On locally self-similar fractional random fields indexed by a manifold

Jacques Istas, Céline Lacaux

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Abstract

Local self-similarity for Euclidean random fields has been introduced since a while. In this paper, we extend it for manifold indexed random fields. We then give examples and derive some properties of the tangent field. In the Gaussian ($\alpha = 2$) and α-stable ($0 < \alpha < 2$) cases, we obtain the expected relations between the fractional index H and the stability index α.

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1 Introduction

Self-similar random fields are widely used to model natural phenomena in Internet traffic, hydrology, geophysics or financial markets, e.g. [1, 13]. The most famous and classical self-similar model is the fractional Brownian motion, see [7, 8]. However, self-similarity is a global property and is then too restrictive for some applications, see [12] and references therein. Therefore, [2, 9] has introduced a weakened property, called local asymptotically self-similarity (in short lass) at point x_0. This property has been defined for random fields indexed by the Euclidean space $\mathbb{R}^n$ and many examples have been developped to illustrate it, see [3] and references therein.

Observe that the self-similarity property is well-defined on Euclidean spaces but it is not on a metric space or even on a manifold equipped with a distance. Then, only weak versions of self-similarity, that still be global properties, have been defined, e.g. [5, 6]. However, in this paper, we see that the local self-similarity, introduced for Euclidean fields in [2, 9], can also be extended in the manifold indexed field realm.

We then illustrate this property by some Gaussian and stable examples. Since moving average fractional fields are on common use for modelling purpose, see [11], we focus on spherical and hyperbolic moving average fractional fields. We also consider multifractional random fields. Then, we give some expected properties of tangent fields of lass random fields indexed by a manifold. Let us emphasize that in the manifold framework, tangent fields may not be defined on whole the tangent space and are self-similar but not at any scale. Moreover, if the manifold indexed field has weak stationary increments, so has the tangent field. Some expected consequences are derived. In particular, if the tangent field is Gaussian, then it is a restriction of an Euclidean fractional Brownian motion.

In Section 2, notation in the framework of manifold indexed lass random fields are given. Section 3 is devoted to the definition of the lass property. Examples are developed in Section 4. Then, some properties of the tangent field of a lass field are studied in Section 5.

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2 Preliminaries and Notation

Throughout this note, $(\mathcal{M}, g)$ is a $\mathcal{C}^\infty$ Riemannian manifold of dimension $n \in \mathbb{N}\backslash\{0\}$ and for any $M_1, M_2 \in \mathcal{M}$, the distance $d(M_1, M_2)$ is the length of the shortest curve between M_1 and M_2.

Let $M_0 \in \mathcal{M}$ and denote by $T_{M_0}\mathcal{M}$ the tangent space to $\mathcal{M}$ at M_0. Then, there exists a neighborhood $\mathcal{V}(M_0)$ of M_0 and $\delta \in (0, +\infty]$, see for example [4], such that

1. for all $M \in \mathcal{V}(M_0)$, there exists an unique minimal geodesic between M and M_0,
2. the exponential map $\exp_{M_0}$ at point M_0 is a diffeomorphism between the open ball $\mathcal{B}(0, \delta) \subset T_{M_0}\mathcal{M}$ and $\mathcal{V}(M_0)$.

In the following, Π_{M_0} denotes the inverse of the exponential map $\exp_{M_0}$ at point M_0.

3 Local asymptotic self-similarity property

In the sense of [2, 9], a random field $(X(x))_{x\in\mathbb{R}^n}$ is locally asymptotically self-similar at point x_0 with index H if

$$\lim_{\varepsilon\to 0_+} \left(\frac{X(x_0 + \varepsilon x) - X(x_0)}{\varepsilon^H} \right)_{x\in\mathbb{R}^n} \stackrel{(d)}{=} (Z_{x_0}(x))_{x\in\mathbb{R}^n} \tag{1}$$

where $\stackrel{(d)}{=}$ stands for equality of finite dimensional distributions and Z_{x_0} is a non degenerate field. To extend this property to fields indexed by the manifold $\mathcal{M}$, we have to interpret $x_0 + \varepsilon x$ as a point of $\mathbb{R}^n$ without the help of the addition. To this aim, let us observe that $x_0 + \varepsilon x$ is the shift of x_0 by the vector $\varepsilon x \in T_{x_0}\mathbb{R}^n \approx \mathbb{R}^n$. Also, since the geodesics in $\mathbb{R}^n$ are the segments, $x_0 + \varepsilon x = \exp_{x_0}(\varepsilon x)$. Then, we propose to replace in (1) the point x_0 by M_0 and its translate $x_0 + \varepsilon x$ by

$$M_0 + \varepsilon v \stackrel{def}{=} \exp_{M_0}(\varepsilon v). \tag{2}$$

Definition 3.1. *A random field $X = (X(M))_{M\in\mathcal{M}}$ is locally asymptotically self-similar at point M_0 with index $H > 0$ if*

$$\lim_{\varepsilon\to 0_+} \left(\frac{X(M_0 + \varepsilon v) - X(M_0)}{\varepsilon^H} \right)_{v\in B(0,\delta)} \stackrel{(d)}{=} (Z_{M_0}(v))_{v\in B(0,\delta)}$$

with $M_0 + \varepsilon v$ defined by (2), δ introduced in Section 2 and Z_{M_0} a non degenerate field, which means that for almost all ω, there exists v, such that $Z_{x_0}(v, \omega) \neq 0$. The random field Z_{M_0} is called tangent field at point M_0 of X.

