frac{h_T^2}{c_{\mathbf{K},T}}$$

for all $T \in \mathcal{T}_h$, where C_P is the constant from the Poincaré inequality

$$\| \varphi - \Pi_0(\varphi) \|_{0,T}^2 \leq C_P h_T^2 \| \nabla \varphi \|_{0,T}^2 \quad \forall \varphi \in H^1(T), \quad (3.8)$$

which can be evaluated as $1/\pi^2$ owing to the convexity of $T \in \mathcal{T}_h$, cf. [23, 5]. We define the *residual estimator* $\eta_{\text{R},T}$ by

$$\eta_{\text{R},T} := m_{T,\mathbf{K}} \| f - \Pi_l(f) \|_{0,T}. \quad (3.9)$$

We then have the following a posteriori error estimate:

Theorem 3.7 (Locally computable a posteriori error estimate in the pure diffusion case). *Let $\beta = \mu = 0$, let u be the unique solution of (2.5), and let u_h be its discontinuous Galerkin approximation given by (2.9). Then*

$$\| \|u - u_h\| \| \leq \left\{ \sum_{T \in \mathcal{T}_h} \{ \eta_{\text{NC},T}^2 + (\eta_{\text{R},T} + \eta_{\text{DF},T})^2 \} \right\}^{1/2}.$$

Proof. Put $s = \mathcal{I}_{\text{Os}}(u_h)$ and $\mathbf{t} = \mathbf{t}_h$ in Theorem 3.2. Note that, for each $T \in \mathcal{T}_h$,

$$\begin{aligned} |(f - \nabla \cdot \mathbf{t}_h, \varphi)_{0,T}| &= |(f - \nabla \cdot \mathbf{t}_h, \varphi - \Pi_0(\varphi))_{0,T}| \\ &= |(f - \Pi_l(f), \varphi - \Pi_0(\varphi))_{0,T}| \leq \eta_{\mathbf{R},T} \|\varphi\|_T, \end{aligned} \quad (3.10)$$

using Lemma 3.6, the Poincaré inequality 3.8, and the definition (2.4) of the energy norm (note that this step holds true for both $l = k$ or $l = k - 1$). Next, $|(\mathbf{K}\nabla_h u_h + \mathbf{t}, \nabla \varphi)_{0,T}| \leq \eta_{\text{DF},T} \|\varphi\|_T$ is immediate. Hence it now suffices to use the Cauchy–Schwarz inequality and to notice that $\|\varphi\| = 1$ in order to conclude the proof. \square

Remark 3.8 (Properties of the estimate of Theorem 3.7). *The following properties of the estimate of Theorem 3.7 can be mentioned:*

- It gives a guaranteed upper bound, i.e., features no undetermined constant.
- The residual estimator $\eta_{\mathbf{R},T}$ coincides with the classical (properly weighted) “data oscillation term”, whence it represents a major improvement of the classical residual estimator, which is of the form $c_{\mathbf{K}} h_T \|f + \nabla \cdot (\mathbf{K}\nabla_h u_h)\|_{0,T}$. Also, although it represents a higher-order term for piecewise smooth f , it shall not be neglected as it can be important on coarse grids or for highly varying \mathbf{K} .
- The Poincaré constant $C_{\mathbf{P}}$ does not depend on the shape-regularity of the mesh, whence the present estimate is valid also on anisotropic meshes.
- The Poincaré constant $C_{\mathbf{P}}$ does not depend on the polynomial degree of u_h , so that, in contrast to the estimates of [16, 6], the present estimate is valid uniformly with respect to k .
- No assumption on the polynomial form of f is needed at this stage.
- Letting $\eta_{\mathbf{R},T} = m_{T,\mathbf{K}} \|f - \nabla \cdot \mathbf{t}_h\|_{0,T}$, the estimate is valid for any $\mathbf{t}_h \in \mathbf{H}(\text{div}, \Omega)$ such that $(\nabla \cdot \mathbf{t}_h, 1)_{0,T} = (f, 1)_{0,T}$ for all $T \in \mathcal{T}_h$, which is a local (conservativity) property, in contrast to the global Galerkin orthogonality used traditionally for conforming finite element methods.

4 Efficiency of the estimates in the pure diffusion case

In this section, we first rapidly check the (global) efficiency of the abstract framework of Theorem 3.2. We then investigate in detail the (local) efficiency of the a posteriori error estimate of Theorem 3.7.

4.1 Global efficiency of the abstract estimate

Theorem 4.1 (Global efficiency of the abstract estimate in the pure diffusion case). *Let $\beta = \mu = 0$, let u be the unique solution of (2.5), and let $u_h \in H^1(\mathcal{T}_h)$ be arbitrary. Let the a posteriori error estimate be given by Theorem 3.2. Then*

$$\begin{aligned} \inf_{s \in H_0^1(\Omega)} \|u_h - s\|^2 + \inf_{\mathbf{t} \in \mathbf{H}(\text{div}, \Omega)} \sup_{\varphi \in H_0^1(\Omega), \|\varphi\|=1} ((f - \nabla \cdot \mathbf{t}, \varphi) - (\mathbf{K}\nabla_h u_h + \mathbf{t}, \nabla \varphi))^2 \\ \leq 2 \|u - u_h\|^2. \end{aligned}$$

Proof. It suffices to put $s = u$, $\mathbf{t} = -\mathbf{K}\nabla u$, and to use the Cauchy–Schwarz inequality and the fact that $\|\varphi\| = 1$. \square

Remark 4.2 (Global effectivity index). *It follows from Theorem 4.1 that the abstract a posteriori error estimate of Theorem 3.2 is quasi-exact in the sense that the effectivity index, i.e., the ratio of the estimated to the actual error, is equal to $\sqrt{2}$. The effectivity index can be improved to 1 when either $u_h \in H_0^1(\Omega)$ or $-\mathbf{K}\nabla_h u_h \in \mathbf{H}(\text{div}, \Omega)$, but this is not to be expected apart from particular cases. A possible remedy is presented in Section 5 below.*

Remark 4.3 (Robustness with respect to data, polynomial degree, and meshes). *It follows from Theorem 4.1 that the abstract a posteriori error estimate of Theorem 3.2 is fully robust with respect to \mathbf{K} without any assumption on its distribution, with respect to f (no polynomial form needed), with respect to the polynomial degree k , and finally with respect to the mesh (which can be anisotropic), in the sense that the effectivity index does not depend on these quantities.*

4.2 Local efficiency of the locally computable estimate

We now investigate how the quasi-optimal abstract global efficiency of the previous section persists for our particular choices of the conforming reconstructions of the discrete solution and of its diffusive flux. To this purpose, we restrict the class of interior-penalty DG schemes by the following assumptions: there exist constants C_1 , C_2 , and C_3 , independent of \mathbf{K} , such that

$$C_1 \min(c_{\mathbf{K},T^+(F)}, c_{\mathbf{K},T^-(F)}) \leq \gamma_{\mathbf{K},F} \leq C_2 \min(C_{\mathbf{K},T^+(F)}, C_{\mathbf{K},T^-(F)}) \quad \forall F \in \mathcal{F}_h^{\text{int}}, \quad (4.1)$$

$$C_1 c_{\mathbf{K},T(F)} \leq \gamma_{\mathbf{K},F} \leq C_2 C_{\mathbf{K},T(F)} \quad \forall F \in \mathcal{F}_h^{\text{ext}}, \quad (4.2)$$

$$c_{\mathbf{K},T^\mp(F)} \omega_{T^\mp(F),F} \leq C_3 \gamma_{\mathbf{K},F} \quad \forall F \in \mathcal{F}_h^{\text{int}}. \quad (4.3)$$

An example of a DG scheme satisfying (4.1)–(4.3) with $C_1 = 1/2$, $C_2 = 1$, and $C_3 = 1$ is that recently derived by Ern *et al.* [14]. It consists of setting

$$\gamma_{\mathbf{K},F} := \frac{\delta_{\mathbf{K},F+} \delta_{\mathbf{K},F-}}{\delta_{\mathbf{K},F+} + \delta_{\mathbf{K},F-}} \quad \forall F \in \mathcal{F}_h^{\text{int}}, \quad (4.4)$$

$$\gamma_{\mathbf{K},F} := \delta_{\mathbf{K},F} \quad \forall F \in \mathcal{F}_h^{\text{ext}}, \quad (4.5)$$

where $\delta_{\mathbf{K},F^\mp} = \mathbf{n}_F^t \mathbf{K}|_{T^\mp(F)} \mathbf{n}_F$ if $F \in \mathcal{F}_h^{\text{int}}$ and $\delta_{\mathbf{K},F} = \mathbf{n}_F^t \mathbf{K}|_{T(F)} \mathbf{n}_F$ if $F \in \mathcal{F}_h^{\text{ext}}$, while the weights are chosen so that

$$\omega_{T^-(F),F} := \frac{\delta_{\mathbf{K},F+}}{\delta_{\mathbf{K},F+} + \delta_{\mathbf{K},F-}}, \quad \omega_{T^+(F),F} := \frac{\delta_{\mathbf{K},F-}}{\delta_{\mathbf{K},F+} + \delta_{\mathbf{K},F-}} \quad \forall F \in \mathcal{F}_h^{\text{int}}. \quad (4.6)$$

Theorem 4.4 (Local efficiency of the locally computable estimate in the pure diffusion case). *Let $\beta = \mu = 0$, let \mathcal{T}_h be shape-regular, let f be a piecewise polynomial of degree m , let u be the unique solution of (2.5), and let u_h be its discontinuous Galerkin approximation given by (2.9) with the weights $\omega_{T,F}$ and penalty parameters $\gamma_{\mathbf{K},F}$ (for the simplicity supposed facewise constant) satisfying (4.1)–(4.3). Let next the a posteriori error estimate be given by Theorem 3.7, with in particular $\eta_{\text{NC},T}$ given by (3.6) and $\eta_{\text{DF},T}$ given by (3.7). Let us put*

$$c_{\mathbf{K},\tilde{\mathcal{T}}_T} := \min_{T' \in \tilde{\mathcal{T}}_T} c_{\mathbf{K},T'}, \quad (4.7)$$

and

$$\|v\|_{*,\mathcal{F}}^2 := \sum_{F \in \mathcal{F}} \|\gamma_F^{1/2} [v]\|_{0,F}^2 \quad v \in H^1(\mathcal{T}_h), \quad (4.8)$$

where we will either take $\mathcal{F} = \mathcal{F}_h$, $\mathcal{F} = \mathcal{F}_T$, or $\mathcal{F} = \tilde{\mathcal{F}}_T$. Then, for each $T \in \mathcal{T}_h$, there holds

$$\eta_{\text{NC},T} \leq C \frac{C_{\mathbf{K},T}^{1/2}}{c_{\mathbf{K},\tilde{\mathcal{F}}_T}^{1/2}} \| \|u - u_h\| \|_{*,\tilde{\mathcal{F}}_T}, \quad (4.9)$$

$$\eta_{\text{DF},T} \leq \tilde{C} \max_{T' \in \mathcal{T}_T} \left\{ \frac{C_{\mathbf{K},T'}}{c_{\mathbf{K},T'}} \right\} (\| \|u - u_h\| \|_{\mathcal{T}_T} + \| \|u - u_h\| \|_{*,\mathcal{F}_T}), \quad (4.10)$$

where the constant C depends only on the space dimension d , on the maximal polynomial degree k , on the DG parameters α_F , on the shape regularity parameter κ_T , and on the constant C_1 from (4.1)–(4.2) and \tilde{C} in addition depends on the polynomial degree m of f , on the DG parameter θ , and on the constants C_2 – C_3 from (4.1)–(4.3).

