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Adaptive estimators for nonparametric heteroscedastic regression models

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Abstract

This paper deals with the estimation of a regression function at a given point in nonparametric heteroscedastic regression models with Gaussian noise. An adaptive kernel estimator which attains the minimax rate is constructed for the minimax absolute error risk.

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1 Introduction

A lot of research studies are devoted to the problem of estimating a regression function and specially a function belonging to a Hölder class. In this direction it is known from [24] that the linear minimax estimator is a kernel estimator in the case of a quasi-Hölder regression function estimated at a single point with squared error loss. This estimator is within 17 percent of asymptotically minimax over all procedures (see [7]). Furthermore it is proved in [18] and in [4] that a kernel estimator is asymptotically efficient when the Hölder regression function or its k th derivative is estimated with the sup-norm global loss. The reader is referred for instance to [15], [4], [6] or [8] for other regression works.

Our regression problem is the following. Suppose we observe data from:

$$y_k = S(x_k) + g(x_k, S)\xi_k, \quad k \in \{1, \dots, n\}, \quad (1)$$

where $x_k = k/n$, $(\xi_k)_{k \in \{1, \dots, n\}}$ are independent identically distributed standard Gaussian random variables. We are interested in the estimation of the regression function S at a given point $z_0 \in]0; 1[$. We point out the fact that in this heteroscedastic regression model the variance of the noises depends on the unknown function S and on the regressors x_k . This kind of model is used in financial analysis or in medical research (see, e.g., [14]).

We assume that the regression function belongs to a Hölder class but its smoothness parameter β remains unknown. We consider the absolute error loss and for the corresponding risk, we aim at constructing an adaptive estimator which attains the minimax rate. Because of the adaptation, this rate differs from the one in the case where β is known. Many papers deal with adaptive problems, see for instance [19]–[21], [1] or [12]. Our construction is based on those one can find in [19] and in [11] for adaptive estimation of the drift coefficient in diffusion processes. We proceed with the method developed in [13] in the homoscedastic and non adaptive case. In this paper the risk of an estimator is defined as the supremum of the absolute error loss taken over a neighborhood (called weak Hölder class) of functions that allows an arbitrary large derivative but has an additional weak Hölder condition (see (2)). It has led to the heteroscedastic case studied in [3]. In these cases it is shown that a kernel estimator is asymptotically efficient.

The paper is laid out as follows. Section 2 gives the description of the problem with all assumptions needed and all definitions of necessary mathematical objects. In section 3 we construct an adaptive estimator for which an upper bound of the risk is found. The lower bound of the minimax risk is given in section 4. Appendix A contains technical results for our proofs.

2 Statement of the problem

Consider model (1) where $S \in C^1([0; 1], \mathbb{R})$ and $g : [0; 1] \times C^1([0; 1], \mathbb{R}) \longrightarrow \mathbb{R}_+^*$ are unknown functions. We want to estimate the regression function S at a fixed point $z_0 \in]0; 1[$. Assume that

$$S \in \bigcup_{M, K > 0} \mathcal{H}(M, K, \beta) = \mathcal{H}(\beta), \text{ où } \beta \in [\beta_*; \beta^*] \subset]1; 2[,$$

$$\mathcal{H}(M, K, \beta) = \left\{ S \in C^1[0; 1] : \|S'\| \leq M, \sup_{x, y \in [0; 1]} \frac{|S'(y) - S'(x)|}{|x - y|^\alpha} \leq K \right\},$$

with $\beta = 1 + \alpha$, $\|f\| = \sup_{x \in [0;1]} |f(x)|$.

The smoothness parameter β is supposed to be unknown whereas the interval $[\beta_\star; \beta^\star]$ is considered as known.

In order to measure the risk taken by using an estimator of $S(z_0)$ we introduce the local weak Hölder class at the point z_0 for the true value of the parameter β :

$$\mathcal{U}_{z_0, \delta, n} = \left\{ S \in \mathcal{H}(\beta) : \|S'\| \leq \delta^{-1}; \forall h \geq 0, \left| \int_{-1}^1 (S(z_0 + hu) - S(z_0)) du \right| \leq \delta h^\beta \right\}, \quad (2)$$

where $\delta \in]0; 1[$, $n \geq 1$. Notice that

$$\int_{-1}^1 (S(z_0 + hu) - S(z_0)) du = \int_{-1}^1 \left(\int_{z_0}^{z_0+uh} (S'(t) - S'(z_0)) dt \right) du, \quad (3)$$

so we have for all $S \in \mathcal{H}(M, K, \beta)$

$$\left| \int_{-1}^1 (S(z_0 + hu) - S(z_0)) du \right| \leq \frac{2K}{\beta(\beta+1)} h^\beta.$$

That is why the class $\mathcal{U}_{z_0, \delta, n}$ is called a weak Hölder class.

The risk of an estimator \hat{S} of $S(z_0)$ is defined over the neighborhood $\mathcal{U}_{z_0, \delta, n}$ by

$$\mathcal{R}_{z_0, \delta, n}(\hat{S}) = \sup_{\beta \in [\beta_\star; \beta^\star]} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \frac{N(\beta)}{g(z_0, S)} \mathbb{E}_S |\hat{S} - S(z_0)|, \quad \text{where } N(\beta) = \left(\frac{n}{\ln n} \right)^{\beta/(2\beta+1)}.$$

We assume that there exists two known constants $g_\star > 0$ and $g^\star < \infty$ such that

$$g_\star \leq \inf_{0 \leq x \leq 1} \inf_{S \in C^1([0;1])} g(x, S) \leq \sup_{0 \leq x \leq 1} \sup_{S \in C^1([0;1])} g(x, S) \leq g^\star.$$

Moreover we suppose that the function g is differentiable in the Frechet sense with respect to S in $C^1([0; 1])$ uniformly over $x \in [0; 1]$, i.e. for any $S, S_0 \in C^1([0; 1])$

$$g(x, S) = g(x, S_0) + L_{x, S_0}(S - S_0) + \Gamma_{x, S_0}(S - S_0), \quad (4)$$

where the linear operator L_{x, S_0} is bounded uniformly over $x \in [0; 1]$ in $C^1([0; 1])$, i.e. for any $S_0 \in C^1([0; 1])$ there exists some positive constant C_{S_0} such that

$$\sup_{x \in [0;1]} \sup_{S \in C^1([0;1]), S \neq 0} \frac{|L_{x, S_0}(S)|}{\|S\|} \leq C_{S_0} \quad (5)$$

and the residual term $\Gamma_{x, S_0}(S)$ satisfies the property

$$\lim_{\|S\| \rightarrow 0} \sup_{x \in [0;1]} \frac{\Gamma_{x, S_0}(S)}{\|S\|} = 0. \quad (6)$$

3 Upper bound

The paper [13] handles the homoscedastic non adaptive case, considering the kernel estimator

$$S_h^*(z_0) = \frac{1}{q_n(h)} \sum_{k=1}^n Q\left(\frac{x_k - z_0}{h}\right) y_k,$$

where $Q = \mathbb{I}_{[-1,1]}$, $h = n^{-1/(2\beta+1)}$ and $q_n(h) = \sum_{k=1}^n Q\left(\frac{x_k - z_0}{h}\right)$. Taking into account the fact that β is unknown we can not use such an estimator because the bandwidth h depends on β . That is the reason why we create a partition of the interval $[\beta_*, \beta^*]$ in the following way:

$$\beta_l = \beta_* + l \frac{\beta^* - \beta_*}{\ln n}, \quad l = 0, \dots, [\ln n],$$

where $[a]$ denotes the integral part of a number a , and we define the corresponding bandwidths $h_l = h(\beta_l) = \left(\frac{n}{\ln n}\right)^{-1/(2\beta_l+1)}$. Then we set

$$\hat{l} = \max \left\{ 0 \leq l \leq [\ln n] : \max_{0 \leq j \leq l} \left(\left| S_{h_l}^*(z_0) - S_{h_j}^*(z_0) \right| - \frac{\lambda}{N_j} \right) \leq 0 \right\},$$

where $N_j = N(\beta_j)$ and $\lambda = 2 + 2\sqrt{2}g^* \left(\frac{\beta^*}{2\beta^* + 1} - \frac{\beta_*}{2\beta_* + 1} \right)^{1/2}$.

