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# LIMIT LAWS FOR BIASED RANDOM WALKS ON A GALTON-WATSON TREE WITH LEAVES

ALEXANDER FRIBERGH AND NINA GANTERT

ABSTRACT. We consider an outwardly  $\beta$ -biased random walk  $X_n$  on a Galton-Watson tree with leaves in the sub-ballistic regime. We prove that  $X_n/n^\gamma$  converges in law and we characterize the limit law. The exponent  $\gamma \in (0,1)$  is explicit and is a decreasing function of  $\beta$ . Key tools for the proof are classical decomposition results for Galton-Watson trees, a new variant of regeneration times and the careful analysis of the time the walker spends in leaves.

## 1. INTRODUCTION

Consider a supercritical Galton-Watson branching process with generating function  $f(z) = \sum_{k \geq 0} p_k z^k$ , i.e. the offspring of all individuals are i.i.d. copies of  $Z$ , where  $\mathbf{P}[Z = k] = p_k$ . We furthermore assume that  $p_0 > 0$ , and that  $\mathbf{m} := \mathbf{E}[Z] = f'(1) \in (1, \infty)$ , that is the tree is supercritical and has leaves. We denote by  $q$  the extinction probability, which is characterized by  $q < 1$  and  $f(q) = q$ . Starting from a single progenitor called 0, this process yields a random infinite tree  $T$  called a Galton-Watson tree, on the event of non-extinction. We denote  $(\Omega, \mathbf{P})$  the associated probability space.

We shall use some classical notations on trees as  $\overleftarrow{u}$  for the parent of  $u$ ,  $u \wedge v$  for the first common ancestor of  $u$  and  $v$  and  $|u| = d(0, u)$  for the level of  $u$ . Here we used  $d(\cdot, \cdot)$  to denote the usual graph distance.

On this Galton-Watson tree  $T$ , we perform for  $\beta > 1$ , a  $\beta$ -outwardly-biased random walk. It is a Markov chain  $(X_n)_{n \in \mathbb{N}}$  on the vertices of  $T$ , such that if  $u \neq 0$  and  $u$  has  $k$  children  $v_1, \dots, v_k$  then

- (i)  $P[X_{n+1} = \overleftarrow{u} | X_n = u] = \frac{1}{1+\beta k}$ ,
- (ii)  $P[X_{n+1} = v_i | X_n = u] = \frac{\beta}{1+\beta k}$ , for  $1 \leq i \leq k$ ,

and from 0 all transitions to its children are equally likely. This is a reversible Markov chain, and as such, can be described as an electrical network with conductances  $c(\overleftarrow{x}, x) := \beta^{|x|-1}$  on every edge of the tree. We refer to [10] for background on electrical networks.

On a configuration  $\omega$  (which is a Galton-Watson tree conditioned on non-extinction) we can define a random walk according to the preceding rules, and we use  $P^\omega[\cdot]$  to denote the corresponding probability measure with  $X_0 = 0$ . Then we define the annealed law as the semi-direct product  $\mathbb{P} = \mathbf{P} \times P^\omega$ .

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Many interesting facts are known about this walk (see [11]). As one could expect, it is transient. Its limiting speed  $v$  exists and is constant a.s. and satisfies with  $\beta_c = 1/\mathbf{f}'(q)$ ,

- (i) if  $\beta < \beta_c$ , then  $v > 0$ ,
- (ii) if  $\beta \geq \beta_c$ , then  $v = 0$ .

The reason for the sub-ballistic regime (ii) is that the walk loses time in the leaves of the tree, from where it cannot go to infinity without having to go for a long time against the drift. The hypothesis  $p_0 > 0$  is crucial for this to happen. Similar phenomena are observed on the infinite percolation cluster of  $\mathbb{Z}^d$ , see [2] and [14].

As in all sub-ballistic models a natural question comes up concerning the order of magnitude of the random walk. Such questions have arisen for example in the case of random walks in a random environment on  $\mathbb{Z}$ , for the preceding results on transience and recurrence and conditions for zero speed on  $\mathbb{Z}$ , see [13] or [16]. In the zero-speed regime, it was shown in [9] that depending on a certain parameter  $\kappa \in (0, 1]$ , we have

- (i) if  $\kappa < 1$ , then  $\frac{X_n}{n^\kappa}$  converges to a stable law,
- (ii) if  $\kappa = 1$ , then  $\frac{\ln n X_n}{n}$  converges to a stable law.

Very recently and using a precise description of the environment, a new result proven in [4] and [5] refined this last theorem by describing all the parameters of the stable law, in case (i).

Lately progress has been made towards finding the order of magnitude of random walks in the more complicated medium of random trees. Some results were proven for random walks on trees under ellipticity assumptions, for example for transient random walks on a regular tree (see [8]) and on Galton-Watson trees without leaves (see [1] and [12]).

In this paper we drop this ellipticity assumptions by imposing  $p_0 > 0$ , which changes drastically the behaviour of the random walk. The methods used in [9] for the one-dimensional case do not seem to apply easily on Galton-Watson trees. Nevertheless, we are able to find the order of magnitude of the walk by using the methods similar to [5]: we find up to the level  $n$ , the parts of the tree where the walker is likely to get trapped for a large amount of time. In this way, we get a rather explicit description of the limit law. A sketch of our proof is given in Section 5.

In comparison to [5], an additional difficulty arises from the fact the traps met depend not only on the environment but also on the walk. Moreover one has to take into account the number of times the walker enters a trap, which is a complicated matter because of the inhomogeneity of the tree. This major technical difficulty can be overcome by decomposing the tree and the walk into independent parts, which we do using a new variant of regeneration times.

The paper is organized as follows: we state our results in Section 2, then in Section 3 and Section 4 we explain how to decompose the tree and the walk, thereafter in Section 5 we give a sketch of the proof of the main result, and the rest of the paper is devoted to the proof.

## 2. STATEMENT OF THE RESULT

For  $a \in (0, 1)$ , let  $\mathcal{S}_a$  be a completely asymmetric stable random variable of index  $a$ , which is characterized by its Laplace transform,

$$\forall \lambda > 0, E[e^{-\lambda \mathcal{S}_a}] = e^{-\lambda^a}.$$

Then we define

$$(2.1) \quad \gamma := \frac{-\ln \mathbf{f}'(q)}{\ln \beta} = \frac{\ln \beta_c}{\ln \beta},$$

and for  $\lambda \geq 0$ , we set

$$\lambda_n := \frac{\lambda}{n^{1/\gamma}}.$$

We denote the time needed to reach level  $n$  by

$$\Delta_n = \inf\{i \geq 0 \mid |X_i| = n\}.$$

Then our main result is the following

**Theorem 2.1.** *Suppose  $\mathbf{E}[Z^2] < \infty$  and  $\beta \mathbf{f}'(q) > 1$ , then for any  $\lambda \geq 0$ , we have*

$$(2.2) \quad \mathbb{E}[e^{-\lambda_n \Delta_n}] \rightarrow e^{-\zeta \lambda^\gamma},$$

where

$$\zeta := \rho C_a C_b,$$

hence

$$\frac{\Delta_n}{n^{1/\gamma}} \xrightarrow{\text{law}} \zeta^{1/\gamma} S_\gamma.$$

Thus by a classical inversion argument

$$\frac{|X_n|}{n^\gamma} \xrightarrow{\text{law}} \frac{1}{\zeta} \left( \frac{1}{S_\gamma} \right)^\gamma.$$

The constants in Theorem 2.1 are not completely explicit. But, we are able to derive expressions for them

$$C_a := \alpha q \frac{\mathbf{m} - \mathbf{f}'(q)}{1 - q},$$

moreover

$$C_b := \frac{\pi \gamma}{\sin(\pi \gamma)} \frac{\overline{\mathbf{E}}[S_\infty^\gamma]}{(1 - \beta^{-1})^\gamma} \mathbb{E} \left[ \nu \left( \frac{1}{1 - \nu} \right)^\gamma \right],$$

where

$$(2.3) \quad \nu := \frac{(1 - \beta^{-1}) W_\infty}{1 - \beta^{-1} W_\infty}.$$

Furthermore the constant  $\rho$  will be defined in (3.2), whereas  $\alpha$  is defined in Lemma 4.3. The description of the random variables  $S_\infty$  and  $W_\infty$  is postponed to (4.5) and Proposition 7.1 respectively.

**Remark 2.1.** *Intuitively  $\nu$  is the probability of being trapped at the bottom of a leaf, see Section 5 for an explanation, and  $1/(1-\nu)$  is the expected number of times that this happens.*

**Remark 2.2.** *The constant  $C_b$  is reminiscent of the constants appearing in [5], indeed we have the expectation of two random variables raised to the power  $\gamma$  ( $S_\infty$  and  $1/(1-\nu)$ ). A new feature in the case of trees is the factor  $\nu$ , which is necessarily equal to 1 on  $\mathbb{Z}$ .*

**Remark 2.3.** *The hypothesis  $\mathbf{E}[Z^2] < \infty$  could probably be relaxed but we do not find any weaker form appealing. The only place where this hypothesis is used is in the proof of Lemma 8.2.*

**Remark 2.4.** *We also point out that this results advocates the general belief that (in the sub-ballistic regime!) increasing the drift slows down the walk:  $\gamma$  is a decreasing function of  $\beta$ . In the physics literature (see for instance [3]) a similar limit law is conjectured to hold for biased random walks on the infinite percolation cluster of  $\mathbb{Z}^d$ . In fact the exponent is believed to have a similar form as in (2.1) with the corresponding critical bias (whose existence is not proven yet mathematically!).*

**Remark 2.5.** *We emphasize that the critical case  $\beta\mathbf{f}'(q) = 1$  is left open. A similar result of convergence to a stable law is expected to hold with an order of magnitude of  $n/\ln n$ .*

Let us give some conventions about constants. The parameters  $\beta$  and  $(p_k)_{k \geq 0}$  will remain fixed so we will usually not point out that constants depend on them. Most constants will be denoted  $c$  or  $C$  and their value may change from line to line to ease notations. Specific constants will have a subscript as for example  $C_a$ . Moreover we will use  $p_\infty = 1 - \beta^{-1}$  to denote the escape probability of a  $\beta$ -biased random walk on  $\mathbb{N}$ .

We will always denote by  $G_a$  a geometric random variable of parameter  $a$ , with law given by  $P[G_a \geq k] = (1-a)^{k-1}$  for  $k \geq 1$ .

### 3. CONSTRUCTING THE ENVIRONMENT AND THE WALK IN THE APPROPRIATE WAY

In order to understand properly the way the walk is slowed down, we need to construct the environment in a way adapted to our problem. Set

$$(3.1) \quad \mathbf{g}(s) = \frac{\mathbf{f}(s) - \mathbf{f}(sq)}{1-q} \quad \text{and} \quad \mathbf{h}(s) = \frac{\mathbf{f}(qs)}{q}.$$

It is known (see [10] Section 11.5 or [11]), that a  $\mathbf{f}$ -Galton-Watson tree (with  $p_0 > 0$ ) can be generated by

- (i) growing a  $\mathbf{g}$ -Galton-Watson tree  $T_{\mathbf{g}}$  called the backbone, where all vertices have an infinite line of descent,
- (ii) attaching on each vertex  $x$  of  $T_{\mathbf{g}}$  a random number  $N_x$  of  $\mathbf{h}$ -Galton-Watson trees,

where  $N_x$  has a distribution depending only on  $\deg_{T_{\mathbf{g}}}(x)$  and given  $T_{\mathbf{g}}$  and  $N_x$  the leaves are i.i.d..

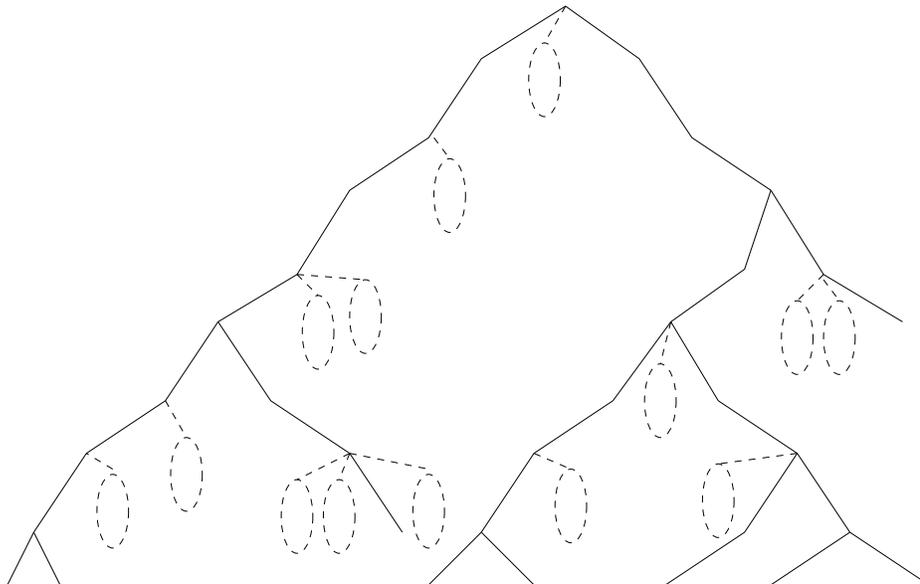


FIGURE 1. The Galton-Watson tree is decomposed into the backbone (solid lines) and the leaves (dashed lines).

We will call bud a vertex at distance exactly one of the backbone. It is important to consider the backbone together with the buds to understand the number of visits to leaves.