Proof. Combine Lemmas 4.10 and 4.12 given below. \square

Remark 4.5 (Local efficiency norm and global efficiency with respect to the energy seminorm). *The local efficiency stated in Theorem 4.4 is given for the energy seminorm augmented by the natural DG discrete energy norm $\| \cdot \|_{*,\mathcal{F}_h}$. Owing to the result of Ainsworth [3, Theorem 3], global efficiency of our nonconformity and diffusive flux estimators in the energy seminorm $\| \cdot \|$ follows from (4.9)–(4.10) for sufficiently large stabilization parameters α_F in the case $d = 2$, $k = 1$, $\mathbf{K} = \text{Id}$, and $\theta = 1$.*

Remark 4.6 (Efficiency of the residual estimator). *We recall that the residual estimator $\eta_{\text{R},T}$ coincides with the usual “data oscillation term” and is in general of higher order, whence no efficiency is to be shown.*

Remark 4.7 (Robustness with respect to inhomogeneities and anisotropies). *Owing to (4.10), our diffusive flux estimator $\eta_{\text{DF},T}$ is fully robust with respect to diffusion inhomogeneities. This is an important property in practical applications, e.g., when dealing with underground flows. The design conditions (4.1)–(4.3) play a crucial role in this respect, cf. the proof of Lemma 4.12 below. A similar result was proved recently in [13] in the context of residual-based a posteriori error estimates for DG methods with diffusivity-dependent weights and penalty parameter based on the harmonic average of the normal diffusivity, see (4.4)–(4.6). We next point out that under the assumption of “monotonicity around vertices” distribution of the inhomogeneities and using the concepts of, e.g., Ainsworth [2], also the nonconformity estimator $\eta_{\text{NC},T}$ may be shown robust with respect to inhomogeneities. Finally, no robustness with respect to anisotropies is achieved by our estimators $\eta_{\text{DF},T}$ and $\eta_{\text{NC},T}$, but, at least, the local efficiency estimates only depend on local, elementwise, anisotropies.*

Remark 4.8 (Generalization to other DG schemes). *Making appropriate changes in the proof of Theorem 4.4 below, all the presented results (up to the robustness with respect to inhomogeneities) extend appropriately to all the DG schemes included in the setting (2.8)–(2.9), even if the design conditions (4.1)–(4.3) are not satisfied.*

Remark 4.9 (Lower bound for the classical estimator). *The proof of Lemma 4.10 below shows that the nonconformity estimator $\eta_{\text{NC},T}$ represents a lower bound for the classical nonconformity estimator $\left\{ \sum_{F \in \tilde{\mathcal{F}}_T} h_F^{-1} \| \|u_h\| \|_{0,F}^2 \right\}^{1/2}$. Similarly, the proof of Lemma 4.12 below shows that the diffusive flux estimator $\eta_{\text{DF},T}$ represents a lower bound for the classical gradient jump estimator $\left\{ \sum_{F \in \mathcal{F}_T} \| \mathbf{n}_F^t \| \| \mathbf{K} \nabla_h u_h \| \|_{0,F}^2 \right\}^{1/2}$ plus again the classical nonconformity estimator.*

As already indicated, the proof of Theorem 4.4 is decomposed into several parts:

Lemma 4.10 (Local efficiency of the nonconformity estimator). *Let the assumptions of Theorem 4.4 be verified. Then (4.9) holds true.*

Proof. We have

$$\begin{aligned}
\eta_{\text{NC},T}^2 &= \|u_h - \mathcal{I}_{\text{Os}}(u_h)\|_T^2 \leq C_{\mathbf{K},T} \|\nabla(u_h - \mathcal{I}_{\text{Os}}(u_h))\|_{0,T}^2 \\
&\leq C C_{\mathbf{K},T} \sum_{F \in \tilde{\mathcal{F}}_T} h_F^{-1} \|[u_h]\|_{0,F}^2 = C \sum_{F \in \tilde{\mathcal{F}}_T} \frac{C_{\mathbf{K},T}}{\alpha_F \gamma_{\mathbf{K},F}} \alpha_F \frac{\gamma_{\mathbf{K},F}}{h_F} \|[u_h]\|_{0,F}^2 \\
&\leq \frac{C}{C_1} \left(\min_{F \in \tilde{\mathcal{F}}_T} \alpha_F \right)^{-1} \frac{C_{\mathbf{K},T}}{c_{\mathbf{K},\tilde{\mathcal{F}}_T}} \sum_{F \in \tilde{\mathcal{F}}_T} \alpha_F \frac{\gamma_{\mathbf{K},F}}{h_F} \|[u - u_h]\|_{0,F}^2,
\end{aligned} \tag{4.11}$$

using Lemma 3.5, the lower bound in (4.1)–(4.2), and the fact that $[u_h - u] = [u_h]$. Recall from Lemma 3.5 that C depends only on d , k , and $\kappa_{\mathcal{T}}$. \square

Lemma 4.11 (Norm estimate for the \mathbf{RTN}_T^l space). *Let \mathcal{T}_h be shape-regular. Then there exists a constant C , depending only on d , k , and $\kappa_{\mathcal{T}}$ such that for all $\mathbf{v}_h \in \mathbf{RTN}_T^l$, there holds*

$$\|\mathbf{v}_h\|_{0,T}^2 \leq C \left\{ h_T \sum_{F \in \mathcal{F}_T} \|\mathbf{v}_h \cdot \mathbf{n}_F\|_{0,F}^2 + \left(\sup_{\mathbf{r}_h \in \mathbb{P}_{l-1}^d(T)} \frac{(\mathbf{v}_h, \mathbf{r}_h)_{0,T}}{\|\mathbf{r}_h\|_{0,T}} \right)^2 \right\}.$$

Proof. Use norm equivalence on finite-dimensional spaces, the Piola transformation, and scaling arguments. \square

Lemma 4.12 (Local efficiency of the diffusive flux estimator). *Let the assumptions of Theorem 4.4 be verified. Then (4.10) holds true.*

Proof. Throughout this proof, let C denote a general constant not necessarily the same at each occurrence, depending only on d , k , and $\kappa_{\mathcal{T}}$. We put $\mathbf{v}_h := (\mathbf{K}\nabla u_h + \mathbf{t}_h)|_T \in \mathbf{RTN}_T^l$ and notice that, for $\mathbf{r}_h \in \mathbb{P}_{l-1}^d(T)$,

$$(\mathbf{v}_h, \mathbf{r}_h)_{0,T} = \theta \sum_{F \in \mathcal{F}_T} \omega_{T,F} (\mathbf{n}_F^t \mathbf{K} \mathbf{r}_h, [u_h])_{0,F} \leq |\theta| C h_T^{-\frac{1}{2}} C_{\mathbf{K},T} \|\mathbf{r}_h\|_{0,T} \sum_{F \in \mathcal{F}_T} \omega_{T,F} \|[u_h]\|_{0,F},$$

owing to the definition (3.4), the Cauchy–Schwarz inequality, and the inverse inequality $\|\mathbf{r}_h\|_{0,F} \leq C h_T^{-1/2} \|\mathbf{r}_h\|_{0,T}$. Hence, using the definition (3.7) of $\eta_{\text{DF},T}$, the previous lemma, the definition of \mathbf{t}_h by (3.3)–(3.4), and the above inequality leads to

$$\begin{aligned}
\eta_{\text{DF},T}^2 &\leq \frac{1}{c_{\mathbf{K},T}} \|\mathbf{v}_h\|_{0,T}^2 \leq \frac{C}{c_{\mathbf{K},T}} \left\{ h_T \sum_{F \in \mathcal{F}_T} \|\mathbf{v}_h \cdot \mathbf{n}_F\|_{0,F}^2 + \left(\sup_{\mathbf{r}_h \in \mathbb{P}_{l-1}^d(T)} \frac{(\mathbf{v}_h, \mathbf{r}_h)_{0,T}}{\|\mathbf{r}_h\|_{0,T}} \right)^2 \right\} \\
&\leq \frac{C}{c_{\mathbf{K},T}} \left\{ h_T \sum_{F \in \mathcal{F}_T} \left\| \bar{\omega}_{T,F} \mathbf{n}_F^t [\mathbf{K}\nabla_h u_h] + \alpha_F \frac{\gamma_{\mathbf{K},F}}{h_F} \Pi_l([u_h]) \right\|_{0,F}^2 \right. \\
&\quad \left. + \theta^2 h_T^{-1} C_{\mathbf{K},T}^2 \sum_{F \in \mathcal{F}_T} \omega_{T,F}^2 \|[u_h]\|_{0,F}^2 \right\}.
\end{aligned}$$

We next study the three resulting terms separately. First of all,

$$\begin{aligned}
& \frac{h_T}{c_{\mathbf{K},T}} \sum_{F \in \mathcal{F}_T} \left\| \alpha_F \frac{\gamma_{\mathbf{K},F}}{h_F} \Pi_l(\llbracket u_h \rrbracket) \right\|_{0,F}^2 \leq C \sum_{F \in \mathcal{F}_T} \alpha_F \frac{\gamma_{\mathbf{K},F}}{c_{\mathbf{K},T}} \alpha_F \frac{\gamma_{\mathbf{K},F}}{h_F} \|\Pi_l(\llbracket u_h \rrbracket)\|_{0,F}^2 \\
& \leq CC_2 \max_{F \in \mathcal{F}_T} \alpha_F \sum_{F \in \mathcal{F}_T} \frac{\min\{C_{\mathbf{K},T^-}, C_{\mathbf{K},T^+}\}}{c_{\mathbf{K},T}} \alpha_F \frac{\gamma_{\mathbf{K},F}}{h_F} \|\llbracket u_h \rrbracket\|_{0,F}^2 \\
& \leq CC_2 \max_{F \in \mathcal{F}_T} \alpha_F \max_{T' \in \mathcal{T}_T} \left\{ \frac{C_{\mathbf{K},T'}}{c_{\mathbf{K},T'}} \right\} \sum_{F \in \mathcal{F}_T} \alpha_F \frac{\gamma_{\mathbf{K},F}}{h_F} \|\llbracket u - u_h \rrbracket\|_{0,F}^2,
\end{aligned}$$

where we have used (4.1) (the modification at the boundary has been skipped for simplicity). Similarly, employing $0 \leq \omega_{T,F} \leq 1$ and (4.3) yields

$$\begin{aligned}
& \theta^2 \frac{C_{\mathbf{K},T}^2}{h_T c_{\mathbf{K},T}} \sum_{F \in \mathcal{F}_T} \omega_{T,F}^2 \|\llbracket u_h \rrbracket\|_{0,F}^2 \leq \theta^2 \frac{C_{\mathbf{K},T}^2}{h_T c_{\mathbf{K},T}^2} \sum_{F \in \mathcal{F}_T} c_{\mathbf{K},T} \omega_{T,F} \|\llbracket u_h \rrbracket\|_{0,F}^2 \\
& \leq CC_3 \theta^2 \frac{C_{\mathbf{K},T}^2}{c_{\mathbf{K},T}^2} \left(\min_{F \in \mathcal{F}_T} \alpha_F \right)^{-1} \sum_{F \in \mathcal{F}_T} \alpha_F \frac{\gamma_{\mathbf{K},F}}{h_F} \|\llbracket u - u_h \rrbracket\|_{0,F}^2.
\end{aligned}$$