Remark 3.2. *As one could expect, the definition 3.1 coincide with the definitions of [2, 9] in the framework of random fields indexed by $\mathbb{R}^n$ choosing $\delta = +\infty$.*

Remark 3.3. *Let us fix $M \in \mathcal{V}(M_0)$ and consider the unique $v \in \mathcal{B}(0, \delta)$ such that $M_0 + v = \exp_{M_0}(v) = M$. Then, $M_0 + \varepsilon v$ describes the geodesic between M_0 and M as ε varies in $[0, 1]$. In addition, $M_0 + \varepsilon v$ tends to M_0 in the direction given by this geodesic as ε tends to zero.*

4 Examples

Let us now give examples of locally asymptotically self-similar random fields indexed by manifold $\mathcal{M}$, which may differ from the Euclidean space $\mathbb{R}^n$. Most of our examples are random fields indexed or by the n-dimensional unit sphere $\mathbb{S}_n = \{x_1, x_2, \ldots, x_{n+1} \in \mathbb{R},\ \sum_{i=1}^{n+1} x_i^2 = 1\}$ or by the n-dimensional hyperbolic space $\mathbb{H}_n = \{x_1, x_2, \ldots, x_{n+1} \in \mathbb{R},\ -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = 1\}$.

4.1 Fractional Brownian motions indexed by a manifold

Let $X_H = (X_H(M))_{M\in\mathcal{M}}$ be a fractional Brownian motion indexed by the manifold $\mathcal{M}$ and with index H. This random field, introduced in the indexed manifold realm by [5], is a Gaussian centered random field such that

$$\exists O \in \mathcal{M},\ \forall (M,N)\in\mathcal{M}^2,\ \mathbb{E}(X_H(M)X_H(N)) = \frac{1}{2}\big(d^{2H}(O,M) + d^{2H}(O,N) - d^{2H}(M,N)\big). \quad (3)$$

Moreover, X_H exists for $H \in (0, \beta_{\mathcal{M}}]$, with $\beta_{\mathcal{M}}$ a constant depending on the manifold $\mathcal{M}$ (see [5, 6]).

Let us now fix $M_0 \in \mathcal{M}$ and $u, v \in \mathcal{B}(0,\delta) \subset T_{M_0}\mathcal{M}$. For every $\varepsilon \in (0,1]$, let

$$R(M_0+\varepsilon v, M_0+\varepsilon w) = \varepsilon^{-2H}\mathrm{Cov}(X_H(M_0+\varepsilon v) - X_H(M_0), X_H(M_0+\varepsilon w) - X_H(M_0)).$$

Then, using (3) and $d(M_0+u, M_0) = \|u\|$ for $u \in B(0,\delta)$, one easily obtains that

$$R(M_0+\varepsilon v, M_0+\varepsilon w) \;=\; \frac{1}{2}\left(\|v\|^{2H} + \|w\|^{2H} - \frac{d^{2H}(M_0+\varepsilon v, M_0+\varepsilon w)}{\varepsilon^{2H}}\right)$$

Let us now give the behaviour of $d(M_0+\varepsilon v, M_0+\varepsilon w)$ as ε tends to 0_+ (see [10, Chapter 5]).

Lemma 4.1. *For any $v, w \in \mathcal{B}(0,\delta)$, $\lim\limits_{\varepsilon\to 0_+} \varepsilon^{-1} d(M_0+\varepsilon v, M_0+\varepsilon w) = \|v-w\|$.*

Since X_H is a centered Gaussian random field, one easily derives from the previous lemma that X_H is locally asymptotically self-similar at point M_0 with index H and with tangent field an Euclidean fractional Brownian motion B_H with index H restricted to $\mathcal{B}(0,\delta)$.