Notice that \hat{l} really exists because the set above contains the index 0.

The adaptive estimator is now defined as $\hat{S}_n = S_{h_{\hat{l}}}^*(z_0)$. Furthermore we associate with the unknown parameter β the unique integer $l(\beta) \in \{0, \dots, [\ln n] - 1\}$ such that $\beta_{l(\beta)} \leq \beta < \beta_{l(\beta)+1}$.

The following result gives an upper bound for the risk of the adaptive estimator.

Theorem 3.1 *One has*

$$\limsup_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mathcal{R}_{z_0, \delta, n}(\hat{S}_n) \leq \lambda \frac{e^{\beta^* - \beta_*}}{g_*}.$$

PROOF: Fix $\delta \in]0; 1[$ and write

$$|\hat{S}_n - S(z_0)| = |\hat{S}_n - S(z_0)| \mathbb{I}_{\{\hat{l} \geq l(\beta)\}} + |\hat{S}_n - S(z_0)| \mathbb{I}_{\{\hat{l} < l(\beta)\}} =: I_1 + I_2,$$

where $I_1 = \sum_{j=l(\beta)}^{[\ln n]} |\hat{S}_n - S(z_0)| \mathbb{I}_{\{\hat{l}=j\}} = \sum_{j=l(\beta)}^{[\ln n]} |S_{h_j}^*(z_0) - S(z_0)| \mathbb{I}_{\{\hat{l}=j\}}$.

For all $j = 0, \dots, [\ln n]$, we note $S_{h_j}^*(z_0) - S(z_0) = \frac{\zeta_n(\beta_j)}{\sqrt{q_n(\beta_j)}} + B_n(\beta_j)$, with

$$\begin{aligned} q_n(\beta_j) &= q_n(h(\beta_j)), \\ \zeta_n(\beta_j) &= \frac{1}{\sqrt{q_n(\beta_j)}} \sum_{k=1}^n Q\left(\frac{x_k - z_0}{h_j}\right) g(x_k, S) \xi_k, \\ B_n(\beta_j) &= \frac{1}{q_n(\beta_j)} \sum_{k=1}^n Q\left(\frac{x_k - z_0}{h_j}\right) (S(x_k) - S(z_0)). \end{aligned}$$

The variance of $\zeta_n(\beta_j)$ is then $\sigma_n^2(\beta_j, S) := \frac{1}{q_n(\beta_j)} \sum_{k=1}^n Q\left(\frac{x_k - z_0}{h_j}\right) g^2(x_k, S)$.

One has $\left| \frac{\zeta_n(\beta_j)}{\sqrt{q_n(\beta_j)}} \right| = |Z_n(\beta_j)| \frac{\sigma_n(\beta_j, S)}{\sqrt{q_n(\beta_j)}}$, with $Z_n(\beta_j) \sim \mathcal{N}(0, 1)$.

In addition, denoting $u_k = \frac{x_k - z_0}{h_j}$ and $\Delta u_k = \frac{1}{nh_j}$, one has

$$\begin{aligned} B_n(\beta_j) &= \frac{1}{q_n(\beta_j)} \sum_{k=1}^n Q\left(\frac{x_k - z_0}{h_j}\right) (S(x_k) - S(z_0)) \\ &= \frac{nh_j}{q_n(\beta_j)} \sum_{k=1}^n Q(u_k) (S(z_0 + h_j u_k) - S(z_0)) \Delta u_k \\ &= \frac{nh_j}{q_n(\beta_j)} \int_{-1}^1 (S(z_0 + h_j u) - S(z_0)) du + \frac{nh_j}{q_n(\beta_j)} R_n(\beta_j) \end{aligned}$$

where

$$\begin{aligned} R_n(\beta_j) &= \sum_{k=1}^n Q(u_k) (S(z_0 + h_j u_k) - S(z_0)) \Delta u_k - \int_{-1}^1 (S(z_0 + h_j u) - S(z_0)) du \\ &= \sum_{k=k_*}^{k^*} \int_{u_{k-1}}^{u_k} (S(z_0 + h_j u_k) - S(z_0 + h_j u)) du \\ &\quad - \int_{u_{k_*}}^1 (S(z_0 + h_j u) - S(z_0)) du + \int_{u_{k_*-1}}^{-1} (S(z_0 + h_j u) - S(z_0)) du, \end{aligned}$$

and $k^* = [n(z_0 + h_j)]$ et $k_* = [n(z_0 - h_j)] + 1$.

As the derivative of any function S in $\mathcal{U}_{z_0, \delta, n}$ is bounded by δ^{-1} , we get

$$\begin{aligned} |R_n(\beta_j)| &\leq \sum_{k=k_*}^{k^*} \int_{u_{k-1}}^{u_k} h_j (u_k - u) \delta^{-1} du + \int_{u_{k_*}}^1 h_j \delta^{-1} u du + \int_{u_{k_*-1}}^{-1} h_j \delta^{-1} |u| du \\ &\leq h_j \delta^{-1} \left(\sum_{k=k_*}^{k^*} (u_k - u_{k-1}) \frac{1}{nh_j} + (1 - u_{k_*}) + 2(-1 - u_{k_*-1}) \right) \leq \frac{6\delta^{-1}}{n}. \end{aligned}$$

Using the weak Hölder condition we obtain $|B_n(\beta_j)| \leq \delta \frac{nh_j^{1+\beta}}{q_n(\beta_j)} + 6\delta^{-1} \frac{h_j}{q_n(\beta_j)}$, and then

$$|S_{h_j}^*(z_0) - S(z_0)| \leq \delta \frac{nh_j^{1+\beta}}{q_n(\beta_j)} + 6\delta^{-1} \frac{h_j}{q_n(\beta_j)} + |Z_n(\beta_j)| \frac{\sigma_n(\beta_j, S)}{\sqrt{q_n(\beta_j)}}. \quad (7)$$

Hence

$$\begin{aligned}
I_1 &\leq \sum_{j=l(\beta)}^{\lfloor \ln n \rfloor} \left(|S_{h_j}^*(z_0) - S_{h_{l(\beta)}}^*(z_0)| \mathbb{I}_{\{\hat{l}=j\}} \right) + |S_{h_{l(\beta)}}^*(z_0) - S(z_0)| \mathbb{I}_{\{\hat{l} \geq l(\beta)\}} \\
&\leq \sum_{j=l(\beta)}^{\lfloor \ln n \rfloor} \frac{\lambda}{N_j} \mathbb{I}_{\{\hat{l}=j\}} + |S_{h_{l(\beta)}}^*(z_0) - S(z_0)| \\
&\leq \frac{\lambda}{N_{l(\beta)}} + |S_{h_{l(\beta)}}^*(z_0) - S(z_0)| \\
&\leq \frac{\lambda}{N(\beta)} e^{\beta^* - \beta_*} + \delta \frac{nh(\beta_{l(\beta)})^{1+\beta}}{q_n(\beta_{l(\beta)})} + 6\delta^{-1} \frac{h(\beta_{l(\beta)})}{q_n(\beta_{l(\beta)})} + |Z_n(\beta_{l(\beta)})| \frac{\sigma_n(\beta_{l(\beta)}, S)}{\sqrt{q_n(\beta_{l(\beta)})}}.
\end{aligned}$$

