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## The Twofold Role of Diagrams in Euclid's Plane Geometry

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Proposition I.1 of Euclid's *Elements* requires to 'construct' an equilateral triangle on a 'given finite straight line', or segment, in modern parlance. To achieve this, Euclid takes this segment to be **AB** (fig. 1), then describes two circles with centre in the two extremities **A** and **B** of this segment, respectively, and takes for granted that these circles intersect each other in a point **C**. This is not licensed by his postulates. Hence, either his argument is flawed, or it is warranted on other grounds.

According to a classical view 'the Principle of Continuity' provides an appropriate ground for this argument, insofar as it ensures 'the actual existence of points of intersection' of lines<sup>2</sup>. M. Friedman<sup>3</sup> has rightly remarked, however, that in the *Elements* 'the notion of "continuity" [...] is not logically analyzed' and thus there is no room for a 'valid *sylogistic inference* of the form:  $C_1$  is continuous [,]  $C_2$  is continuous [, then]  $C$  exists', where  $C_1$  and  $C_2$  are the two circles involved in this proposition.

A possible solution of the difficulty is to admit that Euclid's argument is diagram-based and that continuity provides a ground for it insofar as it is understood as a property of diagrams.

Proposition I.1 is, by far, the most popular example used to justify the thesis that many of Euclid's geometric arguments are diagram-based. Many scholars

have recently articulated this thesis in different ways and argued for it<sup>4</sup>. My purpose is to reformulate it in a quite general way, by describing what I take to be the twofold role that diagrams play in Euclid's plane geometry<sup>5</sup> (EPG, from now on).

Euclid's arguments<sup>6</sup> are object-dependent. They are about geometric objects<sup>7</sup>: the objects of EPG, as I shall call them from now on (points, segments, circles, angles, polygons, and their combinations or parts. Hence, they cannot be diagram-based unless diagrams are supposed to have an appropriate relation with these objects.

I shall take this relation to be a quite peculiar sort of representation<sup>8</sup>. Its peculiarity depends on the two following claims that I shall argue for:

*C.i*) To provide the conditions of identify of the objects of EPG is the same as to provide the identity conditions for the diagrams that represent them or—in the case of angles—of appropriate equivalence classes of diagrams that represent them;

*C.ii*) The objects of EPG inherit some properties and relations from the diagrams.

For short, I shall say that diagrams play a global and a local role in EPG to mean, respectively, that they are such that claims (*C.i*) and (*C.ii*) hold<sup>9</sup>.

Geometric objects are abstract. By contrast, I take diagrams to be concrete—by which I mean physical—objects<sup>10</sup>. Hence, I do not take them to be abstract types (like the diagram annexed to proposition I.1 of the *Elements*), but concrete tokens (like the diagram that I have just drawn)<sup>11</sup>. So understood, a diagram is a

configuration of physical lines and points drawn on physical flat surfaces<sup>12</sup>. Non-contour-closed physical lines have extremities. For reasons of linguistic simplicity, it is convenient to suppose that a single physical contour-closed line is an elementary diagram (a diagram that does not include other diagrams), whereas a single physical non-contour-closed line is a diagram composed by three distinct diagrams: the line and two points providing its extremities.

According to Aristotelian conception of continuity, the lines that compose a diagram and the surface on which they are drawn can be understood as continuous. I argue that this is a necessary condition for diagrams to be able to play their twofold role in EPG. This is because I shall begin my enquiry by expounding—in section I—(my interpretation of) Aristotle's conception of continuity.

The crucial function I assign to this conception of continuity suggests that my account of EPG could be understood as an Aristotelian interpretation of it. This understanding is more generally suggested by my construal of the relation of representation that links objects of EPG to diagrams. According to it, there is room to assert that the objects of EPG result from the diagrams that represent them through Aristotelian abstraction (which, broadly speaking, consists in isolating some properties of certain objects and taking them as if they were objects in turn).

This contrasts with the usual understating of the objects of EPG as ideal objects in Platonic sense<sup>13</sup>. According to this understanding, EPG proceeds from contemplation of eternal truths and provides a paradigm for the most classical

form of Platonism in philosophy of mathematics. Because of the crucial role that EPG has played in the evolution of mathematics, this conception has strongly affected our understanding of mathematics, in general. One of the purposes of my paper is to suggest—based on a single example—a quite different understanding, according to which mathematical objects and truths are not contemplated but constituted.

The *Elements* offer no explicit support for the Platonic interpretation. They offer, likewise, no explicit support for any alternative interpretation. Hence, all that I can do is to detail my interpretation and to apply it to a relevant fragment of the *Elements*. I shall do the former in section II, and the latter in section III. I hope to show, in this way, that my interpretation is plausible, at least. The plausibility of my interpretation should not only result from the consistence and appropriateness of the reconstruction I shall offer of a small part of the *Elements*. It should also result from my interpretation's aid in solving an open interpretative problem: that of the status of the first definitions of book I. I shall consider this matter in section III.1.

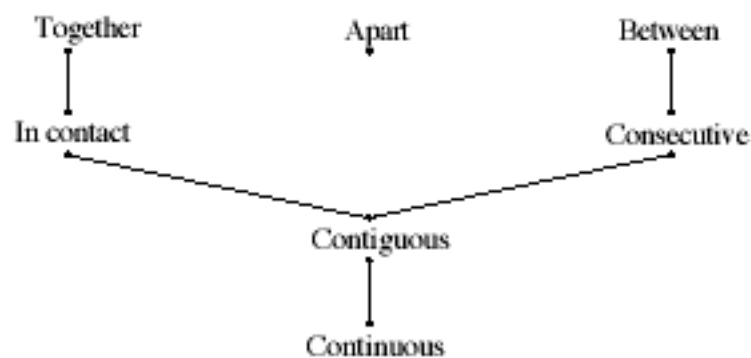
### **I. Aristotle on continuity**

The Aristotelian conception of continuity is not about the continuum as an object; it is rather about the concept of continuity<sup>14</sup>. It is moreover twofold, since continuity applies both to the first movement and to physical objects and particular movements. The latter are continuous insofar as they are generated by the former. Still, any physical object exists as a particular individual only if its

generative motion stopped. Hence continuity is at the same time present and irremediably lost in the latter.

To avoid contradiction, one should distinguish two senses in which Aristotle speaks of continuity: a global sense, according to which only the first movement is continuous; and a local sense, according to which physical objects and particular movements can be continuous. I shall limit myself to this local sense, established by an appropriate definition occurring in *Physics* V, 3.

This is the last one in a sequence of seven definitions, the first six of which concern relations: the relations of being together, apart, in contact, between, consecutive, and contiguous. Five of these definitions form a tree, and the definition of continuity is the unique root of this tree:



Despite the fact that continuity is not a relation, this allows Aristotle to define it as the property of something that is complying with a relational condition: ‘that which is continuous is like that which is contiguous’<sup>15</sup> (227a, 10); something is contiguous if it is ‘consecutive and in contact’ (227a, 6-7); and it is continuous if it is moreover such that ‘the extremity of each one of the two [parts] at which [these parts] touch themselves becomes one and the same [thing], and (as the word tell [us]) holds jointly’ (227a, 11-12).

A classical way to reformulate this definition is the following: something is continuous if the extremities of any pair of its consecutive parts are not merely ‘together [ἄμα]’, but also ‘one [ἓν]’<sup>16</sup>. Aristotle himself entitles such a reformulation: ‘continuous are [the things whose] extremities [are] one’, he claims later (231*a*, 22).

This reformulation is open to two difficulties, however. According to Aristotle, ‘one says [to be] together in place<sup>17</sup> those [things that] are in a [the] same proper place’ (226*b*, 21-22), and ‘the proper place [is] neither smaller nor greater [than the thing itself]’ (211*a*, 2). The difficulties are the following: *a*) Aristotle’s does not define the relation of being together, in general, but only that of being together in place; hence, either such a reformulation is unclear, or does not apply to entities as movements that have a temporal determination that is supposed to be relevant for their continuity; *b*) when the definition of being together in place is compared with the characterisation of a proper place, it is difficult to understand how things that are not one (that is, two distinct things) could be together in space, and thus how the condition of being one could be an additional condition for things that are together in space.

Let us begin with (*b*). Ross suggests two possible solutions<sup>18</sup>: *i*) ‘one thing, occupying one place, may be two things in the sense that it discharges two functions’, hence, two distinct things are together in space if they are distinct for their functions, though being the same thing; *ii*) two (extensionally distinct) things are together in space ‘if they are in one place which contains nothing but *the two*, i.e. where there is nothing between them’.

According to (i), things are together in space without being one if they are the same thing with different functions. Thus, the condition of being one is an additional condition for things that are together in space insofar as it requires that no functional distinction occurs. For a physical object to be continuous, it would then not be enough that the final extremity of any part of it be also the initial extremity of the consecutive part. It would be also necessary that such an extremity have not distinct functions. But if there are two actual parts and they share an extremity, this cannot but be, at the same time, the final extremity of a part and the initial extremity of the other. Hence, if a physical object had actual parts, it could not be continuous: its continuity would be the same as it having no actual part at all. A physical objects would thus be continuous insofar as it is, so to say, intrinsically one: ‘actually undivided’, though, possibly, not ‘potentially undivided’, that is, capable of being divided (*De anima*, 430b, 6-8). It would not be composed by parts, though it would possibly be capable to be decomposed in parts.

According to (ii), things are together in space without being one if there is nothing between them, though they are not the same thing. Thus, to require that two things that are together in space be one would be the same as to require that they be the same thing, possibly discharging different functions (if so, there would certainly be nothing between them, indeed). For a physical object to be continuous, it would then be just enough that the final extremity of any parts of it be also the initial part of the consecutive part. Such a physical object could have actual parts, but these parts could not be spatially separated from each other.



The solution (i) has an important advantage over (ii): the definition of continuity that results from (i) can easily be generalised to motions. According to (i), a motion could, indeed, be said to be continuous, insofar as it has no actual part, or it is intrinsically one. For this condition to be understandable, it would be necessary to establish what counts as an actual part of a motion. But this would be a problem also for physical objects. Hence, if (i) were admitted, the difficulty (a) would *ipso facto* solved, but a new difficulty would arise, that of specifying the notion of actual part both for physical objects and motions.

If (ii) were admitted, the difficulty (a) would not be solved, instead, and motions could then be said to be continuous only if their continuity would be supposed to depend on their trajectory alone. This is not only implausible; it is also inconsistent with Aristotle's arguments of *Physics*, VIII.8, according to which a motion that stops somewhere is not continuous (262a, 13-14).

Moreover, according to these same arguments, a circular motion is continuous since it never changes direction (264b, 9-13), whereas a recliner motion that turns backs cannot be so since it inverts direction (261b, 31-34). It seems thus natural to admit that for Aristotle a motion along a broken line is not continuous, and, thus, that neither its trajectory is so. But, according to (ii), angles do not entail discontinuity. Hence, if (ii) were admitted, there would be no room for concluding that this is so. This seems to me a strong enough reason for rejecting (ii).

Let us resume. If we admit that for Aristotle something is continuous if the extremities of any pair of its consecutive parts are not merely together, but also one, and interpret this condition according to (ii), we reach implausible

conclusions that are moreover inconsistent with the arguments of *Physics*, VIII.8. As there is strong textual evidence for making such an admission, this suggests interpreting this condition according to (i). It follows that for Aristotle something is continuous if it has no actual parts, that is, if it is intrinsically one. This definition leaves open the problem of establishing what counts as an actual part of something that is possibly continuous. Still, the same arguments of *Physics* VIII.8 suggest that, according to Aristotle's conception of continuity, angles mark the extremities of different actual parts of a physical line.

For the purpose of my paper, it is important to notice that such an understating of the notion of continuity does not provide it with any sort of logical analysis. The concept of continuity is not, according to this understanding, reduced to other concepts or defined in terms of them: to say of something that it is intrinsically one, or that it has no actual part is only to exclude the possibility that it is composed by other things each of which is intrinsically one. Thus, according to Aristotle's conception, continuity is a primitive property of physical objects and movements: a property that can only be displayed by showing a physical object or a motion.

## **II. The Double Role of Diagrams in EPG**

Aristotle's definition of local continuity is not a definition of an abstract—and, *a fortiori*, a mathematical—object. Still, it is apt to provide a basis for a mathematical theory of some local continua. I argue that EPG is such a theory<sup>19</sup>.

J. Klein<sup>20</sup> has argued that ‘modern mathematics’ is ‘symbolic’, whereas ‘Greek science’, and especially Euclid’s mathematics, are not. This is because the latter ‘represents the whole complex of those “natural” cognitions which are implied in a prescientific activity’ and its concepts ‘are formed in continual dependence on “natural” prescientific experience’. If I understand well, this means that we have acquaintance of the objects of EPG through our experience of concrete objects, or, as Kline says, that we have an ‘immediate insight’ of them.

Though the objects of EPG can certainly be viewed as forms of physical objects, I do not think that EPG is devoid of any symbolic component. It relies on the capacity of operating with physical objects like diagrams, but these are not its objects: EPG is neither an empirical, nor a contentual theory in Hilbertian sense<sup>21</sup>. It rather relies on diagrams to deal with abstract objects, so that the former can be understood as symbols of the latter.

Still, in EPG the relation between diagrams and the geometric objects they represent is quite peculiar<sup>22</sup>. Its peculiarity results from the facts described in previous claims *C.i* and *C.ii*. Because of the first of these facts, diagrams contribute to fix the objects of EPG; in my parlance, this is the global role of diagrams. Because of the second, they contribute to assign properties to these objects; in my parlance, this is the local role of diagrams. In the two following sections, I shall try to describe these roles better<sup>23</sup>.

## II.1. THE GLOBAL ROLE OF DIAGRAMS

Like any other mathematical theory, EPG relies on stipulations. These can be understood as prescriptions addressed to the members of a relevant community that are supposed to have appropriate cognitive abilities for understanding and applying them<sup>24</sup>.

Some of these prescriptions are supposed to provide appropriate conditions for recognising different sorts of geometric objects, others are supposed to provide appropriate conditions for recognising distinct geometric objects of each sort. The former are conditions of application of appropriate concepts; the latter are identity conditions for the objects of a certain sort. Suppose that the  $X$ 's are the objects of a certain sort. For short, instead of saying 'to provide the conditions of application of the concept of the  $X$ 's' and 'to provide the identity conditions for the  $X$ 's', I shall say, respectively: 'to characterise the  $X$ 's' and 'to identify the  $X$ 's'.

Under a first and very general classification, the objects of EPG are points, segments of straight lines<sup>25</sup> (segments *tout court*, from now on), circles, plane angles (angles *tout court*, from now on), polygons, and their combinations and parts. As segments and circles can be understood as lines, all these objects, with the exception of angles and the combinations they belong to, can be understood as configurations of points and lines (I take, of course, a point or a line to be an elementary configuration of points and lines). Angles can be understood, instead, as that which is common to the elements of an equivalence class of pairs of lines<sup>26</sup>. Circles and polygons can also be understood as portions of plan, or figures

in the Euclidean sense<sup>27</sup>. But for the limited purpose of my paper the first understating will suffice.

If this understating is the only one admitted, then characterising the objects of a certain sort is the same as providing the conditions that a configuration of geometric points and lines have to satisfy in order to be—or, in case of angles, to determine—an object of this sort. I suggest that this consists in establishing the conditions that a certain diagram has to meet in order to be appropriate for representing such an object<sup>28</sup>.

In general, we could say that: geometrical points are the geometrical objects represented by physical points (the extremities of physical lines); segments are the geometrical objects represented by physical straight lines; circles are the geometrical objects represented by physical circles; angles are the geometrical objects represented by pairs of physical lines that represent segments or circles; polygons are the geometrical objects represented by contour-closer physical broken lines.

To make EPG, it is not enough to characterize appropriate sorts of its objects, however. It is also necessary to identify these objects.

Typically, an identity condition for the objects of a certain sort is an instance of the schema ' $x = y$  IFF  $C(x, y)$ ', where ' $x$ ' and ' $y$ ' are distinct names or descriptions for single objects of this sort, and ' $C(x, y)$ ' is another condition where these names or descriptions occur. An identity condition establishes that a condition like ' $C(x, y)$ ' holds if and only if ' $x$ ' and ' $y$ ' apply to the same object. It follows that, in

order to identify the objects of a certain sort, it is necessary that single objects of this sort are set out, and appropriate names or descriptions for them are available.

