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Stability, convergence to the steady state and elastic limit for the Boltzmann equation for diffusively excited granular media

S. MISCHLER¹, C. MOUHOT²

Abstract

We consider a space-homogeneous gas of *inelastic hard spheres*, with a *diffusive term* representing a random background forcing (in the framework of so-called *constant normal restitution coefficients* $\alpha \in [0, 1]$ for the inelasticity). In the physical regime of a small inelasticity (that is $\alpha \in [\alpha_*, 1)$ for some constructive $\alpha_* \in [0, 1)$) we prove uniqueness of the stationary solution for given values of the restitution coefficient $\alpha \in [\alpha_*, 1)$, the mass and the momentum, and we give various results on the linear stability and nonlinear stability of this stationary solution.

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Keywords: Inelastic Boltzmann equation; granular gases; random forcing; hard spheres; stationary solution; uniqueness; stability; small inelasticity; elastic limit; degenerated perturbation; spectrum.

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1 Introduction

1.1 The model and main result

We consider the steady states of the spatially homogeneous inelastic Boltzmann equation for hard spheres with thermal bath forcing. More precisely, we consider a gas which is described by the distribution density of particles $f = f(v) \geq 0$ with velocity $v \in \mathbb{R}^N$ ($N \geq 2$) and such that f satisfies the stationary equation

$$(1.1) \quad Q_\alpha(f, f) + \tau \Delta_v f = 0 \quad \text{in} \quad \mathbb{R}^N,$$

with a constant $\tau > 0$, and given mass and vanishing momentum:

$$(1.2) \quad \int_{\mathbb{R}^N} f \, dv = \rho \in (0, \infty), \quad \int_{\mathbb{R}^N} f \, v \, dv = 0,$$

The term $\mu \Delta_v f$, with constant $\mu \in (0, \infty)$, represents the effect of the heat bath. The quadratic collision operator $Q_\alpha(f, f)$ models the interaction of particles by means of inelastic binary collisions with a constant normal restitution coefficient $\alpha \in [0, 1)$ (see [6, 1, 8, 9]) which preserves mass and momentum but dissipates kinetic energy. We define the collision operator by its action on test functions, or

observables. Taking $\psi = \psi(v)$ to be a suitably regular test function, we introduce the following weak formulation of the collision operator

$$(1.3) \quad \int_{\mathbb{R}^N} Q_\alpha(g, f) \psi dv = \iint \int_{\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{N-1}} b |u| g_* f (\psi' - \psi) d\sigma dv dv_*,$$

where we use the shorthand notations $f := f(v)$, $g_* := g(v_*)$, $\psi' := \psi(v')$, etc. Here and below $u = v - v_*$ denotes the relative velocity and v', v'_* denotes the possible post-collisional velocities (which encapsule the inelasticity of the collision operator in terms of α). They are defined by

$$(1.4) \quad v' = \frac{w}{2} + \frac{u'}{2}, \quad v'_* = \frac{w}{2} - \frac{u'}{2},$$

with

$$w = v + v_*, \quad u' = \left(\frac{1-\alpha}{2} \right) u + \left(\frac{1+\alpha}{2} \right) |u| \sigma.$$

We also introduce the notation $\hat{x} = x/|x|$ for any $x \in \mathbb{R}^N$, $x \neq 0$. The function $b = b(\hat{u} \cdot \sigma)$ in (1.3) is (up to a multiplicative factor) the *differential collisional cross-section*. We assume that

$$(1.5) \quad b \text{ is Lipschitz, non-decreasing and convex on } (-1, 1)$$

and that

$$(1.6) \quad \exists b_m, b_M \in (0, \infty) \quad \text{s.t.} \quad \forall x \in [-1, 1], \quad b_m \leq b(x) \leq b_M.$$

Note that the “physical” cross-section for hard spheres is given by (see [6, 4])

$$(1.7) \quad b(x) = b'_0 (1 - x)^{-\frac{N-3}{2}}, \quad b'_0 \in (0, \infty),$$

so that it fulfills the above hypothesis (1.5,1.6) when $N = 3$. These hypothesis are needed in the proof of moments estimates (see [8, Proposition 3.2] and [9, Proposition 3.1]).

We also define the symmetrized (or polar form of the) bilinear collisional operator \tilde{Q}_α by setting

$$(1.8) \quad \begin{cases} \int_{\mathbb{R}^N} \tilde{Q}_\alpha(g, h) \psi dv = \frac{1}{2} \iint \int_{\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{N-1}} b |u| g_* h \Delta_\psi d\sigma dv dv_*, \\ \text{with } \Delta_\psi = (\psi' + \psi'_* - \psi - \psi_*). \end{cases}$$

In other words, $\tilde{Q}_\alpha(g, h) = (Q_\alpha(g, h) + Q_\alpha(h, g))/2$. The formula (1.3) suggests the natural splitting $Q_\alpha = Q_\alpha^+ - Q_\alpha^-$ between gain and loss part. The loss part Q_α^- can be defined in strong form noticing that

$$\langle Q_\alpha^-(g, f), \psi \rangle = \iint \int_{\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{N-1}} b |u| g_* f \psi d\sigma dv dv_* =: \langle f L(g), \psi \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product in L^2 and L is the convolution operator

$$(1.9) \quad L(g)(v) = (b_0 |\cdot| * g)(v) = b_0 \int_{\mathbb{R}^N} g(v_*) |v - v_*| dv_*, \quad \text{with } b_0 = \int_{S^{N-1}} b(\sigma_1) d\sigma.$$

In particular note that L and $Q_\alpha^- = Q^-$ are indeed independent of the normal restitution coefficient α .

As explained in [6], the operator (1.3) preserves mass and momentum, and since the Laplacian also does so, the mass and momentum of a stationary solution can be prescribed. However energy is not preserved neither by the collisional operator (which tends to cool down the gas) nor by the diffusive operator (which harms it up). Competition between these two effects gives rise to a balance equation (obtained after having multiplied equation (1.1) by $|v|^2$ and integrated) which reads as follows

$$(1.10) \quad (1 - \alpha^2) D_\mathcal{E}(f) = \mu 2 N \rho.$$

The *energy dissipation functional* is given by

$$D_\mathcal{E}(f) := b_1 \int \int_{\mathbb{R}^N \times \mathbb{R}^N} f f_* |u|^3 dv dv_*,$$

where b_1 is (up to a multiplicative factor) the angular momentum defined by

$$(1.11) \quad b_1 := \frac{1}{8} \int_{S^{N-1}} (1 - (\hat{u} \cdot \sigma)) b(\hat{u} \cdot \sigma) d\sigma.$$

In order to establish (1.10) we have used (1.8) and the elementary computation

$$\Delta_{|\cdot|^2}(v, v_*, \sigma) = -\frac{1 - \alpha^2}{4} (1 - (\hat{u} \cdot \sigma)) |u|^2.$$

Existence and qualitative properties of the steady solutions as well of the solutions to the associated evolution equation (Cauchy theory) was done in [6] (see also [1]). Among others, it is proved the following in these papers:

Theorem 1.1 ([6, Theorem 5.2, Lemma 7.2], [1, Theorem 1]) *For any given inelastic coefficient $\alpha \in (0, 1)$ and any given mass $\rho \in (0, \infty)$ there exists at least one solution $F \in \mathcal{S}(\mathbb{R}^N)$ to the stationary equation (1.1)-(1.2) with mass ρ and vanishing momentum which furthermore satisfies*

$$(1.12) \quad F(v) \geq a_1 e^{-a_2 |v|^{3/2}} \quad \forall v \in \mathbb{R}^N \quad \text{and} \quad \int_{\mathbb{R}^N} F(v) e^{a_3 |v|^{3/2}} dv < \infty,$$

for some constants $a_1, a_2, a_3 \in (0, \infty)$.

Our main result gives a partial answer concerning uniqueness of solutions obtained by Theorem 1.1:

Theorem 1.2 *There is some constructive $\alpha_* \in (0, 1)$ such that for any $\alpha \in [\alpha_*, 1]$, and any given mass $\rho \in (0, \infty)$ the stationary equation (1.1)-(1.2) admits a unique solution F with mass ρ and vanishing momentum in the class of functions given by the existence Theorem 1.1.*

1.2 Rescaled variables and elastic limit $\alpha \rightarrow 1$

Let us start with a remark. For any solution F to the equation (1.1) and any constant $\lambda \in (0, \infty)$, the rescaled function g associated to f by the relation

$$g(v) = \lambda^N f(\lambda v).$$

satisfies

$$(1.13) \quad Q_\alpha(g, g) + \lambda^{-3} \tau \Delta_v g = 0.$$

In order to prove (1.13) we have just use the homogeneity properties $Q_\alpha(g, g)(v) = \lambda^{N-1} Q_\alpha(f, f)(\lambda v)$ and $\Delta_v g = \lambda^{N+2} (\Delta_v f)(\lambda v)$.

This elementary remark shows that we may choose $\tau \in (0, \infty)$ arbitrarily in equation (1.1) and we now make the choice

$$(1.14) \quad \tau = \tau_\alpha = \rho(1 - \alpha),$$

and denote by F_α a solution to equation (1.1) with $\tau = \tau_\alpha$:

$$(1.15) \quad Q_\alpha(F_\alpha, F_\alpha) + \tau_\alpha \Delta F_\alpha = 0 \quad \text{in } \mathbb{R}^N.$$

At a formal level, it is immediate that with this choice of scaling, in the elastic limit $\alpha \rightarrow 1$, the equations (1.1)-(1.2) become

$$(1.16) \quad \begin{cases} Q_1(F_1, F_1) = 0 & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} F_1 dv = \rho, \quad \int_{\mathbb{R}^N} F_1 v dv = 0, & 0 \leq F_1 \in \mathcal{S}(\mathbb{R}^N). \end{cases}$$

Moreover, from (1.10), one gets

$$(1.17) \quad 2N(1 - \alpha)\rho^2 - (1 - \alpha^2)D_{\mathcal{E}}(F_\alpha) = 0.$$

Dividing the above equation by $(1 - \alpha)$ and passing to the limit $\alpha \rightarrow 1$, one obtains

$$(1.18) \quad N\rho^2 - D_{\mathcal{E}}(F_1) = 0.$$

It is straightforward that the only function satisfying the constraints (1.16) and (1.18) is the Maxwellian function

$$(1.19) \quad \bar{F}_1 := M_{\bar{\theta}_1} = M_{\rho, 0, \bar{\theta}_1}$$

where, for any $\rho, \theta > 0$, $u \in \mathbb{R}^N$, the function $M_{\rho, u, \theta}$ denotes the Maxwellian with mass ρ , momentum u and temperature θ given by

$$(1.20) \quad M_{\rho, u, \theta}(v) := \frac{\rho}{(2\pi\theta)^{N/2}} e^{-\frac{|v-u|^2}{2\theta}},$$

and where the temperature $\bar{\theta}_1 \in (0, \infty)$ is given by (we recall that b_1 is defined in (1.11))

$$(1.21) \quad \bar{\theta}_1 = \frac{1}{2} \frac{N^{2/3}}{b_1^{2/3}} \left(\int_{\mathbb{R}^N} M_{1,0,1}(v) |v|^3 dv \right)^{-2/3}.$$

For instance in dimension $N = 3$ we obtain $\bar{\theta}_1 = (3^2 \pi)^{1/3} / (2^{10} b_1^2)^{1/3}$. Moreover, in the particular case of the hard-spheres cross-section (1.7) in dimension 3, we find $b_1 = b'_0(4\pi)/3$ and therefore $\bar{\theta}_1 = 3^{4/3} / (2^{14} \pi b'_0{}^2)^{1/3}$.

1.3 Physical and mathematical motivation

In this short paper we shall not review the physical and mathematical on the kinetic theory for granular gases. Let us only refer to some key references where the reader can find all the desired details: for a detailed physical introduction we refer to [2, 4]; and for a short mathematical introduction see [14]; finally apart from the many works on “pseudo-maxwell molecules” (see the bibliography of [14]) let us only mention the works directly related to inelastic hard spheres (freely cooling or in thermal bath): [6, 1, 8, 9, 10] (in particular see the introductions of these papers for some mathematical discussions about the models and their connection to physics). This paper is inspired from the models and questions arised in [6] and the new mathematical tools developed in [10].

1.4 Notation

Throughout the paper we shall use the notation $\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}$. We denote, for any $p \in [1, +\infty]$, $q \in \mathbb{R}$ and weight function $\omega : \mathbb{R}^N \rightarrow \mathbb{R}_+$, the weighted Lebesgue space $L_q^p(\omega)$ by

$$L_q^p(\omega) := \{f : \mathbb{R}^N \mapsto \mathbb{R} \text{ measurable ; } \|f\|_{L_q^p(\omega)} < +\infty\},$$

with, for $p < +\infty$,

$$\|f\|_{L_q^p(\omega)} = \left[\int_{\mathbb{R}^N} |f(v)|^p \langle v \rangle^{pq} \omega(v) dv \right]^{1/p}$$

and, for $p = +\infty$,

$$\|f\|_{L_q^\infty(\mathbb{R}^N)} = \sup_{v \in \mathbb{R}^N} |f(v)| \langle v \rangle^q \omega(v).$$

We shall in particular use the exponential weight functions

$$(1.22) \quad m = m_{s,a}(v) := e^{-a|v|^s} \quad \text{for } a \in (0, \infty), s \in (0, 1),$$

or a smooth version $m(v) := e^{-\zeta(|v|^2)}$ with $\zeta \in C^\infty$ is a positive function such that $\zeta(r) = r^{s/2}$ for any $r \geq 1$, with $s \in (0, 1)$.