4.2 Moving average fractional random fields

Let $\alpha \in (0,2]$ and let $W_{\alpha,\mathcal{M}}$ be a symmetric real α-stable random measure on the manifold $\mathcal{M}$ with control measure the uniform measure $\sigma_{\mathcal{M}}$ on $\mathcal{M}$ (see [11] for details on such random measures). Let us recall that the stochastic integral $\int_{\mathcal{M}} g dW_{\alpha,\mathcal{M}}$ is well-defined iff $g \in L^\alpha(\mathcal{M}, d\sigma_{\mathcal{M}})$. Moreover, if $g \in L^\alpha(\mathcal{M}, d\sigma_{\mathcal{M}})$, then

$$\forall \lambda \in \mathbb{R},\ \mathbb{E}\Big(\mathrm{e}^{i\lambda\int_{\mathcal{M}} g dW_{\alpha,\mathcal{M}}}\Big) = \exp\left(-|\lambda|^\alpha \int_{\mathcal{M}} |g|^\alpha d\sigma_{\mathcal{M}}\right). \quad (4)$$

Let $H \in \mathbb{R}$ such that $H \neq n/\alpha$ and $f : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function. As soon as

$$X_{H,\alpha,\mathcal{M}}(M) = \int_{\mathcal{M}} d\big(M,M'\big)^{H-n/\alpha} f\big(d(M,M')\big) dW_{\alpha,\mathcal{M}}(M'), \quad M \in \mathcal{M}, \quad (5)$$

is well-defined, with convention $0^\beta = 0$ for $\beta < 0$, $X_{H,\alpha,\mathcal{M}}$ is called moving average fractional α-stable random field. Note that $X_{H,2,\mathcal{M}}$ is a Gaussian field.

To ensure that the singularity of the kernel at $M = M'$ is given by $d(M,M')^{H-n/\alpha}$ we assume that $f(0) \neq 0$. Then for the sake of simplicity, we choose f so that $f(0) = 1$.

Let us now observe that $X_{H,\alpha,\mathcal{M}}(M)$ is well-defined iff

$$I_{\mathcal{M}}(M) := \int_{\mathcal{M}} d\big(M,M'\big)^{\alpha H-n} \big|f\big(d(M,M')\big)\big|^\alpha d\sigma_{\mathcal{M}}(M) < +\infty$$

that is iff

$$I_{\mathcal{M}}(M) = \int_{\mathbb{S}_{n-1}} \int_0^{\rho_{\mathcal{M}}(\theta)} r^{\alpha H-1} |f(r)|^\alpha J_{\mathcal{M}}(\theta,r) dr\, d\sigma_{\mathbb{S}_{n-1}}(\theta) < +\infty \quad (6)$$

with $J_{\mathcal{M}}$ the Jacobian of the exponential map $\exp_{\mathcal{M}}$ at point M and $\rho_{\mathcal{M}}(\theta)$ the distance between M and the cut locus in the direction θ (see [4]). The choice of f is then closely linked to the behavior of $J_{\mathcal{M}}$. Moreover, to prove the local asymptotic self-similarity, we use the dominated convergence theorem and then we may choose a quite simple function f. The next proposition state the behaviour of the increments around M_0.

Proposition 4.2. *Let $H > 0$ such that $H \neq n/\alpha$. Assume that $\mathcal{M} = \mathbb{S}_n$ or $\mathbb{H}_n$. If $\mathcal{M} = \mathbb{H}_n$, we assume that $f(x) = \mathrm{e}^{-\gamma x}$. Observe that $\delta = \pi$ if $\mathcal{M} = \mathbb{S}_n$ and $\delta = +\infty$ if $\mathcal{M} = \mathbb{H}_n$.*

1. *(a) The random field $X_{H,\alpha,\mathbb{S}_n}$ is well-defined if and only if $H > 0$.*
   (b) The random field $X_{H,\alpha,\mathbb{H}_n}$ is well-defined if and only if ($H > 0$ and $\gamma > (n-1)/\alpha$) or ($0 < H < (n-1)/\alpha$ and $\gamma = (n-1)/\alpha$).

2. *Assume that $X_{H,\alpha,\mathcal{M}}$ is well-defined on $\mathcal{M}$ and that f is $\mathcal{C}^1$. Define $g : [0, +\infty) \to \mathbb{R}$ by*

$$g(x) = x^{H-n/\alpha} f(x) \tag{7}$$

with convention $0^\beta = 0$ for $\beta < 0$.

(a) If $H \in (0, 1)$, then for every $M_0 \in \mathcal{M}$,

$$\lim_{\varepsilon \to 0_+} \left(\frac{X_{H,\alpha,\mathcal{M}}(M_0 + \varepsilon v) - X_{H,\alpha,\mathcal{M}}(M_0)}{\varepsilon^H} \right)_{v \in B(0,\delta)} \overset{(d)}{=} \left(B_{H,\alpha}(v)\right)_{v \in B(0,\delta)},$$

where $B_{H,\alpha}$ is an Euclidean moving average α-stable random field with index H, that is

$$B_{H,\alpha}(v) = \int_{\mathbb{R}^n} \Big(\|v - x\|^{H-n/\alpha} - \|x\|^{H-n/\alpha} \Big) W_{\alpha,\mathbb{R}^n}(dx). \tag{8}$$