I argue that in EPG a single object is set out if and only if it is given. Hence, in my view, EPG includes only identity conditions for objects that are, or are supposed to be, given<sup>29</sup>. To understand the nature of these conditions is thus necessary to understand what ‘given’ means in EPG<sup>30</sup>.

Though in the *Elements*, geometric objects are often taken as given, the conditions under which an object is given, are never explicitly stated. And this is neither done in the *Data*, whose definitions 1, 3 and 4 establish, rather, under which conditions appropriate geometric objects are given-in-magnitude, given-in-form, and given-in-position, respectively<sup>31</sup>.

In his recent commentary of the *Data*, C. M. Taisbak has discussed these definitions. He has argued that the term ‘given [δεδομένοϛ]’ means in them what it usually means: ‘that an object is given to us means that it is, in some relevant sense and scope, put at our disposal’<sup>32</sup>. According to him, the term ‘given’ occurs in these definitions as ‘a primitive needing no definition’, and ‘the very concept of *given* remains undefined’<sup>33</sup>. In his view, these definitions merely establish the conditions under which ‘some objects are *also* given (in the said respect), besides [...] those that are already given’<sup>34</sup>.

Take the example of definition 1: ‘Given in magnitude is said of figures and lines or angles for which we can provide equals’<sup>35</sup>. According to Taisbak, this definition establishes that an appropriate object *x* is given-in-magnitude if and only if ‘we can provide’ its equal, and this is equivalent to state that an

appropriate object  $x$  is given-in-magnitude if and only if it is equal to an already given object  $a$ <sup>36</sup>.

This interpretation leaves the crucial question open: what does it mean that a geometric object is ‘put at our disposal’ in some ‘sense and scope’, and that ‘we can provide’ an already given object  $a$ —that is, a geometric object  $a$  that is already at our disposal—which is equal to another geometric object  $x$ ? This is not because Taisbak’s interpretation is deficient. It is rather because Euclid’s definition does not aim to establish what ‘given’ means, in general.

Taisbak seems to suggest that our possibility of providing an object  $a$  which is equal to  $x$  is the same as the fact that  $x$  is provably equal to  $a$ , provided that  $a$  is already given. In the definition 1 of the *Data*, Euclid would thus not employ the verb ‘to provide [πορτίζω]’ as a synonymous of the verb ‘to give [δίδωμι]’. Strictly speaking, the object  $a$  would not be provided, but provided-as-equal, and this would merely mean that it is given and proved to be equal to  $x$ .

But what about the verb ‘to give’?

Taisbak’s relies on Plato’s account of *Republic* VII, 527a6-b6 to argue that ‘when mathematicians are doing geometry, describing circles, constructing triangles, producing straight lines, they are not really *creating* these items, but only *drawing pictures* of them’<sup>37</sup>. For him the giving of geometric object concerns the ‘Realm of Intelligence’, where ‘The Helping Hand [...] takes care that lines are drawn, points are taken, circles described, perpendiculars dropped, etc.’ and keeps these operations ‘free from contamination of our mortal fingers’<sup>38</sup>. Taisbak makes the example of the postulate I.1 of the *Elements*, that licenses ‘to draw [a]

straight line from any point to any point'. According to him, such a postulate should be understood as follows: 'whenever there are two points, there is also one (and only one) straight line joining them', and the geometer is 'permitted to behave accordingly, that is to conceive a picture of this line'<sup>39</sup>.

According to Taisbak, a geometric object would thus be given insofar as it results from an act of selection: it would be put at our disposal if it is selected among other objects that are already there as the inhabitants of an eternal realm of abstract objects<sup>40</sup>. Diagrams would thus be nothing but pictures that geometers use, for their convenience, to denote the objects they successively select.

I have a quite different interpretation from Taisbak's.

I begin by suggesting that definition I of the *Data* could also be understood the other way around: as stating that an appropriate object  $a$  is given-in-magnitude if and only if it is given and we can provide another object  $x$  of which it is possible to show that it is equal to  $a$ . This interpretation differs from Taisbak's in two respects: it admits that the verb 'to provide' occurs in this definition as a synonymous of the verb 'to give'; it assigns a crucial role to the modal operator that occurs in Euclid's definition<sup>41</sup>.

I shall come back to the second point in a moment. Before that, it is necessary to clarify my understanding of the notions of being given, or provided. I suggest that in EPG it is not possible to give more than one object at once, and that an object is given<sup>42</sup> if and only if a diagram appropriate to represent it is canonically drawn. I shall say in a moment what 'canonically' means. At present it is enough to notice that, whatever it means, from this condition—and from the fact that, as I



have argued below, EPG includes only identity conditions for objects that are, or are supposed to be, given—it follows that these conditions applies only to objects that are, or are supposed<sup>43</sup> to be, actually represented by appropriate diagrams. This fits quite well with the previous claim, according to which in EPG to characterise the objects of a certain sort is the same as to establish the conditions that a certain diagram has to meet in order to be appropriate for representing an object of this sort.

But suppose that ‘ $x$ ’ and ‘ $y$ ’ are names or descriptions for two given objects of EPG. Under which condition  $x$  is the same object as  $y$ ? I suggest that the right answer is the following: if  $x$  and  $y$  are points, segments, circles or polygons, then  $x$  is the same object as  $y$  if and only if they are represented by the same diagram; if  $x$  and  $y$  are angles, then  $x$  is the same object as  $y$  if and only if they are represented by diagrams belonging to the same appropriate equivalence class. Hence, the identification of an object of EPG goes together with the identification of a diagram that represents this object, as *C.i* asserts.

With this in mind, we can come back to the second respect in which my interpretation of the definition 1 of the *Data* differs from Taisbak’s. This will allow me to clarify what I mean by saying that a diagram is canonically drawn in EPG. Under my understanding of the notion of being given, the definition 1 of the *Data* establishes, for example, that if a certain physical line, appropriate to represent a segment, has been canonically drawn (so that this segment is given), and we can canonically draw a new physical line appropriate to represent another segment equal to the first, then the former segment is given-in-magnitude.

Take the example of the proposition 4 of the *Data*: ‘if a given magnitude be subtracted from a given magnitude, the remainder will be given’<sup>44</sup>. A given magnitude is a geometric object given-in-magnitude, and this is also the case of the remainder. Euclid’s proof begins as follows: ‘For, since **AB** is given, it is possible to provide a [magnitude] equal to it. Let it have been provided, and let it be **DZ**’<sup>45</sup>. Then Euclid continues by repeating the same argument for the second pair of magnitudes—namely **AC** and **DE**—and concludes that as  $AB = DZ$  and  $AC = DE$ , the remainders are equal and that the remainder of **AB** and **AC** is thus a given magnitude.

Notice that Euclid’s does not say ‘since **AB** is given, it is equal to another magnitude **DZ**’. He separates the claim ‘it is possible to provide a magnitude equal to **AB**’, from the claim ‘let it have been provided’. Under my interpretation ‘to provide’ means the same as ‘to give’. If so, Euclid separates a claim like ‘ $x$  can be given’ from a claim like ‘let  $x$  be given’. His argument is general, but it is illustrated by a diagram where **AB** and **DZ** are depicted as segments (fig. 2). Suppose they are segments. I suggest, then, to interpret Euclid’s argument as follows: **AB** is a given segment represented by an appropriate physical line; it is then possible to draw canonically another physical line that represents another segment equal to it; let this line be drawn and let **DZ** be the segment that it represents; **DZ** is thus given.

The crucial question is the following: what does it mean that a diagram can be canonically drawn, and thus a geometric object can be given?

The example of the proposition 4 of the *Data* cannot help us in responding this question, since, in this proposition, Euclid is reasoning in general, and thus he is only supposing that certain geometric objects be given or can be given. To understand what this means is rather appropriate to consider proposition I.3 of the *Elements*: ‘Given two unequal segments, to cut off from the greater [a] segment equal to the less’. This is a problem. To solve it, Euclid refers to a diagram (fig. 3) including two separate dashes representing the given segments **AB** and **C**. The diagram also includes a third dash representing a segment **AD** equal to **C** that is placed at the point **A** according to proposition I.2. Finally, the diagram includes a contour-closed line drawn around **A** and passing from **D** representing the circle with centre **A** and radius **AD** described according to postulate I.3. Euclid tacitly admits that this circle intersects **AB** in a point **E**, and concludes that this point cuts **AB** as required. I suggest that proposition I.3 of the *Elements* is a particular case of proposition 4 of the *Data*, and that in the former Euclid shows how to give the remainder of the given segments **AB** and **C** (assuming that this remainder is the same as that of the segments **AB** and any segment equal to **C**).

If this is so, the response to the previous questions becomes natural. That a diagram can be canonically drawn means that a certain procedure, that starts from some other given diagrams and, if applied, results in the drawing of the former diagram, is authorised by the stipulations of EPG, or that these same stipulations authorise that such a diagram be taken as a starting point of such a procedure. Consequently, that a geometric object can be given means that a certain procedure, that starts from certain diagrams representing certain given geometric

objects and, if applied, results in a diagram representing such an object, is authorised by these stipulations, or that these same stipulations authorise that a diagram that represent such an object be taken as a starting point of such a procedure.

The plausibility of this interpretation depends on the nature of the authorised procedures for drawing diagrams<sup>46</sup>. I shall consider this crucial point in section III.2. Here, I only need to say that this procedure is what in EPG is usually called ‘construction [κατασκευή]’. Accordingly, I suggest that in EPG a construction is an authorised procedure for drawing diagrams<sup>47</sup>, and that a diagram is canonically drawn in EPG if and only if it results from an appropriate construction or is an authorised starting point of a construction<sup>48</sup>.

As a matter of fact, in the *Elements* Euclid uses the verb ‘to give’ to refer to geometric objects that are taken as given, as in expressions of the form ‘a given  $x$ ’ or ‘given  $x$ , to do that and that’, where  $x$  is a geometric object. He uses instead different verbs when he requires that some objects be set out, or he claims that these objects that have been set out. Five of these verbs occur, for example, in postulates I.1-3 and propositions I.1-3: in Heath’s translation, they are the verbs ‘to draw [ἄγω]’, ‘to produce [ἐκβάλλω]’, ‘to describe [γράφω]’, ‘to construct [συνίστημι]’ and ‘to place [λέγω]’. In my view, these verbs are used to require that particular appropriate procedures be applied, so as to obtain the giving of certain geometric objects. They would thus be particular specifications of the verb ‘to give’. This fits quite well with my interpretation of the verb ‘to provide’ in the definition 1 of the *Data*.

As a matter of fact, Euclid does not use modal operators in the *Elements*, unless we interpret modally the verb ‘to postulate [ $\alpha\lambda\tau\acute{\epsilon}\omega$ ]’ occurring at the perfect imperative middle-passive in postulate I.1 and implied in the other postulates. From Euclid’s practice is however clear that not every construction that in any particular situation could be applied is actually applied. If the notion of being given is understood as I have suggested, the appeal to a modal operator applied to such a notion is quite useful for accounting for this practice<sup>49</sup>. For simplicity, I shall use the verb ‘to be susceptible of being given’ and their cognates to refer to such an operator. Hence I shall say of a geometric object that it is susceptible of being given to mean that it can be given, in the sense that I have previously explained<sup>50</sup>.

Before concluding my account of the global role of diagrams in EPG, a last remark is useful. Also admitting that what I have argued for is right, one could wonder why EPG needs diagrams playing their global role. I suggest the following response: EPG relies on diagrams playing their global role in order to reduce the conditions of application of the concepts of its objects and the identity conditions for these same objects to conditions of application and identity relative to physical objects. This reduction is just what makes the former conditions available in EPG.

## II.2. THE LOCAL ROLE OF DIAGRAMS

Once this reduction is admitted, Aristotelian local continuity can apply to EPG and provide the conditions under which each of its objects is intrinsically one.

A diagram is a compositional object: it can be formed by other diagrams. This is the same for the objects of EPG. Thus, to draw a diagram in EPG is possibly the same as to draw distinct (sub-)diagrams, representing distinct objects or distinct configurations of objects. But how are diagrams and, consequently, geometric objects distinct?

Many distinctions depend on specific stipulations. For example, that an appropriate configuration of three segments be taken as a single object—a triangle—depends on the definition of triangles, whereas that a configuration of two triangles, external to each other but sharing a vertex, is never taken as a single object depends on the lack of an appropriate definition. But not any distinction can depend on a specific stipulation: for a specific stipulation to work, elementary objects have to be detected. I suggest that in EPG this is done according to a criterion that fits with Aristotle's conception of continuity: angles in physical lines, and *a fortiori*, spatial gaps among these lines entail separation between distinct geometrical lines that are intrinsically ones; hence, a physical broken line or a pair of lines that mutually intersect or do not touch to each other are not intrinsically ones, that is, they are not elementary diagrams and represent configurations of distinct elementary geometric objects.

This is a consequence of a more general fact concerning the objects of EPG: they inherit the property of continuity from the diagrams that represent them. This is part of the content of *C.ii*. To understand this point it is thus necessary to elucidate this claim.

For reasons of linguistic simplicity, let us admit that the term ‘attribute’ refers either to properties or to relations, and that to say that some objects have a certain attribute is the same as to say either that one or more objects have a certain property or that some objects stay in a certain relation.

This being admitted, consider a certain sort of abstract objects and a certain sort of physical objects—let us say, the  $\mathcal{G}$ ’s and the  $\mathcal{D}$ ’s, respectively—and suppose that: the  $\mathcal{G}$ ’s are linked with the  $\mathcal{D}$ ’s by an application  $\varphi$ ;  $x, \dots, z$  and  $x', \dots, z'$  are  $\mathcal{D}$ ’s, and  $\varphi(x), \dots, \varphi(z)$  and  $\varphi(x'), \dots, \varphi(z')$  are  $\mathcal{G}$ ’s; ‘=’ denotes identity. I say that the  $\mathcal{G}$ ’s inherit a certain attribute—let say  $P$ —from the  $\mathcal{D}$ ’s if and only if  $P$  applies to the  $\mathcal{D}$ ’s, and it is admitted that: *i*) some  $\mathcal{G}$ ’s have a certain attribute and there is no other way to explain what it means that they have this attribute besides saying that they have  $P$ ; *ii*)  $\varphi(x), \dots, \varphi(z)$  have this attribute (that is, according to *i*), they have  $P$ ) if and only if any  $x', \dots, z'$  such that  $\varphi(x') = \varphi(x), \dots, \varphi(z') = \varphi(z)$  have  $P$ ; *iii*) if  $\varphi(x), \dots, \varphi(z)$  have this attribute (that is, if they have  $P$ ), and  $Q$  is an attribute that applies to the  $\mathcal{D}$ ’s, complies with the conditions *i*) and *ii*), and is such that any  $x', \dots, z'$  such that  $\varphi(x') = \varphi(x), \dots, \varphi(z') = \varphi(z)$  have  $Q$  if they have  $P$ , then  $\varphi(x), \dots, \varphi(z)$  have the attribute corresponding to  $Q$  (that is, they have  $Q$ ).

Suppose now that the  $\mathcal{G}$ ’s are the objects of EPG, the  $\mathcal{D}$ ’s are diagrams, and  $\varphi$  is the application that associates diagrams to the objects they represent. Then the objects of EPG inherit an attribute  $P$  from the diagrams if and only if  $P$  applies to diagrams and it is admitted that: *i*) some objects of EPG have a certain attribute and there is no other way to explain what it means that they have this attribute besides saying that they have  $P$ ; *ii*) the objects of EPG have this attribute (that is,

according to (i), they have  $P$ ) if and only if the diagrams that represent these objects have  $P$ ; *iii*) if some objects of EPG have this attribute (that is, they have  $P$ ), and  $Q$  is an attribute that applies to diagrams, complies with the conditions (i) and (ii), and is such that all the diagrams that represent these objects have  $Q$  if they have  $P$ , then these objects have the attribute corresponding to  $Q$  (that is, they have  $Q$ ).

Let us suppose now that the objects of EPG inherit a certain attribute  $P$  from the diagrams. I shall say, for short, that  $P$  is a diagrammatic attribute (by implicitly implying that it is an attribute that applies to the objects of EPG).

What I argue is thus that continuity is a diagrammatic property. This means that continuity applies to diagrams, and it is admitted that: *i*) some objects of EPG enjoy a certain property and there is no other way to explain what it means that they enjoy this property besides saying that they are continuous; *ii*) an object of EPG enjoys this property (that is, according to (i), it is continuous) if and only if the diagrams<sup>51</sup> that represent this object are continuous<sup>52</sup>; *iii*) if some objects of EPG enjoy this property (that is, they are continuous), and  $Q$  is an attribute that applies to diagrams, complies with the conditions (i) and (ii), and is such that all the diagrams that represent these objects have  $Q$  if they are continuous, then these objects have the attribute corresponding to  $Q$  (that is, they have  $Q$ ).