In the same way, the weighted Sobolev space $W_q^{k,p}(\omega)$ ($k \in \mathbb{N}$) is defined by the norm

$$\|f\|_{W_q^{k,p}(\omega)} = \left[\sum_{|s| \leq k} \|\partial^s f(v)\|_{L_q^p(\omega)}^p \right]^{1/p},$$

and as usual in the case $p = 2$ we denote $H_q^k(\omega) = W_q^{k,2}(\omega)$. The weight ω shall be omitted when it is 1. Finally, for $f \in L_{2k}^1$, with $k \geq 0$, we introduce the following notation for the homogeneous moment of order $2k$

$$\mathbf{m}_k(f) := \int_{\mathbb{R}^N} f |v|^{2k} dv,$$

and we also denote by $\rho(f) = \mathbf{m}_0(f)$ the mass of f , $\mathcal{E}(f) = \mathbf{m}_1(f)$ the energy of f and by $\theta(f) = \mathcal{E}(f)/(\rho(f) N)$ the temperature associated to f (when the distribution f has 0 mean). For any $\rho, \mathcal{E} \in (0, \infty)$, $u \in \mathbb{R}^N$ we then introduce the subsets of L^1 of functions of given mass, mean velocity and energy

$$\begin{aligned}\mathcal{C}_{\rho,u} &:= \left\{ h \in L^1_1; \int_{\mathbb{R}^N} h \, dv = \rho, \int_{\mathbb{R}^N} h v \, dv = \rho u \right\}, \\ \mathcal{C}_{\rho,u,\mathcal{E}} &:= \left\{ h \in L^1_2; \int_{\mathbb{R}^N} h \, dv = \rho, \int_{\mathbb{R}^N} h v \, dv = \rho u, \int_{\mathbb{R}^N} h |v|^2 \, dv = \mathcal{E} \right\}.\end{aligned}$$

For any (smooth version of) exponential weight function m and any $k, q \geq 0$ we introduce the Banach spaces

$$(1.23) \quad \mathbb{L}^1(m^{-1}) = L^1(m^{-1}) \cap \mathcal{C}_{0,0} \quad \text{and} \quad \mathbb{W}_q^{k,1}(m^{-1}) = W_q^{k,1}(m^{-1}) \cap \mathcal{C}_{0,0}.$$

1.5 Summary of the results

Our results, that we state now, deal with the evolution equation

$$(1.24) \quad \frac{\partial f}{\partial t} = Q_\alpha(f, f) + \tau_\alpha \Delta f, \quad f(0, \cdot) = f_{\text{in}} \in \mathcal{C}_{\rho,0},$$

where τ_α is defined in (1.14), and with the associated stationary equation (1.15) with $F \in \mathcal{C}_{\rho,0}$.

Theorem 1.3 *There is some constructive $\alpha_* \in (0, 1)$ such that for $\alpha \in [\alpha_*, 1]$, and any given mass $\rho \in (0, \infty)$, we have:*

- (i) *For any $\tau > 0$, the equation (1.1) admits a unique non-negative stationary solution with mass ρ and vanishing momentum. We denote by \bar{F}_α the stationary solution obtained by fixing $\tau = \tau_\alpha$ (defined by (1.14)).*
- (ii) *Let define $\bar{F}_1 = M_{\rho,0,\bar{\theta}_1}$ the Maxwellian distribution with mass ρ , momentum 0 and “diffusive thermodynamical temperature” $\bar{\theta}_1$ defined in (1.21). The path of stationary solutions $\alpha \rightarrow \bar{F}_\alpha$ parametrized by the normal restitution coefficient is C^1 from $[\alpha_*, 1]$ into $W^{k,1} \cap L^1(e^{a|v|})$ for any $k \in \mathbb{N}$ and some $a \in (0, \infty)$.*
- (iii) *For any $\alpha \in [\alpha_*, 1]$, the linearized collision operator*

$$(1.25) \quad h \mapsto \mathcal{L}_\alpha h := 2\tilde{Q}_\alpha(\bar{F}_\alpha, h) + \tau_\alpha \Delta h$$

is well-defined and closed on $\mathbb{L}^1(m^{-1})$ for any exponential weight function m with exponent $s \in (0, 1)$ (defined in (1.22)). Its spectrum decomposes between a part which lies in the half-plane $\{\text{Re } \xi \leq \bar{\mu}\}$ for some constructive $\bar{\mu} < 0$,

and some remaining discrete eigenvalue μ_α . This eigenvalue is real negative and satisfies

$$(1.26) \quad \mu_\alpha = -\frac{3}{\theta_1} \rho (1 - \alpha) + \mathcal{O}(1 - \alpha)^2 \quad \text{when } \alpha \rightarrow 1.$$

The associated eigenspace is of dimension 1 and then denoting by $\phi_\alpha = \phi_\alpha(v)$ the unique associated eigenfunction such that $\|\phi_\alpha\|_{L_2^1} = 1$ and $\phi_\alpha(0) < 0$, there holds $\phi_\alpha \in \mathcal{S}(\mathbb{R}^N)$ (with bounds of regularity independent of α) and

$$(1.27) \quad \phi_\alpha \rightarrow \phi_1 := c_0 (|v|^2 - N \bar{\theta}_1) \bar{F}_1 \quad \text{as } \alpha \rightarrow 1,$$

where c_0 is the positive constant such that $\|\phi_1\|_{L_2^1} = 1$. Finally one has constructive decay estimates on the semigroup associated to this spectral decomposition in this Banach space (see the key Theorem 5.1 and the following point).

- (iv) The stationary solution \bar{F}_α is globally attractive on bounded subsets of L_3^1 under some smallness condition on the inelasticity in the following sense. For any $\rho, \mathcal{E}_0, M_0 \in (0, \infty)$ there exists $\alpha_{**} \in (\alpha_*, 1)$, $C_* \in (0, \infty)$ and $\eta \in (0, 1)$, such that for any initial datum satisfying

$$0 \leq f_{\text{in}} \in L_3^1 \cap \mathcal{C}_{\rho, 0, \mathcal{E}_0}, \quad \|f_{\text{in}}\|_{L_3^1} \leq M_0,$$

the solution g to (1.24) satisfies

$$(1.28) \quad \|f_t - \bar{F}_\alpha\|_{L_2^1} \leq e^{(1-\eta)\mu_\alpha t}.$$

- (v) Moreover, under smoothness condition on the initial datum one may prove a more precise asymptotic decomposition, and construct a Liapunov functional for the equation (1.24). More precisely, there exists $k_* \in \mathbb{N}$ and, for any exponential weight m as defined in (1.22) and any $\rho, \mathcal{E}_0, M_0 \in (0, \infty)$, there exists $\alpha_{**} \in (\alpha_*, 1)$ and a constructive functional $\mathcal{H} : H^{k_*} \cap L^1(m^{-1}) \rightarrow \mathbb{R}$ such that, first, for any initial datum $0 \leq f_{\text{in}} \in H^{k_*} \cap L^1(m^{-1}) \cap \mathcal{C}_{\rho, 0, \mathcal{E}_0}$ satisfying

$$\|f_{\text{in}}\|_{H^{k_*} \cap L^1(m^{-1})} \leq M_0,$$

the solution f to (1.24) satisfies

$$(1.29) \quad f(t, \cdot) = \bar{F}_\alpha + c_\alpha(t) \phi_\alpha + r_\alpha(t, \cdot),$$

with $c_\alpha(t) \in \mathbb{R}$ and $r_\alpha(t, \cdot) \in L_2^1(\mathbb{R}^N)$ such that

$$(1.30) \quad |c_\alpha(t)| \leq C_* e^{\mu_\alpha t}, \quad \|r_\alpha(t, \cdot)\|_{L_2^1} \leq C_* e^{(3/2)\mu_\alpha t}.$$

And second when the initial datum satisfies additionally

$$f_{\text{in}} \geq M_0^{-1} e^{-M_0 |v|^8},$$

the solution satisfies also

$$t \mapsto \mathcal{H}(g(t, \cdot)) \quad \text{is strictly decreasing}$$

(up to reach the stationary state \bar{F}_α).

Remark 1.4 *All the constants appearing in this theorem are constructive, which means that they can be made explicit, and in particular that the proof does not use any compactness argument. Unless otherwise mentioned, these constants will depend on b , on the dimension N , and on some bounds on the initial datum but never on the inelasticity parameter $\alpha \in (0, 1]$. All the other remarks made in [10, Section 1.6] also apply here.*

1.6 Method of proof and plan of the paper

The general structure of the proof is inspired from [10]. We have packed into the appendix the technical results from this previous paper which are used here also, as well as a technical result of lower bound on the diffusive inelastic Boltzmann equation. Therefore the overall structure of the proof is likely to be more visible than in [10]. In the whole, we mainly prove here the few points which differs from [10] (due to the replacement of the anti-drift term by a diffusive term) and refer to [10] for more details.

The first main idea of our method is to consider the rescaled equations (1.15)-(1.14) with an inelasticity dependent diffusion coefficient τ_α which exactly “compensates” the loss of elasticity of the collision operator (in the sense that it compensates its loss of kinetic energy). This scaling allows to prove uniform bounds according to α for the family of stationary solutions F_α to the equation (1.15) (recall that in this scaling, the diffusion is evanescent in the elastic limit).

The second main idea consists in decoupling the variations along the “energy direction” and its “orthogonal direction”. This decoupling makes it possible to identify the limit of different objects as $\alpha \rightarrow 1$ (among them the limit of F_α).

The third main idea is to use systematically the knowledges on the elastic limit problem, once it has been identified thanks to the previous arguments. In particular we use the spectral study of the linearized problem and the dissipation entropy-entropy inequality for the elastic problem. This allows to argue by perturbative method. Let us emphasize that this perturbation is singular in the classical sense because of the addition of a (limit vanishing) second-order derivative operator, but also because of the gain of one more conservative quantity at the limit.

In Section 2, we use the regularity properties of the collision operator in order to establish on the one hand that the family (F_α) is bounded in $H^\infty \cap L^1(m^{-1})$ uniformly according to the inelastic parameter α (the key argument being the use of the entropy functional which provides uniform lower bound on the energy of F_α) and on the other hand that the difference of two stationary solutions in any strong norm may be bounded by the difference of these ones in weak norm (the key idea is a bootstrap argument). This last point shall allow to deal with the loss of derivatives and weights in the operator norms used in the sequel of the paper.

In Section 3, we prove that $F_\alpha \rightarrow \bar{F}_1$ when $\alpha \rightarrow 1$ with explicit “Hölder” rate. The cornerstone of the proof is again the decoupling of the variation $F_\alpha - \bar{F}_1$ between the “energy direction” and its “orthogonal direction”.

Finally in Section 4, we prove uniqueness of the profile \bar{F}_α for small inelasticity by a variation around the implicit function theorem, in Section 5 we prove results on the localization of the spectrum of the linearized equation for small inelasticity, and in Section 6 we prove some (semi)-global stability results by combining the previous linearized study with entropy production estimates.

2 Estimates on the steady states

In this section we prove various regularity and decay estimates on the stationary solutions (or the differences of stationary solutions), uniform as $\alpha \rightarrow 1$, which shall be useful in the sequel.

2.1 Uniform estimates on the steady states

For any $\alpha \in (0, 1)$ we consider \mathcal{F}_α the (not empty) set of all the solutions given by Theorem 1.1 of the inelastic diffusive stationary Boltzmann equation (1.15) with inelasticity coefficient α , given mass $\rho \in (0, +\infty)$ and finite energy. More precisely, we define \mathcal{F}_α as the following set of functions

$$\mathcal{F}_\alpha := \left\{ F \in \mathcal{S}(\mathbb{R}^N) \text{ satisfying (1.15), (1.2), (1.12)} \right\}.$$

For some fixed $\alpha_0 \in (0, 1)$, we also define

$$\mathcal{F} = \cup_{\alpha \in [\alpha_0, 1)} \mathcal{F}_\alpha.$$

We show that for any stationary solution $F_\alpha \in \mathcal{F}$ the decay estimates, the pointwise lower bound and the regularity estimates can be made uniform according to the inelasticity coefficient $\alpha \in [\alpha_0, 1)$. Let us emphasize once again that the choice of the rescaling parameter $\tau_\alpha = \rho(1 - \alpha)$ in (1.15) is fundamental in order to get uniformity of these bounds in the limit $\alpha \rightarrow 1$. Let us also mention that our choice of scaling for the equation (1.15) is mass invariant, that is F with density $\rho(F)$ satisfies the equation if and only if $F/\rho(F)$ satisfies the equation with $\rho = 1$. Therefore all the estimates on the profiles are homogeneous in terms of the density ρ .