We define a leaf to be a graph  $(x \cup V, [x, y] \cup E)$ , where  $x$  is a vertex of the backbone,  $y$  is a bud adjacent to  $x$  and  $V$  (resp.  $E$ ) are the vertices (resp. edges) of the descendants of  $y$ . The leaves can be constructed as a portion of  $\mathbb{Z}$  called the spine, to which smaller trees called subleaves are added, this construction is presented in detail in Section 4

Let us now construct the random walk. We need to consider the walk on the backbone and on the buds, to this end we introduce

- (i)  $\sigma_0 = \sigma'_0 = 0$ ,
- (ii)  $\sigma_{n+1} = \inf\{i > \sigma_n | X_{i-1}, X_i \in \text{backbone}\}$ ,
- (iii)  $\sigma'_{n+1} = \inf\{i > \sigma'_n | X_{i-1}, X_i \in \text{backbone} \cup \text{buds}\}$ ,

and we define  $Y_n = X_{\sigma_n}$  the sample of the walk on the backbone, respectively  $Y'_n = X_{\sigma'_n}$  the walk on the backbone and the buds.

Moreover define  $\Delta_n^Y = \#\{i \geq 0 | \sigma_i \leq \Delta_n\}$  the time spent on the backbone to reach level  $n$  and similarly  $\Delta_n^{Y'} = \#\{i \geq 0 | \sigma'_i \leq \Delta_n\}$ .

Furthermore we define for a set  $A$  in the tree  $T_A^+ = \min\{n \geq 1 | X_n \in A\}$ ,  $T_y^+ := T_{\{y\}}^+$ ,  $T_A := \min\{n \geq 0 | X_n \in A\}$ , and  $T_y := T_{\{y\}}$ .

Note that the process  $(Y_n)_{n \geq 0}$  is a Markov chain on the backbone, which is independent of the leaves and the time spent in the leaves. Hence, in order to generate  $Y_n$  we use a sequence of i.i.d. random variables  $U_i$  uniformly distributed on  $[0, 1]$ . If  $Y_j = w$  with  $Z_1^*$  children on the backbone, then

- (i)  $Y_{i+1} = \overleftarrow{w}$ , if  $U_i \in \left[0, \frac{1}{Z_1^* \beta + 1}\right]$ ,

(ii)  $Y_{i+1}$  = the  $j^{\text{th}}$ -child of  $w$ , if  $U_i \in \left[1 - \frac{j\beta}{Z_1^*\beta+1}, 1 - \frac{(j-1)\beta}{Z_1^*\beta+1}\right]$ .

For background on regeneration times we refer to [15] or [16]. In the case of a  $\beta$ -biased random walk  $\tilde{Y}_n$  on  $\mathbb{Z}$ , a time  $t$  is a regeneration time if

$$\tilde{Y}_t > \max_{s < t} \tilde{Y}_s \text{ and } \tilde{Y}_t < \min_{s > t} \tilde{Y}_s.$$

**Definition 3.1.** *A time  $t$  is a super-regeneration time for  $Y_n$ , if  $t$  is a regeneration time for the coupled  $\beta$ -biased random walk on  $\mathbb{Z}$  defined by*

- (i)  $\tilde{Y}_0 = 0$ ,
- (ii)  $\tilde{Y}_{n+1} = \tilde{Y}_n - 1$ , if  $U_n \in \left[0, \frac{1}{\beta+1}\right]$ ,
- (iii)  $\tilde{Y}_{n+1} = \tilde{Y}_n + 1$  otherwise.

We denote  $t - \text{SR}$  the event that  $t$  is a super-regeneration time for  $Y_n$ .

It is obvious that a super-regeneration time for  $Y_n$  is a regeneration time for  $Y_n$  in the usual sense (the converse is false).

The walk can then be decomposed between the successive super-regeneration times

- (i)  $\tau_0 = 0$ ,
- (ii)  $\tau_{i+1} = \inf\{j \geq \tau_i \mid j - \text{SR}\}$ .

Since the regeneration times of a  $\beta$ -biased random walk on  $\mathbb{Z}$  have some exponential moments, there exists  $a > 1$  such that  $\mathbb{E}[a^{\tau_2 - \tau_1}] < \infty$  and  $\mathbb{E}[a^{\tau_1}] < \infty$ .

**Remark 3.1.** *The advantage of a super-regeneration time over a classical regeneration time, is that the presence of a super-regeneration time does not depend the environment, but only the sequence  $(U_i)_{i \geq 0}$ .*

**Remark 3.2.** *The drawback of super-regeneration is that the event that  $t$  is a super-regeneration time depends on the  $(U_i)_{i \geq 0}$  and not only on the trajectory of the random walk  $(Y_n)_{n \geq 0}$ .*

Denoting for  $k \geq 1$ , the  $\sigma$ -field

$$\mathcal{G}_k = \sigma(\tau_1, \dots, \tau_k, (Y_{n \wedge \tau_k})_{n \geq 0}, \{x \in T(\omega), x \text{ is not a descendent of } Y_{\tau_k}\}).$$

We have the following proposition

**Proposition 3.1.** *For  $k \geq 1$ ,*

$$\begin{aligned} & \mathbb{P}[(Y_{\tau_k+n} - Y_{\tau_k})_{n \geq 0} \in \cdot, \{x \in T(\omega), x \text{ is a descendent of } Y_{\tau_k}\} \in \cdot \mid \mathcal{G}_k] \\ &= \mathbb{P}[(Y_n)_{n \geq 0} \in \cdot, T(\omega) \in \cdot \mid 0 - \text{SR}]. \end{aligned}$$

**Remark 3.3.** *The conditioning  $0 - \text{SR}$  refers only to the walk on the backbone, hence it is obvious that the behaviour of the walk in the leaves and the number of times the walker enters a leaf is independent of that event.*

We skip the proof of this Propositions for it is standard. A consequence of the proposition is that the environment and the walk can be subdivided into super-regeneration

blocks which are i.i.d. (except for the first one). As a consequence we have the existence of

$$(3.2) \quad \rho = \lim \frac{\#\{Y_1, \dots, Y_{\Delta_n^Y}\}}{n},$$

which is the average number of vertices per level visited by  $Y_n$ . This quantity is finite since it is lower than  $1/v(\beta)$ , where  $v(\beta)$  is the speed of  $Y_n$  which is positive by a comparison to the  $\beta$ -biased random walk on  $\mathbb{Z}$ .

When applying the previous proposition, it will be convenient to use the time-shift for the random walk, which we will denote by  $\theta$ .

#### 4. GROWING THE LEAF

In the decomposition theorem of Galton-Watson trees, we attach  $\mathbf{h}$ -Galton-Watson trees. We will denote their distribution with  $\overline{\mathbf{P}}$ , hence  $\overline{\mathbf{P}}[Z = k] = q_k := p_k q^{k-1}$ , where  $Z$  denotes the number of children of a given vertex. As stated before the object we will denote a leaf is slightly different: to describe a leaf  $\ell$  we take a vertex called  $\overrightarrow{root}$  (or  $root(\ell)$  to emphasize the leaf), link it to another vertex (denoted  $\overrightarrow{root}(\ell)$ ), which is the actual root of a random  $\mathbf{h}$ -Galton-Watson tree.

When we use random variables associated to a leaf, we refer to the random part of that leaf (the  $\mathbf{h}$ -Galton-Watson tree). For example the notation  $Z_n$  is the number of children at the generation  $n$  with  $\overrightarrow{root}$  being generation 0. In this context we introduce the height of a leaf

$$(4.1) \quad H = \max\{n \geq 0, Z_n > 0\},$$

and we say a leaf has height  $k$  if  $H(\ell) = k$ , i.e. the distance between  $\overrightarrow{root}$  and the bottom point of the leaf is  $k$ .

This way of denoting the random variables has the advantage that  $Z_n$  (resp.  $H$ ) are distributed under  $\overline{\mathbf{P}}$ , as the number of children at generation  $n$  (resp. the height) of a  $\mathbf{h}$ -Galton-Watson tree.

The biggest leaves seen up to level  $n$  are of size  $-\ln n / \ln \mathbf{f}'(q)$ , therefore a leaf will be considered big if its height is greater or equal to

$$(4.2) \quad h_n = (1 - \varepsilon) \frac{\ln n}{-\ln \mathbf{f}'(q)},$$

for some  $\varepsilon > 0$  which will eventually be chosen small enough. Such a leaf will be called a  $h_n$ -leaf or a big leaf. It is in those leaves that the walker will spend the majority of his time and that is why it is important to have a good description of them.

The leaves are essentially subcritical Galton-Watson trees, as such, they can be grown from the bottom following a procedure described in [6], that we recall for completeness. We will denote by  $\delta$  the starting point of the procedure, corresponding to the left-most bottom point of the leaf, this last notation will be kept for the whole paper.

With a slight abuse of notation, we will denote by  $\bar{\mathbf{P}}$  a probability measure on an enlarged probability space containing the following additional information.

We denote by  $(\phi_{n+1}, \psi_{n+1})$  with  $n \geq 0$ , a sequence of i.i.d pairs of random variables with joint law given by

$$(4.3) \quad \bar{\mathbf{P}}[\phi_{n+1} = j, \psi_{n+1} = k] = c_n q_k \bar{\mathbf{P}}(Z_n = 0)^{j-1} \bar{\mathbf{P}}(Z_{n+1} = 0)^{k-j}, \quad 1 \leq j \leq k, \quad k \geq 1,$$

$$\text{where } c_n = \frac{\bar{\mathbf{P}}[H=n]}{\bar{\mathbf{P}}[H=n+1]}.$$

Set  $\mathcal{T}_0 = \{\delta\}$ . Inductively construct  $\mathcal{T}_{n+1}$ ,  $n \geq 0$  as follows

- (i) let the first generation size of  $\mathcal{T}_{n+1}$  be  $\psi_{n+1}$ ,
- (ii) let  $\mathcal{T}_n$  be the subtree founded by the  $\phi_{n+1}$ -th first generation vertex of  $\mathcal{T}_{n+1}$ ,
- (iii) attach independent  $\mathbf{h}$ -Galton-Watson trees which are conditioned on having height strictly less than  $n$  to the  $\phi_{n+1} - 1$  siblings to the left of the distinguished first generation vertex,
- (iv) attach independent  $\mathbf{h}$ -Galton-Watson trees which are conditioned on having height strictly less than  $n + 1$  to the  $\psi_{n+1} - \phi_{n+1}$  siblings to the right of the distinguished first generation vertex.

Then  $\mathcal{T}_{n+1}$  has the law of a Galton-Watson tree conditioned to have height  $n + 1$  (see [6]).

We denote  $\mathcal{T}$  the infinite tree asymptotically obtained by this procedure; from this tree we can obviously recover all  $\mathcal{T}_n$ . If we pick independently the height  $H$  of a  $\mathbf{h}$ -Galton-Watson tree and the infinite tree  $\mathcal{T}$  obtained by the previous algorithm, then  $\mathcal{T}_H$  has the same law as a  $\mathbf{h}$ -Galton-Watson tree.

We will call spine of this Galton-Watson tree the ancestors of  $\delta$ . If  $y \neq \delta$  is in the spine,  $\vec{y}$  denotes its only child in the spine. We define a subleaf to be a graph  $(x \cup V, [x, y] \cup E)$ , where  $x$  is a vertex of the spine,  $y$  is a descendent of  $x$  not on the spine and  $V$  (resp.  $E$ ) are the vertices (resp. edges) of the descendents of  $y$ . The vertex  $x$  is called the root of the leaf and we denote  $S_x$  the set of all subleaves rooted at  $x$ .

Independently of  $(\phi_n, \psi_n)_{n \geq 0}$  we denote by  $S_{n+1}^{i,j,k}$  and  $\Pi_{n+1}^{i,j,k}$  with  $n, i, j \geq 0$  and  $k = 1, 2$ , two sequences of independent random variables given by

- (i)  $S_n^{n+1,j,1}$  (resp.  $S_n^{n,j,2}$ ) is the  $j$ -th subleaf conditioned to have height inferior to  $n$  added on the left (resp. right) of the  $n + 1$ -th (resp.  $n$ -th) ancestor of  $\delta$ ,
- (ii)  $\Pi_n^{i,j,k}$  is the weight of  $S_n^{i,j,k}$  under the invariant measure associated to the conductances  $\beta^{i+1}$  between the generation  $i$  and  $i + 1$ , the root of  $S_n^{i,j,k}$  being generation 0,

this describes the subleaves and their weights.

We denote  $\Pi_{-1}^{i,j,k} = 0$  and

$$(4.4) \quad \Lambda_i(\omega) = \sum_{j=1}^{\phi_i-1} \Pi_{i-1}^{i,j,1} + \sum_{j=1}^{\psi_i-\phi_i} \Pi_i^{i,j,2},$$

which is the weight of subleaves added to the  $i$ -th ancestor of  $\delta$ .

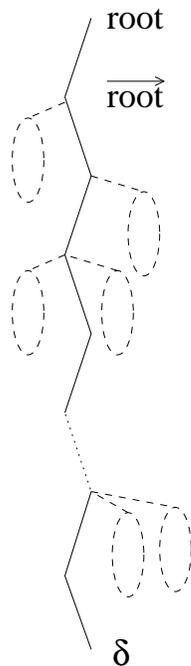


FIGURE 2. The leaf is decomposed into the spine (solide lines) and the subleaves (dashed lines).

Due to the following lemma, the random variables  $\Pi_n^{i,j,k}$  will be important to describe the time spent in leaves.