This explication could appear cumbersome, but reflects the nature of continuity in EPG: when applied to geometric objects, the adjective “continuous” cannot have the same sense as it has when applied to physical objects (so that, strictly speaking, continuity of physical objects and continuity of geometrical objects are



distinct properties), but the sense it has when applied to geometric objects cannot be separately explained for lacking of appropriate (logical) resources<sup>53</sup>.

The same happens with the properties of having extremities and of being contour-closed. These properties are thus diagrammatic, in turn.

Insofar as they are physical objects, diagrams have a position in space, and distinct (sub-)diagrams composing a unique diagram differ from each other with respect to their positions in space. I argue that this is what confers their mutual spatial relations to the corresponding geometric objects<sup>54</sup>. EPG is not concerned with metric properties and relations of its objects. Thus, by speaking of mutual spatial relations of the objects of EPG I mean non-metric relations like the relations of intersecting each other, of being formed by, of being part of, of being inside, of being included in, of being on, or on a certain side of, of passing through, of having an extremity on, of sharing an extremity. I argue that these are diagrammatic relations.

The local role of diagrams—that is, the fact that many properties and relations of the objects of EPG are diagrammatic—allows many arguments in EPG to be diagram-based. An example is provided by the argument related to proposition I.1. This argument depends on the fact that the properties of continuity, of having an extremity, and of being contour-closed, and the relations of intersecting each other, of being inside, and of passing through are diagrammatic. This being admitted, the crucial step in this argument can be reconstructed as a deduction of two conclusions from four premises involving diagrammatic attributes:

Premise *i*) A circle is a contour-closed line, passes through an extremity of a radius of it, and is such that the other extremity of this radius is inside it;

Premise *ii*) A contour-closed line that passes through a point that is inside another contour-closed line intersects this latter line;

Conclusion *i*) The second circle crosses the first;

Premise *iii*) Circles are continuous;

Premise *iv*) Continuous lines are capable of being divided, are actually so when another line intersects them, and, because of this division, an extremity is given where this other line cut them;

Conclusion *ii*) The first circle is divided, and, because of this division, an extremity is given where the second circle intersects it.

Once this last conclusion is admitted, the definition I.3 allows to derive that a point is given.

The presence of premises (*i-iv*) in this argument manifests the local role of diagrams. Still this role does not manifest itself only insofar as many arguments of EPG can be reconstructed as arguments including premises like these. To understand the other ways in which this role manifests itself, it is useful to consider two distinctions introduced by K. Manders<sup>55</sup>.

The first is the distinction between two components of a ‘demonstration’ in EPG: the ‘discursive text’ —that ‘consists of a reason-giving ordered progression of assertions, each with the surface form of an ascription of a feature to a diagram’ —and the diagram itself. According to Manders, a step in the discursive text ‘is licensed by attributions either already in force in the discursive text or

made directly based on the diagram as part of the step, or both' and 'consists in an attribution in the discursive text, or a construction in the diagram or both'<sup>56</sup>.

The second distinction is that between 'exact' and 'co-exact geometric attributes'<sup>57</sup>. The former 'are those which, for at least some continuous variation of the diagram, obtain only in isolated cases'. The latter 'are those [...] which are unaffected by some range of every continuous variation of a specified diagram'<sup>58</sup>.

Manders' crucial claim is that an 'exact attribution is licensed only by prior entries in the discursive text; and may never be "read off" from the diagram', whereas 'co-exact attributions either arise by suitable entries in the discursive text [...] or *are licensed directly by the diagram*'<sup>59</sup>.

I agree with Manders on the fruitfulness of the two previous distinctions. Still, it seems to me that the latter should be better specified, and that co-exact attributes should not be confounded with diagrammatic properties and relations.

Unlike Manders<sup>60</sup>, I take exact and co-exact attributes as attributes of geometric objects. Hence, I argue that they should be distinguished based on what happens in a configuration of geometric objects, rather than in a diagram, and I suggest understanding exact attributes as those attributes of geometric objects that hold or do not hold for these objects based on the properties of non-elementary configurations of these objects that, for at least some variation of these configurations, obtain only in isolated cases; and co-exact attributes as the remaining attributes of geometric objects that hold or do not hold for these objects based on the properties of non-elementary configurations of them.

According to such a characterisation, the distinction of exact and co-exact attributes apply only to relations among geometric objects or to properties of these objects that result of the saturation of  $n-1$  places in a  $n$ -places relation. Hence, it does not apply to properties like straightness, circularity, contour-closure, and, more importantly, continuity. This is enough to conclude that co-exact attributes do not coincide with diagrammatic properties or relations. But neither co-exact attributes other than properties like the latter coincide with diagrammatic properties or relations. For example, the diagrammatic relations of intersecting each other, of being formed by, of being part of, of being inside, of being included in, of being on, and of being on a certain side of are co-exact, whereas the diagrammatic relations of passing through, of having an extremity on, and of sharing an extremity are exact.

Despite the difference between my and Manders' understanding of the distinction between exact and co-exact attributes, it seems to me that he is right in claiming that diagrams cannot license exact attributions. Of course, diagrams cannot licence non-diagrammatic attributions either, whereas the possibility of non-diagrammatic attributions (either exact or co-exact) that are not licensed by diagrams is obvious. Also the possibility of diagrammatic attributions that derive by other diagrammatic attributions in force of the application of some deductive rule is obvious (provided that EPG relies on some form of logic). Call these last attributions 'diagram-neutral'.

All this being admitted, the relevant questions concern co-exact diagrammatic attributions licensed by diagrams and diagrammatic attributions—either exact or

co-exact—that are neither licensed by diagrams nor diagram-neutral. Do they occur in EPG? I argue that they do.

The argument related to proposition I.1 that I have reconstructed above provides evidence for the occurrence of co-exact diagrammatic attributions licensed by diagrams: the co-exact attribution involved in premise (ii) is clearly licensed by diagrams, and the second part of the argument—from conclusion (i) to conclusion (ii)—shows the way as continuity of diagrams<sup>61</sup> enters arguments of EPG to transform a diagrammatic co-exact attribution—like that involved in conclusion (i)—in a conclusion asserting that a certain geometric object is given.

But what about diagrammatic attributions that are neither licensed by diagrams nor diagram-neutral?

Consider the example of proposition I.3 that I have relied on above. The diagrammatic attribution to the circle and the greater segment of the co-exact relation of intersecting each other is certainly not licensed by diagrams. Things go the other way around: it is because the greater segment is greater that the circle intersects it, and the diagram has to be drawn so to display this co-exact diagrammatic relation. Still, there is no deductive inference from the premise ‘the segment  $AB$  is greater than the segment  $AD$  and shares with it the extremity  $A$ ’ to the conclusion ‘the circle whose radius is  $AD$  and whose centre is  $A$  intersects  $AB$ ’. Euclid’s argument seems rather to depend on a reduction of the non-diagrammatic relation of being greater than<sup>62</sup>, applied to two segments that share an extremity, to the diagrammatic relation of intersecting to each other, applied to the greater of these segments and the circle whose the smaller of them is a radius.

This is another relevant aspect of the local role of diagrams in EPG: many co-exact non-diagrammatic attributes are reduced to co-exact diagrammatic attributes, and this allows many theorems to be proved and many problems to be solved<sup>63</sup>.

But suppose now that  $AB$  and  $AD$  are two equal geometrical segments share the extremity  $A$  and that this is the centre of a circle that has  $AD$  as radius. We are not licensed to conclude that this circle passes through the extremity  $B$  of  $AB$  based on the fact this is so for the diagram that represents these objects. Thus the attribution of this exact diagrammatic relation to such a circle and the segment  $AB$  is not diagram-based. But it is neither diagram-neutral. Definition I.15 asserts that a circle is such that their radii are equal but it does not ensure, as such, that if  $AB = AD$  then the circle of centre  $A$  and radius  $AD$  passes through  $B$ . To derive that it so from this definition, one has to suppose that it is not so and conclude that, then, the segments  $AB$  et  $AD$  cannot be equal because one of them is equal to another segment that is cut off from the other or from which this other segment is cut off. This argument by *reductio ad absurdum* involves the derivation of non-diagrammatic attributions from diagrammatic ones. Suppose (fig. 3, again) that the circle of centre  $A$  and radius  $AD$  intersects  $AB$  in a point  $E$ . From the exact diagrammatic attribution to the segments  $AD$  et  $AE$  of the relations of sharing the extremity  $A$  and of being such that the former is the radius of a circle of centre  $A$  that passes through the extremity  $E$  of the other, one derives the exact non-diagrammatic attribution to these same segments of the relation of being equal to each other. Then, from the co-exact diagrammatic attribution to the segments  $AE$

and **AB** of the relation of one being a part of the other, one derives the co-exact non-diagrammatic attribution to these same segments of the relation of one being greater than the other. Both these derivations are licensed by a reduction of a non-diagrammatic relation to a diagrammatic one: whereas the reduction that licenses the latter derivation concerns two co-exact relations, that which licenses the former derivation concerns two exact relations.

It follows that in EPG also exact non-diagrammatic attributes are reduced to exact diagrammatic ones, and this also allows many theorems to be proved and many problems to be solved.

But, it is not enough to remark that in EPG both exact and co-exact non-diagrammatic attributes are reduced to diagrammatic ones, in order to understand the role that diagrams play in arguments that involve such reductions. It is also necessary to notice that these reductions provide, in fact, the most fundamental explication of the former attributes in EPG. To take only an example, what does it mean, in the last resort, in EPG, that two segments are equal to each other? The answer leaves no doubt: it means that an appropriate circle passes through an extremity of one of them; it is enough to study the proposition I.1 and I.2 of the *Elements* to understand that it is so.

### II.3. DETERMINATES AND ACCURANTENS OF DIAGRAMS

EPG can be understood as a close system, that is, a pure argumentative game governed by its own stipulations; but it can also be understood as a mathematical account of some features of physical reality. When conceived in the first way,

EPG requires only that its diagrams be appropriate for playing their global and local roles. But this is possibly not enough for it to be able to provide an appropriate mathematical account of some features of physical reality. This depends of course on the features of physical reality that EPG is supposed to account for. One could require, for example, that its diagrams have some attributes corresponding to non-diagrammatic attributes of its objects, namely that equal segments be represented by physical lines of equal length, or circles be represented by (contour-closed) physical lines with equal curvature.

These two sorts of requirements impose that diagrams have certain features but they are far from determining them completely. Hence, in EPG the shape of diagrams can vary within the limits imposed by these requirements (or even only by the former of them, if EPG is merely understood as a close system), and it varies in fact in the extant manuscripts of the *Elements*, as K. Saito has shown for some Greek and Latin manuscripts<sup>64</sup>.

All that I said and I shall say on the roles of diagrams in EPG is of course intended to be unaffected by these variations, and Saito's results confirm that this is so. The principal phenomenon he notes is 'overspecification': in the manuscripts he has studied, many diagrams are inaccurate 'as metrical representation of geometric objects', since they represent polygons as more regular than the relative propositions require<sup>65</sup>. According to him, this suggests that 'diagrams are not meant to be a strict reproduction of the spatial relationships of geometric objects along the lines of a photograph but are rather meant to be a schematic representation', that he also calls 'topological'<sup>66</sup>. In my language, this



can be said as follows: in EPG, diagrams has to correctly represent diagrammatic attributions but are not supposed to represent correctly metric relations, and, as a matter of fact, in the manuscript studied by Saito, they tend to represent these last relations as if they met some unnecessary conditions of regularity.

But sometimes, both in the Heiberg's edition and in the Greek and Latin manuscripts studied by Saito, diagrams are also inaccurate with respect to diagrammatic attributions. This generally happens in two quite particular circumstances: when proofs by *reductio ad absurdum* or arguments involving distinctions of cases are concerned.

Diagrams related to proofs by *reductio* are of two different types exemplified, respectively, by the propositions I.6 and I.27 of the first book of the *Elements*. Diagrams of the first type are as usual, though they are partially inconsistent with some diagrammatic attributions involved in the proof: in the case of proposition I.6, an angle that should be equal to another—if two appropriate segments were equal to each other—is represented as included in this other angle. Diagrams of the second type include unusual representations of mathematical objects: in the case of proposition I.27, two segments are represented by two broken lines (though the proof could also be conducted based on a usual but metrically inaccurate diagram<sup>67</sup>).

Diagrams related to arguments involving distinctions of cases, are sometimes intended to represent the totality of cases at once, to the effect that they misrepresent some diagrammatic attribution<sup>68</sup>.

The inaccuracy of diagrams occurring in these circumstances depends on their quite particular function, and it does not contradict my account of their role in EPG.

#### II.4. BEING GIVEN, EXISTENCE, AND CONSTRUCTION

In section II.1, I have suggested that EPG includes only identity conditions for given objects or for objects that are supposed to be given, and that in EPG a geometric object is given if and only if a diagram appropriate to represent it is canonically drawn, which entails that in EPG a geometric object is given only if a diagram appropriate to represent it is drawn. Should one admit, thus, that EPG does not provide appropriate resources for identifying those of its objects that are neither actually represented by a certain diagram nor supposed to be so (that is, for providing the conditions of identities of these objects)? If one takes for granted that mathematical—especially, geometrical—objects exist as such, independently of the theories that deal with them, there would be no alternative, I'm afraid. But, why should an account of EPG take for granted that mathematical geometrical objects exist as such? Would it not be possible, instead, to admit that mathematical objects are objects of certain theories, so that their nature depends on the identity conditions provided by (or in) these theories? If this is admitted, then the question of their existence also depends on these identity conditions.

From such a point of view, once it is admitted that in EPG things go as I have suggested on matter of identification and being given, the question of whether EPG does provide resources for identifying those of its objects that are neither

actually represented by a certain diagram nor supposed to be so (and thus neither given, nor supposed to be given) is deprived of any clear meaning. This is because the very questions of whether such objects exist—or whether universal statements of EPG concerns objects like those—are deprived of any clear meaning.

But how should one, then, interpret a theorem in EPG? Is it a universal statement? And if it is, what is the range of the universal quantifiers it includes? Consider the example of the first part of proposition I.5: ‘angles at the bases of isosceles triangles are equal to one other’. I suggest understanding it as follows: ‘if an isosceles triangle is given, then its angles at the base are equal to one other’. One could also rewrite this as follows: ‘angles at the bases of any given isosceles triangle are equal to one other’. But it would then be important to clarify that this does not mean that, among isosceles triangles, those that are given have angles at the base equal to each other, but rather that any time that an isosceles triangle is given it has angles at the base that are equal to each other. Isosceles triangles—as well as any other objects of EPG—are not given once forever: they are given in a context of a particular argument, where they are distinct from any other geometric object occurring in this same argument according to their identity conditions and to the conditions of application of the corresponding concepts. The only thing that can be done once forever, in EPG, concerning isosceles triangles, is simply to prove that they are susceptible of being given, provided that two segments are given<sup>69</sup>, and that, any time that they are given, they have a certain property. This is what a solution of a problem and a proof of a theorem result in, respectively.

To generalise we should claim that in EPG things go as follows: in solving a problem, one proves that geometric objects of a certain sort are susceptible of being given under appropriate conditions<sup>70</sup>; in proving a theorem, one establishes that if such an objects is given, then it is so and so<sup>71</sup>. And this is all that is needed, in fact.

If I'm right, the ontology of EPG is structurally very different from the ontology of a contemporary axiomatic theory. This is because in EPG, geometric objects are not (implicitly) defined at once, through an appropriate system of axioms that assesses, once for all, their mutual relations and warrants their existence as elements of a certain structure (the structure whose elements satisfy these relations)<sup>72</sup>. But EPG is neither founded on a simple recursive definition: its objects do not result from a repeated iteration of a single constructive clause fixed once for all (like in constructive arithmetic, for example). The objects of EPG are rather defined through several steps.

First come what Euclid calls 'definitions'. As a matter of fact, these are not sufficient to define the objects of EPG in our sense of the verb 'to define': they merely provide the conditions of application of an appropriate family of concepts (the concept of some sorts of geometrical objects). These conditions admit an easy interpretation on freely drawn diagrams<sup>73</sup>.