Let us show that

$$\limsup_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{\beta \in [\beta_*, \beta^*]} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \frac{N(\beta)}{g(z_0, S)} \mathbb{E}_S I_1 \leq \lambda \frac{e^{\beta^* - \beta_*}}{g_*}. \quad (8)$$

As $N(\beta)h(\beta_{l(\beta)})^\beta \leq 1$ and $q_n(\beta_{l(\beta)}) \sim 2nh(\beta_{l(\beta)})$, we have

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{\beta \in [\beta_*, \beta^*]} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \delta \frac{N(\beta)}{g(z_0, S)} \frac{nh(\beta_{l(\beta)})^{1+\beta}}{q_n(\beta_{l(\beta)})} = 0. \quad (9)$$

Moreover it is easy to see that

$$\limsup_{n \rightarrow \infty} \sup_{\beta \in [\beta_*, \beta^*]} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \frac{N(\beta)}{g(z_0, S)} 6\delta^{-1} \frac{h(\beta_{l(\beta)})}{q_n(\beta_{l(\beta)})} = 0. \quad (10)$$

Since $\sigma_n(\beta_{l(\beta)}, S)$ is bounded by g^* and

$$\frac{N(\beta)^2}{q_n(\beta_{l(\beta)})} = \frac{nh(\beta_{l(\beta)})}{q_n(\beta_{l(\beta)})} \left(\frac{n}{\ln n} \right)^{2(\beta - \beta_{l(\beta)}) / (2\beta + 1)(2\beta_{l(\beta)} + 1)} \frac{1}{\ln n} \leq \frac{nh(\beta_{l(\beta)})}{q_n(\beta_{l(\beta)})} e^{2(\beta^* - \beta_*)} \frac{1}{\ln n},$$

one has

$$\limsup_{n \rightarrow \infty} \sup_{\beta \in [\beta_*, \beta^*]} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \frac{N(\beta)}{g(z_0, S)} \mathbb{E}_S |Z_n(\beta_{l(\beta)})| \frac{\sigma_n(\beta_{l(\beta)}, S)}{\sqrt{q_n(\beta_{l(\beta)})}} = 0. \quad (11)$$

Finally from (9) – (11), we get (8).

Now recall that $I_2 = |\hat{S}_n - S(z_0)| \mathbb{I}_{\{\hat{l} < l(\beta)\}}$ and let us prove that

$$\lim_{n \rightarrow \infty} \sup_{\beta \in [\beta_*, \beta^*]} \sup_{S \in \mathcal{U}_{z_0, \delta, n}} \frac{N(\beta)}{g(z_0, S)} \mathbb{E}_S I_2 = 0. \quad (12)$$

We have $\{\hat{l} < l(\beta)\} = \bigcup_{j=0}^{l(\beta)-1} \{\hat{l} = j\}$ and by definition of \hat{l} ,

$$\{\hat{l} = j\} \subset \bigcup_{i=0}^{j+1} \left(\left| S_{h_i}^*(z_0) - S_{h_{j+1}}^*(z_0) \right| > \frac{\lambda}{N_i} \right).$$

Remark that for $0 \leq i \leq l \leq l(\beta)$, one has $h_i \leq h_l \leq h(\beta)$ and $q_n(\beta_i) \leq q_n(\beta_l)$. Then denoting $Z_n^* = \max_l |Z_n(\beta_l)|$ and using (7) and lemma (A.1), we obtain for $0 \leq i \leq l \leq l(\beta)$:

$$\begin{aligned} & \left\{ |S_{h_i}^*(z_0) - S_{h_i}^*(z_0)| > \frac{\lambda}{N_i} \right\} \subset \left\{ |S_{h_i}^*(z_0) - S(z_0)| + |S(z_0) - S_{h_i}^*(z_0)| > \frac{\lambda}{N_i} \right\} \\ \subset & \left\{ 2\delta h(\beta)^\beta + \frac{12\delta^{-1}}{nh_i} h(\beta) + \frac{2g^*}{\sqrt{q_n(\beta_i)}} Z_n^* > \frac{\lambda}{N_i} \right\} \\ \subset & \left\{ \frac{2g^*}{\sqrt{q_n(\beta_i)}} Z_n^* > \frac{\lambda - 2\delta - 12\delta^{-1}h(\beta^*)}{N_i} \right\}, \end{aligned}$$

because $nh_i > N_i$ and $N_i < h(\beta)^{-\beta}$.

The precedent inclusions are true for a sufficiently large n and we will consider this case from now to the end of the proof.

As $\frac{q_n(\beta_i)}{N_i^2} = \frac{q_n(\beta_i)}{nh_i} \ln n$, lemma A.1 brings us $\sqrt{\ln n} \leq \frac{\sqrt{q_n(\beta_i)}}{N_i} \leq \sqrt{3 \ln n}$.

Setting $\Lambda_n^* := \lambda - 2\delta - 12\delta^{-1}h(\beta^*)$, we can write

$$(\hat{l} = j) \subset \bigcup_{i=0}^{j+1} \left(|S_{h_i}^*(z_0) - S_{h_{j+1}}^*(z_0)| > \frac{\lambda}{N_i} \right) \subset \left\{ Z_n^* > \frac{\Lambda_n^*}{2g^*} \sqrt{\ln n} \right\} =: A_n^*$$

and

$$I_2 \leq \sum_{j=0}^{l(\beta)-1} |S_{h_j}^*(z_0) - S(z_0)| \mathbb{I}_{\{A_n^*\}}.$$

To get (12) we write

$$\begin{aligned} & \frac{N(\beta)}{g(z_0, S)} \mathbb{E}_S I_2 \leq \frac{N(\beta)}{g(z_0, S)} \sum_{j=0}^{l(\beta)-1} \mathbb{E}_S \left(\frac{\delta n h_j^{1+\beta}}{q_n(\beta_j)} + \frac{6\delta^{-1} h_j}{q_n(\beta_j)} + |Z_n(\beta_j)| \frac{\sigma(\beta_j, S)}{\sqrt{q_n(\beta_j)}} \right) \mathbb{I}_{\{A_n^*\}} \\ \leq & \sum_{j=0}^{l(\beta)-1} \frac{N(\beta)}{g(z_0, S)} \frac{\delta n h_j^{1+\beta}}{q_n(\beta_j)} \mathbb{P}_S(A_n^*) + \sum_{j=0}^{l(\beta)-1} \frac{N(\beta)}{g(z_0, S)} \frac{6\delta^{-1} h_j}{q_n(\beta_j)} \mathbb{P}_S(A_n^*) \\ + & \sum_{j=0}^{l(\beta)-1} \frac{N(\beta)}{g(z_0, S)} \frac{\sigma_n(\beta_j, S)}{\sqrt{q_n(\beta_j)}} \mathbb{E}_S (|Z_n(\beta_j)| \mathbb{I}_{\{A_n^*\}}) \end{aligned}$$

and study the asymptotic behavior of each term. Let $Z \sim \mathcal{N}(0, 1)$ and $A_n := \left\{ |Z| > \frac{\Lambda_n^*}{2g^*} \sqrt{\ln n} \right\}$.