**Lemma 4.1.** *Let  $(G, c(e))$  be a finite electrical network,  $x \in G$  and  $P_x$  the law of the random walk started at  $x$ .*

*If  $\sum_{y \sim x} c([x, y]) = 1$ , then*

$$E_x[T_x^+] = 2 \sum_{e \in G} c(e).$$

*Proof.* Denoting  $\hat{\pi}$  the invariant measure associated with the conductances of the network. Then  $\hat{\pi}(\cdot)/\hat{\pi}(G)$  is the invariant probability of the network and the mean return time formula yields

$$E_x^\omega[T_x^+] = \frac{\hat{\pi}(G)}{\hat{\pi}(x)} = \hat{\pi}(G),$$

since  $\hat{\pi}(x) = \sum_{y \sim x} c([x, y]) = 1$ . Then we simply notice that

$$\hat{\pi}(G) = 2 \sum_{e \in G} c(e).$$

□

Let us introduce another important random variable

$$(4.5) \quad S_\infty = 2 \sum_{i=0}^{\infty} \beta^{-i} (1 + \Lambda_i),$$

which appears in the statement of our theorem. It is the mean return time to  $\delta$  of the walk on the infinite tree  $\mathcal{T}$  described in the algorithm following (4.3).

**Lemma 4.2.** *There exists  $C_\psi$  so that for  $n \geq 0$  and  $k \geq 0$ ,*

$$\bar{\mathbf{P}}[\psi_{n+1} = k] \leq C_\psi k q^k.$$

*In particular  $\sup_{i \in \mathbb{N}} \bar{\mathbf{E}}[\psi_i] < \infty$ .*

*Proof.* Recalling (4.3), we get

$$\begin{aligned} \bar{\mathbf{P}}[\psi_{n+1} = k] &= \sum_{j=1}^k \bar{\mathbf{P}}[\phi_{n+1} = j, \psi_{n+1} = k] \\ &= c_n q_k \sum_{j=1}^k \bar{\mathbf{P}}[Z_n = 0]^{j-1} \bar{\mathbf{P}}[Z_{n+1} = 0]^{k-j} \\ &\leq c_n k q_k. \end{aligned}$$

It is enough to show that the sequence  $(c_n)_{n \geq 0}$  is bounded from above. A Galton-Watson tree of height  $n+1$  can be obtained as *root* having  $j$  children, one of which is attached to a Galton-Watson tree of height  $n$ , the others having no children of their own. Thus

$$1/c_n = \bar{\mathbf{P}}[H = n+1] / \bar{\mathbf{P}}[H = n] \geq q_j q_0^{j-1},$$

for any  $j \geq 1$ . We fix  $j_0 \geq 1$  so that  $q_{j_0} > 0$  and we get

$$\bar{\mathbf{P}}[\psi_{n+1} = k] \leq \frac{1}{q_{j_0} q_0^{j_0-1}} k q^{k-1},$$

where we used  $q_k = p_k q^{k-1} \leq q^{k-1}$ . □

Using this lemma we can get a tail estimate for the height of leaves.

**Lemma 4.3.** *There exists  $\alpha > 0$  such that*

$$\bar{\mathbf{P}}[H \geq n] \sim \alpha \mathbf{f}'(q)^n.$$

*Proof.* It is classical (see [7]) that for a Galton-Watson tree of law  $\mathbf{Q}$  with  $E[Z_1] = m < 1$  expected number of children, we have

$$\lim_{n \rightarrow \infty} \frac{\mathbf{Q}[Z_n > 0]}{m^n} > 0 \iff E_{\mathbf{Q}}[Z_1 \log^+ Z_1] < \infty.$$

The integrability condition is satisfied for  $\bar{\mathbf{P}}$  since  $q_k = p_k q^{k-1} \leq q^{k-1}$ , and the result follows. □

We also recall the following classical upper bound

$$(4.6) \quad \bar{\mathbf{P}}[H \geq n] = \bar{\mathbf{P}}[Z_n > 0] \leq \bar{\mathbf{E}}[Z_n] = \mathbf{f}'(q)^n.$$

The following lemma seems obvious, but not standard, so we include its proof for the convenience of the reader.

**Lemma 4.4.** *We have for  $k \geq 0$ ,*

$$\overline{\mathbf{P}}[Z_1 \leq k | Z_n = 0] \geq \overline{\mathbf{P}}[Z_1 \leq k].$$

*In particular  $\overline{\mathbf{E}}[Z_i | Z_n = 0] \leq \mathbf{f}'(q)^i$ , for any  $i \geq 0$  and  $n \geq 0$ .*

*Proof.* Denoting  $D_n$  a geometric random variable of parameter  $1 - \overline{\mathbf{P}}[Z_{n-1} = 0]$  which is independent of  $Z_1$ , we have  $\overline{\mathbf{P}}[Z_1 \leq k | Z_n = 0] = \overline{\mathbf{P}}[Z_1 \leq k | Z_1 < D_n]$ . Then if we compute

$$\begin{aligned} \overline{\mathbf{P}}[Z_1 \leq k | Z_1 < D_n] &= \frac{\sum_{j=0}^k \overline{\mathbf{P}}[D_n > j] \overline{\mathbf{P}}[Z_1 = j]}{\sum_{j=0}^{\infty} \overline{\mathbf{P}}[D_n > j] \overline{\mathbf{P}}[Z_1 = j]} \\ &= \left( 1 + \frac{\sum_{j=k}^{\infty} \overline{\mathbf{P}}[D_n > j] \overline{\mathbf{P}}[Z_1 = j]}{\sum_{j=0}^k \overline{\mathbf{P}}[D_n > j] \overline{\mathbf{P}}[Z_1 = j]} \right)^{-1}, \end{aligned}$$

using that for all  $j' < k < j$  we have  $\overline{\mathbf{P}}[D_n > j] \leq \overline{\mathbf{P}}[D_n > k] \leq \overline{\mathbf{P}}[D_n > j']$ , yields

$$\overline{\mathbf{P}}[Z_1 \leq k | Z_1 < D_n] \geq \frac{\sum_{j=0}^k \overline{\mathbf{P}}[Z_1 = j]}{\sum_{j=0}^{\infty} \overline{\mathbf{P}}[Z_1 = j]} = \overline{\mathbf{P}}[Z_1 \leq k].$$

□

We can now estimate  $\overline{\mathbf{E}}[\Lambda_i]$ .

**Lemma 4.5.** *There exists a constant  $C_L$  such that, for all  $i \geq 0$*

$$\overline{\mathbf{E}}[\Lambda_i] \leq C_L (\mathbf{f}'(q)\beta)^i.$$

*Proof.* Using (4.4), Lemma 4.2 and Lemma 4.4, we get

$$\overline{\mathbf{E}}[\Lambda_i] = \overline{\mathbf{E}}[\phi_i] \overline{\mathbf{E}}[\Pi_{i-1}] + \overline{\mathbf{E}}[\psi_i - \phi_i] \overline{\mathbf{E}}[\Pi_i] \leq \overline{\mathbf{E}}[\Pi_i] \sup_{i \in \mathbb{N}} \overline{\mathbf{E}}[\psi_i] \leq C \sum_{j=1}^i \mathbf{f}'(q)^j \beta^j,$$

and the result follows immediately, since  $\beta \mathbf{f}'(q) < 1$ . □

## 5. SKETCH OF THE PROOF

In the first step, we show (see Theorem 6.1) that the time is essentially spent in  $h_n$ -leaves.

Then we notice that these  $h_n$ -leaves are far away from each other, and thus the correlation between the time spent in different  $h_n$ -leaves can be neglected. Moreover the number of  $h_n$ -leaves met before the level  $n$  is roughly  $\rho C_a n^\varepsilon$ . At this point we have reduced our problem to estimating

$$\Delta_n \approx \chi_1(n) + \dots + \chi_{\rho C_a n^\varepsilon}(n),$$

where  $\chi_i(n)$  are i.i.d. random variables with the same law as the time spent in the first  $h_n$ -leaf met. Consequently

$$(5.1) \quad \mathbb{E}[e^{-\lambda_n \Delta_n}] \approx \mathbb{E}[e^{-\lambda_n \chi_1(n)}]^{p C_a n^\varepsilon},$$

we refer to Section 8 for details.

Now we decompose the time spent in the first  $h_n$ -leaf according to the number of excursions in it starting from the *root*

$$\chi_1(n) = \sum_{i=0}^{G_{1-W_n}-1} T_0^{(i)},$$

where  $G_{1-W_n}$  denotes a geometric random variable of a certain random parameter  $1 - W_n$  and  $T_0^{(i)}$  an i.i.d. sequence of random variables measuring the time spent during an excursion in a big leaf. Here  $W_n$  is the probability of entering a big leaf.

It is important to notice that the presence of an  $h_n$ -leaf at a vertex gives information on the number of leaves at this vertex, and thus on the geometry of the backbone. So the law of  $W_n$  depends on  $n$ . Nevertheless we show that this dependence can be asymptotically neglected, and that for large  $n$ ,  $W_n$  is close to some random variable  $W_\infty$  (Proposition 7.1).

Now we have essentially no more correlations between what happens on the backbone and on big leaves. The only thing left to understand is the time spent during an excursion in a  $h_n$ -leaf from the *root*. To simplify if the walker does not reach the point  $\delta$  in the leaf (this has probability  $\approx 1 - p_\infty$ ), the time in the leaf can be neglected. Otherwise, the time spent to go to  $\delta$ , and to go directly from  $\delta$  back to the *root* of the leaf can also be neglected, in other words, only the successive excursions from  $\delta$  contribute to the time spent in the leaf. This is developed in Section 9, so we have the following

$$(5.2) \quad \chi_1(n) \approx \sum_{i=0}^{B(G_{1-W_\infty}-1, p_\infty)} \sum_{j=0}^{G-1} T_1^{(i,j)},$$

where  $G$  is the number of excursions from  $\delta$ , a geometric random variable with a parameter which is easily computed,  $B(n, p)$  denotes a binomial law and  $T_1^{(i,j)}$  are i.i.d. random variables, representing the time spent during the successive excursions from  $\delta$  which do not reach *root*.

Now the random variable  $\nu$  appears naturally, indeed

$$\nu = \mathbb{P}[B(G_{1-W_\infty} - 1, p_\infty) \neq 0],$$

is the asymptotic probability that the walker starting from the top of a leaf will reach its bottom.

We now reduced the problem to finding the Laplace transform of the r.h.s. of (5.2). We do this with the help of tail estimates (see Section 10) and a Tauberian theorem (see Section 12). In this way we obtain Theorem 13.1

$$\mathbb{E}[e^{-\lambda_n \chi_1(n)}] = 1 - C_b \lambda^\gamma n^{-\varepsilon} + o(n^{-\varepsilon}),$$

and recalling (5.1), we get

$$\mathbb{E}[e^{-\lambda_n \Delta_n}] \rightarrow e^{-\rho C_a C_b \lambda^\gamma},$$

which is the main result.

Let us explain how the exponent  $\gamma$  appears. In (5.2), we notice that the only random variable that makes  $\Delta_n$  large is the number  $G$  of excursions from  $\delta$ . But roughly  $G = \beta^H$ , where  $H$  is the height of the big leaf and since the largest leaves met are of height  $H \approx -\ln n / \ln \mathbf{f}'(q)$ , we get  $\Delta_n \approx G \approx n^{1/\gamma}$ , hence the order of magnitude.

**Remark 5.1.** *The reasoning fails in the critical case  $\gamma = 1$ , indeed in this case we have to consider a critical height  $h_n$  which is smaller. This causes many problems, in particular in big leaves there can be big subleaves and so, for example, the time to go from the top to the bottom of a leaf cannot be neglected anymore.*

## 6. THE TIME IS ESSENTIALLY SPENT IN BIG LEAVES

We recall that  $h_n = -(1 - \varepsilon) \ln n / \ln \mathbf{f}'(q)$ . We have asymptotic estimates on the probability of being a  $h_n$ -leaf

$$(6.1) \quad \eta_n := \overline{\mathbf{P}}[H \geq h_n] \sim \alpha n^{\varepsilon-1},$$

using Lemma 4.3.

For  $x \in \text{backbone}$ , we denote  $L_x$  the set of leaves rooted at  $x$  (if  $x$  is not in the backbone then  $L_x = \emptyset$ ). Let us denote the vertices in big leaves by  $L(h_n) = \{y \in \text{tree}, y \text{ is in a } h_n\text{-leaf}\}$ .

Our aim for this section is to show the following

**Theorem 6.1.** *For  $\varepsilon > 0$  and any  $\lambda > 0$ , we have*

$$\mathbb{E}[e^{-\lambda_n \Delta_n}] = \mathbb{E}[e^{-\lambda_n \chi(n)}] + o(1),$$

where  $\chi(n) = \#\{1 \leq i \leq \Delta_n | X_{i-1}, X_i \in L(h_n)\}$  is the time spent in the interior of big leaves up to time  $\Delta_n$ .

Define

- (i)  $A_1(n) = \{\Delta_n^Y \leq C_1 n\}$ ,
- (ii)  $A_2(n) = \{\#\cup_{i=1}^{\Delta_n^Y} L_{Y_i} \leq C_2 n\}$
- (iii)  $A_3(n) = \left\{ \max_{\ell \in L_{Y_i}, i \leq \Delta_n^Y} \#\{0 \leq i \leq \Delta_n^Y | Y_i \in \ell, X_{\sigma_{i+1}} \in \ell\} \leq C_3 \ln n \right\}$ ,
- (iv)  $A(n) = A_1(n) \cap A_2(n) \cap A_3(n)$ .

The following lemma tells us that typically the walk spends less than  $C_1 n$  time units before reaching level  $n$ , sees less than  $C_2 n$  leaves and enters each leaf at most  $C_3 \ln n$  times.

**Lemma 6.1.** *For appropriate  $C_1$ ,  $C_2$  and  $C_3$ , we have*

$$\mathbb{P}[A(n)^c] \rightarrow 0.$$

*Proof.* By a comparison to the  $\beta$ -biased random walk on  $\mathbb{Z}$ , the law of large numbers yields

$$\mathbb{P}[A_1(n)^c] \rightarrow 0,$$

for  $C_1$  large enough.