Postulates (or at least some of them) are then employed to codify the procedure for drawing diagrams canonically—that is, for establishing the rules of constructions—and so to allow an alternative interpretation of definitions on canonically drawn diagrams<sup>74</sup>. The identity conditions for the objects of EPG

depend on this last interpretation: as said, these objects are identified insofar as they are represented by diagrams that are canonically drawn.

But such an interpretation is not offered at once. It is rather obtained step by step according to an order whose appropriateness reveals itself only *post festum*. This is the order in which geometric objects are proved to be susceptible of being given: to say of some objects that they are susceptible of being given is the same as to claim that the conditions of application of the corresponding concepts (their definitions, in Euclid's language) have been successfully interpreted according to the postulates.

It has often been claimed that in EPG constructions provide existence proofs<sup>75</sup>. My account allows to clarify and revise this claim: if to prove the existence of an object of EPG means to establish that this object is there among the eternal objects that EPG is about—supposing that these objects are there independently of this theory—, then this claim is flawed; if to prove the existence of an object of EPG means to establish that the corresponding concept is instantiated in EPG, then this claim is correct, I think, and can be easily rephrased by saying that in EPG a geometric object exists, if and only if it is given, and it is proved to be capable of existence under certain conditions, if and only if it is proved to be susceptible of being given under these same conditions<sup>76</sup>.

But, if the connection between construction and existence proof is so understood, then Mueller is right in remarking that this connection provides no evidence for assigning to Euclid a 'constructivist philosophy of mathematics', being rather 'the natural outgrowth of his conception of geometric objects',

according to which ‘geometric objects are treated as isolated entities about which one reasons by bringing other entities into existence and into relation with the original objects and one another’<sup>77</sup>.

This is not the end of the story, however. The matter should be considered, indeed, in the light of a distinction that I have not introduced yet.

As said, the definitions of the objects of EPG admit an interpretation on freely drawn diagrams. These diagrams do not differ from canonically drawn diagrams for their intrinsic nature, but only because the drawing of the latter is submitted to appropriate rules, whereas this is not the case of the former. In the presence of the global role of diagrams in EPG, this circumstance suggests a weaker understanding of the existence of the objects of EPG. According to this understanding, such an object exists if a diagram appropriate for representing it can be freely drawn (that is, it can exist as any physical object is supposed to exist)<sup>78</sup>.

Let us consider a simple example. Suppose that a circle is given in EPG, together with an inscribed and a circumscribed square. According to the local role of diagrams (and the reduction of the relation of being greater applied to figures<sup>79</sup> to the diagrammatic relation of being included in), the circumscribed square is greater than the circle, and the inscribed one is smaller than it. But a square is susceptible of being given, provided that its side is given, and the greater its side, the greater it is. Hence, by continuity, there should exist a square equal to the circle. This argument is not admissible in EPG: the claim ‘by continuity, there should exist a square equal to the circle’ is neither licensed by diagrams, nor by

prior entries in the discursive text. Still this argument seems not completely foreign to EPG. I suggest it is appropriate to establish the existence of the square it is about in the weaker of the two understandings of the existence of an object of EPG that I have distinguished.

These two understandings result in two different notions of existence for the objects of EPG that have to be carefully distinguished. I suggest to term ‘internal’ the existence of an object of EPG insofar as it is understood in the stronger sense, and ‘external’ the existence of such an object insofar as it is understood in the weaker sense.

Now, it seems to me that external existence of an object of EPG is, so to say, a horizon for internal existence: ideally, any object of EPG that exists externally should also be capable of existing internally. But whereas external existence can be established based on an argument as the previous one, capability of internal existence has to be proved in EPG, and this must be done by construction. It is however a matter of fact that Euclid seems often to take external existence of a certain geometric object (that is, the possibility of drawing freely a diagram appropriate to represent such an object) as a sufficient condition for admitting that it susceptible of being given. In many cases (as in proposition I.5<sup>80</sup>) this is a mere shortcut. But sometimes, its operational ease pushes Euclid into arguments that seem difficult to license in EPG (at least if this theory is interpreted as I suggest to do). An example is offered by proposition III.1 that requires to construct the centre of a given circle: the very fact that this problem is advanced suggests that

Euclid is admitting that a circle could be given although its centre is not given, whereas a circle can certainly not be constructed in EPG, if its centre is not given.

### **III. The Construction of a Right Angle**

Up to now I have spoken of EPG in quite general terms. It is time now to consider an example, so as to clarify the previous considerations in *concreto*. The one that I have chosen concerns the fragment of the book I of the *Elements* that leads up to propositions I.11-12. Both of them require constructing a perpendicular: the former to a given segment from a given point on it; the latter to a given straight line from a given point outside it. In EPG, an angle is given when two segments that meet each other are given, and it is taken to be right if and only if it is equal to another one that is adjacent to it. To account for Euclid's solutions of these two problems, it is thus necessary to account for the way in which segments and their intersections are given and for the conditions of equality of angles. This involves a number of fundamental ingredients of EPG, and this is why I have chosen such an example.

#### **III.1. DEFINITIONS I.1-4 AND I.8-10**

Among these ingredients, there are definitions I.1-4. According to them: '[a] point is that of which [there is] no part'; '[a] line [is] breadthless length'; '[the] extremities of [a] line [are] points'; and '[a] straight line is that [line] which lies evenly with respect to [the] points on itself'.



These definitions appeal to notions that are foreign to the argumentative system of EPG, and, as a matter of fact, they openly licence none of the inferences occurring in this system. This led many commentators to argue that these definitions play no effective role in EPG<sup>81</sup>. I argue that this is not so, and suggest that definitions like these aim to specify both which diagrams are appropriate to represent the objects they are supposed to define, and which properties of these diagrams are relevant<sup>82</sup> for them to play their roles in EPG<sup>83</sup>.

Though the notion of a line that lies evenly with respect to the points on itself depends on that of points on a line, the notion of a point does not enter the definition of lines. A line seems rather conceived as an object that has neither parts nor components—though having, possibly, two extremities—, that is, as a local continuous in Aristotle's sense. Definition I.2 seems thus to rely on the cognitive capacity of distinguishing physical lines among other sorts of physical objects, and to state that geometric lines are what physical lines represent, by prescribing, at the same time, that only the properties of physical lines that depend on these lines having a length are relevant.

The case of definition I.1 is different, since the description 'that of which there is no part' is unable to specify a determinate sort of physical objects (on my reading of Aristotle, something that has no part is continuous, but Euclid is certainly not willing to say that points are continuous). It seems thus to be hard to grasp what a point is without relying on the definitions I.3 and/or I.2. Supposing that an extremity of a geometric line is that which is represented by an extremity of a physical line, definition I.3 suggests that (geometric) points are that which

have no parts—both actually and potentially—insofar as they are what extremities of physical line represent<sup>84</sup>, and that only the properties that depend on the fact that these extremities have a spatial location without having parts are relevant. It follows that a point cannot be given if a line is not.

Once it is established what a point is, definition I.4 seems to rely on the cognitive capacity of appreciating evenness in physical lines, and recognising, then, the physical lines that lay evenly with respect to the points on them (that is, I guess, to the points that one can generate on these lines by dividing them in parts). According to this definition, geometric straight lines are thus those represented by physical straight lines, and only the properties that depend on the evenness of these last lines are relevant.

Any physical straight line is limited. It is also so, in general<sup>85</sup>, for geometric straight lines in EPG: they are segments. Hence, in EPG, to give a straight line is, *ipso facto*, to give two points as its extremities.

The definitions I.8-9 can be similarly understood. According to them: '[a] plane angle is the inclination to one another of two lines in [a] plane which meet one another and [which] do not lie in [a] straight line'; and an angle is rectilinear if 'the lines containing [...] [it] are straight'.

Hence, plane angles are not merely pairs of lines that meet one another. Such pairs determine them, but they are not angles. Euclid appeals to the notions of inclination to suggest that what matters is the mutual position of these lines. He seems also to take for granted that two lines cannot meet one another without sharing or generating an extremity. In the second case, only the lines that begin in

this extremity seem to be relevant, and they form an angle provided they do not form a straight line. Let us consider only rectilinear angles. Their definition seems to rely, at least, on two cognitive capacities: that of recognising pairs of physical straight lines that share an extremity without forming another straight line; and that of recognising in each of these pairs a common feature that it shares with all the other pairs of physical lines that share the same extremity and have the same 'inclination', that is, the same mutual position (the position of one of those line with respect to the other). Angles are thus that which is represented by pairs of physical straight lines that share an extremity and that is common for all pairs of physical straight lines that share the same extremity and have the same mutual position. Only the properties of such a pair of physical straight lines that depend on their mutual position and on their sharing a certain extremity are thus relevant. The lengths of these lines are thus not relevant.

Right angles are plane rectilinear angles of a particular sort. Definition I.10 establishes that 'when [a] straight [line] having been set up on [a] straight [line] yields adjacent angles [which are] mutually equal, [then] each of the equal angles is right, and the straight [line] standing upon [the other] is called "perpendicular" to that upon which it stands.' If straight lines and angles are understood as said before, this definition is clear enough, but it leaves open the problem of specifying the conditions under which two (distinct) angles are equal. This is the essential question the construction of a right angle is concerned with.

### III.2. CONSTRUCTIVE CLAUSES AND RULES OF INFERENCE ABOUT BEING GIVEN

Before considering it, a crucial question about construction has to be answered, at least partially: which are the rules that govern it?

Since any construction starts out from some given objects, a first rule should establish which objects can be taken as given without resulting from a previous construction. Postulates I.1 and I.3 suggest that these objects are points. They licence, indeed, respectively, ‘to draw [a] straight line from any point to any point’ and to ‘draw a circle with any centre and interval’<sup>86</sup>. Still, definitions I.1-4 suggest, as we have just seen, that a point cannot be given if a line is not, and that two points are given if a segment is. This suggests that in EPG only segments can be taken as given without resulting from a previous construction, to the effect that any construction has, in principle, to start from segments: namely from any number of segments whose mutual spatial relations obey no specific conditions.

The question is thus the following: supposing that an appropriate number of such segments are given, what are the rules for constructing other objects starting from these segments? According to the global role of diagrams, these rules should apply to already drawn diagrams, establish which other diagrams can be drawn in presence of these diagrams, and specify, thus, which objects are susceptible of being given provided that the objects represented by the already drawn diagrams are given. Each rule would thus be twofold, including both a constructive clause—licensing to draw certain diagrams provided that others have already been drawn—and a rule of inference about being given, warranting that if certain

geometric objects are given and satisfy certain conditions, some other objects are susceptible of being given.

Postulates I.1-3 seems to provide three of these twofold rules<sup>87</sup>.

I have already mentioned postulates I.1 and 1.3. The rules they provide are the following<sup>88</sup>:

R.1) If two points are given, then one and only one physical segment joining the physical points that represent these points can be drawn; hence, if two points are given, one and only one segment joining these points is susceptible of being given.

R.3) If two points are given<sup>89</sup>, then two and only two physical circles can be drawn—each of them having their centre in the physical point that represents one of two given points and passing through the physical point that represents the other—; hence, if two points are given, two and only two circles—each of them having their centre in one of two given points and passing through the other—are susceptible of being given.

Postulate I.2 licences ‘to produce [a] limited straight [line] continuously in [a] straight [line]’. The main difficulty in the understanding of this postulate is concerned with the adverb ‘continuously [κατὰ τὸ συνεχῆς]’. There is no doubt that Euclid is referring to the (diagrammatic) property of continuity. What is not clear is the subject to which this property is attributed. Euclid seems to licence to produce a (given) segment by tracing a new segment from one of its extremities. The grammatical form of the sentence suggests that this property is not attributed to the former segment and it is quite implausible that it be attributed to the latter.

If it were so, one could not avoid of wondering why continuity intervenes so late in Euclid's exposition as an attribute of a line, and why it is explicitly attributed only to this latter segment and not to the former. It is much more plausible that Euclid attributes continuity to the configuration formed by the given segment and its prolongation: this would not merely be a configuration of two contiguous segments, but it might also be taken as something continuous, that is as a unique segment. Supposing that a segment has been given, the postulate I.2 would thus license to produce it so to get two new segments: the segment that produces the given one, and that which results from this latter segment and the given one, taken together. This is the interpretation I suggest.

It would follow that Euclid is admitting that a pair of segments might be understood as something continuous. This would contradict an Aristotelian precept, since it would be the same as admitting that a unique—and thus continuous—segment has two actual parts. This seem to me a necessary condition for EPG to be set forth.

This is not all, however, about postulate I.2. Another relevant question concerns the conditions of its application. Does Euclid licence producing a given segment at will, that is, tracing a new arbitrarily long segment? In many cases (like the in the proposition I.2 and I.5), it seems that it is so. A better scrutiny of Euclid's applications of the postulate I.2 shows however that another interpretation is possible: when Euclid applies such a postulate so as to produce a given segment at will, he merely abridges a possibly more complex procedure, where the given segment would be produced so as to meet a given line. If so, the

application of the postulate I.2 would be submitted to an implicit condition: a given segment could be produced only up to meet a given line. Moreover, this same application would result not only in the construction of two new segments but also in the construction of a new point on a given line and, if this line is a segment, in that of two other segments on it. Still, Euclid does not provide a general criterion to decide whether a given segment can be produced to meet a given line. He simply relies on diagrams to decide whether this is so.

In summary, I suggest that the postulate I.2 provides the following rule:

R.2) If a segment is given and the physical line that represents it is such that its prolongation can meet an already drawn physical line, then this segment can be produced up to meet this line; hence, if a segment  $a$  and another line  $b$  are given, then two other segments—one, let's say  $c$ , producing  $a$  up to meet  $b$ , and the other, let's say  $d$ , formed by  $a$  and  $c$  taken together—and a point on  $b$ —where it meets both with  $c$  and  $d$ —are susceptible of being given.

Not all the rules of inferences about being given that occur in EPG are associated with a constructive clause, and have a modal nature. The construction of a right angle requires two rules of inferences about being given that are not so. One of them is implicit in definition I.3, the other depends on the interpretation of the intersection of two lines as an extremity of other lines resulting from a division of these lines. These rules are the following<sup>90</sup>:

R.4) If a segment is given, two points, consisting in its extremities, are given.

R.5) If two intersecting lines meet each other, then, for any time they meet a point is given where they meet; if both the given lines are segments, then four or two new segments, cut off on the given segments, are also given; if one of the given lines is a segment and the other is a circle, then also two other segments (if the given lines meet each other once) or three other segments (if they meet each other twice), cut on the given segment, are given.

### III.3. COMMON NOTIONS

Not all the rules of inferences occurring in EPG are concerned with being given. Among those that are not, the most relevant concern the relations of equality, of being part of, of being smaller than, and of resulting from the addition or the subtraction of, as applied to segments, polygons and angles. In the *Elements*, these relations are not defined once for all, and they could not have been so defined, since they are quite different from each other according to whether they apply to segments, polygons or angles. Hence, they are successively defined for each of these sorts of objects. Still, whatever sort of geometric objects they apply to, each of these relations has to satisfy appropriate general conditions. In my view, common notions aim to fix some of these conditions<sup>91</sup>.

To say it in modern terms, common notions I.1-1.3 establish that any relation that is supposed to be an equality has to be transitive (provided that its being symmetrical be taken for granted), and conserved under addition and subtraction. Common notions I.4-6 are probably interpolated<sup>92</sup>, since they follow from the



previous ones: they establish, respectively, that any relation that is supposed to be an inequality has to be conserved under addition, and that any relation that is supposed to be an equality is conserved under passage to the double and the half.

Common notions I.7-8 are quite different. They state respectively that two things that coincide each with other are equal, and that the whole is greater than the part. I take them as referring to diagrams and having several functions. Taken alone, the former seems both to state that objects that are represented by the same diagram are equal to each other—which, under my interpretation, is the same as to state that any relation that is supposed to be an inequality has to be reflexive—, and to licence the (non-constructive) argument that Euclid relies on for proving proposition I.4<sup>93</sup>. Taken alone, too, the latter seems to establish that a geometric object represented by a diagram that is composed by, or that includes spatially, another diagram that represents a geometric object of the same sort, is greater than this object. Taken together, these common notions state, moreover, that when defined for the same sort of geometric objects, the relations of being equal to, and of being greater or smaller than are exclusive and respect trichotomy.

#### III.4. EQUALITY OF ANGLES

The solution of any problem in EPG includes two steps: the construction of appropriate objects, and a proof of the appropriateness of these objects (the fact that these objects satisfy the conditions of the problem). To achieve the latter in cases of propositions I.11-12, the equality of two angles has to be proved, and—

for it to be possible—some conditions relative to the equality of equals are needed.