Finally,

$$\frac{h_T}{c_{\mathbf{K},T}} \sum_{F \in \mathcal{F}_T} \bar{\omega}_{T,F}^2 \|\mathbf{n}_F^t \llbracket \mathbf{K} \nabla_h u_h \rrbracket\|_{0,F}^2 \leq CC_2 C_3 \max_{T' \in \mathcal{T}_T} \left\{ \frac{C_{\mathbf{K},T'}}{c_{\mathbf{K},T'}} \right\}^2 \|\llbracket u - u_h \rrbracket\|_{\mathcal{T}_T}^2,$$

which has been shown in [13, Proposition 3.2]. \square

5 A posteriori error estimates for the reconstructed flux

The a posteriori error estimates of Section 3 are given for the DG approximate solution u_h , or, equivalently, taking into account the definition of the energy norm (2.4), for its flux $-\mathbf{K} \nabla_h u_h$. In order to obtain them, we have used its $\mathbf{H}(\text{div}, \Omega)$ -conforming diffusive flux reconstruction \mathbf{t}_h . There arises a natural question whether are we also able to give an a posteriori error estimate for this (supposedly) improved flux, instead of the original estimate. Moreover, Theorem 4.1 indicates that the a posteriori error estimates of Section 3 will only lead to quasi-exactness with effectivity index $\sqrt{2}$. This is quite obvious because both u_h and $-\mathbf{K} \nabla_h u_h$ are nonconforming (in the sense that $u_h \notin H_0^1(\Omega)$ and $-\mathbf{K} \nabla_h u_h \notin \mathbf{H}(\text{div}, \Omega)$) and consequently two estimators appear. We present here a possible remedy to this situation: since $\mathbf{t}_h \in \mathbf{H}(\text{div}, \Omega)$ is such that $\nabla \cdot \mathbf{t}_h = \Pi_l(f)$, it suffices to use the results of [32] in order to obtain the same estimates as for mixed finite elements:

Theorem 5.1 (Abstract a posteriori error estimate for the reconstructed flux). *Let u be the unique solution of (2.5), let u_h be its discontinuous Galerkin approximation given by (2.9), and let \mathbf{t}_h be its diffusive flux reconstruction given by (3.3)–(3.4). Then*

$$\|\mathbf{K}^{-\frac{1}{2}} \mathbf{t}_h + \mathbf{K}^{\frac{1}{2}} \nabla u\|^2 \leq \inf_{s \in H_0^1(\Omega)} \|\mathbf{K}^{-\frac{1}{2}} \mathbf{t}_h + \mathbf{K}^{\frac{1}{2}} \nabla s\|^2 + \sum_{T \in \mathcal{T}_h} m_{T,\mathbf{K}}^2 \|f - \Pi_l(f)\|_{0,T}^2.$$

For practical purposes, a first choice for s in the above theorem is $s = \mathcal{I}_{\text{Os}}(u_h)$. However, more precise reconstructions are suggested and studied in [32] for mixed finite elements.

Concerning the efficiency of this framework, we have the following result, which in contrast to Section 4.1 gives full asymptotic exactness (*i.e.*, effectivity index equal to 1, up to the residual (or data oscillation) term).

Theorem 5.2 (Global efficiency of the abstract estimate for the reconstructed flux). *Let u be the unique solution of (2.5), let u_h be its discontinuous Galerkin approximation given by (2.9), and let \mathbf{t}_h be its diffusive flux reconstruction given by (3.3)–(3.4). Let the a posteriori error estimate for \mathbf{t}_h be given by Theorem 5.1. Then*

$$\begin{aligned} & \inf_{s \in H_0^1(\Omega)} \|\mathbf{K}^{-\frac{1}{2}} \mathbf{t}_h + \mathbf{K}^{\frac{1}{2}} \nabla s\|^2 + \sum_{T \in \mathcal{T}_h} m_{T, \mathbf{K}}^2 \|f - \Pi_l(f)\|_{0, T}^2 \\ & \leq \|\mathbf{K}^{-\frac{1}{2}} \mathbf{t}_h + \mathbf{K}^{\frac{1}{2}} \nabla u\|^2 + \sum_{T \in \mathcal{T}_h} m_{T, \mathbf{K}}^2 \|f - \Pi_l(f)\|_{0, T}^2. \end{aligned}$$

6 Improved energy norm a posteriori error estimates in the general case

We present in this section an extension of our analysis of Section 3 to the general case (1.1a)–(1.1b). Again at this stage, neither assumptions on the data other than those stated in Section 2.2, nor the mesh shape-regularity, are needed.

6.1 Abstract framework

The following general abstract framework has been proved in [31, Lemma 7.1].

Lemma 6.1 (Abstract framework in the general case). *Let $u \in H_0^1(\Omega)$ and $u_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then*

$$\begin{aligned} \| \|u - u_h\| \| \leq & \inf_{s \in H_0^1(\Omega)} \left\{ \| \|u_h - s\| \| + \sup_{\varphi \in H_0^1(\Omega), \| \varphi \| = 1} (\mathcal{B}(u - u_h, \varphi) \right. \\ & \left. + (\boldsymbol{\beta} \cdot \nabla_h(u_h - s) + \frac{1}{2}(\nabla \cdot \boldsymbol{\beta})(u_h - s), \varphi)) \right\}. \end{aligned}$$

Remark 6.2 (Comparison with the abstract framework of Lemma 3.1). *In comparison with the abstract framework of Lemma 3.1, Lemma 6.1 is applicable to the general convection–diffusion–reaction case. In particular, there is an additional contribution from the nonsymmetric part of the bilinear form $\mathcal{B}(\cdot, \cdot)$, which can be evaluated using an arbitrary $s \in H_0^1(\Omega)$. The price for this generality is that Lemma 6.1 yields a triangular-like inequality instead of a Pythagorean-like inequality.*

6.2 Abstract a posteriori error estimate

An abstract form of our a posteriori error estimate now takes the following form (compare with Theorem 3.2).

Theorem 6.3 (Abstract a posteriori error estimate in the general case). *Let u be the unique solution of (2.5) and let $u_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then*

$$\begin{aligned} \| \|u - u_h\| \| \leq & \inf_{s \in H_0^1(\Omega)} \left\{ \| \|u_h - s\| \| \right. \\ & + \inf_{\mathbf{t} \in \mathbf{H}(\text{div}, \Omega)} \sup_{\varphi \in H_0^1(\Omega), \| \varphi \| = 1} \left((f - \nabla \cdot \mathbf{t} - \boldsymbol{\beta} \cdot \nabla s - \mu s, \varphi) \right. \\ & \left. - (\mathbf{K} \nabla_h u_h + \mathbf{t}, \nabla \varphi) + \left((\mu - \frac{1}{2} \nabla \cdot \boldsymbol{\beta})(s - u_h), \varphi \right) \right) \left. \right\}. \end{aligned} \quad (6.1)$$

Proof. We use (2.5) in Lemma 6.1, keep $s \in H_0^1(\Omega)$ arbitrary, introduce an arbitrary $\mathbf{t} \in \mathbf{H}(\text{div}, \Omega)$, and employ the Green theorem to infer

$$\begin{aligned} \mathcal{B}(u - u_h, \varphi) + (\boldsymbol{\beta} \cdot \nabla_h(u_h - s) + \frac{1}{2}(\nabla \cdot \boldsymbol{\beta})(u_h - s), \varphi) &= (f - \nabla \cdot \mathbf{t} - \boldsymbol{\beta} \cdot \nabla_h u_h - \mu u_h, \varphi) \\ -(\mathbf{K} \nabla_h u_h + \mathbf{t}, \nabla \varphi) + (\boldsymbol{\beta} \cdot \nabla_h(u_h - s) + \frac{1}{2}(\nabla \cdot \boldsymbol{\beta})(u_h - s), \varphi), \end{aligned}$$

whence the assertion of the theorem follows easily. \square

Remark 6.4 (A computable a posteriori error estimate). *Similarly as in the pure diffusion case (cf. Remark 3.4), it follows readily from (6.1), using the Friedrichs inequality $\|\varphi\|^2 \leq C_{F,\Omega} h_\Omega^2 \|\nabla \varphi\|^2$ or the inequality $\|\varphi\| \leq \min_{T \in \mathcal{T}_h} \{c_{\boldsymbol{\beta},\mu,T}^{1/2}\}^{-1} \|(\mu - \frac{1}{2} \nabla \cdot \boldsymbol{\beta})^{1/2} \varphi\|$, the Cauchy–Schwarz inequality, the definition (2.4) of the energy seminorm, and the fact that $\|\varphi\| = 1$,*

$$\begin{aligned} \|u - u_h\| &\leq \|u_h - s\| + \min \left\{ \frac{C_{F,\Omega}^{1/2} h_\Omega}{\min_{T \in \mathcal{T}_h} c_{\mathbf{K},T}^{1/2}}, \frac{1}{\min_{T \in \mathcal{T}_h} c_{\boldsymbol{\beta},\mu,T}^{1/2}} \right\} \|f - \nabla \cdot \mathbf{t} - \boldsymbol{\beta} \cdot \nabla s - \mu s\| \\ &\quad + \left(\|\mathbf{K}^{\frac{1}{2}} \nabla_h u_h + \mathbf{K}^{-\frac{1}{2}} \mathbf{t}_h\|^2 + \|(\mu - \frac{1}{2} \nabla \cdot \boldsymbol{\beta})^{\frac{1}{2}}(u_h - s)\|^2 \right)^{1/2} \end{aligned}$$

for any $s \in H_0^1(\Omega)$ and any $\mathbf{t} \in \mathbf{H}(\text{div}, \Omega)$. Again, this is a fully computable and scheme-independent estimate, but all the points from Remark 3.4 apply here as well.

Remark 6.5 (Another form of Theorem 6.3). *The estimate of Theorem 6.3 can be changed into*

$$\begin{aligned} \|u - u_h\| &\leq \inf_{s \in H_0^1(\Omega)} \left\{ \|u_h - s\| \right. \\ &\quad + \inf_{\mathbf{q} \in \mathbf{H}(\text{div}, \Omega)} \inf_{\mathbf{t} \in \mathbf{H}(\text{div}, \Omega)} \sup_{\varphi \in H_0^1(\Omega), \|\varphi\|=1} \left| (f - \nabla \cdot \mathbf{t} - \nabla \cdot \mathbf{q} - (\mu - \nabla \cdot \boldsymbol{\beta})u_h, \varphi) \right. \\ &\quad \left. \left. - (\mathbf{K} \nabla_h u_h + \mathbf{t}, \nabla \varphi) + (\nabla \cdot \mathbf{q} - \nabla \cdot (\boldsymbol{\beta}s), \varphi) - (\frac{1}{2}(\nabla \cdot \boldsymbol{\beta})(u_h - s), \varphi) \right| \right\}. \end{aligned}$$

Again, Theorem 6.3 gives a framework for a quasi-optimal a posteriori error estimate (see Section 7.1 below), which is however not practically computable. In the next sections, we present its locally computable version. To this purpose, we would like to keep the choice of the $H_0^1(\Omega)$ -conforming scalar function s and of the $\mathbf{H}(\text{div}, \Omega)$ -conforming diffusive flux \mathbf{t} the same as in Sections 3.3 and 3.4, respectively. With this choice, however, the residual $f - \nabla \cdot \mathbf{t} - \boldsymbol{\beta} \cdot \nabla s - \mu s$ does not satisfy an orthogonality property as (3.5). It is not even of zero mean necessarily, which would be necessary to obtain a computable estimate as in (3.10). In order to recover (at least partially) these properties, we will employ the form of Theorem 6.3 given in Remark 6.5, where \mathbf{q} will be a $\mathbf{H}(\text{div}, \Omega)$ -conforming convective flux reconstruction.