(b) If $H > 1$, then for every $M_0 \in \mathcal{M}$,

$$\lim_{\varepsilon \to 0_+} \left(\frac{X_{H,\alpha,\mathcal{M}}(M_0 + \varepsilon v) - X_{H,\alpha,\mathcal{M}}(M_0)}{\varepsilon} \right)_{v \in B(0,\delta)} \overset{(d)}{=} \left(< v, S_\alpha >\right)_{v \in B(0,\delta)},$$

where S_α is the the stable random variable defined by

$$S_\alpha = \int_{\mathcal{M}} \frac{\Pi_{M_0}(M')}{\left\| \Pi_{M_0}(M') \right\|} g'\big(d(M_0, M')\big) dW_{\alpha,\mathcal{M}}(M'). \tag{9}$$

Proof. 1. Let us recall that $\rho(\theta) := \rho_{_M}(\theta) = \delta$ and that

$$J(r) := J_{_M}(r, \theta) = \begin{cases} r^{1-n} \sin^{n-1} r & \text{if } \mathcal{M} = \mathbb{S}_n \\ r^{1-n} \sinh^{n-1} r & \text{if } \mathcal{M} = \mathbb{H}_n. \end{cases}$$

Then, replacing in (6) $J_{_M}$ and $\rho_{_M}$ by their value, one easily obtains Assertion 1.

2. Let $M_0 \in \mathcal{M}$ and consider the open ball $\mathcal{B}(0, \delta) \subset T_{M_0}\mathcal{M} \approx \mathbb{R}^n$. Then, let $\varepsilon \in (0, 1]$, $k \in \mathbb{N}\backslash\{0\}$, $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$ and $v = (v_1, \ldots, v_k) \in \mathcal{B}(0, \delta)^k$.

   (a) Assume $H \in (0, 1)$. Before we study the behavior of $X_{H,\alpha,\mathcal{M}}$ around $M_0 \in \mathcal{M}$, let us observe that we can define the random field $Y_{H,\alpha,\mathcal{M}}$ on $\mathcal{M} \times \mathcal{M}$ by

$$Y(M_1, M_2) = \int_{\mathcal{M}} d\big(M_1, M'\big)^{H-n/\alpha} f\big(d\big(M_2, M'\big)\big) dW_{\alpha,\mathcal{M}}\big(M'\big), \quad M_1, M_2 \in \mathcal{M}.$$

   Then, $X_{H,\alpha,\mathcal{M}}(M_0 + \varepsilon w) - X_{H,\alpha,\mathcal{M}}(M_0) = Z_{\varepsilon,1}(w) + Z_{\varepsilon,2}(w)$ with

$$\begin{cases} Z_{\varepsilon,1}(w) &= Y(M_0 + \varepsilon w, M_0) - Y(M_0, M_0) \\ Z_{\varepsilon,2}(w) &= Y(M_0 + \varepsilon w, M_0 + \varepsilon w) - Y(M_0 + \varepsilon w, M_0). \end{cases}$$

We first study the behavior of $Z_{\varepsilon,1}$ as $\varepsilon \to 0_+$. Using (4), the exponential map $\exp_{M_0}$ and the change of variables $\rho = r/\varepsilon$, one obtains that

$$\mathbb{E}\left(\exp\left(i\varepsilon^{-H}\sum_{j=1}^{j}\lambda_j Z_{\varepsilon,1}(v_j)\right)\right) = \exp\left(-\varepsilon^{-\alpha H} I_1(\varepsilon)\right)$$

where $I_1(\varepsilon) = \displaystyle\int_{(0,\delta/\varepsilon)\times\mathbb{S}_{n-1}} f_\varepsilon(\rho,u)d\rho d\sigma_{\mathbb{S}_{n-1}}(u)$ with

$$f_\varepsilon(\rho,u) = \varepsilon^n\left|\sum_{j=1}^{k}\lambda_j\Big(d(M_0+\varepsilon v_j, M_0+\varepsilon\rho u)^{H-n/\alpha} - (\varepsilon\rho)^{H-n/\alpha}\Big)\right|^\alpha |f(\varepsilon\rho)|^\alpha J(\varepsilon\rho)\rho^{n-1}.$$

By definition of J and continuity of f, Lemma 4.1 leads to

$$\lim_{\varepsilon\to 0_+}\varepsilon^{-\alpha H} f_\varepsilon(\rho,u) = \left|\sum_{j=1}^{k}\lambda_j\Big(\|v_j-\rho u\|^{H-n/\alpha} - \rho^{H-n/\alpha}\Big)\right|^\alpha \rho^{n-1}.$$

Moreover, using twice the triangle inequality and $d(M_0+\varepsilon w, M_0) = \varepsilon\|w\|$ (with $\|w\| < \delta$), one obtains that

$$\varepsilon|\|v_j\| - \rho| \le d(M_0+\varepsilon v_j, M_0+\varepsilon\rho u) \le \varepsilon(\|v_j\|+\rho).$$