In both cases, Euclid relies on proposition I.8 that provides a sufficient condition of equality for any pair of angles. This condition establishes that two angles are equal if they are (susceptible of being) included into two triangles with equal homologous sides. Equality of angles is thus reduced to equality of segments (that has, therefore, to be defined beforehand). This reduction goes together with an appropriate justification: proposition I.8 is proved, rather than stated as a pure (consistent) convention. The proof relies on a previous condition provided by proposition I.4. This establishes that two appropriate angles are equal to each other, if two other appropriate angles are so; it is thus, so to speak an implicative particular condition. It can be expressed as follows: if two angles are equal to each other, and they are taken as internal angles of two triangles whose homologous sides that include these angles are also respectively equal to each other, then the two other angles included in these triangles are respectively equal to each other.

But also proposition I.4 has to be proved. Euclid does it by relying on an argument that is generally understood as being foreign to the constructive constraints of EPG, and that supposes, indeed, that a triangle could be rigidly displaced as if the diagram that represents it were replaced a rigid configuration of bars. This argument is thus not only diagram-based, but also mechanical, so to speak<sup>94</sup>. As it is well known, no alternative argument, complying with the

constructive constraints that govern the great majority of the arguments in EPG, may replace it<sup>95</sup>.

Still, once proposition I.4 is admitted, and proposition I.5 is derived from it, the construction performed in the propositions I.11 and I.12 can be proved to be appropriate by relying neither on the proposition I.8 nor on any other sufficient condition of equality for angles. If Euclid prefers to proceed as he does (that is, to rely on proposition I.8 in his proof of the appropriateness of this construction), it is possibly because he wants to replace as soon as possible the implicative particular condition stated by the proposition I.4 with a genuine sufficient condition. This may be understood, from the perspective of the establishment of EPG as a whole. But if the aim is only that of constructing a right angle a simpler argument could be preferred. For short, this is the argument that I shall reconstruct according to my understanding of EPG.

### III.5. PROPOSITIONS I.11-12

Proposition I.1 of the *Elements* requires ‘to construct an equilateral triangle on a given limited straight line’. Provided that segments are defined, and diagrams play their global role, the definitions of triangles in general (definition I.19), and of equilateral, isosceles, and scalene triangles in particular (definition I.20), present no difficulty, apart from the fact that those of equilateral and isosceles triangles rely on equality of segments that is not previously explicitly defined.

Definition I.15 supplies an appropriate basis for providing such a definition, however. It establishes that ‘[a] circle is a plane figure contained by a line, such

that all the straight [lines] falling upon it from one [point among] those lying inside such a figure are equal to one another'. As said, despite this definition, Euclid often considers a circle to be a line rather than a figure, and this is what I have supposed up to now and continue to suppose<sup>96</sup>. Definition I.15 establishes that this line encloses a point—its centre—such that all the segments of which it is an extremity and whose other is on the line itself are equal to each other. When this claim is coupled with postulate I.3 and is understood according to the local role of diagrams, definition I.15 provides a sufficient condition for the equality of segments that share an extremity: these segments are equal if a circle passes through their other extremities.

Relying on this condition, the problem addressed in proposition I.1 can be easily solved through a diagram-based argument that I have already partially reconstructed and discussed: let **AB** (fig. 1) the given segment. According to R.4, the points **A** and **B** are given. Apply R.3 so as to give two circles with radius **AB** and centre **A** and **B**, respectively. According to R.5 and the local role of diagrams, the point **C** and **D** are given. Apply R.1 so as to give four segments joining **A** and **B** to **C** and **D**. For the global role of diagrams, two triangles are thus given, and, for the definition I.15 and the common notion I.1, they are equilateral.

From proposition I.1, it follows that equilateral triangles are susceptible of being given, provided that a segment is given. It would thus be enough to rely on R.1 to give the segment **CD**, and then on R.5 and to the global role of diagrams to conclude that four angles would thus be given at the intersection of **AB** and **CD**. These angles are right, but, without further resources, there would be no way to

prove that it is so. As a matter of fact, these resources are provided by the propositions I.4-5. Hence, were our problem that of constructing a right angle, without any supplementary condition, the conjunction of propositions I.1 and I.4-5 would be enough to solve it. But propositions I.11 and I.12 ask to construct two right angles whose sides meet some conditions that the segments AB and CD do not meet, in general. Thus other constructions are needed to satisfy their requirements.

Propositions I.4-5 are also needed for proving that these other constructions are appropriate (and in fact they are sufficient, too). We have then to consider them.

In the former, it is supposed that two distinct triangles are given and that two sides and the angle they include of one of these triangles are respectively equal to two sides and the angle they include of the other triangle. As said, proposition I.1 ensures that equilateral triangles are susceptible of being given provided that a segment is given. In order to prove that this is also the case for any sort of triangles, provided that two or three appropriate segments are given, one should rely on proposition I.2 that provides a sufficient condition for the equality of any pair of segments. Euclid postpones this proof in proposition I.22, however<sup>97</sup>. Moreover, he does not provide, in advance, any sufficient condition for the equality of angles<sup>98</sup>. Still, once it is admitted that two distinct triangles as those involved in proposition I.4 are susceptible of being given, and that one of them could be rigidly displaced, there is no need to rely on proposition I.2 for proving proposition I.4.

Euclid's proof goes as follows. Let  $ABC$  and  $DEF$  (fig. 4) the given triangles and let  $BA = ED$ ,  $CA = FD$ ,  $\angle B\hat{A}C = \angle EDF$ . According to Euclid, if the triangle  $ABC$  is rigidly displaced so that the respective members of these equalities came to coincide to each other, the points  $B$  and  $C$  come to coincide with the points  $E$  and  $F$  respectively, and thus the side  $BC$  comes to coincide with the side  $EF$ , and, for common notion I.7, these sides will be equal to one another, and thus also the whole triangles and their other angles will be equal each other. This is what proposition I.4 asserts.

The problems with this argument do not depend only on the appeal to rigid displacements of triangles. Even if this is admitted, no explicit stipulation of EPG ensures that when a triangle is so displaced, its sides and angles come to coincide with other segments and angles that are supposed to be equal to them, respectively. For it to be so, the converse of the common notion I.7 would have to hold for segments and angles. Euclid seems to admit implicitly that it is so<sup>99</sup>.

Proposition I.5 is a theorem. It asserts, as already point out, that the angles at the base of any isosceles triangle are equal to one another, as well as their supplementary angles. Once the proposition I.4 is admitted, the proof of this theorem presents no further difficulty.

Let  $ABC$  (fig. 5) be an isosceles triangle with  $BA = CA$ . Euclid applies the postulate I.2 so as to produce these segments on the side of  $B$  and  $C$  up to two points  $D$  and  $E$  taken at random, then takes, once again at random, a point  $F$  on  $DB$  so that  $EA$  be greater than  $FA$ . He relies then on proposition I.3<sup>100</sup>, to cut on  $EA$  a segment  $GA$  equal to  $FA$ .

As what is essential in the proof is merely that  $GA = FA$ , one could avoid indefinite prolongation and points taken at random. It would be enough to apply R.3 so as to give two circles passing through A (fig. 5bis), with centre in B and C, respectively, then to apply R.2 so as to produce BA and CA up to meet these circles in F and G. From the definition I.15, it would just follow that  $GA = FA$ .

Howsoever the points F and G have been given, the rest of Euclid's argument can be reconstructed as follows. Apply R.1 so as to give two segments joining the points F and G to the points C and B, respectively. According to the global role of diagrams, the two triangles ABG and AFC are thus given, and include the same angle at vertex  $\hat{A}$ . Hence, since  $BA = CA$ ,  $GA = FA$  and equality of angles is necessarily reflexive, these triangles satisfy the condition of proposition I.4. Thus the angles  $\angle ABG$  and  $\angle BGA$  are respectively equal to the angles  $\angle FCA$  and  $\angle AFC$  and the segments GB and FC are equal to each other. Moreover, according to the global role of diagrams, the triangles BFC and BGC are also given, and, provided that the non-diagrammatic relation of resulting from the subtraction of is reduced to the diagrammatic relation of being part of, their sides FB and GC can be understood as resulting by subtracting the equal segments BA and CA from the equal segments FA and GA. Hence, for the common notion I.3, these sides are, in turn, equal to each other, and the triangles BFC and BGC also satisfy the condition of proposition I.4, so that the angles  $\angle FCB$  and  $\angle CBG$  are equal to each other, too. Now, according to the global role of diagrams and to the reduction of the relation of resulting from the subtraction of to the relation of

being part of, the angles  $\angle ABC$  and  $\angle BCA$  can be understood as the result of the subtraction of the equal angles  $\angle CBG$  and  $\angle FCB$  from the equal angles  $\angle ABG$  and  $\angle FCA$ . According to common notion I.3, they are thus equal to each other, which was to be demonstrated.

Once propositions I.4 and I.5 are proved, the solution of the problems addressed by propositions I.11-12 presents no further difficulties.

Consider firstly propositions I.11. Euclid's solution can be reconstructed as follows.

Let  $AB$  (fig. 6) be the given segment and  $C$  a given point on it. The problem consists in constructing a perpendicular to  $AB$  from  $C$ . The construction of a segment collinear to  $AB$ , whose middle point is  $C$ , is easy. Euclid suggests taking another point  $D$  at random on  $AB$  so that the circle of centre  $C$  that passes through this point meets  $AB$  in another point  $E$ . The same can be achieved in another way, avoiding points taken at random. It is enough to remark that, according to R.4, the points  $A$  and  $B$  are given, and apply R.3 so as to give the circle of centre  $C$  (fig. 6bis) which passes through one of these points, let's say  $A$ , then, if necessary, to apply R.2 so as to produce  $AB$  on the side of its other extremity—let's say,  $B$ —up to meet this circle in a new point  $G$  that is thus given in turn<sup>101</sup> together with the segment  $AG$ . According to definition I.15,  $C$  is the middle point of this last segment.

Once a segment collinear to  $AB$  whose middle point is  $C$  is constructed, to construct the required perpendicular it is enough to construct, according to



proposition I.1, an equilateral triangle FDE (or FAG in case of fig. 4bis), and to apply R.1 so as to give the segment FC. According to the global role of diagrams, the triangles FDC and FCE (or FAC and FCG) are thus given, together with the adjacent angles  $\angle ACF$  and  $\angle FCB$ . These are equal to each other and, consequently, right. To prove that it is so, Euclid relies on proposition I.8, remarking that the sides DF and DC of FDC are respectively equal to the sides EF and CE of FCE, whereas the side CF is common to these triangles. It is however obvious that propositions I.4 and I.5 are also appropriate, since the triangle FDE is isosceles, and the angles  $\angle FDC$  and  $\angle CEF$  are thus equal to each other.

Let us consider now proposition I.12.

The mutual position of a given segment and a given point that is not on it may of course be such that no perpendicular from the latter to the former is susceptible of being given. Possibly to avoid distinguishing different cases, Euclid supposes to be given an 'unlimited straight line' AB (fig. 7), rather than a segment. Provided that the point C is also given, the proposition requires to construct the perpendicular to AB from C.

Supposing, instead, that what is given is a segment, let's say AG (fig. 7bis), one should rely on R.4 to deduce that its extremities are also given, then apply R.3 so as to give the two circles that pass through these points, and have the given point, let's say C, as centre, and distinguish the case where one of these circles meets twice the segment AG, from that where none of them does. In the latter case (displaced in the fig. 7bis), one should then apply R.2, so as to produce the given

segment, so as to meet one of these circle in a second point, let's say  $E$ , and to give this point together with the segment  $AE$ . Two new cases should then be distinguished: that where the point  $C$  is on  $AE$ , and that where it is not. In the former the problem has no solution; in the latter (displaced in the fig. 7bis), the mutual situation of the point  $C$  and the segment  $AE$  is the same as that of the point  $C$  and the segment  $AG$  in the latter of the two cases previously distinguished, and the problem may thus be solved as in this last case, by replacing  $AG$  with  $AE$ .

Under Euclid's own formulation, the problem is unique. Its solution requires, however, that a point, other than the given one, be taken at random and supposed to be given, in turn. Euclid relies on the diagram, so as to take this other point,  $D$  (fig. 7), 'on the other side' of  $AB$  than  $C$ , then applies postulate I.3, so as to give the circle with centre in  $C$  that passes through  $D$  and—because of the local role of diagrams—meets the given line twice, in  $G$  and  $E$ . At this stage, the situation is the same as in the first and third of the three cases previously distinguished.

The rest of Euclid's argument can be reconstructed as follows (I consider Euclid's diagram: fig. 7).

Apply R.2 so as to construct the segments  $GC$  and  $EC$ . According to the global role of diagrams and to definition I.15, the isosceles triangle  $CGE$  and the angle  $\angle ECG$  are also given. Construct then, according to proposition I.1 the equilateral triangle  $FEG$ , on the other side of  $AB$  than  $C$ , and apply R.1 so as to give the segment  $FC$ . For R.5 and the global role of diagrams, seven new segments— $FG$ ,  $FE$ ,  $FC$ ,  $FH$ ,  $HC$ ,  $GH$ ,  $HE$ —and seven new triangles— $FEG$ ,  $FEH$ ,  $FHG$ ,  $FCG$ ,

FEC, CHE, CGH—, together with their internal angles, are thus given. And, by reasoning on these objects, it is easy to prove that the segment CH is the perpendicular that was to be constructed.

Euclid proves firstly that this segment bisects the angle  $\angle ECG$ , by relying on proposition I.8: the angles  $\angle HCG$  and  $\angle ECH$  are equal since they are homologous angles of the triangles FCG and FEC whose sides are respectively equal. This is the content of proposition I.9, which requires bisecting a given angle. Propositions I.4 and I.5 provide, however, sufficient basis for getting the same conclusion, since the triangles FEG and CGE are isosceles, and thus:  $\angle HGF = \angle FEH$ ,  $\angle CGH = \angle HEC$ , and—according to common notion I.2, and provided that the non-diagrammatic relation of resulting from the addition of is reduced to the diagrammatic relation of being composed by— $\angle CGF = \angle HEC$ .

Secondly, Euclid proves, by relying on proposition I.4, that the point H bisects the segment GE, which corresponds to proposition I.10. This allows him to prove that  $\angle GHC = \angle CHE$  by relying, once again, on proposition I.8. But this last equality can also be proved through proposition I.4, since  $GC = EC$ , and  $\angle HCG = \angle ECH$ .

#### **IV. Conclusions**

The solution of the problems addressed by proposition I.11-12—that I have reconstructed in section III—should provide an example of the roles that diagrams

play in EPG. These roles appear both in proofs and constructions and depend on the relation that diagrams have in EPG with geometric objects.

My account of this relation focuses on two crucial aspects. The first pertains to claim (C.i), that, in sections II.1 and II.4, I have tried to clarify by connecting identification, being given, construction and existence. The second pertains to claim (C.ii), that, in sections II.2 and II.3, I tried to clarify by introducing the notion of a diagrammatic attribute, and comparing my account with the results of philological enquiries about the variations of the shapes of diagrams in the extant manuscripts of the *Elements*.

From a more general point of view, I have also suggested that diagrams are apt to play their roles in EPG since—according to Aristotle’s conception of continuity, that I have discussed in section I—physical lines that compose them are continuous objects, and EPG is (or at least can be viewed as) a mathematical theory of Aristotelian continua. This suggests that the classical interpretation of EPG, according to which it would result from contemplation of something similar to Platonic ideas, can be contrasted and replaced by another interpretation, more Aristotelian (and Kantian) in spirit, according to which EPG is (or results from) a codified practice essentially based on the production and inspection of physical objects, like diagrams.

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<sup>1</sup> Some views expounded in the present paper have been previously presented in Panza 2002, whose first version was written in 1996, during a visiting professorship at the *Universidad Nacional Autónoma de México*. I thank all the people who supported me during my stay there. I also thank, for many suggestions and valuable comments, Carlos Alvarez, Andrew Arana, Jessica Carter, Karine Chemla, Davide Crippa, Massimo Galuzzi, Pierluigi Graziani, Jan Lacki, Danielle Macbeth, Michael Hallett, Ken Manders, Mircea Radu, and Giuseppina Ronziti.

<sup>2</sup> Euclid H, vol. I: 242 and 235. On this matter (in its connection with Book III of the *Elements* and proposition I.12 that I'll consider in section III.5), cf. also Frajese 1968.