6.3 Convective flux reconstruction

Our $\mathbf{H}(\text{div}, \Omega)$ -conforming convective flux reconstruction \mathbf{q}_h will belong to \mathbf{RTN}_h^l with $l = k$ or $l = k - 1$ and we prescribe it locally on all $T \in \mathcal{T}_h$, as follows:

$$(\mathbf{q}_h \cdot \mathbf{n}_F, q_h)_{0,F} = (\boldsymbol{\beta} \cdot \mathbf{n}_F \{u_h\} + \gamma_{\boldsymbol{\beta},F} [u_h], q_h)_{0,F} \quad \forall q_h \in \mathbb{P}_l(F), \forall F \in \mathcal{F}_T, \quad (6.2)$$

$$(\mathbf{q}_h, \mathbf{r}_h)_{0,T} = (u_h, \boldsymbol{\beta} \cdot \mathbf{r}_h)_{0,T} \quad \forall \mathbf{r}_h \in \mathbb{P}_{l-1}^d(T). \quad (6.3)$$

Note in particular that the quantities prescribing the moments of $\mathbf{q}_h \cdot \mathbf{n}_F$ are uniquely defined for each face $F \in \mathcal{F}_h$, whence the continuity of the normal trace of \mathbf{q}_h . By this construction, we have the following generalization of Lemma 3.6:

Lemma 6.6 (Reconstructed convection–diffusion–reaction residual). *There holds*

$$(\nabla \cdot \mathbf{t}_h + \nabla \cdot \mathbf{q}_h + (\mu - \nabla \cdot \boldsymbol{\beta})u_h, \xi_h)_{0,T} = (f, \xi_h)_{0,T} \quad \forall T \in \mathcal{T}_h, \forall \xi_h \in \mathbb{P}_l(T).$$

Moreover, using that $\nabla \cdot \mathbf{t}_h|_T \in \mathbb{P}_l(T)$ and $\nabla \cdot \mathbf{q}_h|_T \in \mathbb{P}_l(T)$ for all $T \in \mathcal{T}_h$,

$$(\nabla \cdot \mathbf{t}_h + \nabla \cdot \mathbf{q}_h + \Pi_l((\mu - \nabla \cdot \boldsymbol{\beta})u_h))|_T = \Pi_l(f)|_T \quad \forall T \in \mathcal{T}_h,$$

and, when in particular μ and $\nabla \cdot \boldsymbol{\beta}$ are elementwise constant and when $l = k$ in the diffusive and convective flux reconstructions,

$$(\nabla \cdot \mathbf{t}_h + \nabla \cdot \mathbf{q}_h + (\mu - \nabla \cdot \boldsymbol{\beta})u_h)|_T = \Pi_k(f)|_T \quad \forall T \in \mathcal{T}_h.$$

Proof. Let $\xi_h \in \mathbb{P}_l(T)$ be arbitrary. Owing to the Green theorem, the fact that $\xi_h|_F \in \mathbb{P}_l(F)$ for all $F \in \mathcal{F}_T$, $\nabla \xi_h \in \mathbb{P}_{l-1}^d(T)$, the definitions (3.3)–(3.4) of \mathbf{t}_h and the definitions (6.2)–(6.3) of \mathbf{q}_h , respectively, and putting $v_h = \xi_h$ on T and $v_h = 0$ otherwise, we infer

$$\begin{aligned} & (\nabla \cdot \mathbf{t}_h + \nabla \cdot \mathbf{q}_h + (\mu - \nabla \cdot \boldsymbol{\beta})u_h, \xi_h)_{0,T} = -(\mathbf{t}_h, \nabla \xi_h)_{0,T} + \sum_{F \in \mathcal{F}_T} (\mathbf{t}_h \cdot \mathbf{n}_T, \xi_h)_{0,F} - (\mathbf{q}_h, \nabla \xi_h)_{0,T} \\ & + \sum_{F \in \mathcal{F}_T} (\mathbf{q}_h \cdot \mathbf{n}_T, \xi_h)_{0,F} + ((\mu - \nabla \cdot \boldsymbol{\beta})u_h, \xi_h)_{0,T} = (\mathbf{K} \nabla_h u_h, \nabla_h v_h)_{0,T} - (u_h, \boldsymbol{\beta} \cdot \nabla_h v_h)_{0,T} \\ & - \sum_{F \in \mathcal{F}_T} \left\{ \theta(\mathbf{n}_F^t \{ \mathbf{K} \nabla_h v_h \}_\omega, \llbracket u_h \rrbracket)_{0,F} + \left(\mathbf{n}_F^t \{ \mathbf{K} \nabla_h u_h \}_\omega - \alpha_F \frac{\gamma_{\mathbf{K},F}}{h_F} \llbracket u_h \rrbracket, \llbracket v_h \rrbracket \right)_{0,F} \right\} \\ & + \sum_{F \in \mathcal{F}_T} (\boldsymbol{\beta} \cdot \mathbf{n}_F \{ u_h \} + \gamma_{\boldsymbol{\beta},F} \llbracket u_h \rrbracket, \llbracket v_h \rrbracket)_{0,F} + ((\mu - \nabla \cdot \boldsymbol{\beta})u_h, v_h)_{0,T} \\ & = \mathcal{B}_h(u_h, v_h) = (f, v_h) = (f, \xi_h)_{0,T}, \end{aligned}$$

employing finally the definition of the DG bilinear form (2.8) and that of the DG approximate solution (2.9). \square

6.4 Locally computable a posteriori error estimate

We are now ready to state our practical locally computable a posteriori error estimate for DG methods and the problem (1.1a)–(1.1b).

While we keep the definitions of the nonconformity estimator $\eta_{\text{NC},T}$ (3.6) and that of the diffusive flux estimator $\eta_{\text{DF},T}$ (3.7) as in Section 3.5, the *residual estimator* $\eta_{\text{R},T}$ will now be defined by

$$\eta_{\text{R},T} := m_{T,\mathbf{K},\boldsymbol{\beta},\mu} \|f - \nabla \cdot \mathbf{t}_h - \nabla \cdot \mathbf{q}_h - (\mu - \nabla \cdot \boldsymbol{\beta})u_h\|_{0,T}, \quad (6.4)$$

where

$$m_{T,\mathbf{K},\boldsymbol{\beta},\mu}^2 := \min \left\{ C_{\text{P}} \frac{h_T^2}{c_{\mathbf{K},T}}, \frac{1}{c_{\boldsymbol{\beta},\mu,T}} \right\}$$

for all $T \in \mathcal{T}_h$; recall that C_{P} is the constant from the Poincaré inequality (3.8). We next define two *convection estimators* $\eta_{\text{C},1,T}$ and $\eta_{\text{C},2,T}$ respectively by

$$\eta_{\text{C},1,T} := m_{T,\mathbf{K},\boldsymbol{\beta},\mu} \|\nabla \cdot (\mathbf{q}_h - \boldsymbol{\beta}s_h) - \Pi_0(\nabla \cdot (\mathbf{q}_h - \boldsymbol{\beta}s_h))\|_{0,T} \quad (6.5)$$

and

$$\eta_{\text{C},2,T} := \frac{1}{c_{\boldsymbol{\beta},\mu,T}^{1/2}} \left\| \frac{1}{2} (\nabla \cdot \boldsymbol{\beta})(u_h - s_h) \right\|_{0,T}, \quad (6.6)$$

with $s_h = \mathcal{I}_{\text{Os}}(u_h)$. Finally, let

$$m_{F,\mathbf{K},\beta,\mu}^2 := \min \left\{ \max_{T; F \in \mathcal{F}_T} \left\{ C_{F,T,F} \frac{|F| h_T^2}{|T| c_{\mathbf{K},T}} \right\}, \max_{T; F \in \mathcal{F}_T} \left\{ \frac{|F|}{|T| c_{\beta,\mu,T}} \right\} \right\} \quad (6.7)$$

for a side $F \in \mathcal{F}_h$. Here $C_{F,T,F}$ is the constant from the generalized Friedrichs inequality, which states that

$$\|\varphi - \Pi_{0,F}(\varphi)\|_{0,T}^2 \leq C_{F,T,F} h_T^2 \|\nabla \varphi\|_{0,T}^2, \quad (6.8)$$

where $\Pi_{l,F}$ is the L^2 -orthogonal projection onto piecewise polynomials of degree l on the face F . It follows from [29, Lemma 4.1] that $C_{F,T,F} = 3d$ for a simplex T and its side F . The *upwinding estimator* $\eta_{U,T}$ is defined by

$$\eta_{U,T} := \sum_{F \in \mathcal{F}_T} m_{F,\mathbf{K},\beta,\mu} \|\Pi_{0,F}((\mathbf{q}_h - \beta s_h) \cdot \mathbf{n}_F)\|_{0,F}. \quad (6.9)$$

We then have the following a posteriori error estimate:

Theorem 6.7 (Locally computable a posteriori error estimate in the general case). *Let u be the unique solution of (2.5) and let u_h be its discontinuous Galerkin approximation given by (2.9). Then*

$$\| \|u - u_h\| \| \leq \left\{ \sum_{T \in \mathcal{T}_h} \eta_{\text{NC},T}^2 \right\}^{1/2} + \left\{ \sum_{T \in \mathcal{T}_h} \left(\eta_{R,T} + (\eta_{\text{DF},T}^2 + \eta_{C,2,T}^2)^{1/2} + \eta_{C,1,T} + \eta_{U,T} \right)^2 \right\}^{1/2}.$$