Using this inequality, one establishes that $\sup_{\varepsilon\in(0,1]}\left|\varepsilon^{-\alpha H} f_\varepsilon\right| \le G$ where G is defined by

$$G(\rho,u) \;=\; c\rho^{n-1}\left[\sum_{j=1}^{k}|\lambda_j|\Big(\Big|\big|\|v_j\|-\rho\big|^{H-n/\alpha} - \rho^{H-n/\alpha}\Big| + + \Big|\big|\|v_j\|+\rho\big|^{H-n/\alpha} - \rho^{H-n/\alpha}\Big|\Big)\right]^\alpha.$$

with c a finite positive constant. Since $H \in (0,1)$, $G \in L^1\Big((0,+\infty)\times\mathbb{S}_{n-1}, d\rho d\sigma_{\mathbb{S}_{n-1}}(u)\Big)$ and the dominated convergence theorem implies that

$$\lim_{\varepsilon\to 0_+}\mathbb{E}\left(i\exp\left(\sum_{j=1}^{k}\lambda_j Z_{\varepsilon,1}(v_j)\right)\right) = \mathbb{E}\left(\exp\left(i\sum_{j=1}^{k}\lambda_j B_{H,\alpha}(v_j)\right)\right) \tag{10}$$

with $B_{H,\alpha}$ defined by (8).
We now study the behavior of $Z_{\varepsilon,2}$ as $\varepsilon \to 0_+$. For any $w \in B(0,\delta)$,

$$\forall\lambda\in\mathbb{R},\ \mathbb{E}\big(\exp\left(i\lambda\varepsilon^{-H}Z_{\varepsilon,2}(w)\right)\big) = \exp\left(-|\lambda|^\alpha\varepsilon^{-\alpha H}I_2(\varepsilon)\right)$$

with $I_2(\varepsilon) = \displaystyle\int_{\mathcal{M}} d\big(M_0+\varepsilon w, M'\big)^{\alpha H-n}\big|f\big(d\big(M_0+\varepsilon w, M'\big)\big) - f\big(d\big(M_0,M'\big)\big)\big|^\alpha d\sigma_{\mathcal{M}}\big(M'\big)$.
Using the mean value theorem and $H < 1$, one checks that

$$\lim_{\varepsilon\to 0_+}\varepsilon^{-H\alpha}I_2(\varepsilon) = 0.$$

Therefore, for any $w \in B(0,\delta)$ $\lim_{\varepsilon\to 0_+}\varepsilon^{-H}Z_{\varepsilon,2}(u) \overset{(d)}{=} 0$ and Assertion 2(a) follows from Equation (10).

(b) Assume that $H > 1$. We only sketch the proof. Let $K = 2\max_{1\le j\le k}\|v_j\|$. The main idea is to write $X_{H,\alpha,\mathcal{M}}(M) = X_{1,\varepsilon}(M) + X_{2,\varepsilon}(M)$ with

$$X_{1,\varepsilon}(M) = \int_{\mathcal{M}} g\big(d(M,M')\big)\mathbf{1}_{d(M_0,M')\le\varepsilon K}\, d\sigma_{\mathcal{M}}(M')$$

and

$$X_{2,\varepsilon}(M) = \int_{\mathcal{M}} g\big(d(M,M')\big)\mathbf{1}_{d(M_0,M')>\varepsilon K}\, d\sigma_{\mathcal{M}}(M').$$

We recall that g is defined by (7). Following the proof of Assertion 2(a), replacing f by $f\mathbf{1}_{d(M_0,M')\le\varepsilon K}$, one proves (even though $H > 1$) that

$$\lim_{\varepsilon\to 0_+}\big(\varepsilon^{-1}(X_{1,\varepsilon}(M_0+\varepsilon v) - X_{\varepsilon,1}(M_0))\big)_{v\in B(0,\delta)} \overset{(d)}{=} 0. \tag{11}$$

Let us now study the asymptotics of $X_{2,\varepsilon}$ as ε tends to 0_+. Let us write

$$\sum_{j=1}^{k}\lambda_j(X_{\varepsilon,2}(M_0+\varepsilon v_j) - X_{\varepsilon,2}(M_0)) = \int_{\mathcal{M}}\tilde{g}_\varepsilon(M')dW_{\alpha,\mathcal{M}}\big(M'\big)$$

with

$$\tilde{g}_\varepsilon(M') := \sum_{j=1}^{k}\lambda_j\big(g\big(d\big(M_0+\varepsilon v_j,M'\big)\big) - g\big(d\big(M_0,M'\big)\big)\big)\mathbf{1}_{d(M_0,M')>\varepsilon K}.$$

For any $M' \notin \mathrm{Cut}_{M_0}\cup\{M_0\}$, where Cut_{M_0} is the cut locus of M_0 in $\mathcal{M}$,

$$\lim_{\varepsilon\to 0_+}\frac{d(M_0+\varepsilon w,M') - d(M_0,M')}{\varepsilon} = -\frac{\langle\Pi_{M_0}(M'),w\rangle}{\left\|\Pi_{M_0}(M')\right\|}.$$