<sup>3</sup> Friedman 1985: 60.

<sup>4</sup> For a survey of recent literature about diagram-based arguments in geometry, cf. Manders 2007a.

<sup>5</sup> With 'Euclid's plane geometry' I mean plane geometry as it is expounded by Euclid, especially in the *Elements* (but the *Data* are also relevant for understanding some crucial feature of this geometry), and was largely practiced up to early modern age. This should be confounded neither with plane Euclidean geometry in general, nor with elementary synthetic plane geometry (Stekeler-

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Weithofer 1992). The text of the *Elements* I refer to is that established by Heiberg (Euclid HM). This leaves open the possibility of confirming or refuting some of my statements based on philological evidences that Heiberg's edition does not reflect.

<sup>6</sup> I use the term 'argument' as a generic term, so as to refer to that which counts for Euclid either as a proof of a theorem or as a solution of a problem.

<sup>7</sup> As it should be clear from that which follows, I take the objects of EPG to be posits resulting from appropriate stipulations. In his *Seventh Letter* (342 a-d), Plato distinguishes between the tools of knowledge—name, definition, and imagine—, the knowledge itself, and its object, and offers a geometric example: that of the circle, its knowledge, its name, its definition, and its diagram. The object is the reason of unity of the other elements: name, definition, imagine and knowledge are its name, its definition, its imagine, and knowledge of it. This unity is for him warranted by the independent existence of the object. To reject this solution—in case of abstract, or more specifically mathematical, objects—is not the same as to maintain that the problem of unity of knowledge and its tools is not a genuine and crucial problem. The admission of objects as posits provides a non-Platonic solutions of this problem. Mathematics can certainly be understood as a codified practice including attributions but not objects. In the case of EPG, this is, for example the suggestion of Ken Manders (cf. Manders 2007a). But such an understating leaves this problem open: what EPG is about? what its singular terms refer to? what their definitions define? and what their diagram represent?

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<sup>8</sup> For short, I shall use the term ‘diagram’ in a quite restricted sense, so as to refer only the particular sort of diagrams that occur in EPG. If the same term is used in its usual larger sense (as I have just do), one should distinguish between ‘intrinsically depictive’ and ‘intrinsically non-depictive’ diagrams (Tennant 1986). The former are those that stand for some objects and display some properties and relations of them. I call ‘representation’ the relation that intrinsically depictive diagrams have with the corresponding objects. If these objects are abstract, intrinsically depictive diagrams cannot, however, be understood as ‘diagrams [...] [that] represent in virtue of a similarity of visual appearance with its objects(s)’ (Norman 2006: 78). The reason is that abstract objects have not visual appearance as such.

<sup>9</sup> Insofar as they occur in the expressions ‘global role of diagrams’ and ‘local role of diagrams’, the adjectives ‘global’ and ‘local’ are thus supposed to have a technical sense that I shall try to clarify (respectively in sections II.1 and II.2). No other sense of them is presupposed, here.

<sup>10</sup> The physical lines and points that enter a diagram have, of course, to be carefully distinguished from the geometric objects that they represent. Anytime that the context is not clear enough to avoid misunderstanding, I shall use the adjectives ‘physical’ and ‘geometric’ to designate the former and the latter, respectively.

<sup>11</sup> Consider two independent diagrams. Is it necessary, for making EPG, to be able to recognise whether these diagrams represent the same geometric objects? Most

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people would certainly answer ‘yes, it is, of course!’ But, I do not think that this is the right answer. After all, we could admit that to make mathematics, and, *a fortiori*, to make EPG, one has only to be able to conduct, in different places and times, certain arguments in a certain—let us say, in a right—way, and that this capacity is completely independent whether these arguments concern the same objects or not. That which is relevant is rather that they concern some objects and that the objects that each argument is about are appropriately distinguished. But also if a more orthodox view were admitted, it would not follow that the objects of EPG are represented by diagram-types. They would rather be represented by tokens complying with appropriate types.

<sup>12</sup> Diagrams have also to be carefully distinguished from figures, in Euclidean sense established in definitions I.14 of the *Elements*, according to which a figure (σχήμα) is ‘that which is contained by any boundary or boundaries’. In this sense, figures are geometric objects. I shall use the term ‘figure’ only in this sense. According to definition 1.15 of the *Elements*, a (geometric) circle is a figure. But in the *Elements*, the term ‘circle’ is also often used to denote a (geometric) line. For the purpose of my paper, there is no danger in admitting this last sense only and use the term ‘circle’ to denote either a geometric or a physical line. I shall do so.

<sup>13</sup> Cf. footnote (7).

<sup>14</sup> Panza 1992.

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<sup>15</sup> The translation of the passages of Aristotle's *Physics* and Euclid's *Elements* that I quote is mine (though that of the latter does very seldom differ from Heath's translation: Euclid H).

<sup>16</sup> Ross 1936: 69; and Düring 1966: 325.

<sup>17</sup> Or perhaps, 'insofar as place is concerned': 'κατὰ τόπον'.

<sup>18</sup> Ross 1936: 627. For an account of other classical interpretations, cf. Heath 1949:122-3.

<sup>19</sup> The term 'theory' is far from being innocent in history and philosophy of mathematics. I use it in a broad sense to refer, more than to a corpus of statements, to a space of authorised acts codified by appropriate stipulations, and resulting in the constitution of a domain of abstract objects and in the establishment of a corpus of results about these same objects.

<sup>20</sup> Klein 1934-36: 119-123.

<sup>21</sup> That is a theory of 'extra-logical discrete objects, which exist intuitively as immediate experience before all thought': cf. Hilbert 1922: 202.

<sup>22</sup> Historically speaking, this relation could be seen as the result of an evolution of the role of *schema* in Thales' geometry as it is described in Caveing 1997: 73-75, 148-149. According to Caveing in this geometry, a *schema* is 'given in visual intuition' and its 'mode of being' is 'the same as that of the decorative drawing', but its 'sense' is 'no more esthetical', being rather that 'of representing a problematic situation', so as 'to work as a field of possibilities' (*ibid.*: 73 and 148). For Caveing (1982), things change radically with Euclid's geometry,

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however, since in this geometry ‘empirical intuition is out of the question’ and the continuum is not—as it was not, already for Aristotle—‘a simple intuitive determination’ (*ibid.*: 155 and 164). My interpretation of EPG and of Aristotle’s conception of continuity is opposite to Caveing’s.

<sup>23</sup> For different, but (at least partially) complementary, insights about the role of diagrams in Euclid’s and, more generally, Greek geometry, cf. Netz 1999: chap. 1; 12-67; Azzouni 2004; Norman 2006; Manders 2007b; Macbeth 2007.

<sup>24</sup> In the case of EPG, these abilities should include, of course, the capacity of dealing with physical lines and points and their configurations (that is, recognising them, distinguish any one of them from another one, etc.), and that of conceiving them as representations of something else.

<sup>25</sup> Euclid usually refers to segments of straight lines through the term ‘straight line [εὐθεῖα γραμμὴ or εὐθεῖα]’ *tout court*. Sometimes (as in postulate I.2 or in proposition I.1), he uses the term ‘limited straight lines [εὐθεῖα πεπερασμένη]’ and quite seldom (as in proposition I.12) he considers ‘unlimited straight lines [εὐθεῖα ἄπειρος]’. Cf. Mueller 1981: 56, note 43.

<sup>26</sup> I shall clarify this matter in section III.1.

<sup>27</sup> Cf. footnote (12).

<sup>28</sup> Notice that for characterising circles and polygons as figures it is necessary (though not sufficient) to characterise them as configurations of lines. Thus, what I shall say about characterisation of the objects of EPG should not be wrong if circles and polygons are understood as figures. It should just be incomplete.

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<sup>29</sup> Cf. footnote (46).

<sup>30</sup> One could argue that in EPG for giving a circle and a polygon as figures it is not necessary to give them as configurations of lines. If it were so, what holds in EPG for the identification of circles and polygons, understood as configurations of lines, could not hold for the identification of circles or polygons, understood as figures. I do not think that this is so, since I maintain that in EPG for giving a circle and a polygon as figures it is necessary to give them as configurations of lines (and thus that things go for the identification of these objects as they go for their characterisation [cf. footnote (28)]). Still, I shall not consider this matter in my paper. To avoid misunderstanding, anyone that do not share my view on it, is invited to take what I shall say about the notion of being given applied to circles and polygons as merely concerned with these objects understood as configurations of lines.

<sup>31</sup> Definition 2 of the *Data* establishes under which condition a ratio is given. The status of ratios in EPG is controversial, but, for my purpose, it is not useful to consider this matter.

<sup>32</sup> Taisbak 2003: 18.

<sup>33</sup> Taisbak 2003: 25 and 22.

<sup>34</sup> Taisbak 2003: 25.

<sup>35</sup> Taisbak 2003: 17.

<sup>36</sup> Taisbak 2003: 29.

<sup>37</sup> Taisbak 2003: 27.

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<sup>38</sup> Taisbak 2003: 28-29.

<sup>39</sup> Taisbak 2003: 28.

<sup>40</sup> Taisbak 2003: 19: ‘The Plane is supposed to be full of points, and one is free to choose among them. The same holds to a certain extent for *lines* and line *segments*’.

<sup>41</sup> According to Taisbak’s interpretation, this operator only occurs in Euclid’s definition *prima facie*, since the logical form of this definition is the following:

$$(\text{Given}(a) \wedge a = x) \Leftrightarrow \text{Given-in-magnitude}(x)$$

I suggest, instead, to interpret Euclid’s definition as follows:

$$(\text{Given}(a) \wedge \diamond \text{Given}(x) \wedge (\text{Given}(x) \Rightarrow a = x)) \Leftrightarrow \text{Given-in-magnitude}(a).$$

I shall can back later to the exact meaning that has to be assigned to the operator ‘ $\diamond$ ’.

<sup>42</sup> Cf. footnote (30).

<sup>43</sup> Cf. footnote (46).

<sup>44</sup> Taisbak 2003: 43.

<sup>45</sup> Taisbak 2003: 44.

<sup>46</sup> There however no need to specify the nature of such a strategy to understand, at this point, what does it mean in EPG that a certain object is supposed to be given, that is, it is supposed to be represented by an appropriate canonically drawn diagram: it means that a diagram appropriate for representing this object is freely drawn—that is, it is drawn without following this procedure—and it is taken as having been canonically drawn. This is the base of analytic arguments in EPG.



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The question related to the nature of these arguments cannot be raised here. The literature on geometrical analysis is quite large. For my views on this matter, cf. Panza 1997 and Panza 2007.

<sup>47</sup> Hartshorne 2000: 19.

<sup>48</sup> Norman 2006: 21 and 33.

<sup>49</sup> The modal nature of EPG has been emphasized in Chihara 2004: 10.

<sup>50</sup> Cf. footnote (70).

<sup>51</sup> In fact, only lines—that is, segments and circles—are continuous in EPG and each line is represented in EPG by a single (elementary) diagram. Hence, if we confine ourselves to lines and to the diagrams that represent them, the relation of representation is associated to an injective application and the plural is here needless.

<sup>52</sup> Of course, neither visual inspection of a physical line nor its material production can ensure that it is continuous (that is, according to Aristotle, it does not present spatial gaps or angles). What matter in EPG are not the real (possibly microscopic) features of a physical line (or generally of a diagram), but the features that are attributed to it and actually recognised in it (cf. Azzouni 2004: 125). Still, I suggest that Norman is wrong when remarks that ‘there is nothing as such about the visual feature of a drawn line that instructs a reasoner to take it as representing a continuous (or non-continuous) geometrical line’ (Norman 2006: 33). In EPG there no other way, indeed, to explain what does it mean that a

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geometrical line is continuous that saying that it has the property that physical lines have if they are continuous.

<sup>53</sup> Friedman 1985: 63.

<sup>54</sup> Is this that Reed (1995: 42) means when he claims that the function of a diagram in EPG is ‘to exhibit the relationship of figures and their parts’? If yes, I agree with him, though I do not see why ‘to ask other things of the diagrams is to misunderstand the nature of Euclid’s demonstrations’ (*ibid.*).

<sup>55</sup> Manders 2007b.

<sup>56</sup> Manders 2007b: 5 of the ms.

<sup>57</sup> I admit that Manders also uses the term ‘attribute’ as referring both to properties and relations.

<sup>58</sup> Manders 2007b: 9 of the ms.

<sup>59</sup> Manders 2007b: 10 of the ms. Of course, attributions are exact or co-exact—or, as I shall say later, diagrammatic or non-diagrammatic—insofar as they are attributions of exact, co-exact, diagrammatic or non-diagrammatic attributes, respectively.

<sup>60</sup> Cf. footnote (7).

<sup>61</sup> In the *Element* only two explicit attributions of continuity occur, namely in postulate I.3 and in proposition 11.1. But in many other cases, like in proposition I.1, these attributions occur implicitly.

<sup>62</sup> It is easy to see that the relation of being greater than—as well as that of being equal to—is not diagrammatic. They meet neither the condition (*i*), nor the

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condition (ii) relative to diagrammatic relations. Consider the example of segments: to explain what does it mean in EPG that two segments are equal to each other or that one of them is greater than the other is certainly not appropriate to say that this the case of the diagrams that represent them, and in EPG it is certainly not necessarily the case that equal segments be represented by equal physical lines and a segment greater than another by a physical line greater than the physical line that represents this other segment.

<sup>63</sup> A large part of Manders' account concerns 'controlling' of 'diagram appearance'. This is because for him, 'diagram-based attribution requires [...] that the feature attributed appear in *appropriately produced* diagrams' (Manders 2007b: 12 of the ms). In my view, this is not a special 'discipline' (the terms is Manders') requiring a specific training. It is just a natural consequence of the local role of diagrams. Take the well-known example (mentioned by Manders: 2007b: 11 of the ms) of the (wrong) argument proving that all triangles are isosceles. Suppose that any triangle is given and trace the bisector of one of its internal angles and the perpendicular bisector of the opposite side. Then, from the intersection point of these bisectors, trace the perpendiculars to the two other sides (or to their prolongations). It is easy to prove that, if both perpendiculars meet these sides inside or outside the triangle, then the given triangle is isosceles. But this provides no evidence for the necessity of a specific discipline aiming to control the appropriateness of the diagrams. It is part of a geometric argument including the previous construction to ask whether these perpendiculars (has to)

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meet the relative sides inside or outside the triangle. This depends on some features of the given triangle. And if these features are not specified, no correct inference can be concerned with the properties of the diagram that depend on the fact that these perpendiculars meet the relative sides inside or outside the given triangle (Norman 2006: 6 and 159).

<sup>64</sup> Cf. Saito's paper in the present collection and Saito 2006, where are reprinted the diagrams relative to the all 48 propositions of the first book of the *Elements* in six principal manuscripts.

<sup>65</sup> Cf. Saito 2006: 82.

<sup>66</sup> Cf. Saito 2006: 82 and his paper in the present volume, ??? (last page of the ms.)

<sup>67</sup> A similar diagram appears, in a bottom margin, in only one of the six manuscripts studied by Saito that also presents a usual diagram where the two parallel line involved in the proof do not meet (2006: 123).

<sup>68</sup> Saito (2006: 84-90) offers a comprehensive study of this phenomenon for proposition III.25 of the *Elements*.

<sup>69</sup> As a matter of fact this proof is (implicitly) offered in the *Elements* only in proposition I.22. Cf. footnote (97).

<sup>70</sup> In EPG, to prove that geometric objects of a certain sort are susceptible of being given under appropriate conditions it is enough to give an object of this sort under these conditions (and without relying on any other supplementary condition). Furthermore, in order for a single geometric object  $x$  to be susceptible of being

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given in EPG, it is enough that it be an object of a certain sort  $X$ , the  $X$ 's be susceptible of being given if appropriate conditions are met, and these conditions are met.

<sup>71</sup> If this view is admitted, the classical generality objection against diagram-based arguments (or, more generally, against arguments involving diagrams) appears to be misleading relatively to EPG. According to this objection, the consideration of a single diagram can, at most, licence a claim concerning with the single object represented by this diagram, but not an universal claim concerned with all the objects of the same sort. But in EPG no such claim is made, I think. And, if the proof of a theorem is only supposed to establish that, if a  $X$  is given, then it is so and so, for such a proof to hold it is sufficient that it relies only on conditions that have to be met for a  $X$  to be given. And this depends on nothing but the conditions of application of the concepts of the  $X$ 's. A similar point—though in the context of a quite different view about the logical nature of the theorems of EPG—is made by Norman 2006: 156-159.