Proposition 2.1 *Let us fix $\alpha_0 \in (0, 1)$. There exists $a_1, a_2, a_3, a_4 \in (0, \infty)$ and, for any $k \in \mathbb{N}$, there exists $C_k \in (0, \infty)$ such that*

$$(2.1) \quad \forall \alpha \in [\alpha_0, 1), \forall F_\alpha \in \mathcal{F}_\alpha, \quad \|F_\alpha\|_{L^1(e^{a_1|v|})} \leq a_2, \quad \|F_\alpha\|_{H^k(\mathbb{R}^N)} \leq C_k.$$

Proof of Proposition 2.1. We split the proof into several steps. We fix $\alpha \in [\alpha_0, 1)$ and $F_\alpha \in \mathcal{F}_\alpha$ for which we will establish the announced bounds. Let emphasize that thanks to the *a priori* bounds satisfied by F_α all the computations we will perform

are rigorously justified. From now we omit the subscript “ α ” when no confusion is possible.

Step 1. Upper bound on the energy using the energy dissipation term. We prove that

$$(2.2) \quad \forall \alpha \in (0, 1] \quad \mathcal{E} \leq \rho \left(\frac{2N}{b_1} \right)^{2/3}.$$

From equation (1.17) on the energy of the profile G there holds

$$(2.3) \quad (1 + \alpha) b_1 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} F F_* |u|^3 dv dv_* = 2N \rho^2.$$

From Jensen’s inequality

$$\int_{\mathbb{R}^N} |u|^3 F_* dv_* \geq \rho |v|^3,$$

and Hölder’s inequality

$$\int_{\mathbb{R}^N} |v|^3 F dv \geq \rho^{-1/2} \left(\int_{\mathbb{R}^N} |v|^2 F dv \right)^{3/2},$$

we get

$$(1 + \alpha) b_1 \rho^{1/2} \mathcal{E}^{3/2} \leq 2N \rho^2$$

from which the bound (2.2) follows.

Step 2. Lower bound on the energy using the entropy. We prove

$$(2.4) \quad \forall \alpha \in (0, 1] \quad \mathcal{E} \geq \rho \left(\frac{\alpha^2 N^2}{\sqrt{2} b_2} \right)^{2/3} \quad \text{with} \quad b_2 := \|b\|_{L^1}.$$

Remark 2.2 *The choice of scaling we have made for the rescaled equation (1.15) becomes clear from this computation: it is chosen such that the energy of the stationary solution does not blow up nor vanishes for $\alpha \rightarrow 1$. The restriction $\alpha \in [\alpha_0, 1)$, $\alpha_0 > 0$, is then made in order to get a uniform estimate from below on the energy.*

By integrating the equation satisfied by F against $\log F$ we find

$$\int_{\mathbb{R}^N} Q(F, F) \log F dv + \rho(1 - \alpha) \int_{\mathbb{R}^N} \log F \Delta_v F dv = 0.$$

Then we write the first term as in [6, Section 1.4] to find

$$\begin{aligned} & \frac{1}{2} \iint \int_{\mathbb{R}^{2N} \times S^{N-1}} F F_* \left(\log \frac{F' F'_*}{F F_*} - \frac{F' F'_*}{F F_*} + 1 \right) B dv dv_* d\sigma \\ & + \frac{1}{2} \iint \int_{\mathbb{R}^{2N} \times S^{N-1}} (F' F'_* - F F_*) B dv dv_* d\sigma - \rho(1 - \alpha) \int_{\mathbb{R}^N} \frac{|\nabla_v F|^2}{F} dv = 0. \end{aligned}$$

Recalling that $x - \log x - 1 \geq 0$ for any $x \geq 0$ and making the change of variables $(v', v'_*) \rightarrow (v, v_*)$ in the second term we obtain

$$(2.5) \quad \rho(1 - \alpha) \int_{\mathbb{R}^N} \frac{|\nabla_v F|^2}{F} dv \leq \frac{1}{2} \left(\frac{1}{\alpha^2} - 1 \right) b_2 \iint_{\mathbb{R}^{2N}} F F_* |u| dv dv_*.$$

On the one hand, from Cauchy-Schwarz's inequality

$$(2.6) \quad \begin{aligned} & \iint_{\mathbb{R}^{2N}} F F_* |u| dv dv_* \leq \\ & \leq \left(\iint_{\mathbb{R}^{2N}} F F_* dv dv_* \right)^{1/2} \left(\iint_{\mathbb{R}^{2N}} F F_* |u|^2 dv dv_* \right)^{1/2} = \sqrt{2} \rho^{3/2} \mathcal{E}^{1/2}. \end{aligned}$$

On the other hand, we compute

$$(2.7) \quad \begin{aligned} 0 & \leq \int_{\mathbb{R}^N} \left| 2 \nabla \sqrt{F} + \frac{\rho N v}{\mathcal{E}} \sqrt{F} \right|^2 dv \\ & = \int_{\mathbb{R}^N} \left(4 |\nabla \sqrt{F}|^2 + 2 \frac{\rho N v}{\mathcal{E}} \cdot \nabla F + \left(\frac{\rho N}{\mathcal{E}} \right)^2 |v|^2 F \right) dv \\ & \leq 2 \int_{\mathbb{R}^N} \frac{|\nabla_v F|^2}{F} dv - 2 \frac{(\rho N)^2}{\mathcal{E}} + \left(\frac{\rho N}{\mathcal{E}} \right)^2 \mathcal{E} = 2 \int_{\mathbb{R}^N} \frac{|\nabla_v F|^2}{F} dv - \frac{(\rho N)^2}{\mathcal{E}}. \end{aligned}$$

Gathering (2.5), (2.6) and (2.7) we get

$$\rho \frac{(\rho N)^2}{2 \mathcal{E}} \leq \frac{1}{2} \frac{1 + \alpha}{\alpha^2} b_2 \sqrt{2} \rho^{3/2} \mathcal{E}^{1/2},$$

from which we deduce (2.4).

Step 3. Upper bound on exponential moments. There exists $A, C > 0$ such that

$$\forall \alpha \in [0, 1), \quad \int_{\mathbb{R}^N} F(v) e^{A|v|} dv \leq C \rho.$$

We refer to [1, Theorem 1] where this bound is obtained as an immediate consequence of the following sharp moment estimates: there exists an explicit $X > 0$ such that

$$(2.8) \quad \forall \alpha \in [0, 1), \quad \mathbf{m}_k = \int_{\mathbb{R}^N} G |v|^k dv \leq \Gamma(k + 1/2) X^{k/2} \rho.$$

It is worth noticing that in [1] the Povzner inequality used in order to get (2.8) is uniform in terms of the normal restitution coefficient $\alpha \in [0, 1]$ appearing in the collisional operator and in terms of the coefficient μ in front of the thermal bath term. Moreover the factor ρ comes from our choice of the scaling variables (in which ρ is involved).

Step 4. Uniform upper bound on the L^2 norm, and H^k norms, $k > 0$. On the basis of the uniform bounds from below and above on the energy, the proof can be done exactly as in [10, Proposition 2.1], since the diffusion term plays no role in the energy estimates (it only helps). \square

After the estimates from above on the profile, let us now state a pointwise bound from below (it is a straightforward consequence of Proposition D.1 in the appendix).

Proposition 2.3 *Let us fix $\alpha_0 \in (0, 1)$. There exists $a_3, a_4 \in (0, \infty)$ such that*

$$(2.9) \quad \forall \alpha \in [\alpha_0, 1), \quad \forall F_\alpha \in \mathcal{F}_\alpha, \quad F_\alpha \geq a_3 e^{-a_4 |v|^8}.$$

2.2 Estimates on the difference of two stationary solutions

In this subsection we take advantage of the mixing effects of the collision operator in order to show that the L^1 norm of their difference of two stationary solutions (corresponding to the same inelasticity coefficient) indeed controls the $H^k \cap L^1(m^{-1})$ norm of their difference for any $k \in \mathbb{N}$ and for some exponential weight function m , uniformly in terms of $\alpha \in [\alpha_0, 1)$.

Proposition 2.4 *For any $k > 0$, there is $m = \exp(-a |v|)$, $a \in (0, \infty)$ and $C_k > 0$ such that for any $\alpha \in [\alpha_0, 1)$ and any $F_\alpha, H_\alpha \in \mathcal{F}_\alpha$ there holds*

$$(2.10) \quad \|H_\alpha - F_\alpha\|_{H^k \cap L^1(m^{-1})} \leq C_k \|H_\alpha - F_\alpha\|_{L^1}.$$

Proof of Proposition 2.4. We proceed in three steps. It is worth mentioning that all the constants in the proof are uniform in terms of the normal restitution coefficient $\alpha \in [\alpha_0, 1)$, as they only depend on the uniform bounds of Proposition 2.1 and some uniform bounds on the collision kernel.

The proof is based on the following three steps, which can all be proved following exactly the arguments in [10, Proposition 2.7].

Step 1. Control of the L^1 moments. We prove first that there exists $A, C \in (0, \infty)$ such that

$$\forall \alpha \in [\alpha_0, 1), \quad \int_{\mathbb{R}^N} |H_\alpha - F_\alpha| e^{A|v|} dv \leq C \int_{\mathbb{R}^N} |H_\alpha - F_\alpha| dv.$$

The proof can be done exactly as in the Step 1 of [10, Proposition 2.7]: the only change is the replacement of the vanishing anti-drift term by the term viscosity term, which both in any case are negligible in the moments estimates (note also that this estimate on the diffusive Boltzmann equation is proved within the paper [1], with constant independent on the viscosity coefficient).

Step 2. Control of the L^2 norms. It is done exactly as in Step 2 of [10, Proposition 2.7], using the regularity theory of the gain term: the only difference is that instead anti-drift term which vanishes in the estimates, one has now a diffusion term

which yields good damping negative in the L^2 energy estimate (which can therefore be dropped).

Step 3. Control of the H^k norms. The proof can be done exactly as in Step 3 of [10, Proposition 2.7]. The anti-drift is replaced by the diffusion term which is also a good negative term in the H^k energy estimates. \square

3 Quantification of the elastic limit $\alpha \rightarrow 1$

We have the following estimate on the distance between F_α and \bar{F}_1 for any stationary solution F_α .

Proposition 3.1 *For any $\varepsilon > 0$ there exists C_ε (independent of the mass ρ) such that*

$$(3.1) \quad \forall \alpha \in [\alpha_0, 1) \quad \sup_{F_\alpha \in \mathcal{F}_\alpha} \|F_\alpha - \bar{F}_1\|_{L_2^1} \leq C_\varepsilon \rho (1 - \alpha)^{\frac{1}{2+\varepsilon}}$$

where we recall that \bar{F}_1 is the Maxwellian function defined by (1.19)–(1.21).

Proof of Proposition 3.1. On the one hand, for any inelasticity coefficient $\alpha \in [\alpha_0, 1)$ and profile F_α , there holds from (2.5) together with Corollary A.4 and the uniform estimates of Proposition 2.1

$$(3.2) \quad D_{H,1}(F_\alpha) \leq D_{H,\alpha}(F_\alpha) + \rho^2 \mathcal{O}(1 - \alpha) \leq \rho^2 \mathcal{O}(1 - \alpha).$$

On the other hand, introducing the Maxwellian function M_θ with the same mass, momentum and temperature as F_α , that is M_θ given by (1.20) with $u = 0$ and $\theta = \mathcal{E}(F_\alpha)/\rho$, and gathering (3.2), (A.9), (A.8) with the uniform estimates of Proposition 2.1 and interpolation inequality, we obtain that for any $q, \varepsilon > 0$ there exists $C_{q,\varepsilon}$ such that

$$(3.3) \quad \forall \alpha \in [\alpha_0, 1) \quad \|F_\alpha - M_\theta\|_{L_q^1}^{2+\varepsilon} \leq C_{q,\varepsilon} \rho^{2+\varepsilon} (1 - \alpha).$$

Next, from (2.3), we have

$$b_1 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} F_\alpha F_{\alpha*} |u|^3 dv dv_* - 2\rho N \int_{\mathbb{R}^N} F_\alpha dv = (1 - \alpha) \frac{b_1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} F_\alpha F_{\alpha*} |u|^3 dv dv_*$$

and then

$$(3.4) \quad |\Psi(\theta)| \leq C_1 \|F_\alpha - M_\theta\|_{L_3^1} + C_2 \rho^2 (1 - \alpha),$$

where we have used that F_α and M_θ are bounded thanks to Proposition 2.1 and we have defined

$$(3.5) \quad \Psi(\theta) = 2\rho N \int_{\mathbb{R}^N} M_\theta dv - b_1 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} M_\theta M_{\theta*} |u|^3 dv dv_*.$$

By elementary changes of variables, this formula simplifies into

$$\Psi(\theta) = k_1 - k_2 \theta^{3/2}$$

with $k_1 = 2 \rho^2 N$ and, using (B.3),

$$k_2 = \rho^2 b_1 \int_{\mathbb{R}^N \times \mathbb{R}^N} M_{1,0,1} (M_{1,0,1})_* |u|^3 dv dv_* = 2^{3/2} \rho^2 b_1 \mathbf{m}_{3/2}(M_{1,0,1}).$$