On  $A_1(n)$ , the number of different vertices visited by  $(Y_i)_{i \geq 0}$  up to time  $\Delta_n^Y$  is lower than  $C_1 n$ . The descendents at each new vertex are drawn independently of the preceding vertices. Moreover at each vertex the mean number of leaves is lower than the mean number of children, thus  $\mathbf{E}[\#L_0] \leq \mathbf{m}/(1-q)$ . The law of large numbers yields for  $C_2 > C_1 \mathbf{m}/(1-q)$  that

$$\mathbb{P}[A_2(n)^c] \leq \mathbb{P} \left[ \sum_{i=0}^{C_1 n} \#L_0^{(i)} > C_2 n \right] + \mathbb{P}[A_1(n)^c] \rightarrow 0,$$

where  $\#L_0^{(i)}$  are i.i.d. random variables with the law  $\#L_0$ . This yields the second part.

For  $A_3(n)$ , we want, given a vertex  $x$  in the backbone and any  $\ell \in L_x$  to give an upper bound on the number of transitions from  $x$  to  $y$ , where  $y$  is the bud associated to  $\ell$ . Moreover let  $z$  be an offspring of  $x$  in the backbone. Then, forgetting all paths not visiting  $y$  or  $z$ , the number of transitions from  $x$  to  $y$  is lower than the number of transitions from  $x$  to  $y$ , if each time the walker is at  $x$  he has to go to  $y$  or  $z$ , both with probability  $1/2$ . Now, for the latter random walk we have that

- (i) the number of transitions from  $x$  to  $y$  before reaching  $z$  is dominated by a geometric random variable of parameter  $1/2$ ,
- (ii) the number of transitions from  $x$  to  $z$  is dominated by a geometric random variable of parameter  $p_\infty$ , since the escape probability from  $z$  is at least  $p_\infty$ .

Consequently the number of transitions from  $x$  to  $y$  is dominated a geometric random variable of parameter  $p_\infty/2$ . Thus

$$\mathbb{P}[A_3(n)^c \cap A_2(n)] \leq C_2 n \mathbb{P} [G_{p_\infty/2} \geq C_3 \ln n] \leq C n^{C_3 \ln(1-p_\infty/2)+1},$$

and if we take  $C_3$  large enough we get the result.  $\square$

Now we can start proving Theorem 6.1, we shall decompose  $\Delta_n$  into

$$(6.2) \quad \Delta_n = \Delta_n^Y + \chi(n) + \sum_{\ell \in \cup_{i=0}^{\Delta_n^Y} L_{Y_i} \setminus L(h_n)} N(\ell),$$

where  $N(\ell) = \{1 \leq i \leq \Delta_n | X_{i-1} \in \ell, X_i \in \ell\}$ .

The distribution of  $N(\ell)$  conditionally on the backbone, the buds and  $(Y'_i)_{i \leq \Delta_n^{Y'}}$ , the walk on the backbone and the buds, is  $\sum_{i=1}^{E_\ell} R_\ell^{(i)}$ . Here we denoted  $E_\ell$  the number

of entries in  $\ell$  and  $R_\ell^{(i)}$  is the return time during the  $i$ -th excursion from the top of  $\ell$ , where  $\ell$  is a leaf conditioned to have height lower than  $h_n$ .

Obviously we get from (6.2) that

$$\mathbb{E}[e^{-\lambda_n \Delta_n}] \leq \mathbb{E}[e^{-\lambda_n \chi(n)}].$$

From (6.2) we get

$$\begin{aligned} \mathbb{E}[e^{-\lambda_n \Delta_n}] &\geq \mathbb{E} \left[ \mathbf{1}_{A(n)} \exp \left( -\lambda_n \left( \chi(n) + \Delta_n^Y + \sum_{\ell \in \cup_{i=0}^{\Delta_n^Y} L_{Y_i} \setminus L(h_n)} N(\ell) \right) \right) \right] \\ &\geq \mathbb{E} \left[ \mathbf{1}_{A(n)} \exp \left( -\lambda_n \left( \chi(n) + \sum_{i=0}^{C_2 n} \sum_{j=0}^{C_3 \ln n} R_\ell^{(i)} \right) \right) \right] + o(1), \end{aligned}$$

since  $\exp(\lambda_n n) = 1 + o(1)$  and  $\Delta_n^Y \leq C_1 n$  on  $A(n)$ .

Now we use that for a fixed  $\ell$ , the  $(R_\ell^{(i)})_{i \geq 0}$  are i.i.d. copies. For  $\ell \neq \ell'$ ,  $R_\ell^{(i)}$  and  $R_{\ell'}^{(j)}$  are independent with respect to the walk and the environment. Finally the random variables  $R_\ell^{(i)}$  are independent of  $\chi(n)$  with respect to the walk and the environment. Hence the last term is

$$\begin{aligned} \mathbb{E}[e^{-\lambda_n \Delta_n}] &\geq \mathbf{E} \left[ E^\omega[e^{-\lambda_n \chi(n)}] \prod_{i=0}^{C_2 n} E^\omega[e^{-\lambda_n R_\ell^{(1)}}]^{C_3 \ln n} \right] + o(1) \\ &\geq \mathbb{E}[e^{-\lambda_n \chi(n)}] \mathbf{E} \left[ E^\omega[e^{-\lambda_n R_1^{(1)}}]^{C_3 \ln n} \right]^{C_2 n} + o(1), \end{aligned}$$

where we used Lemma 6.1.

Then using Jensen's inequality and the fact that  $e^{-x} \geq 1 - x$  for  $x \geq 0$ ,

$$\mathbb{E}[e^{-\lambda_n \Delta_n}] \geq \mathbb{E}[e^{-\lambda_n \chi(n)}] (1 - \lambda_n \mathbb{E}[R_1^{(1)}])^{C_2 C_3 n \ln n} + o(1).$$

Now using Lemma 4.4 and Lemma 4.1, we get

$$\begin{aligned} \mathbb{E}[R_1^{(1)}] &= \overline{\mathbf{E}}[E_{root}^\omega[T_{root}^+ | H < h_n]] = 2 \sum_{i=0}^{h_n-1} \beta^i \overline{\mathbf{E}}[Z_n | H < h_n] \\ &\leq 2 \sum_{i=0}^{h_n-1} (\beta \mathbf{f}'(q))^i \leq C n^{(1-\varepsilon)(-1+1/\gamma)}. \end{aligned}$$

Inserting this in the previous inequality, we get

$$\mathbb{E}[e^{-\lambda_n \Delta_n}] \geq \mathbf{E}[E^\omega[e^{-\lambda_n \chi(n)}]] (1 - C n^{-1/\gamma} n^{(1-\varepsilon)(-1+1/\gamma)})^{n \ln n} + o(1),$$

as  $\gamma < 1$  and  $\varepsilon > 0$ , we get

$$\mathbb{E}[e^{-\lambda_n \Delta_n}] \geq \mathbb{E}[e^{-\lambda_n \chi(n)}] + o(1),$$

thus proving Theorem 6.1.

## 7. ASYMPTOTIC ESCAPE PROBABILITY

We denote  $K_x = \max_{b \in B_x} H(b)$ , the height of the biggest leaf rooted at  $x$  for  $x \in \text{backbone}$ , where we recall that  $H$  denotes the height of the leaf from the bud and not from the *root*.

**Lemma 7.1.** *We have*

$$\mathbf{P}[K_0 \geq h_n] \sim C_a n^{\varepsilon-1},$$

$$\text{where } C_a = \alpha q \frac{\mathbf{m} - \mathbf{f}'(q)}{1-q}.$$

*Proof.* We denote  $Z$  the number of children of a vertex and  $Z^*$  the number of children with an infinite line of descent. Let  $P$  the law of a  $\mathbf{f}$ -Galton Watson which is not conditioned on non-extinction and  $E$  the corresponding expectation.

$$\mathbf{P}[K_0 \geq h_n] = \mathbf{P} \left[ \max_{i=1 \dots Z-Z^*} H^{(i)} \geq h_n \right] = 1 - \frac{E[(1 - \eta_n)^{Z-Z^*} (1 - \mathbf{1}_{Z^*=0})]}{1 - q},$$

the indicator function comes from the conditioning of non-extinction, which corresponds to  $Z^* \neq 0$ . Moreover  $H^{(i)}$  denotes a sequence of independent random variables which have the law of the height of a  $\mathbf{h}$ -Galton-Watson tree.

So

$$\mathbf{P}[K_0 \geq h_n] = 1 - \frac{E[(1 - \eta_n)^{Z-Z^*}] - E[(1 - \eta_n)^{Z-Z^*} 0^{Z^*}]}{1 - q},$$

and using  $E[s^{Z-Z^*} t^{Z^*}] = \mathbf{f}(sq + t(1 - q))$  (see [10] Section 11.5) we get

$$\mathbf{P}[K_0 \geq n] = 1 - \frac{\mathbf{f}((1 - \eta_n)q + 1 - q) - \mathbf{f}(q(1 - \eta_n))}{1 - q}.$$

Now, using Lemma 4.3 and the expansion  $\mathbf{f}(z - x) = \mathbf{f}(z) - \mathbf{f}'(z)x + o(x)$  for  $z \in \{q, 1\}$ , we get the result.  $\square$

Define the first time when we meet the root of a  $h_n$ -leaf using the clock of  $Y_n$ ,

$$(7.1) \quad K(n) = \inf\{i \geq 0 \mid K_{Y_i} \geq h_n\}.$$

We also define  $\ell(n)$  to be a  $h_n$ -leaf rooted at  $Y_{K(n)}$ , if there are several possibilities we choose one leaf according to some predetermined order. We denote  $b(n)$  the associated bud.

We want to describe, on the event  $\{0 - \text{SR}\}$ , starting from  $Y_{K(n)}$  the probability of entering  $\ell(n)$ , this is described by the following random variable,

$$(7.2) \quad W_n = E_0^\omega \left[ P_{Y_{K(n)}}^\omega [T_{b(n)} < \infty \mid 0 - \text{SR}] (U_i)_{i \leq K(n)} \right],$$

where  $\omega$  is chosen under the law  $\mathbf{P}[\cdot]$  and  $(U_i)_{i \leq K(n)}$  under  $P_0^\omega[\cdot \mid 0 - \text{SR}]$ . The conditioning on  $(U_i)_{i \leq K(n)}$  is needed here because of Remark 3.2.

**Proposition 7.1.** *There exists a random variable  $W_\infty$  such that*

$$W_n \xrightarrow{\text{law}} W_\infty.$$

Fix  $n \in \mathbb{N}^*$  and set  $m \geq n$ . We aim at comparing the law of  $W_m$  with that of  $W_n$  and to do that we want to study the behaviour of the random walk starting from the last super-regeneration time before a  $h_n$ -leaf (resp.  $h_m$ -leaf) is seen. This motivates the definition of the last super-regeneration time seen before time  $n$ ,

$$\Sigma(n) := \max\{0 \leq i \leq n, i - \text{SR}\}.$$

For our purpose it is convenient to introduce a modified version of  $W_m$ , which will coincide with high probability with it. For  $m \geq n$ , set

$$\bar{K}(m, n) = \inf\{j \geq 0, K_{Y_j} \geq h_m, \ell(m) \circ \theta_{\Sigma(j)} = \ell(n) \circ \theta_{\Sigma(j)}\},$$

the first time we meet a  $h_m$ -leaf which is the first  $h_n$ -leaf of the current regeneration block, and

$$\bar{W}_{m,n} = E_0^\omega \left[ P_{Y_{\bar{K}(m,n)}}^\omega [T_{b(m)} < \infty \mid 0\text{-SR}] (U_i)_{i \leq \bar{K}(m,n)} \right],$$

where  $\omega$  is chosen under the law  $\mathbf{P}[\cdot]$  and  $(U_i)_{i \leq \bar{K}(m,n)}$  under  $P_0^\omega[\cdot \mid 0\text{-SR}]$ .

**Lemma 7.2.** *For  $m \geq n$  we have that*

$$\bar{W}_{m,n} \stackrel{\text{law}}{=} W_n.$$

*Proof.* To reach a vertex adjacent to a  $h_m$ -leaf, the walker has to reach a vertex adjacent to a  $h_n$ -leaf. We have two cases: either the first  $h_n$ -leaf met is also a  $h_m$ -leaf or it is not.

In the former case, which has probability  $\eta_m/\eta_n > 0$ , since the height of the first  $h_n$ -leaf met is independent of the sequence  $(U_i)_{i \leq K(n)}$ , the random variables  $\bar{W}_{m,n}$  and  $W_n$  coincide.

In the latter case, by its definition,  $\bar{K}_{m,n}$  cannot occur before the next super-regeneration time, hence  $\bar{K}_{m,n} \geq \tau_1 \circ \theta_{K(n)}$ . In this case  $\bar{W}_{m,n} = \bar{W}_{m,n} \circ \theta_{K(n)}$  and then by Proposition 3.1,

$$\bar{W}_{m,n} \circ \theta_{\tau_1 \circ \theta_{K(n)}} \stackrel{\text{law}}{=} \bar{W}_{m,n},$$

and  $\bar{W}_{m,n} \circ \theta_{\tau_1 \circ \theta_{K(n)}}$  is independent of  $(U_i)_{i \leq \tau_1 \circ \theta_{K(n)} - 1}$ .

The scenario repeats itself until the  $h_n$ -leaf reached is in fact a  $h_m$ -leaf, the number of attempts necessary to reach this  $h_m$ -leaf is a geometric random variable of parameter  $\eta_m/\eta_n$  which is independent of the  $(U_i)$ 's.

This means that there is a family  $(W_n^{(i)})_{i \geq 0}$  of i.i.d. random variable with the same law as  $W_n$  such that

$$\bar{W}_{m,n} = W_n^{(G)},$$

where  $G$  is a geometric random variable independent of the  $(W_n^{(i)})_{i \geq 0}$ . Then, note that we have

$$\overline{W}_{m,n} = W_n^{(G)} \stackrel{\text{law}}{=} W_n.$$

□

We need to show that  $\overline{W}_{m,n}$  and  $W_m$  coincide with high probability, so we introduce

$$A_{m,n} = \{\ell(m) = \ell(n) \circ \theta_{\Sigma(K(m))}\},$$

on which clearly those random variables are equal.