For the first term one has

$$\begin{aligned}
& \sum_{j=0}^{l(\beta)-1} \frac{N(\beta)}{g(z_0, S)} \frac{\delta n h_j^{1+\beta}}{q_n(\beta_j)} \mathbb{P}_S(A_n^*) \leq \frac{\delta}{g_\star} \sum_{j=0}^{l(\beta)-1} N(\beta) h_j^\beta \mathbb{P}_S(A_n^*) \\
& \leq \frac{\delta}{g_\star} \sum_{j=0}^{l(\beta)-1} \left(\frac{n}{\ln n} \right)^{2\beta(\beta_j-\beta)/(2\beta_j+1)(2\beta+1)} \mathbb{P}_S(A_n^*) \\
& \leq [\ln n]^2 \frac{\delta}{g_\star} \left(\frac{n}{\ln n} \right)^{2\beta(\beta_{l(\beta)-1}-\beta)/(2\beta_{l(\beta)-1}+1)(2\beta+1)} \mathbb{P}_S(A_n) \\
& \leq [\ln n]^2 \frac{\delta}{g_\star} \left(\frac{n}{\ln n} \right)^{2\beta(\beta_{l(\beta)-1}-\beta)/(2\beta+1)^2} 2 \frac{2g_\star}{\Lambda_n^* \sqrt{\ln n}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(\Lambda_n^*)^2}{4g_\star^2} \ln n} \\
& \leq \frac{4\delta g_\star}{\sqrt{2\pi} g_\star \Lambda_n^*} e^{-2\beta_\star/(2\beta_\star+1)^2} e^{2\beta_\star/(2\beta_\star+1)^2} \frac{[\ln n]^2}{\sqrt{\ln n}} n^{-\frac{(\Lambda_n^*)^2}{4g_\star^2}}.
\end{aligned}$$

Since $\lambda > 2$ the last term tends to zero as n goes to infinity. We handle the second term as well. Using the Cauchy-Schwarz inequality for the third term, we obtain

$$\begin{aligned}
& \frac{N(\beta)}{g(z_0, S)} \sum_{j=0}^{l(\beta)-1} \mathbb{E}_S \left(|Z_n(\beta_j)| \frac{\sigma_n(\beta_j, S)}{\sqrt{q_n(\beta_j)}} \mathbb{I}_{\{A_n^*\}} \right) \leq \frac{g_\star}{g_\star} \sum_{j=0}^{l(\beta)-1} \frac{N(\beta)}{\sqrt{q_n(\beta_j)}} (\mathbb{P}_S(A_n^*))^{1/2} \\
& \leq \frac{g_\star}{g_\star} \sum_{j=0}^{l(\beta)-1} \frac{N(\beta)}{\sqrt{q_n(\beta_j)}} [\ln n]^{1/2} (\mathbb{P}_S(A_n))^{1/2} \leq \frac{2(g_\star)^{3/2}}{g_\star \sqrt{\Lambda_n^* \sqrt{2\pi}}} \frac{[\ln n]^{3/2}}{\sqrt{\sqrt{\ln n} \sqrt{q_n(\beta_\star)}}} N(\beta_\star) n^{-(\Lambda_n^*)^2/8g_\star^2}.
\end{aligned}$$

By definition of λ this term tends to zero as n goes to infinity. Eventually we have proved (12) which, connected to (8), completes the proof.

4 Lower bound

In this section we give the lower bound for the minimax risk. We will consider a family of functions in $\mathcal{U}_{z_0, \delta, n}$ defined with another bandwidth $\tilde{h}(\beta) = n^{-1/2\beta+1} (\ln n)^{-2\beta/2\beta+1}$ such that $n\tilde{h}(\beta) = N^2(\beta)$ and \tilde{h} is a increasing function of β .

Theorem 4.1 *For all $\delta \in]0; 1[$ the following inequality holds*

$$\liminf_{n \rightarrow \infty} \inf_{\tilde{S}} \mathcal{R}_{z_0, \delta, n}(\tilde{S}) \geq \frac{\mathbb{E}|\xi|}{\sqrt{2}}, \quad \xi \sim \mathcal{N}(0, 1),$$

where the infimum is taken over all estimators \tilde{S} of $S(z_0)$.

PROOF: For $\nu \in]0; \frac{1}{4}[$, denote $S_\nu(x) = \frac{1}{N(\beta)} V_\nu \left(\frac{x-z_0}{\tilde{h}(\beta)} \right)$, where the function V_ν is defined by

$$\begin{aligned}
V_\nu(x) &= \frac{1}{\nu} \int_{-\infty}^{+\infty} \tilde{Q}_\nu(u) m \left(\frac{u-x}{\nu} \right) du, \\
\tilde{Q}_\nu(u) &= \mathbb{I}_{\{|u| \leq 1-2\nu\}} + 2\mathbb{I}_{\{1-2\nu \leq |u| \leq 1-\nu\}}
\end{aligned}$$

and m is a nonnegative function, infinitely differentiable on \mathbb{R} , such that $m(z) = 0$ for any $|z| \geq 1$ and $\int_{-1}^1 m(z)dz = 1$. We can easily show that for all $0 < \nu < \frac{1}{4}$, one has $V_\nu(0) = 1$ and $\int_{-1}^1 V_\nu(x)dx = 2$.

Now fix $b > 0$, $\nu \in]0; 1/4[$, $\delta \in]0; 1[$ and denote $S_{\nu,u}(x) = \frac{u}{N(\beta)} V_\nu\left(\frac{x-z_0}{\tilde{h}(\beta)}\right)$ for $x, u \in \mathbb{R}$. Thanks to lemma A.2 we can say that if $|u| \leq b$ then there exists an integer $n_{b,\delta} > 0$ such that $S_{\nu,u} \in \mathcal{U}_{z_0,\delta,n}$ for all $n \geq n_{b,\delta}$. Hence for $n \geq n_{b,\delta}$, we have

$$\begin{aligned} \mathcal{R}_{z_0,\delta,n}(\tilde{S}) &\geq \sup_{\beta \in [\beta_*, \beta^*]} \sup_{|u| \leq b} \frac{N(\beta)}{g(z_0, S_{\nu,u})} \mathbb{E}_{S_{\nu,u}} |\tilde{S}(z_0) - S_{\nu,u}(z_0)| \\ &\geq \sup_{\beta \in [\beta_*, \beta^*]} \frac{1}{2b} \int_{-b}^b \frac{N(\beta)}{g(z_0, S_{\nu,u})} \mathbb{E}_{S_{\nu,u}} v_a \left(\tilde{S}(z_0) - S_{\nu,u}(z_0) \right) du =: \sup_{\beta \in [\beta_*, \beta^*]} I_n(a, b, \beta) \end{aligned}$$

where $v_a(x) = |x| \wedge a$, $a > 0$.

Let $\mathbb{P}_{S_{\nu,u}}$ be the law of $(y_k^{(1)})_{k=1,\dots,n}$, where $y_k^{(1)} = S_{\nu,u}(x_k) + g(x_k, S_{\nu,u})\xi_k$, and \mathbb{P} the law of $(y_k^{(0)})_{k=1,\dots,n}$, where $y_k^{(0)} = g(x_k, S_{\nu,u})\xi_k$. These two measures are equivalent and the corresponding Radon-Nikodym derivative at the point (y_1, \dots, y_n) is

$$\begin{aligned} \rho_n(u, \beta) &= \frac{d\mathbb{P}_{S_{\nu,u}}}{d\mathbb{P}}(y_1, \dots, y_n) = \exp \left\{ -\frac{1}{2} \sum_{k=1}^n \left(\left(\frac{y_k - S_{\nu,u}(x_k)}{g(x_k, S_{\nu,u})} \right)^2 - \left(\frac{y_k}{g(x_k, S_{\nu,u})} \right)^2 \right) \right\} \\ &= \exp \left(u \varsigma_n(\beta) \eta_n(\beta) - \frac{u^2}{2} \varsigma_n^2(\beta) \right) \end{aligned}$$

where $\varsigma_n^2(\beta) = \frac{1}{N^2(\beta)} \sum_{k=1}^n \frac{V_\nu^2\left(\frac{x_k-z_0}{\tilde{h}(\beta)}\right)}{g^2(x_k, S_{\nu,u})}$ and $\eta_n(\beta) = \frac{1}{N(\beta)\varsigma_n(\beta)} \sum_{k=1}^n \frac{V_\nu\left(\frac{x_k-z_0}{\tilde{h}(\beta)}\right)}{g^2(x_k, S_{\nu,u})} y_k$.