We next observe that $\Psi \in C^\infty(0, \infty)$ and Ψ is strictly concave. It is also obvious that the equation $\Psi(\theta) = 0$ for $\theta > 0$ has a unique solution which is $\bar{\theta}_1$ defined in (1.21), and that we have

$$\Psi(\theta) \leq \Psi'(\bar{\theta}_1) (\theta - \bar{\theta}_1) = -C (\theta - \bar{\theta}_1)/2$$

for some explicit constant, as well as

$$(3.6) \quad \Psi(\theta) = k_2 [\bar{\theta}_1^{3/2} - \theta^{3/2}].$$

Plugging this expression for Ψ into (3.4) and using the lower bound (2.4) on the temperature θ and the estimate (3.3) we obtain that for any $\varepsilon > 0$ there is $C_\varepsilon \in (0, \infty)$ such that

$$(3.7) \quad \forall \alpha \in (\alpha_0, 1) \quad \left| \theta^{3/2} - \bar{\theta}_1^{3/2} \right|^{2+\varepsilon} \leq C_\varepsilon (1 - \alpha).$$

Namely, we have thus proved that the temperature of \bar{F}_α converge (with rate) to the expected temperature $\bar{\theta}_1$. In order to come back to the norm of $F_\alpha - \bar{F}_1$, we first write, using Cauchy-Schwarz's inequality,

$$(3.8) \quad \begin{aligned} \|F_\alpha - \bar{F}_1\|_{L^1_{-N}} &\leq \|F_\alpha - M_\theta\|_{L^1_{-N}} + \|M_\theta - \bar{F}_1\|_{L^1_{-N}} \\ &\leq \|F_\alpha - M_\theta\|_{L^1} + C_N \|M_\theta - \bar{F}_1\|_{L^2}, \end{aligned}$$

and we remark that

$$(3.9) \quad \|M_\theta - \bar{F}_1\|_{L^2}^2 \leq C \rho^2 \left| \theta^{3/2} - \bar{\theta}_1^{3/2} \right|.$$

Gathering (3.8) with (3.9), (3.7) and (3.3) we deduce that for any $\varepsilon > 0$ there is $C_\varepsilon \in (0, \infty)$ such that

$$\forall \alpha \in (\alpha_0, 1) \quad \|F_\alpha - \bar{F}_1\|_{L^1_{-N}}^{2+\varepsilon} \leq C_\varepsilon \rho^{2+\varepsilon} (1 - \alpha),$$

and (3.1) follows by interpolation again. \square

4 Uniqueness and continuity of the path of stationary solutions

Theorem 4.1 *There exists a constructive $\alpha_1 \in (0, 1)$ such that the solution F_α of (1.15) is unique for any $\alpha \in [\alpha_1, 1]$. We denote by \bar{F}_α this unique stationary solution.*

That is an immediate consequence of the following result.

Proposition 4.2 *There is a constructive constant $\eta \in (0, 1)$ such that*

$$\left. \begin{array}{l} G, H \in \mathcal{F}_\alpha, \alpha \in (1 - \eta, 1) \\ \|G - \bar{F}_1\|_{L_2^1} \leq \eta, \|H - \bar{F}_1\|_{L_2^1} \leq \eta \end{array} \right\} \quad \text{implies} \quad G = H.$$

Proof of Theorem 4.1. Let us assume that Proposition 4.2 holds. Then Proposition 3.1 implies that there is some explicit $\varepsilon \in (0, 1)$ such that for $\alpha \in (1 - \varepsilon, 1]$ one has

$$\sup_{F_\alpha \in \mathcal{F}_\alpha} \|F_\alpha - \bar{F}_1\|_{L_2^1} \leq \eta$$

where η is defined in the statement of Proposition 4.2. Up to reducing η , it is always possible to take $\eta \leq \varepsilon$, and the proof is completed by applying Proposition 4.2. \square

Proof of Proposition 4.2. Let us consider any exponential weight function m with $s \in (0, 1)$, $a \in (0, +\infty)$, or with $s = 1$ and $a \in (0, \infty)$ small enough. Let us also define the subvector space of $\mathbb{L}^1(m^{-1})$ of functions with zero energy

$$(4.1) \quad \mathcal{O} = \mathcal{C}_{0,0,0} \cap \mathbb{L}^1(m^{-1}),$$

the function $\psi = C(|v|^2 - N)M_{1,0,1}$ such that its mass is zero and its energy is $\mathcal{E}(\psi) = 1$, and Π the following projection

$$\Pi : \mathbb{L}^1(m^{-1}) \rightarrow \mathcal{O}, \quad \Pi(g) = g - \mathcal{E}(g)\psi.$$

Finally, let us introduce Φ the following non-linear functional operator

$$\Phi : [0, 1] \times (W_1^{2,1}(m^{-1}) \cap \mathcal{C}_{\rho,0}) \rightarrow \mathbb{R} \times \mathcal{O},$$

and

$$\Phi(1, \cdot) : (L_1^1(m^{-1}) \cap \mathcal{C}_{\rho,0}) \rightarrow \mathbb{R} \times \mathcal{O},$$

by setting

$$\Phi(\alpha, g) = \left((1 + \alpha)D\mathcal{E}(g) - 2N\rho^2, \Pi\left[Q_\alpha(g, g) + \tau_\alpha \Delta_v g\right] \right).$$

It is straightforward that $\Phi(\alpha, F_\alpha) = 0$ for any $\alpha \in [\alpha_0, 1]$ and $F_\alpha \in \mathcal{F}_\alpha$, and that the equation

$$\Phi(1, g) = (0, 0)$$

has a unique solution, given by $g = \bar{F}_1 = M_{\rho,0,\bar{\theta}_1}$ defined in (1.19), (1.21).

The function Φ is linear and quadratic in its second argument by inspection, and easy computations yield the following formal differential according to the second argument at the point $(1, \bar{F}_1)$:

$$(4.2) \quad D_2 \Phi(1, \bar{F}_1) h =: A h := \left(4 \tilde{D}_{\mathcal{E}}(\bar{F}_1, h), 2 \tilde{Q}_1(\bar{F}_1, h) \right)$$

where \tilde{Q}_α is defined in (1.8) and

$$\tilde{D}_{\mathcal{E}}(g, h) := b_1 \int \int_{\mathbb{R}^N \times \mathbb{R}^N} g h_* |u|^3 dv dv_*.$$

Notice that we can remove the projection on the last argument in (4.2) since the elastic collision operator always has zero energy.

On the basis of Lemma A.7 in the appendix for A , we shall now prove Proposition 4.2. We write

$$(4.3) \quad \begin{aligned} F_\alpha - H_\alpha &= A^{-1} \left[A F_\alpha - \Phi(\alpha, F_\alpha) + \Phi(\alpha, H_\alpha) - A H_\alpha \right] \\ &= A^{-1} (I_1, I_2) \end{aligned}$$

with (recall that the bilinear operators $\tilde{D}_{\mathcal{E}}$ and \tilde{Q}_α are symmetric)

$$\begin{cases} I_1 := 4 \tilde{D}_{\mathcal{E}}(\bar{F}_1, F_\alpha - H_\alpha) - (1 + \alpha) D(F_\alpha) + (1 + \alpha) D(H_\alpha) \\ I_2 := \Pi I_{2,1} + \Pi I_{2,2} \end{cases}$$

and

$$\begin{cases} I_{2,1} := 2 \tilde{Q}_1(\bar{F}_1, F_\alpha - H_\alpha) - Q_\alpha(F_\alpha, F_\alpha) + Q_\alpha(H_\alpha, H_\alpha) \\ I_{2,2} := \rho(1 - \alpha) \Delta(H_\alpha - F_\alpha). \end{cases}$$

On the one hand,

$$I_1 = 2 D(2\bar{F}_1 - (F_\alpha + H_\alpha), F_\alpha - H_\alpha) + (1 - \alpha) D(F_\alpha + H_\alpha, F_\alpha - H_\alpha)$$

so that

$$(4.4) \quad \begin{aligned} |I_1| &\leq C_3 \left(\|\bar{F}_1 - F_\alpha\|_{L_3^1} + \|\bar{F}_1 - H_\alpha\|_{L_3^1} \right. \\ &\quad \left. + (1 - \alpha) \|F_\alpha\|_{L_3^1} + (1 - \alpha) \|H_\alpha\|_{L_3^1} \right) \|F_\alpha - H_\alpha\|_{L_3^1} \\ &\leq \eta_1(\alpha) \|F_\alpha - H_\alpha\|_{L_1^1(m^{-1})} \end{aligned}$$

with $\eta_1(\alpha) \rightarrow 0$ when $\alpha \rightarrow 1$ (with explicit rate, for instance $\eta_1(\alpha) = C_1 (1 - \alpha)^{1/3}$) because of Propositions 2.1 and 3.1.

On the other hand,

$$I_{2,1} = 2 \tilde{Q}_1(\bar{F}_1, F_\alpha - H_\alpha) - 2 \tilde{Q}_\alpha(\bar{F}_1, F_\alpha - H_\alpha) + Q_\alpha(\bar{F}_1 - F_\alpha, F_\alpha - H_\alpha) + Q_\alpha(F_\alpha - H_\alpha, \bar{F}_1 - H_\alpha).$$

From Proposition 2.1 and estimate (A.4) in Proposition A.1 there holds

$$\|\tilde{Q}_1(\bar{F}_1, F_\alpha - H_\alpha) - \tilde{Q}_\alpha(\bar{F}_1, F_\alpha - H_\alpha)\|_{L^1(m^{-1})} \leq C(1 - \alpha) \|F_\alpha - H_\alpha\|_{W_3^{1,3}(m^{-1})}.$$

From estimate (A.1) in Proposition A.1 we have

$$\begin{aligned} & \|Q_\alpha(\bar{F}_1 - F_\alpha, F_\alpha - H_\alpha) + Q_\alpha(F_\alpha - H_\alpha, \bar{F}_1 - H_\alpha)\|_{L^1(m^{-1})} \\ & \leq C_4 \left(\|F_\alpha - \bar{F}_1\|_{L_1^1(m^{-1})} + \|H_\alpha - \bar{F}_1\|_{L_1^1(m^{-1})} \right) \|F_\alpha - H_\alpha\|_{L_1^1(m^{-1})}. \end{aligned}$$

Together with Propositions 3.1 we thus obtain

$$(4.5) \quad \|I_{2,1}\|_{L^1(m^{-1})} \leq \eta(\alpha) \|F_\alpha - H_\alpha\|_{L_1^1(m^{-1})}$$

for some $\eta(\alpha) \rightarrow 0$ as $\alpha \rightarrow 1$. Here we can take for instance (when $s = 1/2$ in the formula of m) $\eta(\alpha) = C(1 - \alpha)^{1/12}$ for some $C \in (0, \infty)$ by picking a suitable ε and interpolating.

Finally from Proposition 2.4 there holds

$$(4.6) \quad \|I_{2,2}\|_{L^1(m^{-1})} \leq C_5(1 - \alpha) \|F_\alpha - H_\alpha\|_{L_1^1(m^{-1})}.$$

Gathering (4.4), (4.5) and (4.6) we obtain from (4.3), Lemma A.7 and Proposition 2.4 again

$$\|F_\alpha - H_\alpha\|_{L_1^1(m^{-1})} \leq \eta(\alpha) \|A^{-1}\| \|F_\alpha - H_\alpha\|_{L_1^1(m^{-1})}$$

for some function η such that $\eta(\alpha) \rightarrow 0$ as $\alpha \rightarrow 1$ (with explicit rate). Hence choosing α_1 close enough to 1 we have $\eta(\alpha) \|A^{-1}\| \leq 1/2$ for any $\alpha \in [\alpha_1, 1)$. This implies $F_\alpha = H_\alpha$ and concludes the proof. \square

Let us state now without proof, because it is completely similar to the corresponding result in [10] and is again a variation around the implicit function theorem, some regularity result on the path of self-similar profiles.

Lemma 4.3 *The map $[\alpha_1, 1] \rightarrow L^1(m^{-1})$, $\alpha \mapsto \bar{F}_\alpha$ is continuous on $[\alpha_1, 1]$ and differentiable at $\alpha = 1$. More precisely, there exists $\bar{F}'_1 \in L^1(m^{-1})$ and for any $\eta \in (1, 2)$ there exists a constructive $C_\eta \in (0, \infty)$ such that*

$$(4.7) \quad \|\bar{F}_\alpha - \bar{F}_1 - (1 - \alpha) \bar{F}'_1\|_{L^1(m^{-1})} \leq C_\eta (1 - \alpha)^\eta \quad \forall \alpha \in (\alpha_0, 1).$$

5 Spectral study of the linearized problem

In this section we shall enounce some results on the geometry of the spectrum of the linearized diffusive inelastic collision operator for a small inelasticity, as well as estimates on its resolvent and on the associated linear semigroup. This is straightforwardly adapted from [10].