Let us denote

$$V_j^i = \{\#\cup_{i=0}^{\tau_1} \{\ell \in L_{Y_i}, \ell \text{ is a } h_j\text{-leaf}\} = i\},$$

and

$$V_j^{i,+} = \{\#\cup_{i=0}^{\tau_1} \{\ell \in L_{Y_i}, \ell \text{ is a } h_j\text{-leaf}\} \geq i\}.$$

Then we have

$$(7.3) \quad \mathbb{P}[A_{m,n}^c | 0 - \text{SR}] \leq \mathbb{P}[V_n^{2,+} | V_m^{1,+}, 0 - \text{SR}].$$

Let us denote  $\#\text{Leaf}$  the number of leaves seen before  $\tau_1$ ,

$$\#\text{Leaf} = \#\{\ell \in \cup_{i=0}^{\tau_1} L_{Y_i}\},$$

and its generating function by

$$\varphi(s) := \mathbb{E}[s^{\#\text{Leaf}} | 0 - \text{SR}].$$

The probability of  $A_{m,n}$  can be estimated with the following lemma.

**Lemma 7.3.** *We have*

$$\forall m \geq n, \quad \mathbb{P}[V_n^{2,+} | V_m^{1,+}, 0 - \text{SR}] \leq \varphi'(1) - \varphi'(1 - \eta_m).$$

Now, because of Remark 3.1, we have  $\mathbb{E}[\#\text{Leaf} | 0 - \text{SR}] \leq \mathbb{E}[\tau_1 | 0 - \text{SR}] \mathbb{E}[\#L_0] < \infty$ , hence  $\varphi'$  is continuous at 1, and we get using (7.3) that

$$(7.4) \quad \sup_{m \geq n} \mathbb{P}[A_{m,n}^c | 0 - \text{SR}] = o(1).$$

*Proof.* Let us make the following remark, set  $i \geq 1$ ,

$$\begin{aligned}
(7.5) \quad \mathbb{P}[V_n^i | V_m^{1,+}, 0 - \text{SR}] &= \frac{\mathbb{P}[V_n^i | 0 - \text{SR}]}{\mathbb{P}[V_m^{1,+} | 0 - \text{SR}]} \mathbb{P}[V_m^{1,+} | V_n^i, 0 - \text{SR}] \\
&\leq \frac{\overline{\mathbb{P}}[H \geq h_m | H \geq h_n]}{\mathbb{P}[V_m^{1,+} | 0 - \text{SR}]} i \mathbb{P}[V_n^i | 0 - \text{SR}] \\
&\leq i \frac{\mathbb{P}[V_n^i | 0 - \text{SR}]}{\eta_n} \frac{\eta_m}{\mathbb{P}[V_m^{1,+} | 0 - \text{SR}]} \\
&\leq i \frac{\mathbb{P}[V_n^i | 0 - \text{SR}]}{\eta_n}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\sum_{i \geq 2} i \mathbb{P}[V_n^i | 0 - \text{SR}] \\
&= \sum_{i \geq 2} \sum_{j \geq i} \mathbb{P}[\#\text{Leaf} = j | 0 - \text{SR}] i \binom{j}{i} \eta_n^i (1 - \eta_n)^{j-i} \\
&= \sum_{j \geq 0} j \mathbb{P}[\#\text{Leaf} = j | 0 - \text{SR}] \sum_{i \geq 2}^j \binom{j-1}{i-1} \eta_n^i (1 - \eta_n)^{j-i} \\
&= \eta_n \sum_{j \geq 0} j \mathbb{P}[\#\text{Leaf} = j | 0 - \text{SR}] \sum_{i \geq 1}^{j-1} \binom{j-1}{i} \eta_n^i (1 - \eta_n)^{(j-1)-i} \\
&= \eta_n \sum_{j \geq 0} j \mathbb{P}[\#\text{Leaf} = j | 0 - \text{SR}] (1 - (1 - \eta_n)^{j-1}) \\
&= \eta_n (\varphi'(1) - \varphi'(1 - \eta_n)).
\end{aligned}$$

Inserting this in (7.5) we get

$$\mathbb{P}[V_n^{2,+} | V_m^{1,+}, 0 - \text{SR}] \leq \frac{1}{\eta_n} \sum_{i \geq 2} i \mathbb{P}[V_n^i | 0 - \text{SR}] = \varphi'(1) - \varphi'(1 - \eta_n),$$

which concludes the proof of Lemma 7.3.  $\square$

Now applying Lemma 7.3, Lemma 7.2 and (7.4) we get,

$$\begin{aligned}
\mathbb{P}[W_m \geq y | 0 - \text{SR}] &= \mathbb{P}[A_{m,n}, \overline{W}_{m,n} \geq y | 0 - \text{SR}] + o(m, n) \\
&= \mathbb{P}[\overline{W}_{m,n} \geq y | 0 - \text{SR}] + o(m, n) \\
&= \mathbb{P}[W_n \geq y | 0 - \text{SR}] + o(m, n),
\end{aligned}$$

where the  $\sup_{m \geq n} o(m, n) \rightarrow 0$  as  $n$  goes to infinity.

The law of a random variable  $W_\infty$  can be defined as a limit of some subsequence of  $(\mathbb{P}[W_m \geq y | 0 - \text{SR}])_{m \geq 0}$ , since for all  $m$  we have  $W_m \in [0, 1]$  a.s.. Then taking  $m$  to infinity along this subsequence in the preceding equation yields

$$\forall y \in [0, 1], \mathbb{P}[W_\infty \geq y] = \mathbb{P}[W_n \geq y | 0 - \text{SR}] + o(1).$$

This proves Proposition 7.1.

We will need the following bounds for the random variables  $(W_n)_{1 \leq n \leq \infty}$ .

**Lemma 7.4.** *We have  $W_n \in [0, 1 - p_\infty/3]$  a.s. for  $n \in \mathbb{N} \cup \{\infty\}$ .*

*Proof.* Indeed, for  $n \in \mathbb{N}$ , starting from any point  $x$  of the backbone, the walker has probability at least  $1/3$  to go to an offspring of  $x$  on the backbone before going to  $b(n)$  or  $\overleftarrow{x}$ . From that point the walker has probability at least  $p_\infty$  to be at a super-regeneration time. The result follows for  $n \in \mathbb{N}$ , the case  $n = \infty$  then follows from Proposition 7.1.  $\square$

## 8. THE TIME SPENT IN DIFFERENT LEAVES IS ASYMPTOTICALLY INDEPENDENT

In order to show the asymptotic independence of the time spent in different big leaves we shall use super-regeneration times. First we show that with overwhelming probability there is no big leaf in the first super-regeneration block.

Define

- (i)  $B_1(n) = \{\#\{Y_1, \dots, Y_{\tau_1}\} \leq n^\varepsilon\}$ ,
- (ii)  $B_2(n) = \{\forall i \leq \tau_1, \#L_{Y_i} \leq n^{2\varepsilon}\}$ ,
- (iii)  $B_3(n) = \{\forall i \leq \tau_1, \forall \ell \in L_{Y_i}, \ell \text{ is not a } h_n\text{-leaf}\}$ ,
- (iv)  $B(n) = B_1(n) \cap B_2(n) \cap B_3(n)$ .

**Lemma 8.1.** *For  $\varepsilon < 1/3$ , we have*

$$\mathbb{P}[B(n)^c] \rightarrow 0.$$

*Proof.* Since  $\tau_1$  has some positive exponential moments and  $B_1(n) \subset \{\tau_1 \geq n^\varepsilon\}$ ,

$$\mathbb{P}[B_1(n)^c] = o(1).$$

Using the fact that the number leaves at different vertices is independent

$$\mathbb{P}[B_2(n)^c] \leq \mathbb{P}[B_1(n)^c] + n^\varepsilon \mathbf{P}[\#L_0 \geq n^{2\varepsilon}] \leq o(1) + n^{-\varepsilon} \frac{\mathbf{m}}{1-q} = o(1),$$

where we used Chebyshev's inequality and  $\mathbf{E}[\#L_0] \leq \mathbf{E}[Z_1] \leq \mathbf{m}/(1-q)$ .

Then we have

$$\mathbb{P}[B_3(n)^c] \leq \mathbb{P}[B_2(n)^c] + n^{2\varepsilon} \eta_n = o(1),$$

yielding the result using (6.1), since  $\varepsilon < 1/3$ .  $\square$

We define  $R(n) = \#\{Y_1, \dots, Y_{\Delta_n^Y}\}$  and the number of vertices adjacent to a  $h_n$ -leaf  $l_n = \#\{i \in [0, \Delta_n^Y], L_{Y_i} \text{ contains a } h_n\text{-leaf}\}$ . Set  $\varepsilon' > 0$ , recall 3.2 and define

- (i)  $C_1(n) = \{(1 - \varepsilon'/2)\rho n \leq R(n) \leq (1 + \varepsilon'/2)\rho n\}$ ,
- (ii)  $C_2(n) = \{(1 - \varepsilon')\rho C_a n^\varepsilon \leq l_n \leq (1 + \varepsilon')\rho C_a n^\varepsilon\}$ ,
- (iii)  $C_3(n) = \{\forall 1 \leq i \leq \Delta_n^Y, \#\{\ell \in L_{Y_i} | \ell \text{ is a } h_n\text{-leaf}\} \leq 1\}$ ,
- (iv)  $C(n) = C_1(n) \cap C_2(n) \cap C_3(n)$ .

We have

**Lemma 8.2.** *For any  $\varepsilon' > 0$  and  $\varepsilon < 1/3$ , we have*

$$\mathbb{P}[C(n)] \rightarrow 1.$$

*Proof.* The asymptotics for  $C_1(n)$  and  $C_2(n)$  are obtained by applying the law of large numbers, we recall that  $C_a$  is defined in Lemma 7.1.

On  $C_2(n)$ , there are at most  $Cn^\varepsilon$  vertices adjacent to at least a  $h_n$ -leaf, we only need to prove that those vertices do not contain more than two  $h_n$ -leaves with probability going to 1. Using the same reasoning as in Lemma 7.3 we get

$$\begin{aligned} & \mathbf{P}[0 \text{ has at least two } h_n\text{-leaves} | 0 \text{ has at least one } h_n\text{-leaf}] \\ & \leq \frac{1}{1-q}(\mathbf{f}'(1) - \mathbf{f}'(1 - \eta_n)) \leq C\eta_n \end{aligned}$$

where we used that  $\mathbf{E}[Z^2] < \infty$ , which implies that  $\mathbf{f}''(1) < \infty$ .

The result follows from the fact that  $\eta_n = o(n^{-\varepsilon})$  for  $\varepsilon < 1/3$ .  $\square$

Let us denote  $\chi^*(n) = \{1 \leq i \leq \sigma_{\Delta_n^Y}, X_{i-1}, X_i \in L(h_n)\}$  the time spent in big leaves before crossing level  $n$  on the backbone.

On  $B(n)$  there is no big leaf in the first super-regeneration block, on  $C(n)$  all big leaves are met in distinct super-regeneration blocks and  $C_2(n)$  tells us the asymptotic number of such blocks. Hence using Lemma 8.1, Lemma 8.2, Proposition 3.1 and Remark 3.3, we get

$$\mathbb{E}[e^{-\lambda_n \chi^*(n)}] \in \left[ \mathbb{E}[e^{-\lambda_n \chi_1(n)} | 0\text{-SR}]^{(1+\varepsilon')\rho C_a n^\varepsilon} + o(1), \mathbb{E}[e^{-\lambda_n \chi_1(n)} | 0\text{-SR}]^{(1-\varepsilon')\rho C_a n^\varepsilon} + o(1) \right],$$

with

$$(8.1) \quad \chi_1(n) = \#\{i \geq 1 | X_{i-1}, X_i \in \ell(n)\},$$

where  $\ell(n)$  was defined right after (7.1).

We want to show that  $\chi(n)$  (defined in Theorem 6.1) and  $\chi^*(n)$  are close. Let us denote

$$D(n) = \left\{ \ell \in \cup_{i=0, \dots, \Delta_n^Y} L_{Y_i} \mid H(\ell) \geq \frac{2 \ln n}{-\ln \mathbf{f}'(q)} \right\}.$$

**Lemma 8.3.** *We have*

$$\mathbb{P}[D(n)] \rightarrow 1.$$

*Proof.* Due to (4.6), we know that  $\overline{\mathbf{P}}[H \geq \frac{2 \ln n}{-\ln \mathbf{f}'(q)}] \leq n^{-2}$ , so using Lemma 6.1

$$\mathbb{P}[D(n)^c] \leq \mathbb{P}[A_2(n)^c] + \mathbb{P}[A_2(n) \cap D(n)^c] \leq o(1) + Cn^{-1} = o(1),$$

which concludes the proof.  $\square$

On  $D(n)$ , we know that to cross level  $n$  on a leaf, it has to be rooted after level  $n - (-2 \ln n / \ln \mathbf{f}'(q))$ , thus we have the following inequality

$$\chi^* \left( n - \frac{2 \ln n}{-\ln \mathbf{f}'(q)} \right) \leq \chi(n) \leq \chi^*(n),$$

and consequently we get for all  $\varepsilon' > 0$

$$(8.2) \quad \mathbb{E}[e^{-\lambda_n \chi(n)}] \in \left[ \mathbb{E}[e^{-\lambda_n \chi_1(n)} | 0\text{-SR}]^{(1+2\varepsilon')\rho C_a n^\varepsilon} + o(1), \mathbb{E}[e^{-\lambda_n \chi_1(n)} | 0\text{-SR}]^{(1-2\varepsilon')\rho C_a n^\varepsilon} + o(1) \right],$$

for  $n$  large enough.