Proof. We start by putting $s = s_h = \mathcal{I}_{\text{Os}}(u_h)$, $\mathbf{t} = \mathbf{t}_h$, and $\mathbf{q} = \mathbf{q}_h$ in the abstract estimate of Remark 6.5. We next write

$$\begin{aligned} & (f - \nabla \cdot \mathbf{t}_h - \nabla \cdot \mathbf{q}_h - (\mu - \nabla \cdot \beta) u_h, \varphi) - (\mathbf{K} \nabla_h u_h + \mathbf{t}_h, \nabla \varphi) + (\nabla \cdot \mathbf{q}_h - \nabla \cdot (\beta s_h), \varphi) \\ & - (\tfrac{1}{2} (\nabla \cdot \beta)(u_h - s_h), \varphi) = \sum_{T \in \mathcal{T}_h} \left\{ (f - \nabla \cdot \mathbf{t}_h - \nabla \cdot \mathbf{q}_h - (\mu - \nabla \cdot \beta) u_h, \varphi - \Pi_0(\varphi))_{0,T} \right. \\ & - (\mathbf{K} \nabla u_h + \mathbf{t}_h, \nabla \varphi)_{0,T} - (\tfrac{1}{2} (\nabla \cdot \beta)(u_h - s_h), \varphi)_{0,T} + (\nabla \cdot (\mathbf{q}_h - \beta s_h), \varphi - \Pi_0(\varphi))_{0,T} \\ & \left. + \sum_{F \in \mathcal{F}_T} ((\mathbf{q}_h - \beta s_h) \cdot \mathbf{n}_T, \Pi_0(\varphi))_{0,F} \right\}, \end{aligned}$$

using Lemma 6.6 in the first term and subtracting $(\nabla \cdot (\mathbf{q}_h - \beta s_h), \Pi_0(\varphi))_{0,T}$ and adding the same quantity rewritten using the Green theorem in the last but one term. Next note that in this last term, we may replace $\nabla \cdot (\mathbf{q}_h - \beta s_h)$ by $\nabla \cdot (\mathbf{q}_h - \beta s_h) - \Pi_0(\nabla \cdot (\mathbf{q}_h - \beta s_h))$, and similarly in the last term, we may replace $(\mathbf{q}_h - \beta s_h) \cdot \mathbf{n}_T$ by $\Pi_{0,F}((\mathbf{q}_h - \beta s_h) \cdot \mathbf{n}_T)$. The above expression is thus bounded, using the Cauchy–Schwarz inequality, the inequality

$$\|\varphi - \Pi_0(\varphi)\|_{0,T} \leq m_{T,\mathbf{K},\beta,\mu} \|\varphi\|_T,$$

which follows from the Poincaré inequality (3.8) and from the definition of the energy norm (2.4), and finally using [30, Lemma 4.5] for the last term, by

$$\sum_{T \in \mathcal{T}_h} \left(\eta_{R,T} + (\eta_{\text{DF},T}^2 + \eta_{C,2,T}^2)^{1/2} + \eta_{C,1,T} + \eta_{U,T} \right) \|\varphi\|_T.$$

Using the Cauchy–Schwarz inequality, noticing that $\|\varphi\| = 1$, and adding the nonconformity estimator, which appears directly as the first term in the estimate of Remark 6.5, concludes the proof. \square

Remark 6.8 (Properties of the estimate of Theorem 6.7). *As in the pure diffusion case, we remark that the estimate of Theorem 6.7 yields a guaranteed upper bound, the residual represents a higher-order term, neither C_P nor $C_{F,T,F}$ depend on the polynomial degree of u_h , whence the estimate is valid uniformly with respect to k , no polynomial data form is needed at this stage, and, finally, the estimate is valid for any $\mathbf{t}_h, \mathbf{q}_h \in \mathbf{H}(\text{div}, \Omega)$ such that $(\nabla \cdot \mathbf{t}_h + \nabla \cdot \mathbf{q}_h + (\mu - \nabla \cdot \boldsymbol{\beta})u_h, 1)_{0,T} = (f, 1)_{0,T}$ for all $T \in \mathcal{T}_h$.*

Remark 6.9 (Mean values in $\eta_{C,1,T}$ and $\eta_{U,T}$). *Since $\|g - \Pi_0(g)\| \leq \|g\|$ and $\|\Pi_0(g)\| \leq \|g\|$, where $\|\cdot\|$ denotes the L^2 -norm for a square-integrable function g , the estimators $\eta_{C,1,T}$ and $\eta_{U,T}$ may be considerably smaller in the convection-dominated case when compared to the situation where the piecewise constant projection is not subtracted/used.*

7 Efficiency of the estimates in the general case

In this section, we first rapidly check the (global) efficiency of the abstract framework of Theorem 6.3. We then investigate in detail the (local) efficiency of the a posteriori error estimate of Theorem 6.7.

7.1 Global efficiency of the abstract estimate

Theorem 7.1 (Global efficiency of the abstract estimate in the general case). *Let u be the unique solution of (2.5) and let $u_h \in H^1(\mathcal{T}_h)$ be arbitrary. Let the a posteriori error estimate be given by Theorem 6.3. Then*

$$\begin{aligned} & \inf_{s \in H_0^1(\Omega)} \left\{ \| \|u_h - s\| \| + \inf_{\mathbf{t} \in \mathbf{H}(\text{div}, \Omega)} \sup_{\varphi \in H_0^1(\Omega), \| \varphi \| = 1} \left((f - \nabla \cdot \mathbf{t} - \boldsymbol{\beta} \cdot \nabla s - \mu s, \varphi) \right. \right. \\ & \quad \left. \left. - (\mathbf{K} \nabla_h u_h + \mathbf{t}, \nabla \varphi) + \left((\mu - \frac{1}{2} \nabla \cdot \boldsymbol{\beta})(s - u_h), \varphi \right) \right) \right\} \\ & \leq 2 \| \|u - u_h\| \|. \end{aligned}$$

Proof. It suffices to put $s = u$ and $\mathbf{t} = -\mathbf{K} \nabla u$ in Theorem 6.3 and to use the Cauchy–Schwarz inequality and the fact that $\| \varphi \| = 1$. \square

Before we start to investigate the local efficiency of the locally computable estimate, we note that the same global efficiency result holds true also for the form of the estimate of Remark 6.5—it suffices to put in addition $\mathbf{q} = \boldsymbol{\beta}u$. Also, similar observations to those given after Theorem 4.1 hold true also in the present case.

7.2 Local efficiency of the locally computable estimate

The following theorem is a generalization of Theorem 4.4:

Theorem 7.2 (Local efficiency of the locally computable estimate in the general case). *Let \mathcal{T}_h be shape-regular, let f be a piecewise polynomial of degree m , and let, for the sake of simplicity, $\nabla \cdot (\mathbf{q}_h - \boldsymbol{\beta} s_h) \in \mathbb{P}_l$ on all $T \in \mathcal{T}_h$. Let next u be the unique solution of (2.5) and let u_h be its discontinuous Galerkin approximation given by (2.9). Assume (4.1)–(4.3) for the weights $\omega_{T,F}$ and for the penalty parameters $\gamma_{\mathbf{K},F}$ and that $\gamma_{\boldsymbol{\beta},F} \leq \| \boldsymbol{\beta} \|_{\infty, T}$ for all $T \in \mathcal{T}_h$ and $F \in \mathcal{F}_T$ (both $\gamma_{\mathbf{K},F}$ and $\gamma_{\boldsymbol{\beta},F}$ are for the simplicity supposed facewise constant). Let finally the a posteriori error estimate be given by Theorem 6.7, with in particular $\eta_{NC,T}$ given by (3.6), $\eta_{R,T}$ by (6.4), $\eta_{DF,T}$*

by (3.7), $\eta_{C,1,T}$ by (6.5), $\eta_{C,2,T}$ by (6.6), and $\eta_{U,T}$ by (6.9). Recall the notation $c_{\mathbf{K},\tilde{\mathcal{T}}_T}$ from (4.7) and put

$$c_{\beta,\mu,\tilde{\mathcal{T}}_T} := \min_{T' \in \tilde{\mathcal{T}}_T} c_{\beta,\mu,T'}, \quad c_{\beta,\tilde{\mathcal{F}}_T} := \min_{F \in \tilde{\mathcal{F}}_T} \gamma_{\beta,F}, \quad (7.1)$$

and $R(u_h) := f + \nabla \cdot (\mathbf{K} \nabla_h u_h) - \beta \cdot \nabla_h u_h - \mu u_h$. Then, for each $T \in \mathcal{T}_h$, there holds

$$\eta_{NC,T} \leq C \left(\frac{C_{\mathbf{K},T}^{1/2}}{c_{\mathbf{K},\tilde{\mathcal{T}}_T}^{1/2}} + \|\mu - \frac{1}{2} \nabla \cdot \beta\|_{\infty,T}^{1/2} \min \left\{ \frac{h_T}{c_{\mathbf{K},\tilde{\mathcal{T}}_T}^{1/2}}, \frac{h_T^{1/2}}{c_{\beta,\tilde{\mathcal{F}}_T}^{1/2}} \right\} \right) \|u - u_h\|_{*,\tilde{\mathcal{F}}_T}, \quad (7.2)$$

$$\eta_{C,2,T} \leq C \frac{|\frac{1}{2} \nabla \cdot \beta|_{1/2}}{c_{\beta,\mu,T}^{1/2}} \min \left\{ \frac{h_T}{c_{\mathbf{K},\tilde{\mathcal{T}}_T}^{1/2}}, \frac{h_T^{1/2}}{c_{\beta,\tilde{\mathcal{F}}_T}^{1/2}} \right\} \|u - u_h\|_{*,\tilde{\mathcal{F}}_T}, \quad (7.3)$$

$$\eta_{U,T} \leq C \min \left\{ \frac{1}{c_{\mathbf{K},\mathcal{T}_T}^{1/2}}, \frac{1}{h_T c_{\beta,\mu,\mathcal{T}_T}^{1/2}} \right\} \|\beta\|_{\infty,T} \min \left\{ \frac{h_T}{c_{\mathbf{K},\tilde{\mathcal{T}}_T}^{1/2}}, \frac{h_T^{1/2}}{c_{\beta,\tilde{\mathcal{F}}_T}^{1/2}} \right\} \|u - u_h\|_{*,\tilde{\mathcal{F}}_T}, \quad (7.4)$$

$$\eta_{C,1,T} \leq C \min \left\{ \frac{1}{c_{\mathbf{K},\mathcal{T}_T}^{1/2}}, \frac{1}{h_T c_{\beta,\mu,\mathcal{T}_T}^{1/2}} \right\} \|\beta\|_{\infty,T} \min \left\{ \frac{h_T}{c_{\mathbf{K},\tilde{\mathcal{T}}_T}^{1/2}}, \frac{h_T^{1/2}}{c_{\beta,\tilde{\mathcal{F}}_T}^{1/2}} \right\} \|u - u_h\|_{*,\tilde{\mathcal{F}}_T}, \quad (7.5)$$

$$\begin{aligned} \eta_{DF,T} &\leq \tilde{C} \max_{T' \in \mathcal{T}_T} \left\{ \frac{C_{\mathbf{K},T'}}{c_{\mathbf{K},T'}} \right\} \|u - u_h\|_{*,\mathcal{F}_T} + \tilde{C} \max_{T' \in \mathcal{T}_T} \left\{ \frac{C_{\mathbf{K},T'}}{c_{\mathbf{K},T'}} \right\}^{1/2} \\ &\quad \left(\sum_{T' \in \mathcal{T}_T} \frac{h_{T'}}{c_{\mathbf{K},T'}^{1/2}} \|R(u_h)\|_{0,T'} + \sum_{T' \in \mathcal{T}_T} \left\{ \max \left\{ 1, \frac{\|\mu\|_{\infty,T'}}{c_{\beta,\mu,T'}} \right\} \right. \right. \\ &\quad \left. \left. + \left(1 + \frac{\|\mu - \frac{1}{2} \nabla \cdot \beta\|_{\infty,T'}^{1/2}}{c_{\mathbf{K},T'}^{1/2}} h_{T'} \right) + \frac{\|\beta\|_{\infty,T'}}{c_{\mathbf{K},T'}} h_{T'} \right\} \|u - u_h\|_{\mathcal{T}_T} \right), \end{aligned} \quad (7.6)$$

and

$$\begin{aligned} \|R(u_h)\|_{0,T} &\leq \bar{C} \left(\frac{C_{\mathbf{K},T}^{1/2}}{h_T} + \min \left\{ \frac{\|\mu\|_{\infty,T}}{c_{\beta,\mu,T}^{1/2}} + \frac{\|\beta\|_{\infty,T}}{c_{\mathbf{K},T}^{1/2}}, \frac{\|\mu - \nabla \cdot \beta\|_{\infty,T}}{c_{\beta,\mu,T}^{1/2}} + \frac{\|\beta\|_{\infty,T}}{c_{\beta,\mu,T}^{1/2} h_T} \right\} \right) \\ &\quad \times \|u - u_h\|_T. \end{aligned} \quad (7.7)$$

Here, the constants C depend only on the space dimension d , on the maximal polynomial degree k , on the shape regularity parameter $\kappa_{\mathcal{T}}$, on the DG parameters α_F , and on the constant C_1 from (4.1)–(4.2), \tilde{C} in addition depends on the polynomial degree m of f , on the DG parameter θ , and on the constant C_3 from (4.3), and \bar{C} depends only on d , k , m , and $\kappa_{\mathcal{T}}$.