We thus consider the operator

$$\mathcal{L}_\alpha : f \mapsto Q_\alpha(f, f) + \tau_\alpha \Delta f$$

and some fluctuations h around the stationary solution \bar{F}_α : that means $f = \bar{F}_\alpha + h$ with $f \in L^1(m^{-1})$ where m is a fixed smooth exponential weight function, as defined in (1.22). The corresponding linearized unbounded operator \mathcal{L}_α is acting on $L^1(m^{-1})$ with domain $\text{dom}(\mathcal{L}_\alpha) = W_1^{2,1}(m^{-1})$ if $\alpha \neq 1$ and $\text{dom}(\mathcal{L}_1) = L_1^1(m^{-1})$ (it is straightforward to check that it is closed in this space). Since the equation in self-similar variables preserves mass and the zero momentum, the correct spectral study of \mathcal{L}_α requires to restrict this operator to zero mean and centered distributions (which are preserved as well), that means to work in $\mathbb{L}^1(m^{-1})$. When restricted to this space, the operator \mathcal{L}_α is denoted by $\hat{\mathcal{L}}_\alpha$. We denote by $R(\hat{\mathcal{L}}_\alpha)$ the resolvent set of $\hat{\mathcal{L}}_\alpha$, and by $\mathcal{R}_\alpha(\xi) = (\hat{\mathcal{L}}_\alpha - \xi)^{-1}$ its resolvent operator for any $\xi \in R(\hat{\mathcal{L}}_\alpha)$.

5.1 The result on the spectrum and resolvent

The result proved in this section is a translation of [10, Theorem 5.2] for the diffusive inelastic Boltzmann equation. Let us define for any $x \in \mathbb{R}$ the half-plane Δ_x by

$$\Delta_x = \{\xi \in \mathbb{C}, \Re \xi \geq x\}.$$

Theorem 5.1 *Let us fix $\bar{\mu} \in (\mu_2, 0)$, with $\mu_2 < 0$ defined in Theorem A.6, $k, q \in \mathbb{N}$ and m a smooth weight exponential function with $s \in (0, 1)$. Then there exists $\alpha_2 \in (\alpha_1, 1)$ such that for any $\alpha \in [\alpha_2, 1]$ the following holds:*

- (i) *The unbounded operator $\hat{\mathcal{L}}_\alpha$ is well defined and closed in $\mathbb{W}_q^{k,1}(m^{-1})$ ($\forall k \geq 0, \forall q \geq 0$). Its spectrum $\Sigma(\hat{\mathcal{L}}_\alpha)$ satisfies*

$$\Sigma(\hat{\mathcal{L}}_\alpha) \subset \Delta_{\bar{\mu}}^c \cup \{\mu_\alpha\},$$

where μ_α is a real eigenvalue which does not depend on the choice of the space $\mathbb{W}_q^{k,1}(m^{-1})$ and satisfies (with explicit bounds)

$$(5.1) \quad \mu_\alpha = -3\rho(1 - \alpha) + \mathcal{O}(1 - \alpha)^2 \quad \text{when } \alpha \rightarrow 1.$$

Moreover, μ_α is a 1-dimensional eigenvalue, and more precisely, the eigenspace associated to μ_α is $\mathbb{R}\phi_\alpha$ with $\|\phi_\alpha\|_{L_2^1} = 1$ and $\|\phi_\alpha - \phi_1\|_{\mathbb{W}_q^{k,1}(m^{-1})} \leq C(1 - \alpha)$, where $\phi_1 := c_0(|v|^2 - N\bar{\theta}_1)\bar{F}_1$ is the “energy eigenfunction” associated to the linearized elastic Boltzmann operator.

- (ii) *The resolvent $\mathcal{R}_\alpha(\xi)$ in $\mathbb{W}_q^{k,1}(m^{-1})$ is holomorphic on a neighborhood of $\Delta_{\bar{\mu}} \setminus \{\mu_\alpha\}$ and there are explicit constants C_1, C_2 such that*

$$\sup_{z \in \mathbb{C}, \Re z = \bar{\mu}} \|\mathcal{R}_\alpha(z)\|_{\mathbb{W}_q^{k,1}(m^{-1}) \rightarrow \mathbb{W}_q^{k,1}(m^{-1})} \leq C_1$$

and

$$\|\mathcal{R}_\alpha(\bar{\mu} + is)\|_{\mathbb{W}_{q+1}^{k+2,1}(m^{-1}) \rightarrow \mathbb{W}_q^{k,1}(m^{-1})} \leq \frac{C_2}{1 + |s|}.$$

(iii) The linear semigroup $S_\alpha(t)$ associated to $\hat{\mathcal{L}}_\alpha$ in $\mathbb{W}_q^{k,1}(m^{-1})$ writes

$$S_\alpha(t) = e^{\mu_\alpha t} \Pi_\alpha + R_\alpha(t),$$

where Π_α is the projection on the (1-dimensional) eigenspace associated to μ_α and where $R_\alpha(t)$ is a semigroup which satisfies

$$(5.2) \quad \|R_\alpha(t)\|_{\mathbb{W}_{q+2}^{k+2,1}(m^{-1}) \rightarrow \mathbb{W}_q^{k,1}(m^{-1})} \leq C_k e^{\bar{\mu} t}$$

with explicit bounds.

5.2 Sketch of proof

The proof is straightforwardly adapted from [10, Section 5]. We shall only mention the main steps of the proof and we emphasize the few points which differs here (due to the replacement of the anti-drift term by a diffusive term).

The proof is based on results on the elastic collision operator together with the decomposition for $\xi \in \mathbb{C}$

$$(5.3) \quad \mathcal{L}_\alpha - \xi = A_\delta - B_{\alpha,\delta}$$

where

$$(5.4) \quad A_\delta = \mathcal{L}_{1,\delta}^+ - \mathcal{L}^*, \quad B_{\alpha,\delta} = \nu + \xi + (\mathcal{L}_{1,\delta}^+ - \mathcal{L}_1^+) + P_\alpha$$

with $P_\alpha = \mathcal{L}_1 - \mathcal{L}_\alpha = \mathcal{L}_1^+ - \mathcal{L}_\alpha^+ + \tau_\alpha \Delta$, and where $\mathcal{L}_1 = \mathcal{L}^+ - \mathcal{L}^* - \nu$ is the usual decomposition of the elastic collision operator and $\mathcal{L}_{1,\delta}^+$ is the regularized truncation of the “gain” part introduced in [11].

The differences with the similar decomposition in [10] only lies in the $B_{\alpha,\delta}$ and P_α operators. But the key technical estimates are still true for this operator. Indeed it can be checked straightforwardly that the following holds (let us point out that one loses now two indices of regularity instead of one, which explains the minor change in the statement of the estimate on the resolvent).

Lemma 5.2 (See Lemmas 5.8 & 5.9 in [10]) *Let us fix $k, q \geq 0$ and an exponential weight function m .*

(i) *There exists some constant C such that for any $\alpha \in (\alpha_0, 1]$*

$$\|\mathcal{L}_\alpha\|_{W_{q+1}^{k+2,1}(m^{-1}) \rightarrow W_q^{k,1}(m^{-1})} \leq C, \quad \|\mathcal{L}_1 - \mathcal{L}_\alpha\|_{W_3^{3,1}(m^{-1}) \rightarrow L^1(m^{-1})} \leq C(1 - \alpha).$$

(ii) *There exists some constants $\delta^* > 0$ and $\alpha_2 \in (\alpha_1, 1)$ such that for any $\xi \in \Delta_{\mu_2}$, $\delta \in [0, \delta^*]$ and $\alpha \in [\alpha_2, 1]$, the operator*

$$B_{\alpha,\delta} : W_{q+1}^{k+2,1}(m^{-1}) \rightarrow W_q^{k,1}(m^{-1})$$

is invertible and the inverse operator $B_{\alpha,\delta}(\xi)^{-1}$ satisfies

$$\|B_{\alpha,\delta}(\xi)^{-1}\|_{W_q^{k,1}(m^{-1}) \rightarrow W_{q+1}^{k+2,1}(m^{-1})} \leq \frac{C_1}{\text{dist}(\Re \xi, \nu(\mathbb{R}^N))}$$

and

$$\|B_{\alpha,\delta}(\xi)^{-1}\|_{W_q^{k,1}(m^{-1}) \rightarrow W_q^{k+2,1}(m^{-1})} \leq \frac{C_2}{\text{dist}(\xi, \nu(\mathbb{R}^N))}$$

for some explicit constants $C_1, C_2 > 0$ depending on k, q, δ^*, α_2 .

(iii) As a consequence, the resolvent operator $\mathcal{R}_\alpha(\xi)$ satisfies for any $\alpha \in [\alpha_2, 1]$ and any $\xi \in \Delta_{\mu_2}$

$$\|\mathcal{R}_\alpha(\xi)\|_{\mathbb{W}_q^{k,1}(m^{-1})} \leq \frac{C_3 + C_4 \|\mathcal{R}_1(\xi)\|_{W_{q+1}^{k+1,1}(m^{-1})}}{1 - C_5 (1 - \alpha) \|\mathcal{R}_1(\xi)\|_{W_{q+1}^{k+1,1}(m^{-1})}},$$

for some constants $C_i, i = 3, 4, 5$.

The rest of the proof is done the same, and we recall the main steps.

Step 1: Structure of the spectrum and rough localization

We decompose

$$\hat{\mathcal{L}}_\alpha = T + K$$

with $T = \tau_\alpha \Delta - L(\bar{F}_\alpha)$, $K = 2\tilde{Q}_\alpha^+(\bar{F}_\alpha, \cdot) - \bar{F}_\alpha L(\cdot)$. Since the spectrum of T is easily seen to be included in $\Delta_{\mu_2 + \mathcal{O}(1-\alpha)}^c$ (see the definition of $\mu_2 < 0$ in the statement of Theorem A.6 and using the regularity estimates on the stationary solutions of Proposition 2.1), and since the operator K is T -compact (it is the only point where we use the sharp estimate “in the norm of the graph” stated in Propositions A.2 & A.3), we conclude (for a small enough inelasticity) thanks to Weyl’s theorem that $\Sigma(\hat{\mathcal{L}}_\alpha) \cap \Delta_{\bar{\mu}}$ only contains discrete spectrum (that is isolated and finite multiplicity spectrum values, or, in other words, eigenvalues), a set that we denote by $\Sigma_d(\hat{\mathcal{L}}_\alpha)$. Moreover, thanks to point (iii) in Lemma 5.2 and to the geometry of the spectrum of $\hat{\mathcal{L}}_1$ as stated in Theorem A.6 (more particularly point (iii)) we deduce that $\Sigma(\hat{\mathcal{L}}_\alpha) \cap \Delta_{\bar{\mu}}$ is confined to a disc $B(0, c(1 - \alpha))$. To sum up, we have yet proved

$$\Sigma(\hat{\mathcal{L}}_\alpha) \cap \Delta_{\bar{\mu}} \subset \Sigma_d(\hat{\mathcal{L}}_\alpha) \cap B(0, c(1 - \alpha)).$$

Step 2: The “energy eigenvalue”

First, we infer (for instance, from the decomposition (5.3)) that any eigenfunction associated to an eigenvalue in Δ_{μ_2} is smooth: more precisely, there exists $C = C_{k,q,m}$ such that $\forall (\lambda, \psi)$ with $\lambda \in \Delta_{\mu_2}$, $\psi \in \mathbb{W}_1^2$ satisfying $\hat{\mathcal{L}}_\alpha \psi = \lambda \psi$, there holds $\|\psi\|_{W_q^{k,1}(m^{-1})} \leq C \|\psi\|_{L_2^1}$.

Second, we introduce the projector Π_α on the eigenspace associated to the spectrum included in Δ_{μ_2} . It is given by

$$\Pi_\alpha := -\frac{1}{2\pi i} \int_{\{\zeta \in \mathbb{C}, |\zeta|=r\}} \mathcal{R}_\alpha(\zeta) d\zeta$$

In particular the operator Π_1 is the projection on the energy eigenline $\mathbb{R} \phi_1$, where ϕ_1 is the energy eigenfunction defined in Theorem A.6. Thanks to the smoothness estimate on eigenfunctions just mentioned above and the estimate on $\mathcal{L}_1 - \mathcal{L}_\alpha$ in Lemma 5.2 (i) we easily deduce that $\|\Pi_1 - \Pi_\alpha\| < 1$ for α close enough to 1, and then that $\text{rang } \Pi_\alpha = \text{rang } \Pi_1$. We may sum up as

$$\Sigma(\hat{\mathcal{L}}_\alpha) \cap \Delta_{\bar{\mu}} = \{\mu_\alpha\}, \quad \mu_\alpha \in \mathbb{R}, \quad |\mu_\alpha| \leq c(1 - \alpha), \quad \text{Nul}(\mu_\alpha - \hat{\mathcal{L}}_\alpha) = \mathbb{R} \phi_\alpha.$$

Third, we prove that $\|\phi_\alpha - \phi_1\|_{W_q^{k,1}(m^{-1})} \leq C(1 - \alpha)$ exactly as in [10].