<sup>72</sup> On this matter, and especially on the difference between 'Euclid's and Hilbert's geometry', cf. Mueller 1981: 14-15.

<sup>73</sup> Cf. footnote (46).

<sup>74</sup> As we shall see at section III.2, postulates are in fact not enough to codify this procedure completely. For that, other implicit presuppositions are needed.

<sup>75</sup> Zeuthen's emphasis on this claim is well known, as well known are the more recent objections against his theses. Cf., for instance, Zeuthen 1896; Mueller

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1981: 14-15 and 27-29; Knorr 1983; Euclid V: vol. 1, 170-172; Hartshorne 2000: 18-19; Harari 2003.

<sup>76</sup> Harari (2003) has argued for an alternative view about construction in EPG. According to her (*ibid*: 21), constructions serve in EPG both ‘as means of measurement’ allowing to deduce ‘quantitative relations’ between objects, and as ‘means of exhibiting qualitative relations, i.e., the order or the position of geometric figures’, so as to place ‘the elementary geometric figures (*i.e.*, lines) in different spatial relations’. This view is however perfectly consistent with my understanding of the claim that in EPG constructions provides existence proofs.

<sup>77</sup> Mueller 1981: 14-15. Mueller adds that in the *Elements* ‘the emphasis on construction [...] is connected with the absence of absolute existence assertions’: ‘the existence of one object is always inferred from the existence of another’, and construction is the means used for making this inference.

<sup>78</sup> This understanding of the existence of the objects of EPG seems to me to correspond to an understanding of these objects as forms of physical objects.

<sup>79</sup> Cf. footnotes (12), (28), (30).

<sup>80</sup> Cf. footnotes (69) and (97).

<sup>81</sup> Russo (1998) has even claimed that the definitions I.1-1.7 are due to Heron and were interpolated in the *Elements*.

<sup>82</sup> A similar view is argued for in Azzouni 2004: 126.

<sup>83</sup> But, if it is so and diagrams are supposed to play their global and local roles, why—one could wonder—no diagram is associated to definitions in the

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*Elements*? I suggest two different reasons. The first of them depends on the global role of diagrams: according to this role, a diagram represents a single configuration of geometric objects, whereas definitions are supposed to define sorts of geometric objects not single ones (they provide the conditions of application of appropriate concepts, not the identity conditions for the objects falling under these concepts). The second reason depends on the fact that in EPG diagrams represent given objects only if they are canonically drawn. It would thus be inappropriate to draw diagrams and suppose that they represent (given) geometric objects before establishing the rules governing a construction.

<sup>84</sup> According to B. Levi (1947: 94), the right understanding of the definition I.3 is this: ‘point is what of which it is absurd to conceive parts’. To avoid appealing to the definition I.3 for specifying what a point is, Proclus (*Commentary*: 93.6-94.7; Proclo CM: 76) admits that the subject matter of geometry is established in advance and maintains that, according to the definition I.1, a point is that which has no part ‘in geometric matter’. This is clearly unsatisfactory, however.

<sup>85</sup> Cf. footnote (25).

<sup>86</sup> Cf. footnote (12).

<sup>87</sup> P. Mäenpää and J. von Plato capture the twofold nature of the rules provided by postulates I.1-3, by rendering them as rules of introduction, according to the ‘general pattern of natural deduction rules used in intuitionistic type theory’ (1990: 281). For example postulate I.1 is rendered as following (I write

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‘Segment’ instead of ‘Line’ to adapt Mäenpää’s and von Plato’s rule to my language):

$$\frac{a : \text{Point} \quad b : \text{Point}}{l(a, b) : \text{Segment}}$$

Any segment so introduced is a particular value of function defined on points. Hence, the rule specifies which segment can be introduced for any pair of particular points. But, in Mäenpää’s and von Plato’s system there is no place for diagrams, to the effect that geometric objects are ultimately identified only as that which appropriate terms of the language of this same system refer. Hence, in my view, does not account for many essential features of EPG: namely it allows describing construction involved in proposition I.1 of the *Elements*, but does not account for its grounds: cf. Mäenpää and von Plato 1990: 288-289.

<sup>88</sup> Cf. Azzouni 2004: 123-124, that also understand postulates I.1-3 are rules for drawing diagrams.

<sup>89</sup> Postulate I.3 literally speaks of a point and an ‘interval [διάστημα]’. The latter can be understand both as a segment having an extremity in this point, or merely as the distance between to given points. According to postulate I.1 and definition I.3 these two interpretations are equivalent, in fact.

<sup>90</sup> Cf. the second and third ways ‘in which points enter into arguments in the *Elements*’, according to Mäenpää and von Plato (1990: 286).

<sup>91</sup> A similar interpretation of Euclid’s common notions is also suggested by Stekeler-Weithofer (1992: 136).



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<sup>92</sup> This is in any case Heath's view: he is so sure that it omits them. My numeration is of course Heiberg's one.

<sup>93</sup> I shall come back to this argument in section III.5.

<sup>94</sup> This argument has been, along the centuries, the object matter of a great number of discussions. For two opposed views about it, cf. Euclid H: vol. I, 225-231 and 249-250, and Mueller 1981: 21-26.

<sup>95</sup> This situation originates in the fact that Euclid wants to prove a proposition involving equality of angles without having introduced any explicit stipulation concerning this equality (apt to play the same rule played by definition I.15 for equality of segments). This led Hilbert to include a weaker version of the proposition I.4 among the postulates of his own version of Euclidean plane geometry: cf. Hilbert 1899, post. IV.6 (or III.6, or III.5, in other editions of Hilbert's treatise).

<sup>96</sup> Cf. footnotes (12), (28) and (30).

<sup>97</sup> This proposition relies of course on proposition I.2 through proposition I.3 that is, in fact, merely used to simplify the construction. On the construction of any sort of triangle, cf. also Proclus, *Commentary*, 218.12-220.6, and Proclo CM: 171-172.

<sup>98</sup> Cf. footnote (95).

<sup>99</sup> Cf. Hartshorne 2000: 34.

<sup>100</sup> Cf. section II.1.

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<sup>101</sup> Euclid seems implicitly to admit that what is given is not a segment, but a straight line, in our sense. He is thus compelled to take on it a point at random, but avoids distinguishing the cases where the point **B** is inside, outside or on the circle that he draws.

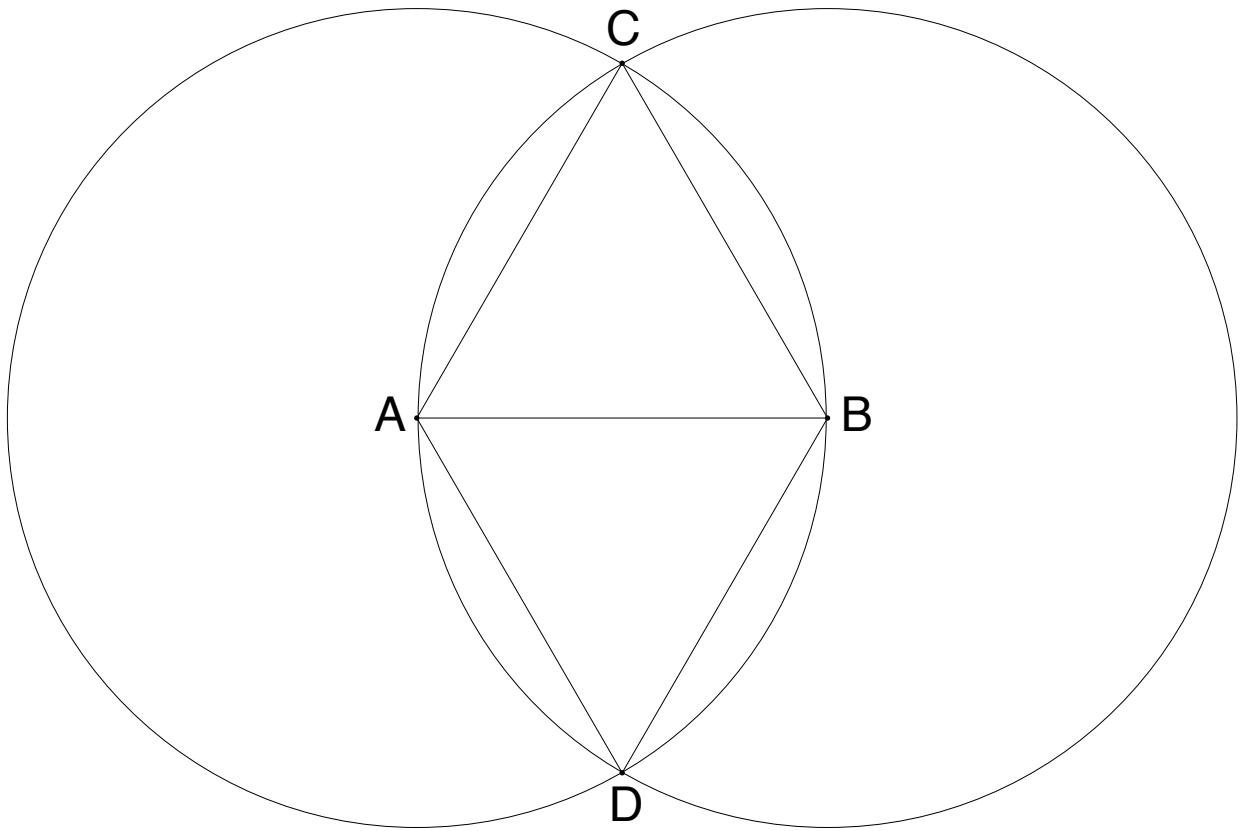


Fig. 1



Fig. 2

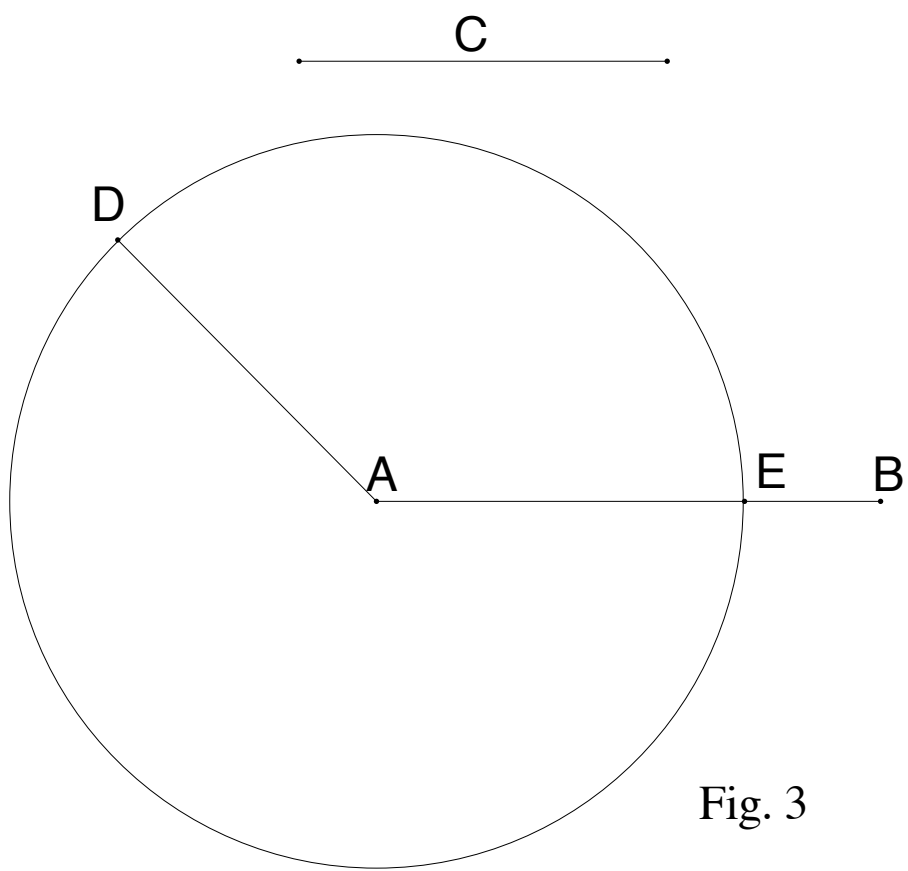


Fig. 3

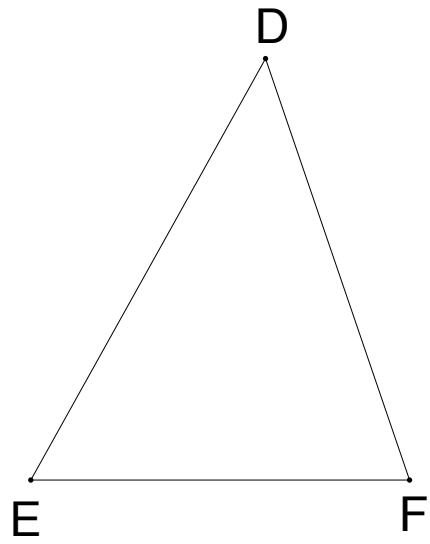
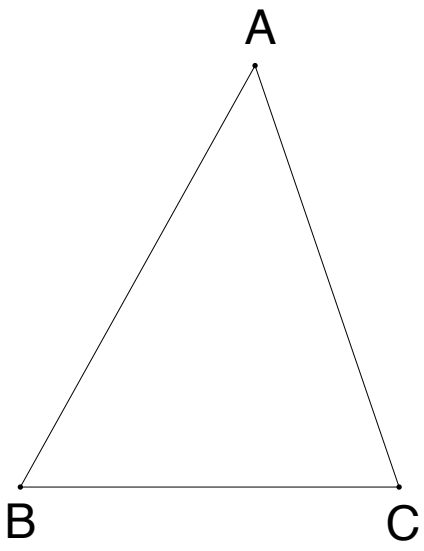