Under the law \mathbb{P} , $\eta_n(\beta)$ is a standard Gaussian random variable.

Setting $\sigma_\nu^2 = \int_{-1}^1 \frac{V_\nu^2(z)}{g^2(z_0, 0)} dz$ and since the square root is continuous at σ_ν^2 , lemma A.3 implies

$$\sup_{\beta \in [\beta_*, \beta^*]} |\varsigma_n(\beta) - \sigma_\nu| \xrightarrow{n \rightarrow \infty} 0. \quad (13)$$

Rewrite $\rho_n(u, \beta) = \exp \left\{ u \sigma_\nu \eta_n(\beta) - \frac{u^2 \sigma_\nu^2}{2} + r_n(\beta) \right\}$, where $r_n(\beta)$ is a Gaussian random variable with expectation $\frac{u^2}{2} (\sigma_\nu^2 - \varsigma_n^2(\beta))$ and variance $u^2 (\varsigma_n(\beta) - \sigma_\nu)^2$. Then using (13) and Tchebychev inequality we get for $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \sup_{\beta \in [\beta_*, \beta^*]} \mathbb{P}(|r_n(\beta)| > \varepsilon) = 0. \quad (14)$$

As a consequence we can show that $\rho_n(u, \beta) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \rho_\infty(u) := \exp \left(u \sigma_\nu \eta - \frac{u^2 \sigma_\nu^2}{2} \right)$ uniformly in β and

$$\lim_{n \rightarrow \infty} \sup_{\beta \in [\beta_*, \beta^*]} \mathbb{P} \left(|e^{r_n(\beta)} - 1| > \varepsilon \right) = 0.$$

Denoting $\psi_{a,n}(\tilde{S}, S_{\nu,u}) = v_a(N(\beta)(\tilde{S}_n(z_0) - S_{\nu,u}(z_0)))$ and \mathbb{E} the expectation for the probability measure \mathbb{P} , one has

$$I_n(a, b, \beta) \geq \frac{1}{2b} \int_{-b}^b \mathbb{E} \mathbb{I}_{B_d(\beta)} \frac{\psi_{a,n}(\tilde{S}, S_{\nu,u})}{g(z_0, S_{\nu,u})} \varrho_n(u, \beta) du + \delta_n(a, b, \beta) \quad (15)$$

$$=: J_n(a, b, \beta) + \delta_n(a, b, \beta), \quad (16)$$

where

$$B_d(\beta) = \{|\eta_n(\beta)| \leq d\} \text{ and } d = \sigma_\nu(b - \sqrt{b}), b > 1,$$

$$\varrho_n(u, \beta) = \exp\left(u\sigma_\nu\eta_n(\beta) - \frac{u^2\sigma_\nu^2}{2}\right),$$

$$\delta_n(a, b, \beta) = \frac{1}{2b} \int_{-b}^b \mathbb{E} \mathbb{I}_{B_d(\beta)} \frac{\psi_{a,n}(\tilde{S}, S_{\nu,u})}{g(z_0, S_{\nu,u})} \theta_n(u, \beta) du,$$

$$\theta_n(u, \beta) = \rho_n(u, \beta) - \varrho_n(u, \beta) = \exp\left(u\sigma_\nu\eta_n(\beta) - \frac{u^2\sigma_\nu^2}{2}\right) (e^{r_n(\beta)} - 1).$$

Let us show that

$$\inf_{\tilde{S}} \sup_{\beta \in [\beta_*, \beta^*]} \delta_n(a, b, \beta) \xrightarrow{n \rightarrow \infty} 0. \quad (17)$$

We can easily prove that $\mathbb{E}\rho_\infty(u) = 1$ and $\mathbb{E}\rho_n(u, \beta) = 1$. Then the sequence $\{\rho_n(u, \beta), n \geq 1\}$ is uniformly integrable (see [2, p.32]). As $\varrho_n(u, \beta)$ is bounded on $B_d(\beta)$, we obtain the uniform integrability of $\left\{ \mathbb{I}_{B_d(\beta)} \frac{\psi_{a,n}(\tilde{S}, S_{\nu,u})}{g(z_0, S_{\nu,u})} \theta_n(u, \beta), n \geq 1 \right\}$.

We remark that $\exp\left(u\sigma_\nu\eta_n(\beta) - \frac{u^2\sigma_\nu^2}{2}\right)$ is bounded by a constant independent of β on $B_d(\beta)$ and that $\psi_{a,n}(\tilde{S}, S_{\nu,u})$ is bounded by a . Then

$$\lim_{n \rightarrow \infty} \sup_{\beta \in [\beta_*, \beta^*]} \mathbb{P} \left(\mathbb{I}_{B_d(\beta)} \frac{\psi_{a,n}(\tilde{S}, S_{\nu,u})}{g(z_0, S_{\nu,u})} |\theta_n(u, \beta)| > \varepsilon \right) = 0.$$

It follows from this and from the uniform integrability of $\left\{ \mathbb{I}_{B_d(\beta)} \frac{\psi_{a,n}(\tilde{S}, S_{\nu,u})}{g(z_0, S_{\nu,u})} \theta_n(u, \beta), n \geq 1 \right\}$ that

$$\lim_{n \rightarrow \infty} \sup_{\beta \in [\beta_*, \beta^*]} \mathbb{E} \left| \mathbb{I}_{B_d(\beta)} \frac{\psi_{a,n}(\tilde{S}, S_{\nu,u})}{g(z_0, S_{\nu,u})} \theta_n(u, \beta) \right| = 0.$$

Furthermore there exists a constant $K > 0$ such that

$$\sup_{\beta \in [\beta_*, \beta^*]} \left| \mathbb{E} \frac{\mathbb{I}_{B_d(\beta)}}{g(z_0, S_{\nu,u})} \psi_{a,n}(\tilde{S}, S_{\nu,u}) \theta_n(u, \beta) \right| \leq K \sup_{\beta \in [\beta_*, \beta^*]} \mathbb{E} |e^{r_n(\beta)} - 1| \leq K \exp(u^2 \sigma_n \nu^2).$$

By bounded convergence we obtain finally

$$\sup_{\beta \in [\beta_*, \beta^*]} \delta_n(a, b, \beta) \leq \frac{1}{2b} \int_{-b}^b \sup_{\beta \in [\beta_*, \beta^*]} \mathbb{E} \left| \frac{\mathbb{I}_{B_d(\beta)}}{g(z_0, S_{\nu,u})} \psi_{a,n}(\tilde{S}, S_{\nu,u}) \theta_n(u) \right| du \xrightarrow{n \rightarrow \infty} 0,$$

and then (17).

Now we are interested in the term $J_n(a, b, \beta)$ in (16).