Fourth and last, we obtain a first order expansion of μ_α thanks to the following computation. By integrating the eigenvalue equation related μ_α

$$\hat{\mathcal{L}}_\alpha \phi_\alpha = \mu_\alpha \phi_\alpha$$

against $|v|^2$ and dividing it by $(1 - \alpha)$, we get

$$\frac{\mu_\alpha}{1 - \alpha} \mathcal{E}(\phi_\alpha) = \rho 2 N \rho(\phi_\alpha) - 2(1 + \alpha) \tilde{D}(\bar{F}_\alpha, \phi_\alpha).$$

Using convergence of $\bar{F}_\alpha \rightarrow \bar{F}_1$ and $\phi_\alpha \rightarrow \phi_1$ established before, and the fact that the mass of ϕ_1 is zero, we deduce that

$$(5.5) \quad \frac{\mu_\alpha}{1 - \alpha} \mathcal{E}(\phi_1) = -4 \tilde{D}(\bar{F}_1, \phi_1) + \mathcal{O}(1 - \alpha).$$

Then we compute thanks to (B.1) and (B.2)

$$(5.6) \quad \mathcal{E}(\phi_1) = 2 N c_0 \rho \bar{\theta}_1^2,$$

where c_0 is still the normalizing constant in (1.27) such that $\|\phi_1\|_{L_2^1} = 1$. Similarly, using (B.3), (B.4) and the relation (1.21) which make a link between b_1 and $\bar{\theta}_1$, we find

$$(5.7) \quad \tilde{D}(\bar{F}_1, \phi_1) = \frac{3}{2} N c_0 \rho^2 \bar{\theta}_1.$$

We conclude gathering (5.5), (5.6) and (5.7).

6 Convergence to the stationary solution

In this section, we consider the nonlinear evolution equation (1.15) and we prove the convergence of its solutions to the stationary solution.

6.1 The results

We first state a local linearized stability result.

Proposition 6.1 *For any $\alpha \in [\alpha_3, 1)$, the stationary solution \bar{F}_α is locally asymptotically stable, with domain of stability uniform according to $\alpha \in [\alpha_3, 1)$.*

More precisely, let us fix $\rho \in (0, \infty)$ and some exponential weight function m as in (1.22). There is $k_1, q_1 \in \mathbb{N}^$ such that for any $M_0 \in (0, \infty)$ there exists $C, \varepsilon \in (0, \infty)$ such that for any $\alpha \in [\alpha_3, 1]$, for any $f_{\text{in}} \in H^{k_1} \cap L^1(m^{-q_1})$ with mass ρ , momentum 0 satisfying*

$$(6.1) \quad \|f_{\text{in}}\|_{H^{k_1} \cap L^1(m^{-q_1})} \leq M_0, \quad \|f_{\text{in}} - \bar{F}_\alpha\|_{L^1(m^{-1})} \leq \varepsilon,$$

the solution f to the equation (1.15) with initial datum f_{in} satisfies

$$(6.2) \quad \forall t \geq 0, \quad \|\Pi_\alpha(f_t - \bar{F}_\alpha)\|_{L^1(m^{-1})} \leq C \|f_{\text{in}} - \bar{F}_\alpha\|_{L^1(m^{-1})} e^{\mu_\alpha t},$$

$$(6.3) \quad \forall t \geq 0, \quad \|(\text{Id} - \Pi_\alpha)(f_t - \bar{F}_\alpha)\|_{L^1(m^{-1})} \leq C \|f_{\text{in}} - \bar{F}_\alpha\|_{L^1(m^{-1})} e^{(3/2)\mu_\alpha t}.$$

Then we prove that when the inelasticity is small, depending on the size of the initial datum (but not necessarily close to the stationary solution), the equation (1.15) is stable. This mainly relies on the fact that the entropy production timescale is of a different order (much faster) than the energy dissipation timescale as $\alpha \rightarrow 1$.

Proposition 6.2 *Define $k_2 := \max\{k_0, k_1\}$, $q_2 := \max\{q_0, q_1, 3\}$, where k_i and q_i are defined in Theorem A.5 and Corollary A.4. For any ρ, \mathcal{E}_0, M_0 there exists $\alpha_4 \in [\alpha_3, 1)$, $c_1 \in (0, \infty)$ and for any $\alpha \in [\alpha_4, 1]$ there exist $\varphi = \varphi(\alpha)$ with $\varphi(\alpha) \rightarrow 0$ as $\alpha \rightarrow 1$ and $T = T(\alpha)$ (possibly blowing-up as $\alpha \rightarrow 1$) such that for any initial datum $0 \leq f_{\text{in}} \in L^1_{q_2} \cap H^{k_2} \cap \mathcal{C}_{\rho, 0, \mathcal{E}_0}$ with*

$$\|f_{\text{in}}\|_{L^1_{q_2} \cap H^{k_2}} \leq M_0,$$

the solution f associated to the rescaled equation (1.24) satisfies

$$\forall t \geq 0, \quad \mathcal{E}(f_t) \geq c_1$$

and for all $\alpha' \in [\alpha_4, 1)$ and then all $\alpha \in [\alpha', 1]$

$$(6.4) \quad \forall t \geq T(\alpha'), \quad \|f_t - \bar{F}_\alpha\|_{L^1_2} \leq \varphi(\alpha').$$

Then the proof of the global convergence for smooth initial data only amounts to connect the two previous results of Propositions 6.1 and 6.2 by choosing α such that $\varphi(\alpha) \leq \varepsilon$ where ε is the size of the attraction domain in Proposition 6.1 and $\varphi(\alpha)$ is defined in Propositions 6.2. More precisely, we have straightforwardly the

Corollary 6.3 *Let us fix an exponential weight function m as in (1.22), with exponent $s \in (0, 1)$. Then for any ρ, \mathcal{E}_0, M_0 there exists C and $\alpha_5 \in [\alpha_4, 1)$ (depending on $\rho, \mathcal{E}_0, M_0, m$) such that for any $\alpha \in [\alpha_5, 1)$ and any initial datum $0 \leq f_{\text{in}} \in L^1(m^{-q_2}) \cap H^{k_2}$ satisfying*

$$f_{\text{in}} \in \mathcal{C}_{\rho, 0, \mathcal{E}_0}, \quad \|f_{\text{in}}\|_{L^1(m^{-q_2}) \cap H^{k_2}} \leq M_0,$$

the solution f associated to the rescaled equation (1.24) satisfies

$$\begin{aligned} \forall t \geq 0, \quad \|\Pi_\alpha(f_t - \bar{F}_\alpha)\|_{L^1(m^{-1})} &\leq C e^{\mu_\alpha t}, \\ \forall t \geq 0, \quad \|(\text{Id} - \Pi_\alpha)(f_t - \bar{F}_\alpha)\|_{L^1(m^{-1})} &\leq C e^{(3/2)\mu_\alpha t}. \end{aligned}$$

As a by-product of the previous propositions, we state and prove a result which provides a partial answer to the question (important from the physical viewpoint) of finding Liapunov functionals for this particles system. Let us define the required objects. We consider a fixed mass ρ and some restitution coefficient α whose range will be specified below. At initial times, non-linear effects dominate and therefore we define

$$\mathcal{H}_1(f) := H(g|M[f]) + (\mathcal{E} - \bar{\mathcal{E}}_\alpha)^2$$

where $\bar{\mathcal{E}}_\alpha = \mathcal{E}(\bar{F}_\alpha)$ is the energy of the self-similar profile corresponding to α and the mass ρ . At eventual times, linearized effects dominate. Therefore we define a quite natural candidate from the spectral study:

$$\mathcal{H}_2(f) := \|h^1\|_{L^1(m^{-1})}^2 + (1 - \alpha) \int_0^{+\infty} \|\mathcal{R}_\alpha(s) h^2\|_{L^2}^2 ds,$$

with $h^1 = \Pi_\alpha h$, $h^2 = \Pi_\alpha^\perp h$ and $h = g - \bar{G}_\alpha$.

Proposition 6.4 *There is $k_4 \in \mathbb{N}$ big enough (this value is specified in the proof) such that for any exponential weight function m as defined in (1.22), any time $t_0 \in (0, \infty)$ and any $\rho, \mathcal{E}_0, M_0 \in (0, \infty)$, there exists $\kappa_* \in (0, \infty)$ and $\alpha_6 \in [\alpha_5, 1)$ such that for any $\alpha \in [\alpha_6, 1]$ and initial datum $f_{\text{in}} \in H^{k_4} \cap L^1(m^{-1})$ satisfying*

$$f_{\text{in}} \in \mathcal{C}_{\rho, 0, \mathcal{E}_0}, \quad \|f_{\text{in}}\|_{H^{k_4} \cap L^1(m^{-1})} \leq M_0, \quad f_{\text{in}}(v) \geq M_0^{-1} e^{-M_0 |v|^8},$$

the solution g to the rescaled equation (1.24) with initial datum f_{in} is such that the functional

$$\mathcal{H}(f_t) = \mathcal{H}_1(f_t) \mathbf{1}_{\{\mathcal{H}_1(f_t) \geq \kappa_*\}} + \mathcal{H}_2(f_t) \mathbf{1}_{\{\mathcal{H}_1(f_t) \leq \kappa_*\}}$$

is decreasing for all times $t \in [0, +\infty)$. Moreover, $\mathcal{H}(f(t, \cdot))$ is strictly decreasing as long as $f(t, \cdot)$ has not reached the self-similar state \bar{F}_α .

Then the previous results can be extended to general initial data by a study of the decay of singularities in the same spirit as in [10]. Indeed one can easily prove the following decomposition result.

Lemma 6.5 *Consider $f_{\text{in}} \in L_3^1$ and the associated solution $f \in C([0, \infty); L_3^1)$ to the rescaled equation (1.24). Assume that for some constant $\rho, c_1, M_1, T \in (0, \infty)$ there holds*

$$(6.5) \quad f_{\text{in}} \in \mathcal{C}_{\rho, 0}, \quad \|f_{\text{in}}\|_{L_3^1} \leq M_1, \quad \forall t \in [0, T], \quad \mathcal{E}(f(t, \cdot)) \geq c_1.$$

Then, there are $\alpha_7 \in [\alpha_6, 1)$ and $\lambda \in (-\infty, 0)$, and for any exponential weight function m (as defined in (1.22) and any $k \in \mathbb{N}$, there exists a constant K (which

depends on ρ, c_1, M_1, k, m) such that for any $\alpha \in [\alpha_7, 1]$, we may split $f = f^S + f^R$ with

$$(6.6) \quad \forall t \in [0, T], \quad \|f^S(t, \cdot)\|_{H^k \cap L^1(m^{-1})} \leq K, \quad \|f^R(t, \cdot)\|_{L^1_3} \leq K e^{\lambda t}.$$

This last tool together with the L^1 usual stability result of the elastic Boltzmann (stating that the error between two flows grows at most exponentially) allow to conclude to the proof of point (iv) of Theorem 1.3.

6.2 Sketch of the proof

The proof is exactly the same as in [10, Section 6]. Indeed the precise form of the equation was not used in this proof, it only uses:

- on the one hand, for the perturbative argument, the spectral properties obtained before (and valid here), some uniform estimates on moments and regularity in terms of a lower bound on the energy (still valid here) and some estimates from above on the bilinear term (which are the same here);
- on the other hand, for the argument “in the large”, it uses elastic entropy - entropy production estimates (independent of our problem), and the difference of timescales between entropy production (which is $\mathcal{O}(1)$) and temperature thermalization (which is $\mathcal{O}(1 - \alpha)$), also present here.

A Appendix: Functional toolbox on the collision operator

In this first appendix we shall recall some functional results on the collision operator Q which were obtained in [10].

First let us show that the collision operator depends continuously on the inelasticity coefficient $\alpha \in [0, 1]$. Since it is an unbounded operator, this continuous dependency is expressed in the norm of the graph of the operator or in some weaker norm. We start showing that this dependency of the collision operator is Lipschitz, and even $C^{1,\eta}$ for any $\eta \in (0, 1)$, when allowing a loss (in terms of derivatives and weight) in the norm it is expressed. Let us define the formal derivative of the collision operator according to α by

$$Q'_\alpha(g, f) := \nabla_v \cdot \left(\int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} g(v_*(\alpha)) f(v(\alpha)) b |u| \left(\frac{u - |u| \sigma}{4 \alpha^2} \right) d\sigma dv_* \right)$$

or by duality

$$\langle Q'_\alpha(g, f), \psi \rangle := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} g_* f b |u| \left(\frac{|u| \sigma - u}{4} \right) \nabla \psi(v'_\alpha) d\sigma dv_* dv.$$

Proposition A.1 (See Proposition 3.1 in [10]) *Let us fix a smooth exponential weight $m = \exp(-a|v|^s)$, $a \in (0, +\infty)$, $s \in (0, 1)$. Then*

(i) *For any $k, q \in \mathbb{N}$ there exists $C \in (0, \infty)$ such that for any smooth functions f, g (say in $\mathcal{S}(\mathbb{R}^N)$) and any $\alpha \in [0, 1]$ there holds*

$$(A.1) \quad \|Q_\alpha^\pm(g, f)\|_{W_q^{k,1}(m^{-1})} \leq C_{k,m} \|f\|_{W_{q+1}^{k,1}(m^{-1})} \|g\|_{W_{q+1}^{k,1}(m^{-1})}$$

$$(A.2) \quad \|Q'_\alpha(g, f)\|_{W_q^{k,1}(m^{-1})} \leq C_{k,m} \|f\|_{W_{q+2}^{k+1,1}(m^{-1})} \|g\|_{W_{q+2}^{k+1,1}(m^{-1})}.$$

(ii) *Moreover, for any smooth functions f, g and for any $\alpha, \alpha' \in [0, 1]$, there holds*

$$(A.3) \quad \begin{aligned} & \|Q_\alpha^+(g, f) - Q_{\alpha'}^+(g, f) - (\alpha - \alpha') Q'_\alpha(g, f)\|_{W_q^{-2,1}(m^{-1})} \\ & \leq |\alpha - \alpha'|^2 \|f\|_{L_{q+3}^1(m^{-1})} \|g\|_{L_{q+3}^1(m^{-1})}. \end{aligned}$$

(iii) *As a consequence, there holds*

$$(A.4) \quad \|Q_{\alpha'}^+(g, f) - Q_\alpha^+(g, f)\|_{W_q^k(m^{-1})} \leq C |\alpha - \alpha'| \|f\|_{W_{q+3}^{2k+3,1}(m^{-1})} \|g\|_{W_{q+3}^{2k+3,1}(m^{-1})},$$

and for any $\eta \in (1, 2)$, there exists $k_\eta \in \mathbb{N}$, $q_\eta \in \mathbb{N}$ and $C_\eta \in (0, \infty)$ such that

$$(A.5) \quad \begin{aligned} & \|Q_\alpha^+(g, f) - Q_{\alpha'}^+(g, f) - (\alpha - \alpha') Q'_\alpha(g, f)\|_{L^1(m^{-1})} \\ & \leq C_\eta |\alpha - \alpha'|^\eta \|f\|_{W_{q_\eta}^{k_\eta,1}(m^{-1})} \|g\|_{W_{q_\eta}^{k_\eta,1}(m^{-1})}. \end{aligned}$$

We next state a mere (Hölder) continuity dependency on α , which is however stronger than Proposition A.1 in some sense, since it is written in the norm of the graph of the operator for one of the arguments.