Then, for any $M' \notin \mathrm{Cut}_{M_0}\cup\{M_0\}$, a Taylor expansion leads to

$$\tilde{g}\big(M'\big) = \lim_{\varepsilon\to 0_+}\varepsilon^{-1}\tilde{g}_\varepsilon\big(M'\big) = -\frac{g'(d(M_0,M'))\langle\Pi_{M_0}(M'),\sum_{j=1}^{k}\lambda_j v_j\rangle}{\left\|\Pi_{M_0}(M')\right\|}.$$

Moreover applying the mean value Theorem to the function g and noting that

$$d\big(M_0+\varepsilon v_j,M'\big) \ge d\big(M_0,M'\big) - \varepsilon\|v_j\| \ge \frac{d(M_0,M')}{2}$$

for $d(M_0,M') \ge \varepsilon K$, one finds $\tilde{G}\in L^\alpha(\mathcal{M},d\sigma_{\mathcal{M}})$ such that $\sup_{\varepsilon\in(0,1]}\left|\varepsilon^{-1}\tilde{g}_\varepsilon\right| \le \tilde{G}$. Since $\sigma_{\mathcal{M}}(\mathrm{Cut}_{M_0}\cup\{M_0\}) = 0$, the dominated convergence theorem then leads to

$$\lim_{\varepsilon\to 0_+}\varepsilon^{-\alpha}\int_{\mathcal{M}}|\tilde{g}_\varepsilon|^\alpha d\sigma_{\mathcal{M}} = \int_{\mathcal{M}}|\tilde{g}|^\alpha d\sigma_{\mathcal{M}}.$$

Combining this with (4) and (11), one obtains that

$$\lim_{\varepsilon\to 0_+}(X_{H,\alpha,\mathcal{M}}(M_0+\varepsilon v) - X_{H,\alpha,\mathcal{M}}(M_0))_{v\in B(0,\delta)} = (-\langle v,S_\alpha\rangle)_{v\in B(0,\delta)}$$

with S_α defined by (9). Since $S_\alpha \overset{(d)}{=} -S_\alpha$, the proof of Proposition 4.2 is then complete.

□

4.3 Moving average multifractional random fields

As done for Euclidean random fields, to obtain some fields whose index of lass property vary with $M_0 \in \mathcal{M}$, we replace the index H by a function $h(\cdot)$ in the integral representation (5). Hence, let us consider a function $h : \mathcal{M} \longrightarrow (0, +\infty)$ such that $h(M) \neq n/\alpha$ for every $M \in \mathcal{M}$. Then, as soon as

$$X_{h,\alpha,\mathcal{M}}(M) \;=\; \int_{\mathcal{M}} d(M,M')^{h(M)-n/\alpha} f\big(d(M,M')\big) dW_{\alpha,\mathcal{M}}(M'),\; M \in \mathcal{M} \tag{12}$$

is well-defined, $X_{h,\alpha}$ is called moving average multifractional α-stable random field with multifractional function h. Then, replacing H by $h(M_0)$ and assuming h sufficiently smooth, we can state the analoguous of proposition 4.2 for the random field $X_{h,\alpha,\mathcal{M}}$.

Proposition 4.3. *Assume that $\mathcal{M} = \mathbb{S}_n$ or $\mathbb{H}_n$. If $\mathcal{M} = \mathbb{H}_n$, we assume that $f(x) = \mathrm{e}^{-\gamma x}$ with $\gamma > (n-1)/\alpha$. Then, $X_{h,\alpha,\mathcal{M}}$ is well-defined. Moreover, if h and f are $\mathcal{C}^1$ and if $h(M_0) < 1$ for $M_0 \in \mathcal{M}$, then,*

$$\lim_{\varepsilon \to 0_+} \left(\frac{X_{h,\alpha,\mathcal{M}}(M_0 + \varepsilon v) - X(M_0)}{\varepsilon^{h(M_0)}} \right)_{v \in B(0,\delta)} \overset{(d)}{=} \big(B_{h(M_0),\alpha}(v)\big)_{v \in B(0,\delta)},$$

where $B_{h(M_0),\alpha}$ is defined by (8).

Proof of proposition 4.3. By Proposition 4.2, the random field

$$Y(M,H) = \int_{\mathcal{M}} d(M,M')^{H-n/\alpha} f\big(d\big(M,M'\big)\big) dW_{\alpha,\mathcal{M}}(M'),\; M \in \mathcal{M},\; H > 0$$

is well-defined. For every $v \in \mathcal{B}(0,\delta)$ and every $\varepsilon \in (0,1)$, let us write that

$$X_{h,\alpha,\mathcal{M}}(M_0 + \varepsilon v) - X_{h,\alpha,\mathcal{M}}(M_0) = Y(M_0 + \varepsilon v, h(M_0)) - Y(M_0, h(M_0)) + R_{M_0}(M_0 + \varepsilon v)$$

with $R_{M_0}(M) = Y_{\alpha,\mathcal{M}}(M, h(M)) - Y_{\alpha,\mathcal{M}}(M, h(M_0))$. Then, by (4),

$$\mathbb{E}(\exp(iuR_{M_0}(M_0 + \varepsilon v))) = \exp\left(-|u|^\alpha I(\varepsilon)\right)$$

with

$$I(\varepsilon) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^\delta \left| r^{h(M_0+\varepsilon v)-n/\alpha} - r^{h(M_0)-n/\alpha} \right|^\alpha |f(r)|^\alpha dr.$$