Firstly rewrite $\varrho_n(u, \beta) = \zeta_n(\beta)e^{-\sigma_\nu^2(u-\tilde{\eta}_n(\beta))^2/2}$ where $\zeta_n(\beta) = e^{\eta_n^2(\beta)/2}$ and $\tilde{\eta}_n(\beta) = \frac{\eta_n(\beta)}{\sigma_\nu}$, then

$$J_n(a, b, \beta) = \frac{1}{2b} \int_{-b}^b \mathbb{E} \mathbb{I}_{B_d(\beta)} \zeta_n(\beta) \frac{v_a(u - t_n(\beta))}{g(z_0, S_{\nu, u})} \exp\left(-\frac{\sigma_\nu^2}{2}(u - \tilde{\eta}_n(\beta))^2\right) du,$$

with $t_n(\beta) = N(\beta)\tilde{S}_n(z_0)$.

Further if $\xi \sim \mathcal{N}(0, 1)$ and if we set $\tilde{\xi} = \frac{\xi}{\sigma_\nu}$, $\zeta = e^{\xi^2/2}$, $\tilde{B}_d = \{|\xi| \leq d\}$ and $\tilde{\mathbb{E}}$ the expectation for the probability measure of ξ , we have

$$J_n(a, b, \beta) = \tilde{\mathbb{E}} \mathbb{I}_{\tilde{B}_d} \zeta \frac{1}{2b} \int_{-b}^b \frac{v_a(u - t_n(\beta))}{g(z_0, S_{\nu, u})} \exp\left(-\frac{\sigma_\nu^2}{2}(u - \tilde{\xi})^2\right) du.$$

One has the following limit

$$\sup_{\beta \in [\beta_\star; \beta^\star]} \tilde{\mathbb{E}} \mathbb{I}_{\tilde{B}_d} \zeta \frac{1}{2b} \int_{-b}^b v_a(u - t_n(\beta)) \exp\left(-\frac{\sigma_\nu^2}{2}(u - \tilde{\xi})^2\right) \left(\frac{1}{g(z_0, S_{\nu, u})} - \frac{1}{g(z_0, 0)}\right) du \xrightarrow{n \rightarrow \infty} 0.$$

Indeed:

$$\begin{aligned} & \left| \tilde{\mathbb{E}} \mathbb{I}_{\tilde{B}_d} \zeta \frac{1}{2b} \int_{-b}^b v_a(u - t_n(\beta)) \exp\left(-\frac{\sigma_\nu^2}{2}(u - \tilde{\xi})^2\right) \left(\frac{1}{g(z_0, S_{\nu, u})} - \frac{1}{g(z_0, 0)}\right) du \right| \\ & \leq \tilde{\mathbb{E}} \mathbb{I}_{\tilde{B}_d} \zeta \frac{1}{2b} \int_{-b}^b v_a(u - t_n(\beta)) \exp\left(-\frac{\sigma_\nu^2}{2}(u - \tilde{\xi})^2\right) \left| \frac{-L_{z_0, 0}(S_{\nu, u}) + \Gamma_{z_0, 0}(S_{\nu, u})}{g_\star^2} \right| du \\ & \leq \tilde{\mathbb{E}} \mathbb{I}_{\tilde{B}_d} \zeta \frac{1}{2b} \int_{-b}^b a \exp\left(-\frac{\sigma_\nu^2}{2}(u - \tilde{\xi})^2\right) \frac{C_0 \|S_{\nu, u}\| + |\Gamma_{z_0, 0}(S_{\nu, u})|}{g_\star^2} du \\ & \leq \tilde{\mathbb{E}} \mathbb{I}_{\tilde{B}_d} \zeta \frac{1}{2b} \int_{-b}^b a \frac{C_0 \|S_{\nu, u}\| + |\Gamma_{z_0, 0}(S_{\nu, u})|}{g_\star^2} du. \end{aligned}$$

As $\|S_{\nu, u}\|$ tends to zero uniformly in β as $n \rightarrow \infty$, we have the announced result. It follows that

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \inf_{\tilde{S}} \sup_{\beta \in [\beta_\star; \beta^\star]} J_n(a, b, \beta) \geq \\ & \liminf_{n \rightarrow \infty} \inf_{\tilde{S}} \sup_{\beta \in [\beta_\star; \beta^\star]} \tilde{\mathbb{E}} \mathbb{I}_{\tilde{B}_d} \zeta \frac{1}{2b} \int_{-b}^b \frac{v_a(u - t_n(\beta))}{g(z_0, 0)} \exp\left(-\frac{\sigma_\nu^2}{2}(u - \tilde{\xi})^2\right) du. \end{aligned}$$

Moreover we can write

$$\begin{aligned} \tilde{\mathbb{E}} \frac{\mathbb{I}_{\tilde{B}_d} \zeta}{2b} \int_{-b}^b \frac{v_a(u - t_n(\beta))}{g(z_0, 0)} e^{-\frac{\sigma_\nu^2}{2}(u - \tilde{\xi})^2} du & \geq \tilde{\mathbb{E}} \frac{\mathbb{I}_{\tilde{B}_d} \zeta}{2b} \int_{-\sqrt{b}}^{\sqrt{b}} \frac{v_a(t - t_n(\beta) + \tilde{\xi})}{g(z_0, 0)} e^{-\frac{\sigma_\nu^2}{2}t^2} dt \\ & \geq \tilde{\mathbb{E}} \frac{\mathbb{I}_{\tilde{B}_d} \zeta}{2b} \int_{-\sqrt{b}}^{\sqrt{b}} \frac{v_a(t)}{g(z_0, 0)} \exp\left(-\frac{\sigma_\nu^2}{2}t^2\right) dt, \end{aligned}$$

using Anderson's lemma for the last inequality (see [17, Chapter II, Lemma 10.1 and Corollary 10.2]).

Noticing that $\tilde{\mathbb{E}}\mathbb{I}_{\tilde{B}_d}\zeta = \frac{2\sigma_\nu(b - \sqrt{b})}{\sqrt{2\pi}}$ and limiting successively $a \rightarrow \infty$ and $b \rightarrow \infty$, we get

$$\liminf_{b \rightarrow \infty} \liminf_{a \rightarrow \infty} \liminf_{n \rightarrow \infty} \inf_{\tilde{S}} \sup_{\beta \in [\beta_\star; \beta^\star]} J_n(a, b, \beta) \geq \frac{\sigma_\nu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{|t|}{g(z_0, 0)} \exp\left(-\frac{\sigma_\nu^2}{2} t^2\right) dt.$$

Eventually using the fact that $\sigma_\nu^2 \xrightarrow{\nu \rightarrow 0} \frac{2}{g^2(z_0, 0)}$ and combining the result with (17) finish the proof. \square

A Appendix

Lemma A.1 *There exists an integer N_\star such that if $n \geq N_\star$, then*

$$1 \leq \frac{q_n(\beta_i)}{nh_i} \leq 3,$$

for all $i = 0, \dots, [\ln n]$.

PROOF: Writing

$$\frac{q_n(\beta_i)}{nh_i} = \frac{1}{nh_i} \sum_{k=1}^n Q\left(\frac{x_k - z_0}{h_i}\right) = \frac{[n(z_0 + h_i)] - [n(z_0 - h_i)]}{nh_i},$$

the following inequalities

$$\begin{aligned} [n(z_0 + h_i)] &\leq n(z_0 + h_i) \leq [n(z_0 + h_i)] + 1, \\ -1 - [n(z_0 - h_i)] &\leq -n(z_0 - h_i) \leq -[n(z_0 - h_i)], \end{aligned}$$

show that $2 - \frac{1}{nh_i} \leq \frac{q_n(\beta_i)}{nh_i} \leq 2 + \frac{1}{nh_i}$.