Proof. Since $\eta_{R,T}$ is a higher-order term owing to Lemma 6.6 (given by $m_{T,\mathbf{K},\beta,\mu} \|f - \Pi_k(f)\|_{0,T}$ when μ and $\nabla \cdot \beta$ are elementwise constant and when $l = k$), we prove only the efficiency of the other estimates. In this proof, we denote by C a general constant not necessarily the same at each occurrence, depending only on d , k , and $\kappa_{\mathcal{T}}$. Also, recall that it follows from (4.11) that

$$\sum_{F \in \tilde{\mathcal{F}}_T} h_F^{-1} \|[u_h]\|_{0,F}^2 \leq C_1^{-1} (\min_{F \in \tilde{\mathcal{F}}_T} \alpha_F)^{-1} \frac{1}{c_{\mathbf{K},\tilde{\mathcal{T}}_T}} \|u - u_h\|_{*,\tilde{\mathcal{F}}_T}^2. \quad (7.8)$$

Similarly, one obviously has, using (4.8) and (7.1),

$$\sum_{F \in \tilde{\mathcal{F}}_T} \|[u_h]\|_{0,F}^2 \leq \frac{1}{c_{\beta,\tilde{\mathcal{F}}_T}} \|u - u_h\|_{*,\tilde{\mathcal{F}}_T}^2. \quad (7.9)$$

We begin with $\eta_{\text{NC},T}$. As the estimate for its diffusive part is given by (4.11), we only estimate

$$\left\| \left(\mu - \frac{1}{2} \nabla \cdot \boldsymbol{\beta} \right)^{\frac{1}{2}} (u_h - \mathcal{I}_{\text{Os}}(u_h)) \right\|_{0,T}^2 \leq C \left\| \mu - \frac{1}{2} \nabla \cdot \boldsymbol{\beta} \right\|_{\infty,T} \sum_{F \in \tilde{\mathcal{F}}_T} h_F \|[u_h]\|_{0,F}^2,$$

using Lemma 3.5. We can further bound this term either by (7.8) or by (7.9), whence (7.2) follows. Similarly, (7.3) is readily deduced, using

$$\eta_{\text{C},2,T}^2 \leq C \frac{\left| \frac{1}{2} \nabla \cdot \boldsymbol{\beta} \right|^2}{c_{\boldsymbol{\beta},\mu,T}} \sum_{F \in \tilde{\mathcal{F}}_T} h_F \|[u_h]\|_{0,F}^2.$$

Next, we remark that, for all $F \in \mathcal{F}_T$,

$$m_{F,\mathbf{K},\boldsymbol{\beta},\mu}^2 \leq C \min \left\{ \frac{h_T}{c_{\mathbf{K},\mathcal{T}_T}}, \frac{1}{h_T c_{\boldsymbol{\beta},\mu,\mathcal{T}_T}} \right\},$$

whence, employing (6.2) and the fact that $\|\Pi_{0,F}(g)\|_{0,F} \leq \|g\|_{0,F}$, we get, with $s_h = \mathcal{I}_{\text{Os}}(u_h)$,

$$\begin{aligned} & \eta_{\text{U},T} \\ & \leq C \min \left\{ \frac{h_T^{1/2}}{c_{\mathbf{K},\mathcal{T}_T}^{1/2}}, \frac{1}{h_T^{1/2} c_{\boldsymbol{\beta},\mu,\mathcal{T}_T}^{1/2}} \right\} \sum_{F \in \mathcal{F}_T} \|\boldsymbol{\beta} \cdot \mathbf{n}_F \{u_h\} + \gamma_{\boldsymbol{\beta},F} [u_h] - \boldsymbol{\beta} \cdot \mathbf{n}_F s_h\|_{0,F} \\ & \leq C \min \left\{ \frac{h_T^{1/2}}{c_{\mathbf{K},\mathcal{T}_T}^{1/2}}, \frac{1}{h_T^{1/2} c_{\boldsymbol{\beta},\mu,\mathcal{T}_T}^{1/2}} \right\} \sum_{F \in \mathcal{F}_T} \left\{ \|\gamma_{\boldsymbol{\beta},F} [u_h]\|_{0,F} + \frac{1}{2} \sum_{T': F \in \mathcal{F}_{T'}} \|\boldsymbol{\beta} \cdot \mathbf{n}_F (u_h - s_h)\|_{0,F} \right\} \\ & \leq C \min \left\{ \frac{1}{c_{\mathbf{K},\mathcal{T}_T}^{1/2}}, \frac{1}{h_T c_{\boldsymbol{\beta},\mu,\mathcal{T}_T}^{1/2}} \right\} \|\boldsymbol{\beta}\|_{\infty,T} \sum_{F \in \tilde{\mathcal{F}}_T} h_F^{1/2} \|[u_h]\|_{0,F}, \end{aligned}$$

using also the inequality

$$\|u_h - \mathcal{I}_{\text{Os}}(u_h)\|_{0,F} \leq C \sum_{F': F' \cap F \neq \emptyset} \|[u_h]\|_{0,F'}$$

valid for the Oswald interpolation operator.

We next prove the efficiency of $\eta_{\text{C},1,T}$. First of all,

$$\begin{aligned} & m_{T,\mathbf{K},\boldsymbol{\beta},\mu} \|\nabla \cdot (\mathbf{q}_h - \boldsymbol{\beta} s_h) - \Pi_0(\nabla \cdot (\mathbf{q}_h - \boldsymbol{\beta} s_h))\|_{0,T} \\ & \leq m_{T,\mathbf{K},\boldsymbol{\beta},\mu} \|\nabla \cdot (\mathbf{q}_h - \boldsymbol{\beta} s_h)\|_{0,T} = m_{T,\mathbf{K},\boldsymbol{\beta},\mu} \sup_{\xi_h \in \mathbb{P}_l(T)} \frac{(\nabla \cdot (\mathbf{q}_h - \boldsymbol{\beta} s_h), \xi_h)_{0,T}}{\|\xi_h\|_{0,T}}, \end{aligned}$$

using the assumption $\nabla \cdot (\mathbf{q}_h - \boldsymbol{\beta} s_h) \in \mathbb{P}_l$ on all $T \in \mathcal{T}_h$. Next, using the Green theorem, the definition of \mathbf{q}_h (6.2)–(6.3), the inverse inequalities $\|\xi_h\|_{0,F} \leq C h_T^{-1/2} \|\xi_h\|_{0,T}$ and $\|\nabla \xi_h\|_{0,T} \leq$

$Ch_T^{-1}\|\xi_h\|_{0,T}$, and Lemma 3.5,

$$\begin{aligned}
& (\nabla \cdot (\mathbf{q}_h - \beta s_h), \xi_h)_{0,T} \\
= & -(\mathbf{q}_h - \beta s_h, \nabla \xi_h)_{0,T} + \sum_{F \in \mathcal{F}_T} ((\mathbf{q}_h - \beta s_h) \cdot \mathbf{n}_T, \xi_h)_{0,F} \\
= & -(u_h - s_h, \beta \cdot \nabla \xi_h)_{0,T} + \sum_{F \in \mathcal{F}_T} (\beta \cdot \mathbf{n}_T \{u_h\} + \mathbf{n}_F^t \mathbf{n}_F \gamma_{\beta,F} \llbracket u_h \rrbracket - \beta \cdot \mathbf{n}_T s_h, \xi_h)_{0,F} \\
\leq & C \|\beta\|_{\infty,T} h_T^{-1} \|\xi_h\|_{0,T} \sum_{F \in \tilde{\mathcal{F}}_T} h_F^{1/2} \|\llbracket u_h \rrbracket\|_{0,F} \\
& + Ch_T^{-1/2} \|\xi_h\|_{0,T} \sum_{F \in \mathcal{F}_T} \|\beta \cdot \mathbf{n}_T \{u_h\} + \mathbf{n}_F^t \mathbf{n}_F \gamma_{\beta,F} \llbracket u_h \rrbracket - \beta \cdot \mathbf{n}_T s_h\|_{0,F} \\
\leq & C \|\beta\|_{\infty,T} h_T^{-1} \|\xi_h\|_{0,T} \sum_{F \in \tilde{\mathcal{F}}_T} h_F^{1/2} \|\llbracket u_h \rrbracket\|_{0,F},
\end{aligned}$$

using finally the result proved previously for $\eta_{U,T}$, whence (7.5) follows.

According to Lemma 4.12, which holds true also in the general case, the estimate on $\eta_{DF,T}$ can be decomposed into three parts, the first two of which are bounded by

$$\tilde{C} \max_{T' \in \mathcal{T}_T} \left\{ \frac{C_{\mathbf{K},T'}}{c_{\mathbf{K},T'}} \right\} \| \|u - u_h\| \|_{*,\mathcal{F}_T}$$

with \tilde{C} depending on $d, k, \kappa_T, \alpha_F, \theta, C_2$, and C_3 . The estimate for the third term is similar to that of [13] and can be obtained using the edge bubble function technique introduced in [28], yielding altogether (7.6).