Let $\varepsilon_0 > 0$. Then, by continuity of h there exists $a, b \in (0, +\infty)$ such that for $\varepsilon \in [0, \varepsilon_0)$, $a \leq h(M_0 + \varepsilon v) \leq b$. Hence, applying the Taylor-Lagrange inequality, one obtains that for $\varepsilon \in (0, \varepsilon_0)$,

$$I(\varepsilon) \leq \frac{2\pi^{n/2}}{\Gamma(n/2)} |h(M_0 + \varepsilon v) - h(M_0)|^\alpha \int_0^\delta |\ln r| \Big(r^{a-n/\alpha} + r^{b-n/\alpha} \Big)^\alpha |f(r)|^\alpha dr.$$

Since h is a $\mathcal{C}^1$ function and $h(M_0) < 1$, $\lim_{\varepsilon \to 0_+} \varepsilon^{-h(M_0)} R_{M_0}(M_0 + \varepsilon v) \overset{(d)}{=} 0$. Then, one concludes the proof applying Proposition 4.2 to the moving average fractional α-stable field $Y(\cdot, h(M_0))$, whose index is $h(M_0) < 1$. □

5 Classical properties of the tangent field

We give in this section some general properties of fractional fields, especially properties of the tangent fields and range of permissible fractional index. These properties are indeed more or less expected.

The self-similarity, stated in next proposition, of the tangent field as small scales is a direct consequence of the definition of tangent fields.

Proposition 5.1. *If the random field $X = (X(M))_{M\in\mathcal{M}}$ is locally asymptotically self-similar random field at point M_0 with index H and tangent field Z_{M_0}, then $Z_{M_0}(0) = 0$ almost surely and*

$$\forall \lambda \in (0,1],\ (Z_{M_0}(\lambda v))_{v\in B(0,\delta)} \overset{(d)}{=} \lambda^H (Z_{M_0}(v))_{v\in B(0,\delta)}. \tag{13}$$

Under a weak stationary property, defined below, the range of permissible indices H depends on the existence of moments, with a break for moments higher that one. In the case of fields indexed by whole $\mathbb{R}^n$, this property has already been established in [11].

Definition 5.2. *The increments of the field $X = (X(M))_{M\in\mathcal{M}}$ are weakly stationary if for all $(M,N) \in \mathcal{M}^2$, the distribution of $X(M) - X(N)$ only depends on the geodesic distance $d(M,N)$.*

The fractional fields defined in Sections 4.1 and 4.2 are examples of fields with weakly stationary increments. Then, they illustrate the following proposition.

Proposition 5.3. *Let X be a locally asymptotically self-similar random field at point M_0 with index H and tangent field Z_{M_0}. Assume that X has weakly stationary increments.*

1. *Then, Z_{M_0} has weakly stationary increments.*
2. *If for some $u \in \mathcal{B}(0,\delta)\backslash\{0\}$ and some $\gamma > 0$, $\mathbb{P}(Z_{M_0}(u) \neq 0) = 1$ and $\mathbb{E}|Z_{M_0}(u)|^\gamma < +\infty$, then $0 < H < \max\left(\frac{1}{\gamma}, 1\right)$.*
3. *Assume Z_{M_0} is a Gaussian random field. Then $H \in (0,1]$ and there exists a finite constant $C > 0$ such*

$$\forall v, w \in \mathcal{B}(0,\delta),\ \mathbb{E}(Z_{M_0}(v)Z_{M_0}(w)) = \frac{C^2}{2}\Big(\|v\|^{2H} + \|w\|^{2H} - \|v-w\|^{2H}\Big). \tag{14}$$

   Moreover, if $H < 1$, Z_{M_0} is centered and is then an Euclidean fractional Brownian motion restricted to $B(0,\delta)$.

4. *If Z_{M_0} is an α-stable random field, then $0 < H \leq \max\left(1, \frac{1}{\alpha}\right)$.*

These properties are already known for Euclidean random fields (see [11]) and then were expected in the manifold indexed realm. As regards Assertions 2-4, we provide a proof to emphasize where the manifold realm plays a role, and refer to [11] for some details (see proof of Proposition 7.1.10, Corollary 7.1.11 and Lemma 7.2.1).

Proof of Proposition 5.3.