Proposition A.2 (See Proposition 3.2 in [10]) *For any $\alpha, \alpha' \in (0, 1]$, and any $g \in L_1^1(m^{-1})$, $f \in W_1^{1,1}(m^{-1})$, there holds*

$$(A.6) \quad \begin{cases} \|Q_\alpha^+(g, f) - Q_{\alpha'}^+(g, f)\|_{L^1(m^{-1})} \leq \varepsilon(\alpha - \alpha') \|f\|_{W_1^{1,1}(m^{-1})} \|g\|_{L_1^1(m^{-1})}, \\ \|Q_\alpha^+(f, g) - Q_{\alpha'}^+(f, g)\|_{L^1(m^{-1})} \leq \varepsilon(\alpha - \alpha') \|f\|_{W_1^{1,1}(m^{-1})} \|g\|_{L_1^1(m^{-1})}. \end{cases}$$

where $\varepsilon(r) = C r^{\frac{1}{3+4/s}}$ for some constant C (depending only on b).

As a consequence of Propositions A.1 & A.2, together with Lemma 4.3 and Proposition 2.1, we have the

Proposition A.3 (See Proposition 5.8 in [10]) *For any $k, q \geq 0$ and for any exponential weight function m there exists some constant C such that $\forall \alpha \in (\alpha_0, 1]$*

$$\begin{aligned} (i) \quad & \|\mathcal{L}_\alpha^+\|_{W_q^{k,1}(m^{-1}) \rightarrow W_{q+1}^{k,1}(m^{-1})} \leq C, \\ (ii) \quad & \|\mathcal{L}_1^+ - \mathcal{L}_\alpha^+\|_{W_q^{k,1}(m^{-1}) \rightarrow W_{q+3}^{2k+3,1}(m^{-1})} \leq C(1 - \alpha) \\ (iii) \quad & \|\mathcal{L}_1^+ - \mathcal{L}_\alpha^+\|_{W_q^{k,1}(m^{-1}) \rightarrow W_{q+1}^{k,1}(m^{-1})} \leq \varepsilon(1 - \alpha), \end{aligned}$$

where ε is defined as previously (up to a constant).

Second we shall give some estimates on the entropy production functional associated with the elastic collision operator Q_1 . We begin with a simple consequence of Proposition A.1.

Corollary A.4 (See Corollary 3.4 in [10]) *There exists $k_0, q_0 \in \mathbb{N}$ such that for any $a_i \in (0, \infty)$ $i = 1, 2, 3$, there exists an explicit constant $C \in (0, \infty)$ such that for any function g satisfying*

$$\|g\|_{H^{k_0} \cap L^1_{q_0}} \leq a_1, \quad g \geq a_2 e^{-a_3 |v|^8},$$

there holds

$$|D_{H,\alpha}(g) - D_{H,1}(g)| \leq C(1 - \alpha),$$

where we recall that $D_{H,\alpha}$ is defined by

$$(A.7) \quad D_{H,\alpha}(g) = \frac{1}{2} \iint \int_{\mathbb{R}^{2N} \times S^{N-1}} g g_* \left(\frac{g' g'_*}{g g_*} - \log \frac{g' g'_*}{g g_*} - 1 \right) B dv dv_* d\sigma \geq 0.$$

Let us recall now two famous inequalities, namely the Csiszár-Kullback-Pinsker inequality (see [5, 7]) and the so-called entropy-entropy production inequalities (the version we present here is established in [13]) that we will use several time in the sequel.

Theorem A.5 (See Theorem 3.5 in [10]) *(i) For a given function $g \in L^1_2$, let us denote by $M[g]$ the Maxwellian function with the same mass, momentum and temperature as g . For any $0 \leq g \in L^1_2(\mathbb{R}^N)$, there holds*

$$(A.8) \quad \|g - M[g]\|_{L^1}^2 \leq 2\rho(g) \int_{\mathbb{R}^N} g \ln \frac{g}{M[g]} dv.$$

(ii) For any $\varepsilon > 0$ there exists $k_\varepsilon, q_\varepsilon \in \mathbb{N}$ and for any $A \in (0, \infty)$ there exists $C_\varepsilon = C_{\varepsilon,A} \in (0, \infty)$ such that for any $g \in H^{k_\varepsilon} \cap L^1_{q_\varepsilon}$ such that

$$g(v) \geq A^{-1} e^{-A|v|^8}, \quad \|g\|_{H^{k_\varepsilon} \cap L^1_{q_\varepsilon}} \leq A,$$

there holds

$$(A.9) \quad C_\varepsilon \rho(g)^{1-\varepsilon} \left(\int_{\mathbb{R}^N} g \ln \frac{g}{M[g]} dv \right)^{1+\varepsilon} \leq D_{H,1}(g).$$

Third and last we recall some results on the linearized elastic collision operator.

Theorem A.6 (See [11] and Proposition 5.5 & 5.7 in [10]) *Let define $\hat{\mathcal{L}}_1$ as the restriction of \mathcal{L}_1 to the space $L^1_1 \cap \mathcal{C}_{0,0}$. Then for any $k, q \geq 0$ and for any exponential weight function m the following holds.*

(i) The regularized truncation $\mathcal{L}_{1,\delta}^+$ of the “gain” part (see (5.4) and [11]) satisfies

$$\|\mathcal{L}_1^+ - \mathcal{L}_{1,\delta}^+\|_{W_q^{k,1}(m^{-1}) \rightarrow W_{q+1}^{k,1}(m^{-1})} \leq \varepsilon(\delta)$$

where $\varepsilon(\delta) > 0$ is an explicit constant going to 0 as δ goes to 0. For any $\delta > 0$, the linear operator A_δ (see (5.3)) satisfies $A_\delta : L^1 \rightarrow W_{\infty}^{\infty,1}(m^{-1})$ is bounded.

(ii) The unbounded operator $\hat{\mathcal{L}}_1$ is well defined and close in $\mathbb{W}_q^{k,1}(m^{-1})$. Its spectrum $\Sigma(\hat{\mathcal{L}}_1)$ is real and satisfies

$$\Sigma(\hat{\mathcal{L}}_1) \subset \Delta_{\mu_2}^c \cup \{\mu_1\},$$

with $\mu_2 < 0$ and $\mu_1 = 0$ is the 1-dimensional “energy eigenvalue” associated to the “energy eigenfunction” $\phi_1 := c_0(|v|^2 - N\bar{\theta}_1)\bar{F}_1$. In particular, $\hat{\mathcal{L}}_1$ is onto from $\mathcal{O} \cap \mathbb{L}_1^1(m^{-1})$ onto \mathcal{O} .

(iii) The resolvent $\mathcal{R}_1(\xi)$ has a sectorial property for the spectrum subtracted from the “energy” eigenvalue, namely there is a constructive $\lambda \in (\mu_2, 0)$ such that

$$\forall \xi \in \mathcal{A}, \quad \|\mathcal{R}_1(\xi)\|_{\mathbb{W}_q^{k,1}(m^{-1})} \leq a_{k,q} + \frac{b_{k,q}}{|\xi + \lambda|},$$

with some explicit constant $a_{k,q}, b_{k,q} > 0$ and

$$\mathcal{A} = \left\{ \xi \in \mathbb{C}, \quad \arg(\xi + \lambda) \in \left[-\frac{3\pi}{4}, \frac{3\pi}{4} \right] \quad \text{and} \quad \Re \xi \leq \frac{\lambda}{2} \right\}.$$

A quite simple consequence of point (ii) above is the following quantitative invertibility result on a modified version of the linearized elastic collision operator.

Lemma A.7 (See Lemma 4.3 in [10]) *The linear operator $A : \mathbb{L}_1^1(m^{-1}) \rightarrow \mathbb{R} \times \mathcal{O}$ defined thanks to (4.2) is invertible: it is bijective with A^{-1} bounded with explicit estimate.*

B Appendix: Moments of Gaussians

We state here some results on the moments of tensor product of Gaussians (the proof is done in the appendix of [10]).

Lemma B.1 *The following identities hold*

$$(B.1) \quad \int_{\mathbb{R}^N} M_{1,0,1} |v|^2 dv = N,$$

$$(B.2) \quad \int_{\mathbb{R}^N} M_{1,0,1} |v|^4 dv = N(N+2),$$

$$(B.3) \quad \int_{\mathbb{R}^N \times \mathbb{R}^N} M_{1,0,1} (M_{1,0,1})_* |u|^3 dv dv_* = 2^{3/2} \int_{\mathbb{R}^N} M_{1,0,1} |v|^3 dv,$$

$$(B.4) \quad \int_{\mathbb{R}^N \times \mathbb{R}^N} M_{1,0,1} (M_{1,0,1})_* |v|^2 |u|^3 dv dv_* = \sqrt{2} (2N+3) \int_{\mathbb{R}^N} M_{1,0,1}(v) |v|^3 dv.$$

C Appendix: Interpolation inequalities

Again the proof of the following simple interpolation inequality can be found in the appendix of [10].

Lemma C.1 (i) For any $k, k^*, q, q^* \in \mathbb{Z}$ with $k \geq k^*$, $q \geq q^*$ and any $\theta \in (0, 1)$ there is $C \in (0, \infty)$ such that for $h \in W_{q^{**}}^{k^{**}, 1}(m^{-1})$

$$(C.1) \quad \|h\|_{W_q^{k, 1}(m^{-1})} \leq C \|h\|_{W_{q^*}^{k^*, 1}(m^{-1})}^{1-\theta} \|h\|_{W_{q^{**}}^{k^{**}, 1}(m^{-1})}^\theta.$$

with $k^{**}, q^{**} \in \mathbb{Z}$ such that $k = (1 - \theta)k^* + \theta k^{**}$, $q = (1 - \theta)q^* + \theta q^{**}$.

(ii) For any $k, q \in N^*$ and any exponential weight function m as defined in (1.22), there exists $C \in (0, \infty)$ such that for any $h \in H^{k^\dagger} \cap L^1(m^{-12})$ with $k^\dagger := 8k + 7(1 + N/2)$

$$(C.2) \quad \|h\|_{W_q^{k, 1}(m^{-1})} \leq C \|h\|_{H^{k^\dagger}}^{1/4} \|h\|_{L^1(m^{-12})}^{1/4} \|h\|_{L^1(m^{-1})}^{3/4}.$$

D Appendix: Lower bound for the diffusive inelastic Boltzmann equation

In this last appendix, we state and prove a technical result on the pointwise lower bound of the diffusive Boltzmann equation (the new difficulty as compared to previous results is due to the diffusion and not transport nature of the term added to the collision operator).

Proposition D.1 Let $g \in C([0, \infty); L_3^1)$ be a solution of the rescaled evolution equation

$$\partial_t g = Q(g, g) + \tau_\alpha \Delta_v g,$$

with inelasticity parameter $\alpha \in (0, 1)$, and assume that for some $C, T \in (0, \infty)$

$$\sup_{[0, T]} \|g\|_{L^2 \cap L_3^1} \leq C.$$

For any $t_1 \in (0, T)$ there exists $a_1 \in (0, \infty)$ (depending on C , ρ and t_1 but not on T) such that

$$(D.1) \quad \forall t \in [t_1, T], \quad \forall v \in \mathbb{R}^N, \quad g(t, v) \geq a_1^{-1} e^{-a_1 |v|^8}.$$

We closely follow the proof of the Maxwellian lower bound for the solutions of the elastic Boltzmann equation (see [3, 12]) taking advantage of some technical results established in its extension to the solutions of the inelastic Boltzmann equation (see [9, Theorem 4.9] and [10, Lemma 2.6]). The starting point is again the evolution equation satisfied by g written in the form

$$(D.2) \quad \partial_t g - \tau_\alpha \Delta_v g + \lambda(v) g = Q_\alpha^+(g, g) + (\lambda(v) - L(g)) g, \quad \lambda(v) = \kappa(1 + |v|),$$

where the last term in the right hand side term is clearly non-negative for some well-chosen numerical constant $\kappa \in (0, \infty)$. Let us introduce the semigroup S_t associated to the linear evolution equation $\partial_t g - \tau_\alpha \Delta_v g + \lambda(v) g = 0$.