But there exists an integer N_\star such that if $n \geq N_\star$, then $nh(\beta_\star) \geq 1$. As h is an increasing function of β , we get for $n \geq N_\star$:

$$2 - \frac{1}{nh(\beta_\star)} \leq \frac{q_n(\beta_i)}{nh_i} \leq 2 + \frac{1}{nh(\beta_\star)},$$

and then the desired result. \square

Lemma A.2 *The family of functions $\mathcal{S} = \{S_\nu, \nu \in]0; \frac{1}{4}[\}$ satisfies*

$$\mathcal{S} \subset \bigcup_{k \geq 1} \bigcap_{n \geq k} \mathcal{U}_{z_0, \delta, n}.$$

PROOF: Firstly one has $\int_{-1}^1 (S_\nu(z_0 + u\tilde{h}(\beta)) - S_\nu(z_0)) du = 0$. Moreover we can write

$$|S'_\nu(x)| \leq \frac{2}{\nu N(\beta)\tilde{h}(\beta)} \int_{-1}^1 |m'(u)| du \leq \frac{2}{\nu} n^{(1-\beta_*)/(2\beta_*+1)} (\ln n)^{3\beta^*/2\beta^*+1} \int_{-1}^1 |m'(u)| du,$$

since

$$\left| V'_\nu \left(\frac{x-z_0}{\tilde{h}(\beta)} \right) \right| = \left| -\nu^{-2} \int_{-\infty}^{+\infty} \tilde{Q}_\nu(u) m' \left(\frac{u - \frac{x-z_0}{\tilde{h}(\beta)}}{\nu} \right) du \right| \leq \frac{2}{\nu} \int_{-1}^1 |m'(u)| du$$

Hence for any fixed $\delta \in]0; 1[$, there exists $n_\delta \in \mathbb{N}^*$ such that if $n \geq n_\delta$, then $|S'_\nu(x)| \leq \delta^{-1}$ and $S_\nu \in \mathcal{U}_{z_0, \delta, n}$. \square

Lemma A.3 *The following limit holds*

$$\sup_{\beta \in [\beta_*, \beta^*]} \left| \zeta_n^2(\beta) - \int_{-1}^1 \frac{V_\nu^2(z)}{g^2(z_0, 0)} dz \right| \xrightarrow{n \rightarrow \infty} 0. \quad (18)$$

PROOF: There exists an integer n_0 such that for all $n \geq n_0$ and all $\beta \in [\beta_*, \beta^*]$, we have $[z_0 - \tilde{h}(\beta); z_0 + \tilde{h}(\beta)] \subset [0; 1]$. As a consequence for $n \geq n_0$ we get

$$\zeta_n^2(\beta) = \frac{1}{\tilde{h}(\beta)} \int_{z_0 - \tilde{h}(\beta)}^{z_0 + \tilde{h}(\beta)} \frac{V_\nu^2 \left(\frac{x-z_0}{\tilde{h}(\beta)} \right)}{g^2(x, S_{\nu, u})} \mu_n(dx) = \int_0^1 \frac{V_\nu^2 \left(\frac{x-z_0}{\tilde{h}(\beta)} \right)}{g^2(x, S_{\nu, u})} \nu_n(dx)$$

with $\mu_n = \frac{1}{n} \sum_{k=1}^n \delta_{k/n}$ and $\nu_n = \frac{\mathbb{I}_{[z_0 - \tilde{h}(\beta); z_0 + \tilde{h}(\beta)]}}{\tilde{h}(\beta)} \mu_n$.

The hypothesis (4) and (5) on the function g yield for all $x \in [0; 1]$:

$$\begin{aligned} \left| \frac{1}{g^2(x, S_{\nu, u})} - \frac{1}{g^2(x, 0)} \right| &= \left| \frac{g^2(x, 0) - g^2(x, S_{\nu, u})}{g^2(x, S_{\nu, u})g^2(x, 0)} \right| \\ &\leq \frac{1}{g_\star^4} \left| 2g(x, 0)L_{x,0}(S_{\nu, u}) + (L_{x,0}(S_{\nu, u}))^2 + (\Gamma_{x,0}(S_{\nu, u}))^2 \right. \\ &\quad \left. + 2g(x, 0)\Gamma_{x,0}(S_{\nu, u}) + 2L_{x,0}(S_{\nu, u})\Gamma_{x,0}(S_{\nu, u}) \right| \\ &\leq \frac{1}{g_\star^4} \left(2g^\star C_0 \|S_{\nu, u}\| + C_0^2 \|S_{\nu, u}\|^2 + \frac{|\Gamma_{x,0}(S_{\nu, u})|^2}{\|S_{\nu, u}\|^2} \|S_{\nu, u}\|^2 \right. \\ &\quad \left. + 2g^\star \frac{|\Gamma_{x,0}(S_{\nu, u})|}{\|S_{\nu, u}\|} \|S_{\nu, u}\| + 2C_0 \|S_{\nu, u}\| \frac{|\Gamma_{x,0}(S_{\nu, u})|}{\|S_{\nu, u}\|} \|S_{\nu, u}\| \right), \end{aligned}$$

and further

$$\begin{aligned} &\left| \int_0^1 \left(\frac{1}{g^2(x, S_{\nu, u})} - \frac{1}{g^2(x, 0)} \right) \nu_n(dx) \right| \\ &\leq \frac{1}{g_\star^4} \int_0^1 \nu_n(dx) \left(2g^\star C_0 \|S_{\nu, u}\| + C_0^2 \|S_{\nu, u}\|^2 + \left(\sup_{x \in [0; 1]} \frac{|\Gamma_{x,0}(S_{\nu, u})|}{\|S_{\nu, u}\|} \right)^2 \|S_{\nu, u}\|^2 \right. \\ &\quad \left. + 2g^\star \left(\sup_{x \in [0; 1]} \frac{|\Gamma_{x,0}(S_{\nu, u})|}{\|S_{\nu, u}\|} \right) \|S_{\nu, u}\| + 2C_0 \|S_{\nu, u}\| \left(\sup_{x \in [0; 1]} \frac{|\Gamma_{x,0}(S_{\nu, u})|}{\|S_{\nu, u}\|} \right) \|S_{\nu, u}\| \right). \end{aligned}$$

We easily prove that for a real function f defined on \mathbb{R} and continuous at the point z_0 , one has

$$\lim_{n \rightarrow \infty} \sup_{\beta \in [\beta_*, \beta^*]} \left| \int_0^1 f(x) \nu_n(dx) - 2f(z_0) \right| = 0. \quad (19)$$

Writing

$$\begin{aligned} & \int_0^1 \left(\frac{1}{g^2(x, S_{\nu, u})} - \frac{1}{g^2(z_0, 0)} \right) \nu_n(dx) \\ &= \int_0^1 \left(\frac{1}{g^2(x, S_{\nu, u})} - \frac{1}{g^2(x, 0)} \right) \nu_n(dx) + \int_0^1 \left(\frac{1}{g^2(x, 0)} - \frac{1}{g^2(z_0, 0)} \right) \nu_n(dx), \end{aligned}$$

and applying (19) to the functions $x \mapsto 1$ and $x \mapsto \frac{1}{g(x, 0)}$ and using hypothesis (6), we obtain

$$\lim_{n \rightarrow \infty} \sup_{\beta \in [\beta_*, \beta^*]} \left| \int_0^1 \left(\frac{1}{g^2(x, S_{\nu, u})} - \frac{1}{g^2(z_0, 0)} \right) \nu_n(dx) \right| = 0. \quad (20)$$