Finally, the estimate (7.7) was established in [13] using the equivalence of norms on finite-dimensional spaces, inverse inequalities, and the definition of $\| \cdot \|_T$ by (2.4), following the approach given in [28]. \square

Remark 7.3 (Comments on the results of Theorem 7.2). *In comparison with Theorem 7.1, again the crucial advantage of Theorem 7.2 is the confirmation of the localization of the error. However, the efficiency constant is no longer parameter-independent, the major overestimation being produced in the convection-dominated case. Nevertheless, as $h \rightarrow 0$, the estimators $\eta_{C,1,T}$, $\eta_{C,2,T}$, and $\eta_{U,T}$ will completely loose influence and $\eta_{NC,T}$ and $\eta_{DF,T}$ will become optimally efficient, the rapidity being a function of the local Péclet number $\frac{\|\beta\|_{\infty,T}}{c_{\mathbf{K},T}} h_T$ on each $T \in \mathcal{T}_h$. This result is of the same quality as those achieved in [28, 31, 30, 13].*

Remark 7.4 (Efficiency of $\eta_{DF,T}$). *In comparison with the results of [28, 31, 30, 13], the diffusive flux estimator $\eta_{DF,T}$ is not efficient with a constant of the form $c_1 + c_2 \min\{\text{Pe}, \varrho\}$ with ϱ depending on β and \mathbf{K} , but only of the form $c_1 + c_2 \text{Pe}$. The former efficiency can be obtained if integration by parts is performed and $\eta_{DF,T}$ is replaced by a minimum of $\eta_{DF,T}$ and an estimator as that in [13]. We did not perform here such a modification, also in view of the fact that the numerical experiments presented below show that $\eta_{DF,T}$ is actually small in comparison with the other estimators.*

Remark 7.5 (Efficiency of $\eta_{U,T}$ in comparison with finite volumes or mixed finite elements). *In finite volume or mixed finite schemes, upwinding can likewise be used in order to stabilize the schemes in the convection-dominated regime. However, no efficiency of the corresponding upwinding estimator $\eta_{U,T}$ can be proved for these schemes, see [30] and [31], respectively. Contrarily to this situation, $\eta_{U,T}$ in the discontinuous Galerkin method is by (7.4) locally efficient (with a constant depending on the local Péclet number).*

Remark 7.6 (Efficiency of $\eta_{C,1,T}$ and $\eta_{U,T}$). *We remark that the improvements in $\eta_{C,1,T}$ and $\eta_{U,T}$ described in Remark 6.9 were not taken into account in the proof of Theorem 7.2. Hence the actual efficiency of these estimators may be still better.*

8 Numerical experiments

We present in this section the results of several numerical experiments.

8.1 Pure diffusion

For the pure diffusion problem we have examined three different test cases, all posed on the domain $\Omega = \{-1 < x, y < 1\}$ with Dirichlet boundary conditions. The diffusion tensor is isotropic (but heterogeneous in test cases 2 and 3) and can thus be identified with a scalar diffusion coefficient denoted by κ . The discrete solution has been obtained using the weighted interior-penalty DG scheme proposed in [14] with polynomial degree $p = 1$, given by (2.8)–(2.10) with the penalty parameter and the weights given by (4.4)–(4.6). The diffusive flux \mathbf{t}_h was reconstructed using (3.3)–(3.4) for both $l = 0$ or $l = 1$. Next, the piecewise affine Oswald interpolate $\mathcal{I}_{Os}(u_h)$ of the discrete solution u_h was used. In the subsequent tables, sequences of uniformly refined, structured or unstructured meshes are considered to evaluate the convergence rates and N indicates the number of mesh elements. Columns labeled “eff” report the global effectivity index, that is the ratio of the a posteriori error estimate to the actual error, both quantities being computed over all mesh elements. We employ the following notation for the various error estimators: $\eta_{NC} := \{\sum_{T \in \mathcal{T}_h} \eta_{NC,T}^2\}^{1/2}$, $\eta_R := \{\sum_{T \in \mathcal{T}_h} \eta_{R,T}^2\}^{1/2}$, $\eta_{DF} := \{\sum_{T \in \mathcal{T}_h} \eta_{DF,T}^2\}^{1/2}$, and so on.

For test case 1 the exact solution is $u(x, y) = \cos(0.5\pi x) \cos(0.5\pi y)$ and κ is equal to unity. The purpose of this test case is to assess the convergence rate of all the estimators in the case of a smooth solution. Tables 1 and 2 report the results obtained on structured and unstructured meshes, respectively. As expected, the residual estimator η_R converges to order $(l + 2)$, *i.e.*, superconverges with respect to the nonconformity estimator and to the diffusive flux estimator, the latter always dominating the former by a factor between 2 and 3. For $l = 0$, the efficiency index is equal to 1.2. This exceptionally good result is actually below the value derived for the global efficiency of the abstract estimate in Theorem 4.1, namely $\sqrt{2}$. This is not a contradiction since in the present case, it turns out that the Oswald interpolate $\mathcal{I}_{Os}(u_h)$ is closer to the discrete solution u_h than the exact solution u . For $l = 1$, the effectivity index is equal to 1.5 on structured meshes and to 1.3 on unstructured meshes, which confirms the sharpness of the estimate for $l = 1$ also. The effectivity index for $l = 1$ is however slightly larger than for $l = 0$, showing that for the present test case, the lowest-order diffusive flux reconstruction yields the sharpest results (a different conclusion is reached in the two following test cases).

Before moving to the following test cases, it is useful to compare the present error estimators to those previously available in the literature. We focus here on the classical error estimator for the pure diffusion case which consists of four terms: the nonconformity estimator (evaluated using the Oswald interpolate as in this paper) and three additional terms, namely the residual estimator

N	$\ u - u_h\ $	$l = 0$				$l = 1$		
		η_{NC}	η_{R}	η_{DF}	eff.	η_{R}	η_{DF}	eff.
128	3.28e-1	1.89e-1	7.23e-2	3.38e-1	1.2	5.50e-3	4.32e-1	1.4
512	1.62e-1	9.72e-2	1.82e-2	1.69e-1	1.2	6.90e-4	2.22e-1	1.5
2048	8.04e-2	4.89e-2	4.54e-3	8.39e-2	1.2	8.64e-5	1.12e-1	1.5
8192	4.01e-2	2.45e-2	1.14e-3	4.18e-2	1.2	1.08e-5	5.64e-2	1.5
order	1.0	1.0	2.0	1.0	-	3.0	1.0	-

Table 1: Convergence rates of error estimators for test case 1, structured meshes

N	$\ u - u_h\ $	$l = 0$				$l = 1$		
		η_{NC}	η_{R}	η_{DF}	eff.	η_{R}	η_{DF}	eff.
112	3.16e-1	1.25e-1	7.01e-2	3.60e-1	1.2	5.13e-3	3.58e-1	1.2
448	1.58e-1	6.85e-2	1.76e-2	1.82e-1	1.2	6.90e-4	2.22e-1	1.5
1792	7.88e-2	3.53e-2	4.40e-3	9.10e-2	1.2	8.05e-5	9.43e-2	1.3
7168	3.93e-2	1.77e-2	1.10e-3	4.55e-2	1.2	1.01e-5	4.76e-2	1.3
order	1.1	1.1	2.1	1.1	-	3.2	1.1	-

Table 2: Convergence rates of error estimators for test case 1, unstructured meshes

η_{R}^* , the diffusive flux (mass balance) estimator η_{DF}^* , and the jump estimator η_{J}^* defined as follows:

$$\begin{aligned}
(\eta_{\text{R}}^*)^2 &= \sum_{T \in \mathcal{T}_h} m_{T,\mathbf{K}}^2 \|f + \nabla \cdot (\mathbf{K} \nabla u_h)\|_{0,T}^2, \\
(\eta_{\text{DF}}^*)^2 &= \sum_{T \in \mathcal{T}_h} C_T \frac{h_T}{c_{\mathbf{K},T}} \|\mathbf{n}_F^t \llbracket \mathbf{K} \nabla_h u_h \rrbracket\|_{0,\partial T \setminus \partial \Omega}^2, \\
(\eta_{\text{J}}^*)^2 &= \sum_{T \in \mathcal{T}_h} C_T \frac{1}{h_T} \|\gamma_{\mathbf{K},F}^{1/2} \llbracket u_h \rrbracket\|_{0,\partial T}^2,
\end{aligned}$$

where $C_T = 3dh_T|\partial T|/|T|$. Results are presented in Table 3. In particular, the 5th column, which displays the effectivity index, shows that the error is overestimated by a factor of 10. It should be observed that the main source for overestimation is the residual estimator η_{R}^* , which we have transformed into a super-convergent term in the present work. The diffusive flux estimator η_{DF}^* is also observed to be about three-times larger than the estimators η_{DF} evaluated using the present reconstructed flux \mathbf{t}_h either with $l = 0$ or with $l = 1$. For completeness, the last column of Table 3 proposes a comparison with the recent results of [13] where the mean value is subtracted from the residue within each mesh element, while the diffusive flux estimator and the nonconformity estimator are considered together as one unique term. We see that the estimate becomes sharper, though it still overestimates the error by a factor of 4.5, mainly because the reconstructed flux \mathbf{t}_h is not used.

The aim of test cases 2 and 3, which were proposed in [26], is to address the question of diffusion inhomogeneities. The domain Ω is split along the Cartesian axes into four subregions Ω_i . The subregion $\{x > 0, y > 0\} \cap \Omega$ is indicated by Ω_1 and the subsequent numbering is done in a counterclockwise manner. The diffusion coefficient is equal to κ_i in subregion Ω_i where κ_i

N	η_R^*	η_{DF}^*	η_J^*	eff.	eff. [13]
112	1.74	1.38	3.57e-1	7.5	5.2
448	8.73e-1	5.86e-1	2.03e-1	7.2	4.9
1792	4.37e-1	2.75e-1	1.07e-1	7.1	4.7
7168	2.19e-1	1.31e-1	5.42e-2	7.1	4.5
order	1.1	1.1	1.0	-	-

Table 3: Comparison with other error estimators for test case 1, unstructured meshes

is a constant. Taking the forcing term equal to zero, the analytical solution with corresponding nonhomogeneous Dirichlet boundary conditions can be written in polar coordinates as

$$u(r, \phi)|_{\Omega_i} = r^\alpha (a_i \sin(\alpha\phi) + b_i \cos(\alpha\phi)),$$

where the subscript i refers to the corresponding subregion. Owing to the singularity in the origin, the calculated solution converges with order 2α in the L^2 -norm and with order α in the energy (semi)norm. For test case 2, we take $\kappa_1 = \kappa_3 = 5$ and $\kappa_2 = \kappa_4 = 1$, yielding $\alpha = 0.53544095$ and

$$\begin{aligned} a_1 &= 0.44721360; & b_1 &= 1.00000000; \\ a_2 &= -0.74535599; & b_2 &= 2.33333333; \\ a_3 &= -0.94411759; & b_3 &= 0.55555556; \\ a_4 &= -2.40170264; & b_4 &= -0.48148148. \end{aligned}$$

For test case 3, we take $\kappa_1 = \kappa_3 = 100$ and $\kappa_2 = \kappa_4 = 1$, yielding $\alpha = 0.12690207$ and

$$\begin{aligned} a_1 &= 0.10000000; & b_1 &= 1.00000000; \\ a_2 &= -9.60396040; & b_2 &= 2.96039604; \\ a_3 &= -0.48035487; & b_3 &= -0.88275659; \\ a_4 &= 7.70156488; & b_4 &= -6.45646175. \end{aligned}$$

The results for test case 2 are shown in Tables 4 and 5 for structured and unstructured meshes, respectively. Since the forcing term is zero, the residual estimator is also equal to zero, and has not been reported. The interpolation error on nonhomogeneous Dirichlet boundary conditions is not reported either. We observe that the expected convergence rate of order α is obtained for both the nonconformity estimator η_{NC} and for the diffusive flux estimator η_{DF} . Both estimators yield comparable values. The effectivity index is 1.9 for $l = 0$ and 1.8 for $l = 1$; hence, for this test case, employing $l = 1$ for the reconstruction leads to a slightly sharper estimator.