We start establishing some technical results that we need in the proof of Proposition D.1. Let us recall some elementary results extracted from [9] (which are a mere adaptation to the inelastic collision of some result proved in [12]).

Lemma D.2 [9, Lemma 4.6] *Let $0 \leq \ell \in (L_2^1 \cap L^2)(\mathbb{R}^N)$ satisfy*

$$\int_{\mathbb{R}^N} \ell dv = m_0, \quad \int_{\mathbb{R}^N} \ell |v|^2 dv \leq m_1, \quad \int_{\mathbb{R}^N} \ell^2 dv \leq m_2,$$

for some positive real constants m_i . There exists $R > r > 0$ and $\eta > 0$ depending only on m_0, m_1, m_2 , and $(v_i)_{i=1, \dots, 4}$ such that $|v_i| \leq R$, $i = 1, \dots, 4$, $|v_i - v_j| \geq 3r$ for $1 \leq i \neq j \leq 3$, and

$$(D.3) \quad \int_{B(v_i, r)} \ell(v) dv \geq \eta \quad \text{for } i = 1, 2, 3,$$

$$(D.4) \quad \forall w_i \in B(v_i, r), \quad E_{w_3, w_4}^\alpha \cap S_{w_1, w_2}^\alpha \text{ is a sphere of radius larger than } r,$$

where, as in [9, Proposition 1.5], E_{w_3, w_4}^α stands for the hyperplan orthogonal to the vector $w_3 - w_4$ and passing through the point $\Omega(w_3, w_4)$, defined by

$$\Omega(w_3, w_4) := w_3 + (1 - \alpha)(w_3 - w_4)/(1 + \alpha),$$

and S_{w_1, w_2}^α stands for the sphere of all possibles post-collisional velocity v' defined by (1.4) from $(v = w_3, v_ = w_4, \sigma)$.*

Lemma D.3 *Let $0 \leq f, g, h \in L^1(\mathbb{R}^N)$ and $(v_i)_{i=1, \dots, 4}$ satisfy (D.3), (D.4) and $|v_i| \leq R$, $i = 1, \dots, 4$, $|v_i - v_j| \geq 3r$ for $1 \leq i \neq j \leq 3$, for some given constants $R > r > 0$, $\eta > 0$. There exists $T'_0 > 0$, $\delta'_0 > 0$ and $\eta'_0 > 0$ only depending on R , r and η such that*

$$\forall t' \in [0, T_0] \quad Q^+(f, S_{t'} Q^+(g, h)) \geq \eta'_0 \mathbf{1}_{B(v_3, \delta'_0)}.$$

Proof of Lemma D.3. We first recall a convenient formula to handle representations of the iterated gain term. Using Carleman representation [9, Proposition 1.5], for any f, h and ℓ and any $v \in \mathbb{R}^N$ there holds (setting $'v = w$ and $'v_* = w_*$)

$$\begin{aligned} Q^+(f, S_t Q^+(h, \ell))(v) &= C'_b \int_{\mathbb{R}^N} \frac{f(w)}{|v - w|} \left\{ \int_{E_{v, w}} S_t Q^+(g, h)(w_*) dw_* \right\} dw \\ &= \int_{\mathbb{R}^{3N}} f(w) g(z) h(z_*) \left\{ C'_b \frac{|z_* - z|}{|v - w|} \int_{S^{N-1}} (S_t \mathbf{1}_{E_{v, w}^\alpha})(z') b(\sigma \cdot \hat{z}) d\sigma \right\} dz_* dz dw, \end{aligned}$$

where $E_{v, w}^\alpha$ is defined in the statement of Lemma D.2 and z' is defined from (1.4) with $(v = z, v_* = z_*, \sigma)$. Let us define $\bar{f} := f \mathbf{1}_{B(v_4, r)}$, $\bar{g} = g \mathbf{1}_{B(v_1, r)}$, $\bar{h} = h \mathbf{1}_{B(v_2, r)}$

and \tilde{S}_t the semigroup associated to evolution equation $\partial_t g - \tau_\alpha \Delta_v g - \lambda(2R)g = 0$. By the maximum principle, we have for any $v \in B(v_3, r)$

$$Q^+(f, S_t Q^+(h, \ell))(v) \geq \int_{\mathbb{R}^{3N}} \bar{f}(w) \bar{g}(z) \bar{h}(z_*) \left\{ C_b'' \frac{r}{R} \int_{S^{N-1}} (\tilde{S}_t \chi)(z') d\sigma \right\} dz_* dz dw.$$

Taking $v \in B(v_3, r)$, $w \in B(v_4, r)$, $z \in B(v_1, r)$, $z_* \in B(v_2, r)$ and denoting by A the term between brackets, we have thanks to (D.4)

$$A(v, w, z, z_*) := C_b'' \frac{r}{R} \int_{S^{N-1}} \int_{\mathbb{R}^N} \chi(z' - u) e^{-\frac{|u|^2}{2\tau_\alpha t}} d\sigma du \geq \frac{r}{R} C_b C r^{N-2} \frac{1}{2},$$

for any $\alpha \in [0, 1]$ and $t \in [0, T_0]$ with T_0 small enough. We then conclude the proof. \square

For a given $\mu \in (0, 1)$ we consider a function $\chi \in W^{2,\infty}(\mathbb{R}_+)$, which satisfies the following properties: $\chi \geq 0$, $\chi' \leq 0$, $\chi \equiv 1$ on $[0, \mu]$ and $\chi(x) = (1-x)^2$ for any $x \in [\mu', 1]$ with $\mu' := \max\{1-1/N, (1+\mu)/2\}$. Abusing notations we define the radial symmetric functions χ and χ_δ on \mathbb{R}^N by setting $\chi(v) = \chi(|v|)$ and $\chi_\delta(v) = \chi(|v|/\delta)$ for any $\delta > 0$.

Lemma D.4 *For any $\delta_0 > 0$, there exists $C \in (0, \infty)$ such that*

$$\forall \alpha \in (0, 1), \quad \forall R > 0, \quad \forall \bar{v} \in B(0, R), \quad \forall \delta > \delta_0, \quad \forall t \geq 0, \quad S_t(\tau_{\bar{v}} \chi_\delta) \geq e^{-C(1+R+\delta)t} (\tau_{\bar{v}} \chi_\delta).$$

Proof of Lemma D.4. We shall rely on a maximum principle argument for the operator

$$Tf := \tau_\alpha \Delta_v f - \lambda(v) f.$$

First it can be seen easily that if f_0 is a non-negative function and $f = f(t, v)$ satisfies the differential inequality

$$\partial_t f \geq Tf, \quad t \in [0, T],$$

then f_t remains non-negative on this time interval $[0, T]$.

Second let us show that

$$\phi(t, v) := e^{-at} \tau_{\bar{v}} \chi_\delta$$

is a sub-solution for $a = a_{\delta, R}$ big enough, on a small initial time interval $[0, T]$, in the sense that

$$\partial_t \phi \leq T\phi, \quad t \in [0, T].$$

After elementary computations and taking advantage of the fact that χ is radially symmetric, it amounts to show that

$$(D.5) \quad [a - \lambda(\delta w + \bar{v})] \chi + \frac{\tau_\alpha}{\delta^2} \left[\chi'' + \frac{(N-1)}{|w|} \chi' \right] \geq 0 \quad \text{on } \mathbb{R}^N$$

for a big enough. From the definition of χ , it is zero outside of $B(0, 1)$ and therefore equation (D.5) holds on $\mathbb{R}^N \setminus B(0, 1)$. Observing that

$$\chi''(x) + \frac{(N-1)}{x} \chi'(x) \geq 0$$

for any $x \in [\mu', 1]$, we deduce that equation (D.5) holds on $B(0, 1) \setminus B(0, \mu')$ as soon as $a \geq \kappa(1 + \delta + R)$. Finally, using that χ is decreasing and bounded in $W^{2,\infty}(0, 1)$, we see that equation (D.5) holds on $B(0, \mu')$ as soon as

$$[a - \kappa(1 + \delta + R)] \chi(\mu') \geq \frac{\rho}{\delta^2} \left\| \chi''(x) + \frac{(N-1)}{x} \chi'(x) \right\|_{L^\infty(0,1)},$$

and therefore as soon as $a \geq C(1 + \delta + R)$ for some constant $C \in (0, \infty)$ only depending on χ , δ_0 , ρ and N . \square

Proof of Proposition D.1. We split the proof into two steps.

Step 1. Thanks to the Duhamel formula a solution g to the evolution equation (D.2) satisfies

$$(D.6) \quad g(t, \cdot) \geq S_t g(0, \cdot) + \int_0^t S_{t-s} Q^+(g(s, \cdot), g(s, \cdot)) ds.$$

Let us fix $t_0 > 0$ and define $\tilde{g}_0(t, \cdot) := g(t_0 + t, \cdot)$. Using twice the inequality (D.6), we find

$$(D.7) \quad \tilde{g}_0(t, \cdot) \geq \int_0^t \int_0^s S_{t-s} Q^+(S_s \tilde{g}_0, S_{s-s'} Q^+(S_{s'} \tilde{g}_0, S_{s'} \tilde{g}_0)) ds' ds.$$

We first apply Lemma D.3 to $\ell = S_{s'} \tilde{g}_0$ with $\tau = s - s'$ and we next use Lemma D.4 to obtain for any $t \in [0, T_0]$, with T_0 given by Lemma D.3

$$(D.8) \quad \tilde{g}_0(t, \cdot) \geq \int_0^t \int_0^s S_{t-s} \eta \mathbf{1}_{B(\bar{v}, r)} ds' ds \geq \eta \int_0^t e^{-a(1-(t-s))} \tau_{\bar{v}} \chi_r s ds.$$

We have then proved that there exists $T_1 > 0$ and $\bar{v} \in B(0, R)$ and for any $t_1 \in (0, T_1/2]$ there exists $\eta_1 > 0$ such that

$$\forall t \in [0, T_1/2], \quad \tilde{g}_1(t, \cdot) := \tilde{g}_0(t + t_1, \cdot) \geq \eta_1 \tau_{\bar{v}} \chi_{\delta_1} =: \eta_1 \bar{\chi}_{\delta_1}.$$

Step 2. Using again the Duhamel formula (D.6) and the preceding step we have

$$\tilde{g}_1(t, \cdot) \geq \int_0^t S_{t-s} Q^+(\tilde{g}_1(s, \cdot), \tilde{g}_1(s, \cdot)) ds.$$

Now let recall, that on the one hand, from [9, Lemma 4.8], there exists $\kappa \in (0, \infty)$ such that

$$Q_\alpha^+(\mathbf{1}_{B(0,1)}, \mathbf{1}_{B(0,1)}) \geq \kappa' \mathbf{1}_{B(0, \sqrt{5}/2)}$$

and that, on the other hand, the scaling properties of the Q_α^+ term infers that

$$Q_\alpha^+(\phi(\cdot/\delta), \phi(\cdot/\delta))(v) = \delta^{-N-1} Q_\alpha^+(\phi, \phi)(v/\delta)$$

for any function ϕ and scaling coefficient $\delta > 0$. Then, thanks to Lemma D.4, there holds

$$\begin{aligned} \tilde{g}_1(t, \cdot) &\geq \eta_1^2 \int_0^t S_{t-s} Q^+(\bar{\chi}_{\delta_1}, \bar{\chi}_{\delta_1}) ds \\ &\geq \eta_1^2 b e^{-a(1+R+\delta_1)t} \delta_1^{-N-1} \bar{\chi}_{\theta \delta_1} \end{aligned}$$

on $[0, T_2]$ with $T_2 \in (0, T_1/2]$, $\theta \in (1, \sqrt{5}/2)$ as close as we wish to $\sqrt{5}/2$ by choosing μ close to 1 and b a numerical constant (depending on κ', χ). Defining $t_k = t_{k-1} + t_1 2^{-k}$ and repeating the precedent computation we see that

$$g(t, \cdot) \geq \eta_k \bar{\chi}_{\delta_k} \quad \forall t \in [t_k, T],$$

with $\delta_k = \theta^k \delta_1$ and

$$\eta_{k+1} = \eta_k^2 b e^{-a(1+R+\theta^k \delta_1)t_1 2^{-k}} \frac{t_1}{2^k} \delta_1^{-N-1} (\theta^{-N-1})^k = \eta_k^2 A B^k,$$

for some constants $A, B \in (0, \infty)$. Elementary computations yields to

$$\eta_k \geq \eta_1^{2^k} A^{1+2+\dots+2^{k-1}} B^{k+(k-1)2+\dots+2^{k-1}} \geq D^{2^k},$$

for some constant $D \in (0, \infty)$. As a conclusion, using that $\theta^8 > 2$, we have proved

$$\forall t \geq t_1, \forall k \in \mathbb{N}, \quad g(t, v) \geq D^{2^k} \mathbf{1}_{B(\bar{v}, 2^{k/8}(\delta_1/2))}(v),$$

from which we easily conclude. \square

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