1. Let $v, w \in \mathcal{B}(0,\delta)$. Then, by the lass property

$$Z_{M_0}(v) - Z_{M_0}(w) \overset{(d)}{=} \lim_{\varepsilon\to 0_+} \frac{X(M_0+\varepsilon v) - X(M_0+\varepsilon w)}{\varepsilon^H}. \tag{15}$$

   Let us fix $u \in \mathcal{B}(0,\delta)$ such that $u \neq 0$. Then, by continuity of the distance and of the exponential map $\exp_{M_0}$, there exists $\varepsilon_0 > 0$ such that

$$\forall \varepsilon \leq \varepsilon_0,\ d(M_0+\varepsilon v, M_0+\varepsilon w) = d\Big(\exp_{M_0}(\varepsilon v), \exp_{M_0}(\varepsilon w)\Big) < \|u\|.$$

   Therefore, for every $\varepsilon \leq \varepsilon_0$, the point

$$M_\varepsilon = M_0 + \frac{d(M_0+\varepsilon v, M_0+\varepsilon w)}{\|u\|} u = \exp_{M_0}\left(\frac{d(M_0+\varepsilon v, M_0+\varepsilon w)}{\|u\|} u\right)$$

is well-defined and such that $d(M_0, M_\varepsilon) = d(M_0 + \varepsilon v, M_0 + \varepsilon w)$. Hence, by (15) and by the weakly stationarity of the increments of X,

$$Z_{M_0}(v) - Z_{M_0}(w) \overset{(d)}{=} \lim_{\varepsilon \to 0_+} \frac{X(M_\varepsilon) - X(M_0)}{\varepsilon^H}.$$

Observe that $M_\varepsilon = M_0 + \varepsilon' u$ where $\varepsilon' = d(M_0 + \varepsilon v, M_0 + \varepsilon w) \to 0+$ as $\varepsilon \to 0_+$. Hence, applying again the lass property and Lemma 4.1, we obtain that

$$Z_{M_0}(v) - Z_{M_0}(w) \overset{(d)}{=} \|u\|^{-H} \|v - w\|^H Z_{M_0}(u). \tag{16}$$

Since this equality holds for any $v, w \in \mathcal{B}(0, \delta)$, the increments of Z_{M_0} are weakly stationary.

2. Let us remark that $u/2 \in \mathcal{B}(0, \delta)$ and then that $Z_{M_0}(u/2)$ is well-defined. Then, since $Z_{M_0}(0) = 0$, by Equation (16) (applied $(v, w) = (u/2, 0)$), $Z_{M_0}(u/2) \neq 0$ almost surely and by weak stationarity of the increments of Z_{M_0},

$$Z_{M_0}(u) - Z_{M_0}\Big(\frac{u}{2}\Big) \overset{(d)}{=} Z_{M_0}\Big(\frac{u}{2}\Big).$$

Therefore, $\mathbb{P}\big(Z_{M_0}\big(\frac{u}{2}\big) \neq 0,\ Z_{M_0}(u) \neq Z_{M_0}\big(\frac{u}{2}\big)\big) = \mathbb{P}\big(Z_{M_0}(u) \neq Z_{M_0}\big(\frac{u}{2}\big)\big) = 1 > 0$. Then, one can now follow the proof of Proposition 7.1.10 in [11] to establish Assertion 2 (replacing $X(2) - X(1)$ by $Z_{M_0}(u) - Z_{M_0}\big(\frac{u}{2}\big)$).

3. Assume that Z_{M_0} is a centered Gaussian random field. We can not directly proceed as in the proof of Lemma 7.2.1 of [11] since Z_{M_0} is not defined on whole $\mathbb{R}^n$. However, the idea stills be the same.

   Let us first remark that since Z_{M_0} is a non degenerate Gaussian random field, it fulfills the assumptions of Assertion 2 with $\gamma = 2$, which implies that $H \leq 1$. Moreover, by Equation (16) (where we recall that $u \neq 0$ is a fix chosen point of $B(0, \delta)$), for every $v, w \in \mathcal{B}(0, \delta)$,

$$\mathbb{E}(Z_{M_0}(v) - Z_{M_0}(w))^2 = C^2 \|v - w\|^{2H}$$

   with $C = \|u\|^{-H} \sqrt{\mathbb{E}(Z_{M_0}(u))^2} \in (0, +\infty)$. Since $Z_{M_0}(0) = 0$ almost surely, this is equivalent to (14). If $H < 1$, using the self-similarity property (13) with $\lambda = 1/2$ and the weak stationarity property, one checks, as in the proof of Lemma 7.2.1 in [11], that

$$\mathbb{E}(Z_{M_0}(u/2)) = (2^{-H} - 1)\mathbb{E}(Z_{M_0}(u)).$$

   This implies that $\mathbb{E}(Z_{M_0}(u)) = 0$ and then by (16) that Z_{M_0} is centered.

4. Since Z_{M_0} is an α-stable random field, it fullfils the assumptions of Assertion 2 for any $\gamma < \alpha$ and then $H \leq \max(1/\alpha, 1)$.

□

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