The results for test case 3 are shown in Tables 6 and 7 for structured and unstructured meshes, respectively. The order of convergence of the error estimators is close to α , and the error is overestimated by a factor of approximately 3.8. This is because the nonconformity error estimator now dominates over the diffusive flux estimator. Hence, although the diffusive flux estimator is lower for $l = 1$ than for $l = 0$, this difference is scarcely reflected in the effectivity index. Finally, it is worthwhile to notice that in the present setting, the diffusion coefficient is not monotone around the singularity, thus precluding the use of weighted variants of the Oswald interpolate such as that proposed in [2]. On the other hand, one can employ a piecewise quadratic Oswald interpolate as in [31, 30].

N	$\ u - u_h\ $	η_{NC}	$l = 0$		$l = 1$	
			η_{DF}	eff.	η_{DF}	eff.
128	6.61e-01	9.60e-1	8.02e-1	1.9	6.54e-1	1.8
512	4.58e-01	6.68e-1	5.63e-1	1.9	4.63e-1	1.8
2048	3.17e-01	4.62e-1	3.92e-1	1.9	3.23e-1	1.8
8192	2.19e-01	3.20e-1	2.72e-1	1.9	2.25e-1	1.8
order	0.53	0.53	0.53	-	0.53	-

Table 4: Convergence rates of error estimators for test case 2, structured meshes

N	$\ u - u_h\ $	η_{NC}	$l = 0$		$l = 1$	
			η_{DF}	eff.	η_{DF}	eff.
112	6.11e-01	8.70e-1	7.43e-1	1.9	6.00e-1	1.7
448	4.28e-01	6.09e-1	5.35e-1	1.9	4.32e-1	1.7
1792	2.97e-01	4.23e-1	3.74e-1	1.9	3.05e-1	1.8
7168	2.01e-01	2.92e-1	2.60e-1	1.9	2.12e-1	1.8
order	0.53	0.53	0.53	-	0.52	-

Table 5: Convergence rates of error estimators for test case 2, unstructured meshes

N	$\ u - u_h\ $	η_{NC}	$l = 0$		$l = 1$	
			η_{DF}	eff.	η_{DF}	eff.
128	3.49	12.4	2.68	3.6	2.02	3.6
512	3.29	11.9	2.57	3.7	1.95	3.6
2048	3.09	11.3	2.45	3.7	1.86	3.7
8192	2.88	10.7	2.32	3.8	1.76	3.8
order	0.10	0.08	0.08	-	0.08	-

Table 6: Convergence rates of error estimators for test case 3, structured meshes

N	$\ u - u_h\ $	η_{NC}	$l = 0$		$l = 1$	
			η_{DF}	eff.	η_{DF}	eff.
112	3.27	11.8	2.39	3.7	1.89	3.7
448	3.11	11.3	2.33	3.7	1.84	3.7
1792	2.93	10.8	2.23	3.8	1.77	3.7
7168	2.75	10.3	2.12	3.8	1.68	3.8
order	0.09	0.08	0.08	-	0.07	-

Table 7: Convergence rates of error estimators for test case 3, unstructured meshes

N	$\ u - u_h\ $	η_{NC}	eff. $l = 0$	eff. $l = 1$
128	1.95e-3	3.62e-3	13.8	14.4
512	4.01e-4	1.84e-3	11.1	10.9
2048	1.89e-3	8.84e-4	8.10	7.75
order	1.1	1.1	-	-

Table 8: Efficiency of error estimators for test case 4 ($\kappa = 10^{-2}$)

N	η_{R}^*	$l = 0$			$l = 1$			
		η_{R}	η_{DF}	η_{U}	η_{R}	η_{DF}	η_{U}	$\eta_{\text{C},1}$
128	4.91e-2	3.94e-2	8.82e-3	6.35e-2	1.12e-2	8.73e-3	6.35e-2	3.28e-2
512	1.44e-2	9.86e-3	4.93e-3	2.87e-2	1.66e-3	4.73e-3	2.87e-2	7.69e-3
2048	4.63e-3	2.42e-3	2.51e-3	9.77e-3	3.19e-4	2.37e-3	9.77e-3	1.53e-3
order	1.6	2.0	1.0	1.6	2.4	1.0	1.6	2.3

Table 9: Convergence of error estimators for test case 4 ($\kappa = 10^{-2}$)

8.2 Convection–diffusion–reaction

We consider the domain $\Omega = \{0 < x, y < 1\}$, the reaction coefficient $\mu = 1$, the convection field $\beta = (1, 0)^t$, and an isotropic homogeneous diffusion tensor represented by a diffusion coefficient κ . We run tests with $\kappa = 10^{-2}$ (test case 4) and $\kappa = 10^{-4}$ (test case 5). The source term f is chosen so that the exact solution with homogeneous Dirichlet boundary conditions is

$$u = \frac{1}{2}x(x-1)y(y-1)(1 - \tanh(10 - 20x)).$$

For brevity, only results for uniformly refined structured meshes are presented.

Tables 8 and 9 report the results for $\kappa = 10^{-2}$. Table 8 focuses on the global effectivity index when both the diffusive and convective fluxes are reconstructed using $l = 0$ or $l = 1$. Both choices yield comparable results with efficiency indices ranging between 8 and 14 approximately. A more detailed comparison can be found in Table 9. The residual estimator η_{R} super-converges and converges faster for $l = 1$ than for $l = 0$. The classical residual estimator η_{R}^* evaluated using solely the discrete solution is also reported; it takes, as expected, larger values. The diffusive flux estimator η_{DF} yields the smallest contribution among the different terms in the error estimate. The upwinding estimator η_{U} is dominant, along with the first convection estimator $\eta_{\text{C},1}$ for $l = 1$, while this latter estimator vanishes for $l = 0$ since in this case, $\nabla \cdot (\mathbf{q}_h - \beta \mathcal{I}_{\text{Os}}(u_h))$ is by construction piecewise constant. Finally, the second convection estimator $\eta_{\text{C},2}$ vanishes identically because β is divergence-free. All in all, there is little gain in efficiency when going from $l = 0$ to $l = 1$.

Tables 10 and 11 report the results for $\kappa = 10^{-4}$. Table 10 focuses on the global effectivity index for the $l = 0$ and $l = 1$ flux reconstructions. Again, both choices yield similar results, and the efficiency indices are roughly ten-times larger than those observed for $\kappa = 10^{-2}$, in agreement with the cut-off coefficients employed in front of the estimators. A more detailed comparison can be found in Table 11. As for test case 4, the residual estimator η_{R} converges faster for $l = 1$ than for $l = 0$, but this gain is compensated by the first convection estimator $\eta_{\text{C},1}$. The diffusive flux estimator η_{DF} yields the smallest contribution, while the upwinding estimator η_{U} dominates the overall estimate.

N	$\ u - u_h\ $	η_{NC}	eff. $l = 0$	eff. $l = 1$
128	1.72e-3	2.73e-3	80	89
512	5.68e-4	6.74e-4	124	128
2048	2.14e-4	1.66e-4	145	152
order	1.4	2.0	-	-

Table 10: Efficiency of error estimators for test case 5 ($\kappa = 10^{-4}$)

N	η_{R}^*	$l = 0$			$l = 1$			
		η_{R}	η_{DF}	η_{U}	η_{R}	η_{DF}	η_{U}	$\eta_{\text{C},1}$
128	7.77e-2	6.84e-2	1.06e-3	6.98e-2	1.92e-2	1.03e-3	6.55e-2	6.98e-2
512	3.90e-2	3.41e-2	6.20e-4	3.60e-2	3.44e-3	5.71e-4	3.38e-2	3.60e-2
2048	1.87e-2	1.63e-2	3.23e-4	1.47e-2	2.01e-3	2.86e-4	1.46e-2	1.60e-2
order	1.1	1.1	0.9	1.3	0.8	1.0	1.3	1.1

Table 11: Convergence of error estimators for test case 5 ($\kappa = 10^{-4}$)

8.3 Adaptive meshes

We conclude this section by an example on how the error estimator with $l = 0$ can be used to adapt the mesh. Test case 2 is considered. The adaptive mesh refinement procedure flags 5% of the mesh elements yielding the largest error indicators. Results are reported in Table 12. The effectivity index fluctuates between 1.7 and 2 and decreases as finer meshes are constructed. Comparing with Table 5, we observe that the energy norm of the error on an adapted mesh with 494 elements is comparable to that obtained on a uniformly refined mesh with 7168 elements. Finally, Figure 1 presents two meshes obtained within the adaptive refinement procedure, one with 342 elements and one with 494 elements. We see that the adaptive refinement correctly aims at capturing the singularity at the origin.

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N	$\ u - u_h\ $	η_{NC}	η_{DF}	eff.
112	6.11e-1	8.70e-1	7.43e-1	1.87
148	4.58e-1	6.17e-1	5.78e-1	1.84
204	3.59e-1	5.59e-1	4.63e-1	2.02
264	2.96e-1	4.21e-1	3.76e-1	1.91
342	2.50e-1	3.05e-1	3.23e-1	1.78
494	2.10e-1	2.20e-1	2.78e-1	1.68

Table 12: Error as a function of mesh elements

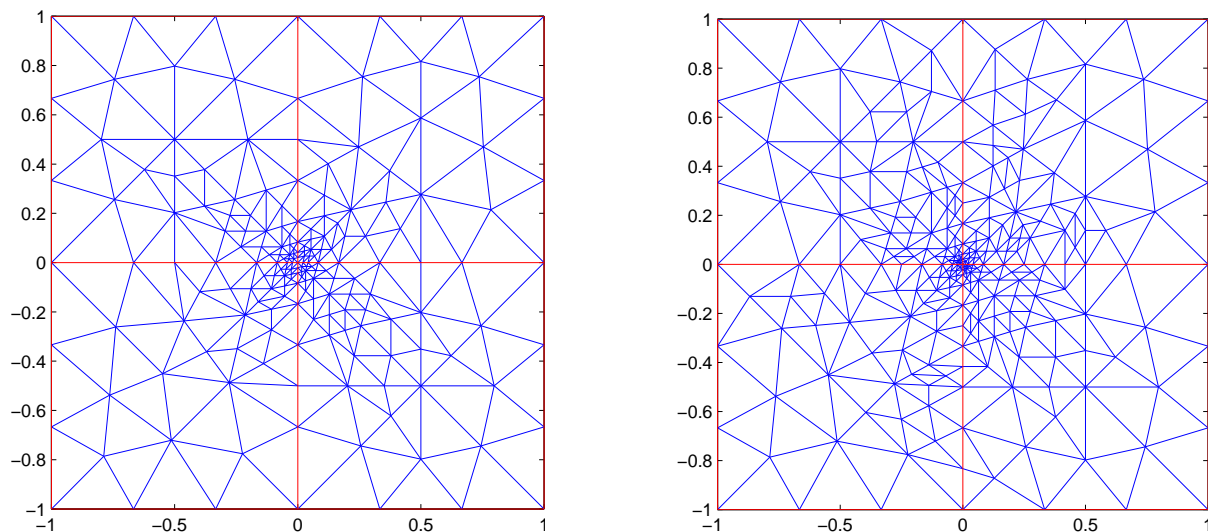


Figure 1: Two meshes successively refined using the error estimator with $l = 0$ reconstruction: 342 elements (left) and 494 elements (right)

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