In the light of Theorem 6.1, our problem reduces to understanding the Laplace transform of  $\chi_1(n)$ .

## 9. THE TIME IS SPENT AT THE BOTTOM OF THE LEAVES

In order to compute  $\chi_1(n)$  (defined in (8.1)), we see that the walker enters  $\ell(n)$  exactly  $G_{1-W_n} - 1$  times, where  $G_{1-W_n}$  is a geometric random variable of parameter  $1 - W_n$  (see (7.2) for the definition). The random variable  $W_n$  is independent of  $\ell(n)$  and of the walk on this big leaf.

In this section we investigate the walk on a big leaf., this means that we are now considering a random walk in a finite random environment. For clarity we will call *root* the vertex  $Y_{K(n)}$  on the backbone where  $\ell(n)$  is attached. Moreover set  $\overline{\mathbf{E}}_n[\cdot] := \overline{\mathbf{E}}[\cdot | H \geq h_n]$ ,  $E^\omega := E_{root}^\omega$  and  $\overline{\mathbf{E}}_n := \overline{\mathbf{E}}_n[E^\omega[\cdot]]$ .

**Remark 9.1.** *To ease notations, we add to all these probability spaces an independent random variable  $W_n$  which is distributed as the former  $W_n$  (see (7.2)) under the law  $\mathbb{P}[\cdot | 0 - SR]$ .*

We will extensively use the description of Section 4, in particular we recall that a leaf is composed of *root* which is linked by an edge to a  $\mathbf{h}$ -Galton-Watson tree.

We want to specify what  $\ell(n)$  looks like. Denoting

$$h'_n = \frac{(1 + \varepsilon) \ln n}{-\ln \mathbf{f}'(q)},$$

consider

- (i)  $A_1(n) = \{H \leq h'_n\}$ ,
- (ii)  $A_2(n) = \{\text{there are less than } n^\varepsilon \text{ subleaves}\}$ ,
- (iii)  $A_3(n) = \{\text{all subleaves of } \ell(n) \text{ have height } \leq h_n\}$ ,
- (iv)  $A(n) = A_1(n) \cap A_2(n) \cap A_3(n)$ .

**Lemma 9.1.** *For  $\varepsilon < 1/3$ , we have*

$$\overline{\mathbf{P}}_n[A(n)^c] = o(n^{-\varepsilon}).$$

*Proof.* First

$$\overline{\mathbf{P}}_n[A_1(n)^c] \leq \frac{\overline{\mathbf{P}}[H \geq h'_n]}{\overline{\mathbf{P}}[H \geq h_n]} \sim n^{-2\varepsilon} = o(n^{-\varepsilon}).$$

Furthermore using Lemma 4.2, Lemma 4.4 and (4.6), we get

$$\begin{aligned} & \overline{\mathbf{P}}_n[A_2(n)^c] \\ & \leq \overline{\mathbf{P}}_n[A_1(n)^c] + \overline{\mathbf{P}}_n[A_1(n)], \text{ there are } n^\varepsilon/h'_n \text{ subleaves on a vertex of the spine} \\ & \leq o(n^{-\varepsilon}) + h'_n C_\psi \frac{n^\varepsilon}{h'_n} q^{n^\varepsilon/h'_n} = o(n^{-\varepsilon}). \end{aligned}$$

Finally

$$\begin{aligned} & \overline{\mathbf{P}}_n[A_3(n)^c] \\ & \leq \overline{\mathbf{P}}_n[A_2(n)^c] + \overline{\mathbf{P}}_n[A_2(n)], \text{ there exists a subleaf of height } \geq h_n \\ & \leq o(n^{-\varepsilon}) + n^\varepsilon \eta_n = o(n^{-\varepsilon}), \end{aligned}$$

where we used Lemma 4.4 and  $\varepsilon < 1/3$ .  $\square$

We recall that we denote by  $\delta = \delta(\omega)$  the left-most bottom point of  $\ell(n)$ . Let us introduce

$$(9.1) \quad T_0^{\ell(n)} := \max\{i \geq 0 \mid X_i = \delta, i < T_{root}^+ \circ \theta_{T_\delta}\} - T_\delta.$$

Starting from  $\delta$ , it is the time spent during all excursions from  $\delta$  before exiting the leaf. In this section we want to show

**Proposition 9.1.** *For  $\varepsilon < 1/4$ ,*

$$\mathbb{E}[e^{-\lambda_n \chi_1(n)} \mid 0\text{-}SR] = \overline{\mathbb{E}}_n \left[ \frac{1}{1 + \frac{W_n(1-\beta^{-1})}{1-W_n} \left(1 - E_\delta^\omega[e^{-\lambda_n T_0^{\ell(n)}}]\right)} \right] + o(n^{-\varepsilon}).$$

Fix an environment  $\omega$ . Each time the walker enters the  $h_n$ -leaf two cases can be distinguished: either the walker will reach  $\delta$ , or he will not before he comes back to root. This yields :

$$\begin{aligned} E^\omega[e^{-\lambda_n T_{root}^+}] &= P^\omega[T_\delta^+ > T_{root}^+] E^\omega[e^{-\lambda_n T_{root}^+} \mid T_\delta^+ > T_{root}^+] \\ &\quad + P^\omega[T_\delta^+ < T_{root}^+] E^\omega[e^{-\lambda_n T_{root}^+} \mid T_\delta^+ < T_{root}^+]. \end{aligned}$$

Let us point out that  $P^\omega[T_\delta^+ < T_{root}^+] = \frac{1-\beta^{-1}}{1-\beta^{-(H+1)}}$ , since we recall that the distance between  $root$  and  $\delta$  is  $1 + H$ .

In particular if  $\omega \in \{H \geq h_n\}$ , we get

$$\left| P^\omega[T_\delta^+ < T_{root}^+] - (1 - \beta^{-1}) \right| \leq C n^{\frac{\varepsilon-1}{\gamma}},$$

where  $C$  does not depend on  $\omega$ . Using  $|(a+x)^k - a^k| \leq k|x|(1+|x|)^{k-1}$  for  $|a| \leq 1$ , we get that

$$(9.2) \quad E^\omega[e^{-\lambda_n T_{root}^+}]^k = \left( \beta^{-1} E^\omega[e^{-\lambda_n T_{root}^+} | T_\delta^+ > T_{root}^+] + (1 - \beta^{-1}) E^\omega[e^{-\lambda_n T_{root}^+} | T_\delta^+ < T_{root}^+] \right)^k + k o_1(n^{-\varepsilon})(1 + o_1(n^{-\varepsilon}))^{k-1},$$

where  $o_1(n^{-\varepsilon}) \leq C n^{\frac{\varepsilon-1}{\gamma}}$  and  $C$  does not depend on  $k$  nor on  $\omega \in \{H \geq h_n\}$ .

To tackle the conditionings that appear, we shall use  $h$ -processes, see [5] and [16] for further references. For a given environment  $\omega$  let us denote  $h^\omega$  the voltage (see [10]) on the leaf when  $h^\omega(root) = 1$  and  $h^\omega(\delta) = 0$ . Then we have the following formula :

$$(9.3) \quad P_y^\omega[X_1 = z | T_\delta^+ > T_{root}^+] = \frac{h^\omega(z)}{h^\omega(y)} P_y^\omega[X_1 = z],$$

for  $y, z$  in the leaf.

We recall that the voltage is harmonic except on  $\delta$  and  $root$ , and can be written as  $h^\omega(z) = P_z^\omega[T_\delta > T_{root}]$ . It can be computed using electrical networks

$$h^\omega(y) = h^\omega(y \wedge \delta) = \beta^{-d(root, y \wedge \delta)} \frac{1 - \beta^{-(H+1-d(root, y \wedge \delta))}}{1 - \beta^{-(H+1)}}.$$

In particular we can make the following remarks for the walk conditioned on  $T_\delta^+ > T_{root}^+$

- (i) the walk remains unchanged on the subleaves,
- (ii) for  $y$  on the spine and  $s \in S_y$ , we have  $P_y^\omega[X_1 = \overleftarrow{y} | T_\delta^+ > T_{root}^+] > P_y^\omega[X_1 \in s | T_\delta^+ > T_{root}^+]$ ,
- (iii) for  $y \neq \{\delta, root\}$  on the spine, we have  $P_y^\omega[X_1 = \overleftarrow{y} | T_\delta^+ > T_{root}^+] > \beta P_y^\omega[X_1 = \overrightarrow{y} | T_\delta^+ > T_{root}^+]$ .

Condition (ii) and (iii) state respectively that the conditioned walk is more likely to go towards  $root$  than to go into a given subleaf and that restricted to the spine the conditioned walk is more than  $\beta$ -drifted towards  $root$ .

We can thus prove the following lemma

**Lemma 9.2.** *Suppose  $\varepsilon < 1/3$ .*

*Let  $z$  be  $\delta$  or  $root$ , there exists a function  $o_2 : \mathbb{R} \rightarrow \mathbb{R}$  such that*

$$1 - \overline{\mathbf{E}}_n[E_z^\omega[e^{-\lambda_n T_{root}^+} | T_\delta^+ > T_{root}^+]] \leq o_2(n^{-\varepsilon}),$$

*and  $o_2(n^{-\varepsilon}) = o(n^{-\varepsilon})$ .*

*Proof.* Set  $\omega \in A(n)$ . As always, we denote by  $G_a$  a geometric random variable with parameter  $a$ .

First we show that the walk cannot visit too often a vertex of the spine. Indeed let  $y$  be a vertex of the spine, using remark (iii), we have  $P_y^\omega[T_y^+ > T_{root}^+ | T_\delta^+ > T_{root}^+] \geq p_\infty$ . So denoting  $N(y) = \#\{n \leq T_{root}^+ | X_n = y\}$ , we have

$$P_z^\omega[N(y) \geq n | T_\delta^+ > T_{root}^+] \leq P[G_{p_\infty} \geq n] = (1 - p_\infty)^{n-1},$$

Furthermore, we cannot visit often a given subleaf  $s(y) \in S_y$ . If we denote the number of entries into  $s(y)$  by  $N(s(y)) = \#\{n \leq T_{root}^+ | X_n = y, X_{n+1} \in s(y)\}$ , we have

$$P_z^\omega[N(s(y)) \geq n | T_\delta^+ > T_{root}^+] \leq P[G_{p_\infty/2} \geq n],$$

where we use remark (ii) and a reasoning similar to the one for the asymptotics on  $A_3(n)$  in Lemma 6.1. So

$$P_z^\omega[N(s(y)) \geq n | T_\delta^+ > T_{root}^+] \leq r^{n-1},$$

for any  $r \in (p_\infty/2, 1)$ .

Denoting  $B(n) = \{N(y) \leq n^\varepsilon, N(s(y)) \leq n^\varepsilon, \text{ for } y \text{ in the spine, and } s(y) \in S_y\}$ , we have easily, with  $A(n)$  as in Lemma 9.1

$$\forall \omega \in A(n), P_z^\omega[B(n)^c] \leq o_3(n^{-\varepsilon}),$$

where  $o_3(n^{-\varepsilon}) = o(n^{-\varepsilon})$  and does not depend on  $\omega$ .

Let us now consider the following decomposition :

$$T_{root}^+ = T_{spine} + \sum_{s \in \text{subleaves}} \sum_{j=1}^{N(s)} R_s^j,$$

where  $T_{spine} = \#\{n \leq T_{root}^+ | X_n \text{ is in the spine}\}$  and  $R_s^j$  is the time spent in the subleaf  $s$  during the  $j$ -th excursion in it. Consequently

$$\begin{aligned} & \bar{\mathbf{E}}_n[\mathbf{1}_{A(n)} E^\omega[e^{-\lambda_n T_{root}^+} | T_\delta^+ > T_{root}^+]] \\ & \geq e^{-\lambda_n n^\varepsilon h'_n} \bar{\mathbf{E}}_n \left[ \mathbf{1}_{A(n)} E^\omega \left[ \mathbf{1}_{B(n)} \exp \left( -\lambda_n \sum_{s \in S} \sum_{j=1}^{N(s)} R_s^j \right) \right] \right] + o(n^{-\varepsilon}) \\ & \geq (1 - \lambda_n n^\varepsilon h'_n) \left( 1 - \lambda_n \bar{\mathbf{E}}_n \left[ \mathbf{1}_{A(n)} E^\omega \left[ \mathbf{1}_{B(n)} \sum_{s \in S} \sum_{j=1}^{N(s)} R_s^j \right] \right] \right) + o(n^{-\varepsilon}) \\ & \geq (1 - \lambda_n n^\varepsilon h'_n) \left( 1 - \lambda_n \bar{\mathbf{E}}_n \left[ \mathbf{1}_{A(n)} \sum_{s \in S} n^\varepsilon E^\omega[R_s^1] \right] \right) + o(n^{-\varepsilon}), \end{aligned}$$

where we used  $e^{-x} \geq 1 - x$  in the second inequality.