Now we want to show that

$$\lim_{n \rightarrow \infty} \sup_{\beta \in [\beta_*, \beta^*]} \left| \int_0^1 V_\nu^2 \left(\frac{x - z_0}{\tilde{h}(\beta)} \right) \nu_n(dx) - \int_{-1}^1 V_\nu^2(y) dy \right| = 0. \quad (21)$$

Let F be a function such that $F' = V_\nu^2$ and denote $u_k = \frac{x_k - z_0}{\tilde{h}(\beta)}$ again. For all $n \geq n_0$, one has

$$\int_{-1}^1 V_\nu^2(y) dy = F(1) - F(-1) = \sum_{k=1}^n (F(u_k) - F(u_{k-1})),$$

then

$$\begin{aligned} & \left| \int_0^1 V_\nu^2 \left(\frac{x - z_0}{\tilde{h}(\beta)} \right) \nu_n(dx) - \int_{-1}^1 V_\nu^2(y) dy \right| = \left| \sum_{k=1}^n \left(F(u_k) - F(u_{k-1}) - \frac{1}{n\tilde{h}(\beta)} F'(u_k) \right) \right| \\ &= \left| \sum_{k=1}^n \frac{-1}{2(n\tilde{h}(\beta))^2} F''(v_k) \right| \leq \frac{1}{2n\tilde{h}^2(\beta_*)} (V_\nu^{2'})_{max}, \end{aligned}$$

with $v_k \in]u_{k-1}; u_k[$ and $(V_\nu^{2'})_{max}$ the maximum of the function $V_\nu^{2'}$. Limiting $n \rightarrow \infty$ yields (21).

Finally, as we have

$$\begin{aligned} & \left| \zeta_n^2(\beta) - \int_{-1}^1 \frac{V_\nu^2(z)}{g^2(z_0, 0)} dz \right| = \left| \int_0^1 \frac{V_\nu^2 \left(\frac{x - z_0}{\tilde{h}(\beta)} \right)}{g^2(x, S_{\nu, u})} \nu_n(dx) - \int_{-1}^1 \frac{V_\nu^2(z)}{g^2(z_0, 0)} dz \right| \\ &\leq \left| \int_0^1 \frac{V_\nu^2 \left(\frac{x - z_0}{\tilde{h}(\beta)} \right)}{g^2(x, S_{\nu, u})} \nu_n(dx) - \int_0^1 \frac{V_\nu^2 \left(\frac{x - z_0}{\tilde{h}(\beta)} \right)}{g^2(z_0, 0)} \nu_n(dx) \right| + \left| \int_0^1 \frac{V_\nu^2 \left(\frac{x - z_0}{\tilde{h}(\beta)} \right)}{g^2(z_0, 0)} \nu_n(dx) - \int_{-1}^1 \frac{V_\nu^2(z)}{g^2(z_0, 0)} dz \right| \\ &\leq V_\nu^{max} \int_0^1 \left| \frac{1}{g^2(x, S_{\nu, u})} - \frac{1}{g^2(z_0, 0)} \right| \nu_n(dx) + \left| \int_0^1 \frac{V_\nu^2 \left(\frac{x - z_0}{\tilde{h}(\beta)} \right)}{g^2(z_0, 0)} \nu_n(dx) - \int_{-1}^1 \frac{V_\nu^2(z)}{g^2(z_0, 0)} dz \right|, \end{aligned}$$

(20) and (21) implies lemma A.3. \square

References

- [1] A. Barron, L. Birgé, and P. Massart. Risk bounds for model selection via penalization. *Probab. Theory Related Fields*, 113:301–413, 1999.
- [2] P. Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999. A Wiley-Interscience Publication.
- [3] J.-Y. Brua. Asymptotically efficient estimators for nonparametric heteroscedastic regression models. 2007.
- [4] D.L. Donoho. Asymptotic minimax risk for sup-norm loss: solution via optimal recovery. *Probab. Theory Related Fields*, 99(2):145–170, 1994.
- [5] D.L. Donoho. Statistical estimation and optimal recovery. *Ann. Statist.*, 22:238–270, 1994.
- [6] D.L. Donoho, I.M. Johnstone, G. Kerkycharian, and D. Picard. Wavelet shrinkage: asymptopia? *J. Roy. Statist. Soc. Ser. B*, 57:301–369, 1995.
- [7] D.L. Donoho and R.C. Liu. Geometrizing rates of convergence. III. *Ann. Statist.*, 19:668–701, 1991.
- [8] S. Efromovich. *Nonparametric Curve Estimation. Methods, Theory and Applications*. Springer, Berlin, New York, 1999.
- [9] S. Efromovich. Sequential design and estimation in heteroscedastic nonparametric regression. *Sequential Anal.*, 26(1):3–25, 2007.
- [10] S. Efromovich and M. Pinsker. Sharp-optimal and adaptive estimation for heteroscedastic nonparametric regression. *Statist. Sinica*, 6(4):925–942, 1996.
- [11] L. Galtchouk and S. Pergamenshchikov. Sequential nonparametric adaptive estimation of the drift coefficient in diffusion processes. *Math. Methods Statist.*, 10(3):316–330, 2001. Meeting on Mathematical Statistics (Marseille, 2000).
- [12] L. Galtchouk and S. Pergamenshchikov. Efficient adaptive nonparametric estimation in heteroscedastic regression models. *Preprint of the Strasbourg Louis Pasteur University, IRMA*, pages available online at <http://hal.archives-ouvertes.fr/hal-00129707/fr/>, 2005.
- [13] Leonid Galtchouk and Serge Pergamenshchikov. Asymptotically efficient estimates for nonparametric regression models. *Statist. Probab. Lett.*, 76:852–860, 2006.
- [14] S.M. Goldfeld and R.E. Quandt. *Nonlinear Methods in Econometrics*. North-Holland, Amsterdam, London, 1972.
- [15] G. K. Golubev. Asymptotically minimax estimation of a regression function in an additive model. *Problems Inform. Trans.*, 28:3–15, 1992.
- [16] G. K. Golubev and M. Nussbaum. Adaptive spline estimates in a non parametric regression model. *Theory Probab. Appl.*, 37:521–529, 1993.

- [17] I.A. Ibragimov and R.Z. Hasminskii. *Statistical Estimation: Asymptotic Theory*. Springer, Berlin, New York, 1981.
- [18] A.P. Korostelev. Exact asymptotically minimax estimator for nonparametric regression in uniform norm. *Theory Probab. Appl.*, 38:737–743, 1993.
- [19] O. V. Lepskii. A problem of adaptive estimation in Gaussian white noise. *Teor. Veroyatnost. i Primenen.*, 35(3):459–470, 1990.
- [20] O. V. Lepskii. Asymptotically minimax adaptive estimation. I. Upper bounds. Optimally adaptive estimates. *Teor. Veroyatnost. i Primenen.*, 36(4):645–659, 1991.
- [21] O. V. Lepskii. Asymptotically minimax adaptive estimation. II. Schemes without optimal adaptation. Adaptive estimates. *Teor. Veroyatnost. i Primenen.*, 37(3):468–481, 1992.
- [22] M. Nussbaum. Spline smoothing in regression models and asymptotic efficiency in L_2 . *Ann. Statist.*, 13:984–997, 1985.
- [23] J. Sacks and W. Strawderman. Improvements on linear minimax estimates. In *Statistical decision theory and related topics, III, Vol. 2 (West Lafayette, Ind., 1981)*, pages 287–304. Academic Press, New York, 1982.
- [24] J. Sacks and D. Ylvisaker. Asymptotically optimum kernels for density estimation at a point. *Ann. Statist.*, 9(2):334–346, 1981.