fig. 4



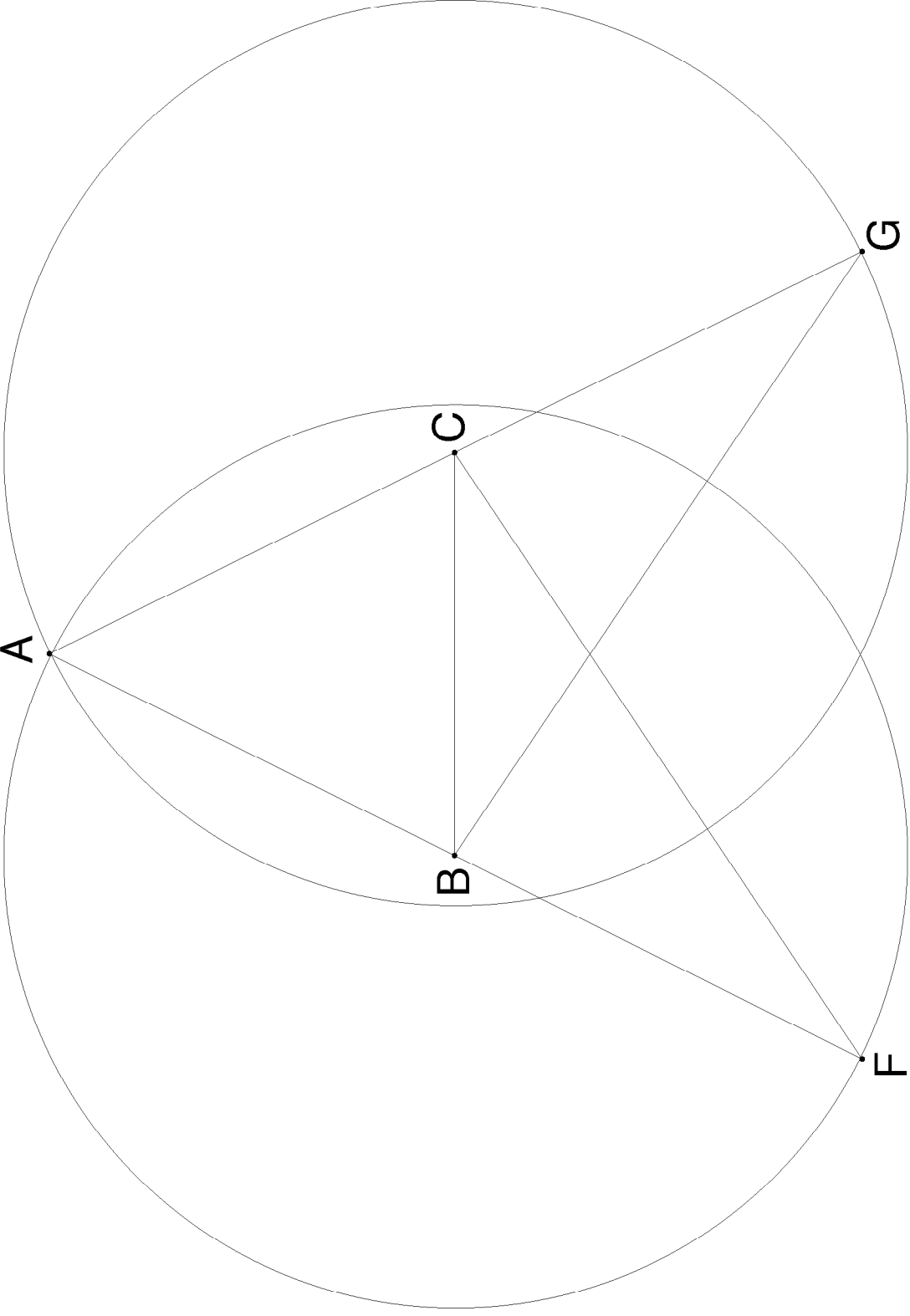


Fig. 3bis



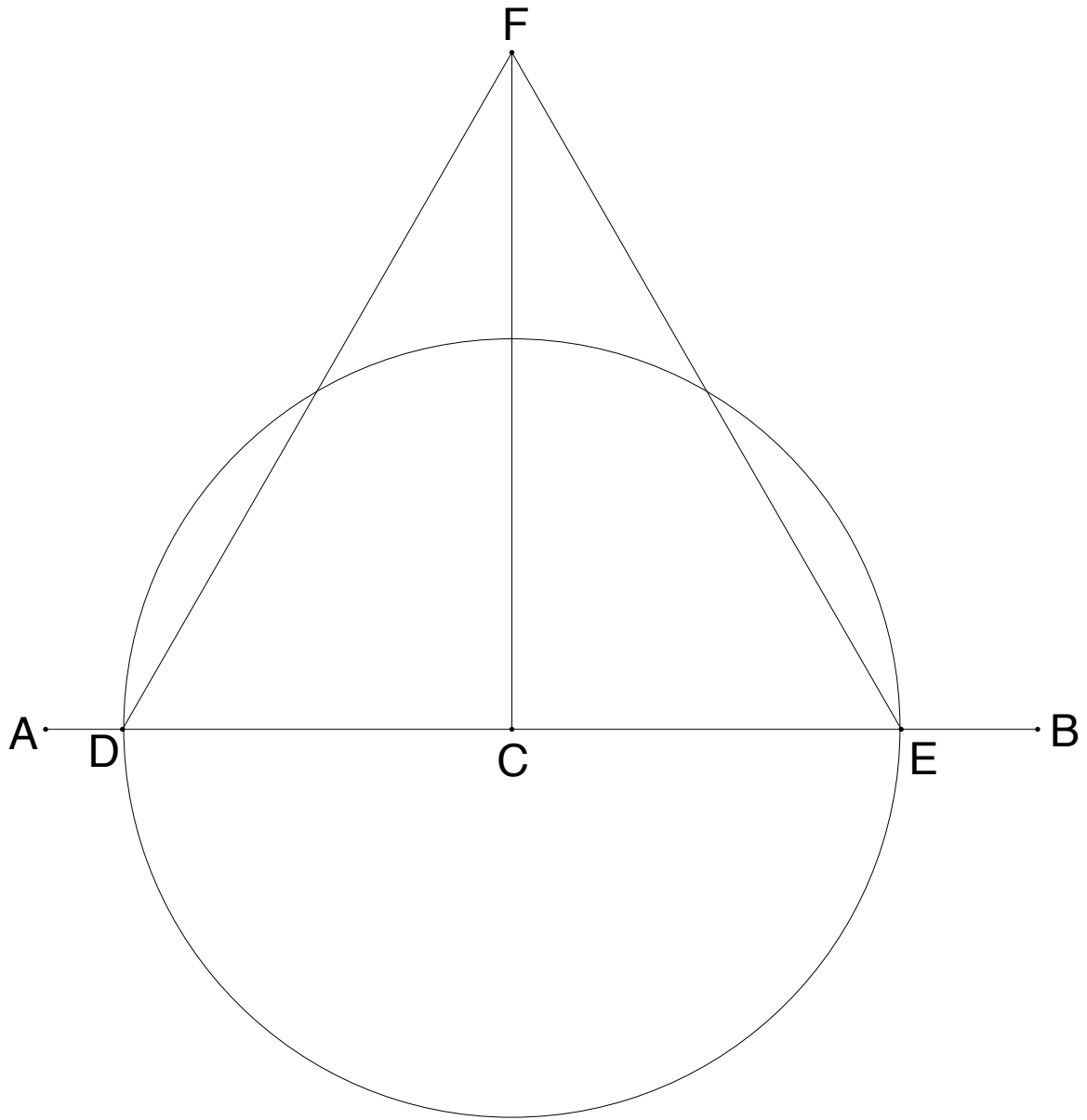


Fig. 6

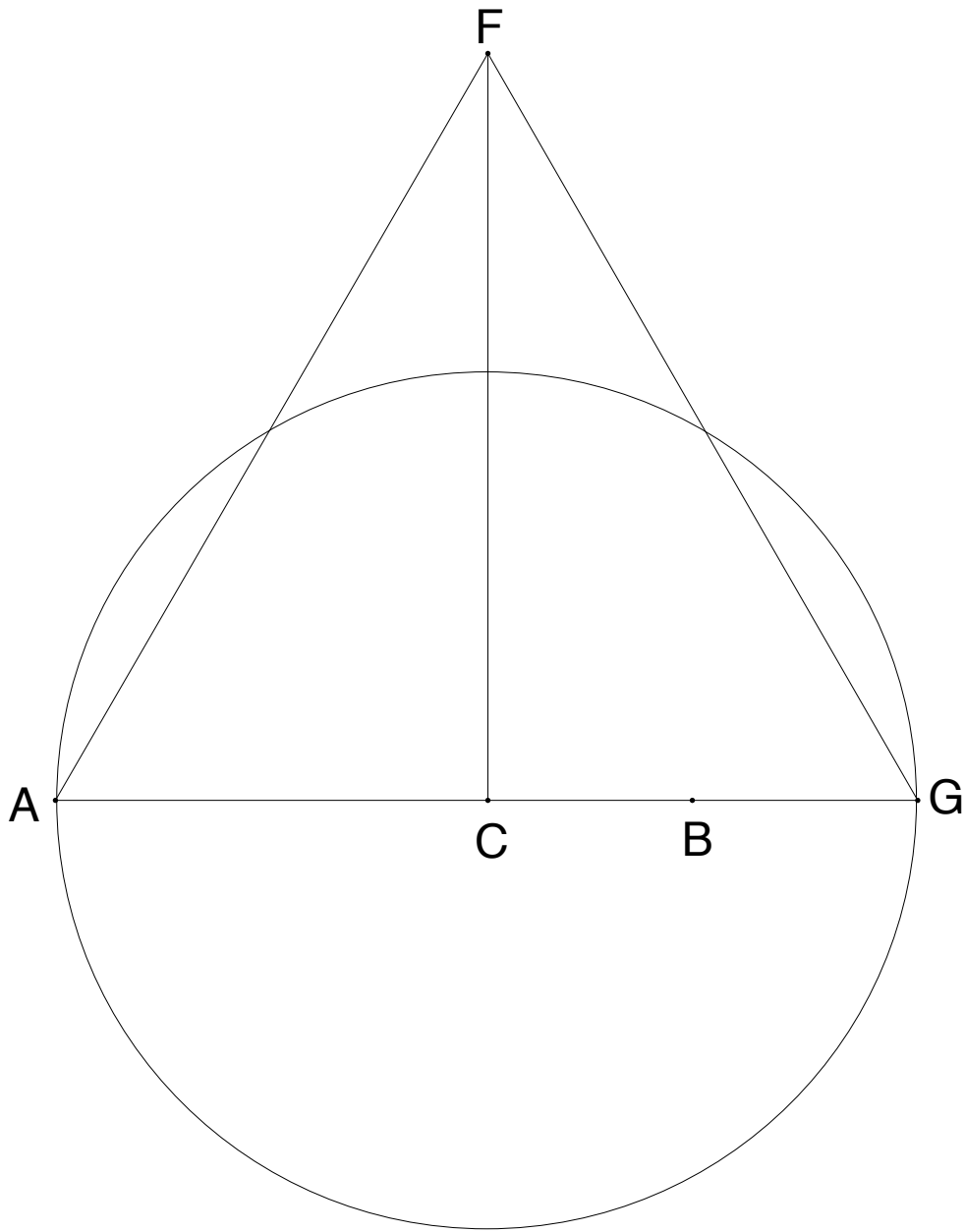


Fig. 6 bis

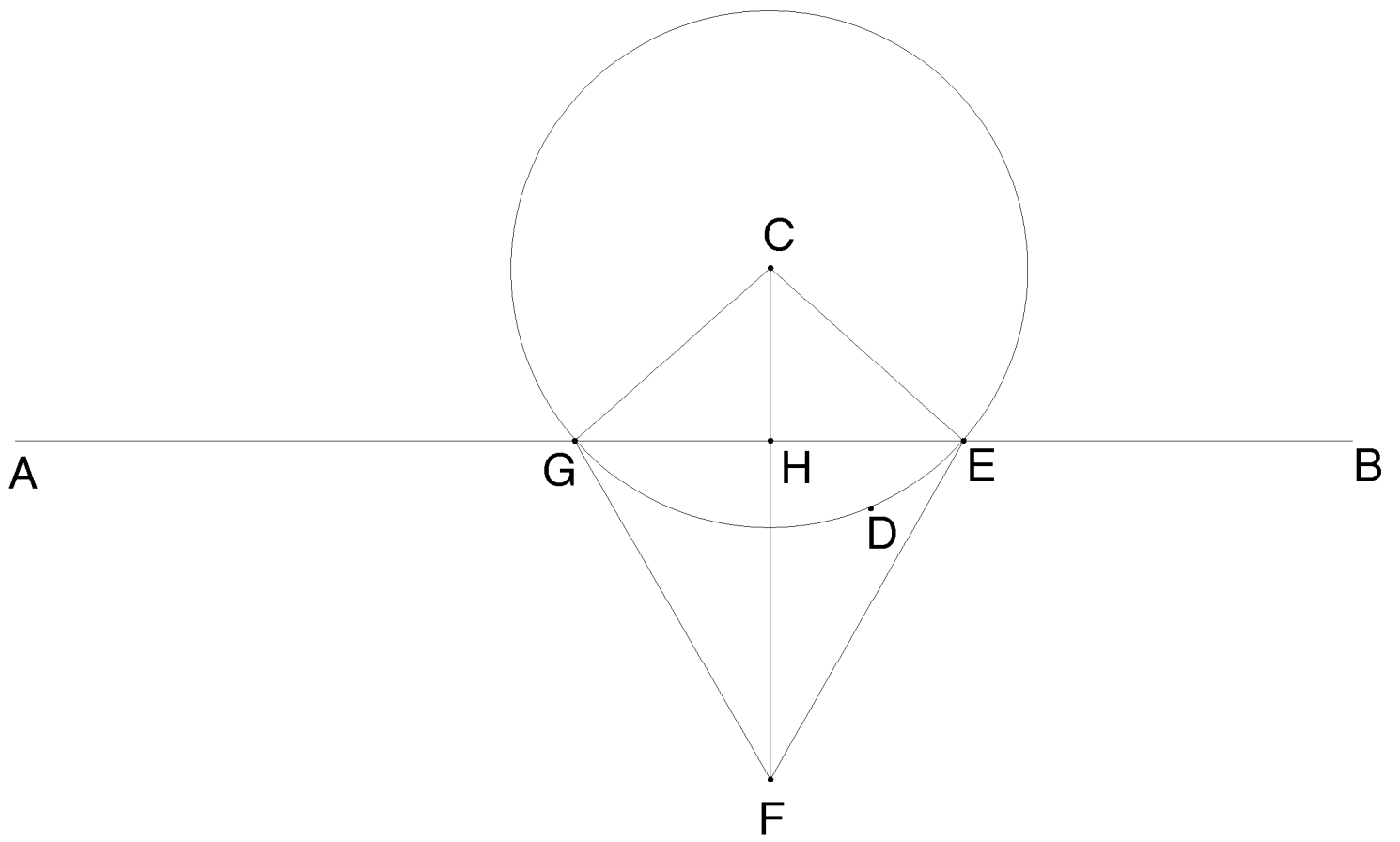


Fig. 7

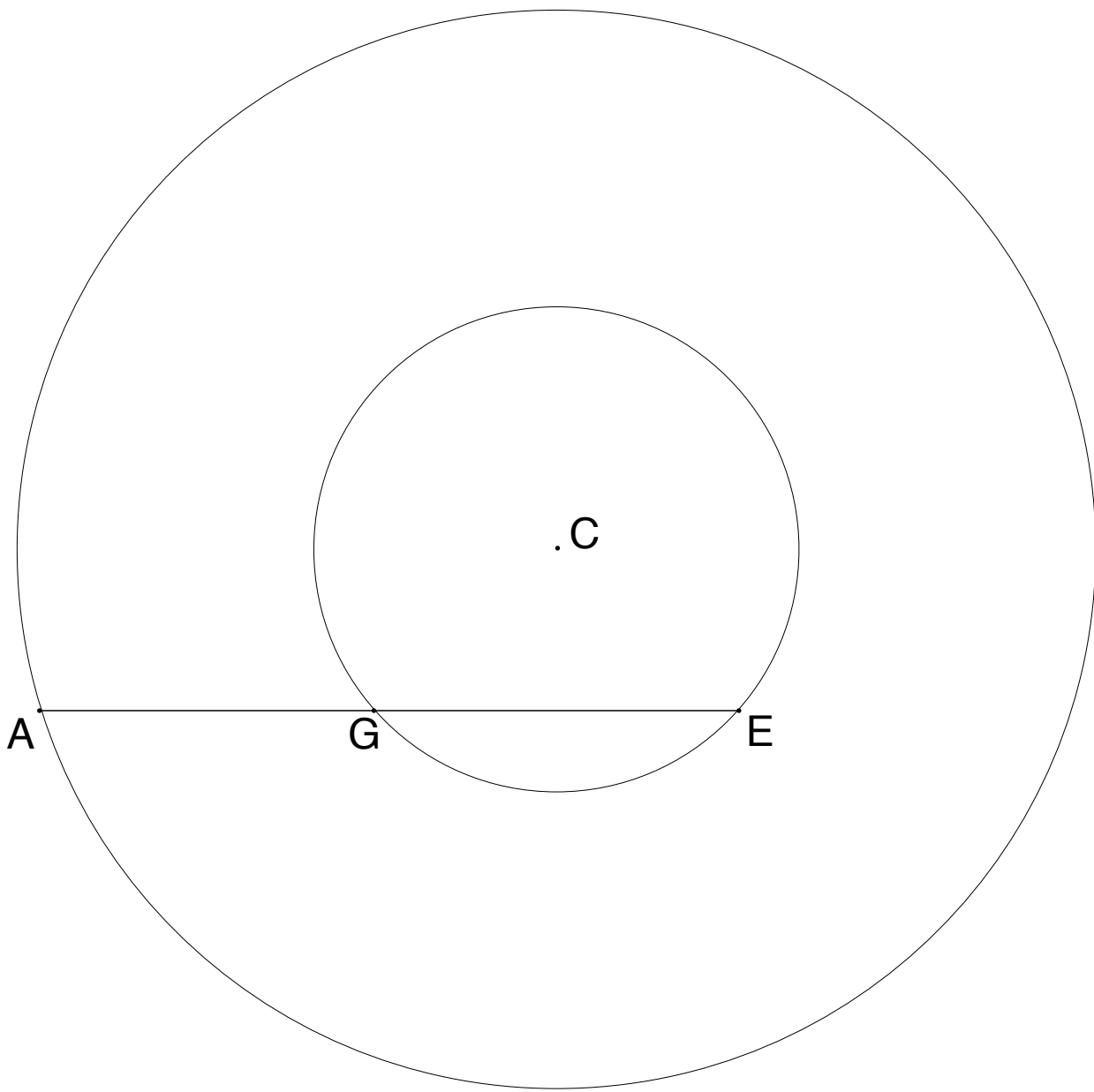


Fig. 7 bis