On  $A(n)$ , the law of any subleaf  $s$  is that of a Galton-Watson conditioned to have height strictly less than  $i + 1$  for some  $i \leq h_n$ . Then Lemma 4.1 implies that for such a subleaf,  $E^\omega[R_s^j]$  has the same law as  $2\Pi_{i+1}^{i,1,1}$ , we can choose independent random variables  $\Pi_1^j, \dots, \Pi_{h(n)}^j$  with the same law as  $\Pi_2^{1,1,1}, \dots, \Pi_{h_n+1}^{h_n,1,1}$  for  $j = 1, \dots, n^\varepsilon$  so that for an arbitrary enumeration of the subleaves  $s_1(\omega), \dots, s_{\#S(\omega)}(\omega)$ , we have

$$\forall 1 \leq i \leq \#S(\omega), E^\omega[R_{s_i(\omega)}^1] \leq 2 \sum_{j=1}^{h_n} \Pi_j^i.$$

On  $A(n)$ ,  $\#S(\omega) \leq n^\varepsilon$  and  $\overline{\mathbf{E}}_n[\Pi_j^i] \leq C(\beta \mathbf{f}'(q))^i$  using Lemma 4.4. Hence, we get

$$\begin{aligned} & \overline{\mathbf{E}}_n[\mathbf{1}_{A(n)} E^\omega[e^{-\lambda_n T_{root}^+} | T_\delta > T_{root}^+]] \\ & \geq (1 - \lambda_n n^\varepsilon h'_n) \left(1 - \lambda_n \overline{\mathbf{E}}_n[\mathbf{1}_{A(n)} \sum_{s \in S} n^\varepsilon E^\omega[R_s^1]]\right) + o(n^{-\varepsilon}) \\ & \geq (1 - \lambda_n n^\varepsilon h'_n) \left(1 - 2\lambda_n n^\varepsilon \overline{\mathbf{E}}_n\left[n^\varepsilon \sum_{j=1}^{h_n} \Pi_j^1\right]\right) + o(n^{-\varepsilon}) \\ & \geq \left(1 - \frac{\lambda_n n^\varepsilon h'_n}{n^{1/\gamma}}\right) (1 - C\lambda_n n^{2\varepsilon} (\beta \mathbf{f}'(q))^{h_n}) + o(n^{-\varepsilon}). \end{aligned}$$

Furthermore we have

$$\frac{n^{2\varepsilon} (\beta \mathbf{f}'(q))^{h_n}}{n^{1/\gamma}} = n^{2\varepsilon} n^{-1+\varepsilon(1-1/\gamma)} = o(n^{2\varepsilon-1}),$$

so that

$$\overline{\mathbf{E}}_n[\mathbf{1}_{A(n)} E^\omega[e^{-\lambda_n T_{root}^+} | T_\delta > T_{root}^+]] \geq (1 - o(n^{\varepsilon-1}))(1 - o(n^{2\varepsilon-1})) + o(n^{-\varepsilon}).$$

Since  $\varepsilon < 1/3$  the result holds.  $\square$

The previous proof is mainly based on the three remarks preceding the statement of Lemma 9.2, thus we can make a very similar proof work in the following case :

**Lemma 9.3.** *Suppose  $\varepsilon < 1/3$ .*

*Let  $z$  be  $\delta$  or  $root$ , there exists a function  $o_2 : \mathbb{R} \rightarrow \mathbb{R}$  such that*

$$1 - \overline{\mathbf{E}}_n[E_z^\omega[e^{-\lambda_n T_\delta^+} | T_\delta^+ < T_{root}^+]] \leq o_2(n^{-\varepsilon}),$$

*where  $o_2(n^{-\varepsilon}) = o(n^{-\varepsilon})$ .*

We can take the same function  $o_2(n)$  for both Lemma 9.2 and Lemma 9.3 to ease notations.

To apply the same methods we only need that the  $h$ -process corresponding to the conditioning  $T_\delta^+ < T_{root}^+$  verifies that

- (i) the walk remains unchanged on the subleaves,
- (ii) for  $y$  on the spine and  $s \in S_y$ , we have  $P_y^\omega[X_1 = \vec{y} | T_\delta^+ < T_{root}^+] > P_y^\omega[X_1 \in s | T_\delta^+ < T_{root}^+]$ ,
- (iii) for  $y \neq \{\delta, root\}$  on the spine, we have  $P_y^\omega[X_1 = \vec{y} | T_\delta^+ < T_{root}^+] > \beta P_y^\omega[X_1 = \overleftarrow{y} | T_\delta^+ < T_{root}^+]$ .

This immediately follows from the computation of the voltage verifying  $\widehat{h}^\omega(root) = 0$  and  $\widehat{h}^\omega(\delta) = 1$ , which gives

$$(9.4) \quad h^\omega(y) = h^\omega(d(y \wedge \delta, \delta)) = \frac{\beta^{H+1} - \beta^{d(y \wedge \delta, \delta)}}{\beta^{H+1} - 1}.$$

From Lemma 9.3 and Lemma 9.2, we deduce the following proposition where  $T_0^{\ell(n)}$  is defined in (9.1)

**Proposition 9.2.** *Given  $k \in \mathbb{N}$  and  $j \leq k$  we have :*

$$\begin{aligned} & \bar{\mathbf{E}}_n[E^\omega[e^{-\lambda_n T_{root}^+} | T_\delta^+ < T_{root}^+]^j E^\omega[e^{-\lambda_n T_{root}^+} | T_\delta^+ > T_{root}^+]^{k-j}] \\ & \geq \bar{\mathbf{E}}_n[E^\omega[e^{-\lambda_n T_0^{\ell(n)}}]^j] - 2k o_2(n). \end{aligned}$$

We point out that it is immediate to obtain the converse inequality without the term  $k o_2(n)$ .

*Proof.* We have to understand  $T_{root}^+$  under  $P^\omega[\cdot | T_\delta^+ < T_{root}^+]$ , which can be decomposed into

$$(9.5) \quad T_{root}^+ \stackrel{\text{law}}{=} R_1 + T_0^{\ell(n)} + R_2,$$

where  $R_1$  is distributed as  $T_\delta^+$  under  $P^\omega[\cdot | T_\delta^+ < T_{root}^+]$ ,  $R_2$  is distributed as  $T_{root}^+$  under  $P_\delta^\omega[\cdot | T_\delta^+ > T_{root}^+]$  and  $R_1$ ,  $R_2$  and  $T_0^{\ell(n)}$  are independent.

For convenience we denote  $R_3$  a random variable of law  $T_{root}^+$  under  $P^\omega[\cdot | T_\delta^+ > T_{root}^+]$ .

Now that the notations are in place, we get

$$\begin{aligned} & \bar{\mathbf{E}}_n[E^\omega[e^{-\lambda_n T_{root}^+} | T_\delta^+ < T_{root}^+]^j E^\omega[e^{-\lambda_n T_{root}^+} | T_\delta^+ > T_{root}^+]^{k-j}] \\ & = \bar{\mathbf{E}}_n[E^\omega[e^{-\lambda_n T_0^{\ell(n)}}]^j E^\omega[e^{-\lambda_n R_1}]^j E^\omega[e^{-\lambda_n R_2}]^j E^\omega[e^{-\lambda_n R_3}]^{k-j}]. \end{aligned}$$

Then

$$\begin{aligned} & \bar{\mathbf{E}}_n[(1 - E^\omega[e^{-\lambda_n R_1}]) E^\omega[e^{-\lambda_n T_0^{\ell(n)}}]^j E^\omega[e^{-\lambda_n R_1}]^{j-1} E^\omega[e^{-\lambda_n R_2}]^j E^\omega[e^{-\lambda_n R_3}]^{k-j}] \\ & \leq \bar{\mathbf{E}}_n[1 - E^\omega[e^{-\lambda_n G_1}]] \leq o_2(n), \end{aligned}$$

where we used Lemma 9.3. Using Lemma 9.2 as well, a simple recursion gets rid of the exponents  $R_1$ ,  $R_2$  and  $R_3$  in  $k + j \leq 2k$  steps, proving Proposition 9.2.  $\square$

We are now able to prove Proposition 9.1. Recalling (8.1), we see that

$$\chi_1(n) = \sum_{i=0}^{G_{1-W_n} - 1} T_{root}^{+, (i)},$$

where  $G_{1-W_n}$  is a geometric random variable of random parameter  $1 - W_n$  and  $T_{root}^{+, (i)}$  is a sequence of i.i.d. random variables distributed as  $T_{root}^+$  under  $\bar{\mathbf{E}}_n[\cdot]$ . Moreover  $W_n$  and the  $T_{root}^{+, (i)}$ 's are independent. So

$$\mathbb{E}[e^{-\lambda_n \chi_1(n)} | 0\text{-SR}] = \sum_{k=0}^{\infty} \bar{\mathbf{E}}_n[(1 - W_n) W_n^k] \bar{\mathbf{E}}_n[E^\omega[e^{-\lambda_n T_{root}^{+, (1)}}]^k].$$

Then we use (9.2) to get

$$\begin{aligned}
& \sum_{k=0}^{\infty} \overline{\mathbb{E}}_n [(1 - W_n) W_n^k] \overline{\mathbb{E}}_n [E^\omega [e^{-\lambda_n T_{root}^{+, (1)}}] ^k] \\
&= \sum_{k=0}^{\infty} \overline{\mathbb{E}}_n [(1 - W_n) W_n^k] \overline{\mathbb{E}}_n \left[ \left( \beta^{-1} E^\omega [e^{-\lambda_n T_{root}^+} | T_\delta^+ > T_{root}^+] \right. \right. \\
&\quad \left. \left. + (1 - \beta^{-1}) E^\omega [e^{-\lambda_n T_{root}^+} | T_\delta^+ < T_{root}^+] \right)^k \right] + k o_1(n^{-\varepsilon}) (1 + o_1(n^{-\varepsilon}))^k.
\end{aligned}$$

Recalling Lemma 7.4 and Remark 9.1, we get that

$$\sum_{k=0}^{\infty} \overline{\mathbb{E}}_n [(1 - W_n) W_n^k] k o_1(n^{-\varepsilon}) (1 + o_1(n^{-\varepsilon}))^k = o(n^{-\varepsilon}).$$

So we have

$$\begin{aligned}
& \mathbb{E}[e^{-\lambda_n \chi_1(n)} | 0\text{-SR}] \\
&= \sum_{k=0}^{\infty} \overline{\mathbb{E}}_n [(1 - W_n) W_n^k] \sum_{j=0}^k \binom{k}{j} \overline{\mathbb{E}}_n [(\beta^{-1} E^\omega [e^{-\lambda_n T_{root}^+} | T_\delta^+ > T_{root}^+])^j \\
&\quad ((1 - \beta^{-1}) E^\omega [e^{-\lambda_n T_{root}^+} | T_\delta^+ < T_{root}^+])^{k-j}] + o(n^{-\varepsilon}),
\end{aligned}$$

which yields using Proposition 9.2 that

$$\begin{aligned}
& \mathbb{E}[e^{-\lambda_n \chi_1(n)} | 0\text{-SR}] \\
&= \sum_{k=0}^{\infty} \overline{\mathbb{E}}_n [(1 - W_n) W_n^k] \sum_{j=0}^k \binom{k}{j} \overline{\mathbb{E}}_n \left[ (\beta^{-1})^j \left( (1 - \beta^{-1}) E_\delta^\omega [e^{-\lambda_n T_0^{\ell(n)}}] \right)^{k-j} \right] \\
&\quad + o(n^{-\varepsilon}),
\end{aligned}$$

since  $\sum_{k=0}^{\infty} \overline{\mathbb{E}}_n [(1 - W_n) W_n^k] k o_2(n^{-\varepsilon}) = o(n^{-\varepsilon})$  by Lemma 7.4. Finally

$$\begin{aligned}
& \mathbb{E}[e^{-\lambda_n \chi_1(n)} | 0\text{-SR}] \\
&= \sum_{k=0}^{\infty} \overline{\mathbb{E}}_n [(1 - W_n) W_n^k] \overline{\mathbb{E}}_n \left[ \left( (1 - \beta^{-1}) E_\delta^\omega [e^{-\lambda_n T_0^{\ell(n)}}] + \beta^{-1} \right)^k \right] + o(n^{-\varepsilon}) \\
&= \overline{\mathbb{E}}_n \left[ \frac{1 - W_n}{1 - W_n \left( (1 - \beta^{-1}) E_\delta^\omega [e^{-\lambda_n T_0^{\ell(n)}}] + \beta^{-1} \right)} \right] + o(n^{-\varepsilon}) \\
&= \overline{\mathbb{E}}_n \left[ \frac{1}{1 + \frac{(1 - \beta^{-1}) W_n}{1 - W_n} \left( 1 - E_\delta^\omega [e^{-\lambda_n T_0^{\ell(n)}}] \right)} \right] + o(n^{-\varepsilon}),
\end{aligned}$$

which proves Proposition 9.1.

## 10. TAIL OF THE TIME SPENT AT THE BOTTOM OF THE LEAVES

For a given configuration  $\omega \in \Omega$ , we decompose  $T_0^{\ell(n)}$  into

$$(10.1) \quad T_0^{\ell(n)} \stackrel{\text{law}}{=} \sum_{i=0}^{G_p-1} T_1^i,$$

where  $G_p$  is a geometric random variable of parameter  $p$  with  $p = p(\omega) := P_\delta[T_\delta^+ > T_{root}^+]$  and we denote  $T_1$  and  $T_1^{(i)}$  i.i.d. random variables independent of  $G_p$ , distributed as  $T_\delta^+$  under  $P_\delta^\omega[\cdot | T_\delta^+ < T_{root}^+]$ .

For our purpose it is not necessary to fully understand  $T_0^{\ell(n)}$ , it is in fact enough to have tail estimates for a certain random variable, namely  $\frac{1-p}{p} E_\delta^\omega[T_1]$  under the law  $\bar{\mathbf{P}}$ .

The random variable  $p$  is function of the height  $H$  and can be computed

$$p = P_\delta^\omega[T_\delta^+ > T_{root}^+] = \frac{1 - \beta^{-1}}{\beta^H - \beta^{-1}}.$$

In particular

$$(10.2) \quad \frac{1-p}{p} = \frac{\beta^H - 1}{1 - \beta^{-1}}.$$

We will show the following theorem

**Theorem 10.1.** *We have the following estimate :*

$$\bar{\mathbf{P}} \left[ \frac{1-p}{p} E_\delta^\omega[T_1] \geq n \right] \sim \frac{\alpha \bar{\mathbf{E}}[S_\infty^\gamma]}{(1 - \beta^{-1})^\gamma} n^{-\gamma}.$$

As the computations of the tail are technical, let us outline a sketch of the proof. We recall that the definition of  $S_\infty$  can be found at (4.5).

- (i) If  $\frac{1-p}{p} E_\delta^\omega[T_1]$  is big, then the height  $H$  of the tree is big.
- (ii) For  $H$  large,  $E_\delta^\omega[T_1]$  does not depend much on  $H$  and is close to  $S_\infty$ .
- (iii) Since  $\frac{1-p}{p}$  only depends on  $H$ , the random variables  $\frac{1-p}{p}$  and  $S_\infty$  are independent and the result follows easily.

First of all we need to understand  $P_\delta^\omega[\cdot | T_\delta^+ < T_{root}^+]$  and to this end we will consider the  $h$ -process associated with this conditioning. Let us compute the voltage  $\hat{h}^\omega$  verifying  $\hat{h}^\omega(\delta) = 1$  and  $\hat{h}^\omega(\text{root}) = 0$ .

We shall enumerate the vertices of the backbone from 0 to  $H + 1$ , starting from  $\delta$  up to  $\text{root}$ . With these new notations formula (9.4) becomes

$$(10.3) \quad \hat{h}^\omega(y) = \hat{h}^\omega(y \wedge \delta) = \frac{\beta^{H+1} - \beta^{y \wedge \delta}}{\beta^{H+1} - 1},$$

where  $y \wedge \delta$  is identified to its number, as it is a vertex of the backbone.

The transition probabilities are then given by a formula similar to (9.3). Obviously they arise from conductances, we may take

- (i)  $\widehat{c}(0, 1) = 1$ ,
- (ii)  $\widehat{c}(i, i+1) = \widehat{c}(i-1, i) \frac{P_i^\omega[X_1=i+1|T_\delta^+ < T_{root}^+]}{P_i^\omega[X_1=i-1|T_\delta^+ < T_{root}^+]}$ , for  $1 \leq i \leq H$ ,
- (iii)  $\widehat{c}(i, z) = c(i, i-1) \frac{P_i^\omega[X_1=z|T_\delta^+ < T_{root}^+]}{P_i^\omega[X_1=i-1|T_\delta^+ < T_{root}^+]}$  for  $i \neq 0$  on the spine and  $z$  one of its descendants which is not on the spine
- (iv)  $\widehat{c}(y, z) = \beta \widehat{c}(\overleftarrow{y}, y)$  for any vertex  $y$  not on the spine and  $z$  one of its descendants.

We can easily deduce from this that for  $y \neq root$  in the leaf and denoting  $z_0 = \delta, \dots, z_n = y$  the geodesic path from  $\delta$  to  $y$  :

$$\widehat{c}(\overleftarrow{y}, y) = \prod_{j=1}^{n-1} \frac{P_{z_j}^\omega[X_1 = z_{j+1} | T_\delta^+ < T_{root}^+]}{P_{z_j}^\omega[X_1 = z_{j-1} | T_\delta^+ < T_{root}^+]},$$

which gives using (10.3) that

$$(10.4) \quad \widehat{c}(i, i+1) = \beta^{-i} \frac{\widehat{h}(i+1)\widehat{h}(i)}{\widehat{h}(1)\widehat{h}(0)} = \beta^{-i} \frac{(1 - \beta^{i-H})(1 - \beta^{i-(H+1)})}{(1 - \beta^{-H})(1 - \beta^{-(H+1)})}.$$

For an vertex  $z$  not on the spine, we have

$$(10.5) \quad \begin{aligned} \widehat{c}(z, \overleftarrow{z}) &= \beta^{d(\overleftarrow{z}, z \wedge \delta)} \frac{\widehat{h}(z \wedge \delta)}{\widehat{h}(z \wedge \delta - 1)} c(z \wedge \delta, \overrightarrow{z \wedge \delta}) \\ &= \beta^{d(\overleftarrow{z}, z \wedge \delta)} \frac{1 - \beta^{z \wedge \delta - (H+1)}}{1 - \beta^{z \wedge \delta - 1 - (H+1)}} c(z \wedge \delta, \overrightarrow{z \wedge \delta}). \end{aligned}$$

This computation and Lemma 4.1 yields,

$$(10.6) \quad E_\delta^\omega[T_1] = 2 \sum_{i=0}^{H-1} \beta^{-i} \frac{(1 - \beta^{i-H})(1 - \beta^{i-(H+1)})}{(1 - \beta^{-H})(1 - \beta^{-(H+1)})} \left( 1 + \frac{1 - \beta^{i-(H+1)}}{1 - \beta^{(i-1)-(H+1)}} \Lambda_i(\omega) \right)$$

where  $\Lambda_i$  was defined at (4.4).

The random variable  $S_\infty$  is the limit of this quantity as  $H$  goes to infinity. Introducing the notation

$$S_n = 2 \sum_{i=0}^{n-1} \beta^{-i} (1 + \Lambda_i),$$

and recalling (4.5), we get the following

**Proposition 10.1.** *We have  $\overline{\mathbf{E}}[S_\infty] < \infty$ , and thus*

$$S_n \rightarrow S_\infty,$$

where the convergence is almost sure and in  $L_1$ .

We can therefore define

$$R_n = 2 \sum_{i=n}^{\infty} \beta^{-i} (1 + \Lambda_i).$$

*Proof.* Recalling Lemma 4.5, we get

$$\bar{\mathbf{E}}[S_\infty] = 2 \sum_{i=0}^{\infty} \beta^{-i} \bar{\mathbf{E}}[1 + \Lambda_i] \leq C_L 2 \sum_{i=0}^{\infty} \beta^{-i} (1 + (\beta \mathbf{f}'(q))^i) < \infty.$$

The  $L^1$  convergence is now obvious, and the almost sure convergence comes from the fact that  $(S_n)_{n \geq 0}$  increases to an integrable limit  $S_\infty$ .  $\square$

Next we prove

**Lemma 10.1.** *For any  $\varepsilon_1 > 0$ , we have for  $n$  large enough*

$$(1 - \varepsilon_1) \bar{\mathbf{P}} \left[ \frac{1-p}{p} S_\infty \geq \frac{1}{1-\varepsilon_1} n \right] \leq \bar{\mathbf{P}} \left[ \frac{1-p}{p} E_\delta^\omega[T_1] \geq n \right] \leq \bar{\mathbf{P}} \left[ \frac{1-p}{p} S_\infty \geq n \right].$$

*Proof.* First we have

$$(10.7) \quad E_\delta^\omega[T_1] \leq S_\infty,$$

which yields the right-most inequality. To prove the left-most inequality, note that

$$(10.8) \quad \begin{aligned} & \bar{\mathbf{P}} \left[ \frac{1-p}{p} E_\delta^\omega[T_1] \geq n \right] \\ & \geq \bar{\mathbf{P}} \left[ \frac{1-p}{p} E_\delta^\omega[T_1] \geq n, E_\delta^\omega[T_1] \geq (1-\varepsilon_1) S_\infty \right] \\ & \geq \bar{\mathbf{P}} \left[ \frac{1-p}{p} S_\infty \geq \frac{1}{1-\varepsilon_1} n, E_\delta^\omega[T_1] \geq (1-\varepsilon_1) S_\infty \right] \\ & \geq \bar{\mathbf{P}} \left[ \frac{1-p}{p} S_\infty \geq \frac{1}{1-\varepsilon_1} n \right] \bar{\mathbf{P}} \left[ E_\delta^\omega[T_1] \geq (1-\varepsilon_1) S_\infty \mid \frac{1-p}{p} S_\infty \geq \frac{1}{1-\varepsilon_1} n \right]. \end{aligned}$$

Let us show

$$(10.9) \quad \bar{\mathbf{P}} \left[ E_\delta^\omega[T_1] \geq (1-\varepsilon_1) S_\infty \mid \frac{1-p}{p} S_\infty \geq \frac{1}{1-\varepsilon_1} n \right] \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Set  $M \in \mathbb{N}$ , it follows from (10.6) that there exists  $y_0$  large enough such that

$$(10.10) \quad H(\omega) \geq y_0 \implies E_\delta^\omega[T_1] \geq \left(1 - \frac{\varepsilon_1}{2}\right) S_M.$$

Consequently

$$\begin{aligned}
& \bar{\mathbf{P}} \left[ E_\delta^\omega [T_1] \geq (1 - \varepsilon_1) S_\infty \mid \frac{1-p}{p} S_\infty \geq \frac{1}{1-\varepsilon_1} n \right] \\
& \geq \bar{\mathbf{P}} \left[ E_\delta^\omega [T_1] \geq (1 - \varepsilon_1) S_\infty, H \geq y_0 \mid \frac{1-p}{p} S_\infty \geq \frac{1}{1-\varepsilon_1} n \right] \\
& \geq \bar{\mathbf{P}} \left[ S_M \geq \frac{1-\varepsilon_1}{1-\varepsilon_1/2} S_\infty, H \geq y_0 \mid \frac{1-p}{p} S_\infty \geq \frac{1}{1-\varepsilon_1} n \right] \\
& \geq \bar{\mathbf{P}} \left[ S_M \geq (1 - \varepsilon_1/2) S_\infty \mid S_\infty \frac{1-p}{p} \geq \frac{1}{1-\varepsilon_1} n \right] \\
& + \bar{\mathbf{P}} \left[ H \geq y_0 \mid \frac{1-p}{p} S_\infty \geq \frac{1}{1-\varepsilon_1} n \right] - 1.
\end{aligned}$$

Set  $\varepsilon_2 > 0$ , to prove that the right-most term of the previous goes to 1 we proceed in two steps.

- (i) The first term can be made larger than  $1 - \varepsilon_2$  independently of  $n$ , provided we pick  $M$  large enough.
- (ii) Once  $M$  is chosen, we can pick an appropriate  $y_0$ , and it will remain to show that for that  $y_0$ , the second term can be made larger than  $1 - \varepsilon_2$  provided  $n$  is big enough.

Let us show (i) by bounding from above the complementary event :

$$\begin{aligned}
(10.11) \quad & \bar{\mathbf{P}} \left[ S_M < (1 - \varepsilon_1/2) S_\infty \mid \frac{1-p}{p} S_\infty \geq \frac{1}{1-\varepsilon_1} n \right] \\
& \leq \frac{\bar{\mathbf{P}} \left[ R_M \geq \frac{\varepsilon_1}{2} S_\infty, \frac{1-p}{p} S_\infty \geq \frac{1}{1-\varepsilon_1} n \right]}{\bar{\mathbf{P}} \left[ \frac{1-p}{p} S_\infty \geq \frac{1}{1-\varepsilon_1} n \right]} \\
& \leq \frac{\bar{\mathbf{P}} \left[ \frac{1-p}{p} \geq \frac{\varepsilon_1}{2(1-\varepsilon_1)} \frac{n}{R_M} \right]}{\bar{\mathbf{P}} \left[ \frac{1-p}{p} \geq \frac{1}{1-\varepsilon_1} n \right]},
\end{aligned}$$

where we used that  $S_\infty \geq 1$ .

Using (10.2) and (4.3), we get the following tail estimate

$$(10.12) \quad \bar{\mathbf{P}} \left[ \frac{1-p}{p} \geq n \right] \sim \frac{\alpha}{(1-\beta^{-1})^\gamma} n^{-\gamma},$$

Thus, there exists  $c, C$  so that

$$cn^{-\gamma} \leq \bar{\mathbf{P}} \left[ \frac{1-p}{p} \geq n \right] \leq Cn^{-\gamma}.$$

Since  $H$  and  $R_M$  are independent, (10.11) becomes

$$\begin{aligned}
\bar{\mathbf{P}} \left[ S_M < (1 - \varepsilon_1/2)S_\infty \mid \frac{1-p}{p}S_\infty \geq \frac{1}{1-\varepsilon_1}n \right] &\leq \frac{\bar{\mathbf{P}} \left[ \frac{1-p}{p} \geq \frac{\varepsilon_1}{2(1-\varepsilon_1)} \frac{n}{R_M} \right]}{\bar{\mathbf{P}} \left[ \frac{1-p}{p} \geq \frac{1}{1-\varepsilon_1}n \right]} \\
&\leq \frac{C(\varepsilon_1)n^{-\gamma} \int t^\gamma \bar{\mathbf{P}}[R_M \in dt]}{c(\varepsilon_1)n^{-\gamma}} \\
&\leq C'(\varepsilon_1) \bar{\mathbf{E}}[R_M^\gamma].
\end{aligned}$$

This upper bound is independent of  $n$  and since  $R_M$  goes to 0 almost surely, we can pick  $M_1$  big enough for the upper bound to be smaller than  $\varepsilon_2$ . This completes step (i).

Fixing now the  $y_1$  corresponding to  $M_1$  in (10.10), we get

$$\begin{aligned}
&\bar{\mathbf{P}} \left[ E_\delta^\omega[T_1] \geq (1 - \varepsilon_1)S_\infty \mid \frac{1-p}{p}S_\infty \geq \frac{1}{1-\varepsilon_1}n \right] \\
&\geq \bar{\mathbf{P}} \left[ H \geq y_1 \mid \frac{1-p}{p}S_\infty \geq \frac{1}{1-\varepsilon_1}n \right] - \varepsilon_2.
\end{aligned}$$

Let us complete step (ii) by showing that the complement of the term on the right side of the previous equation goes to 0 as  $n$  goes to infinity. Keeping in mind (10.2), we get

$$\begin{aligned}
&\bar{\mathbf{P}} \left[ H < y_1 \mid \frac{1-p}{p}S_\infty \geq \frac{1}{1-\varepsilon_1}n \right] \\
\leq &\bar{\mathbf{P}} \left[ \frac{n}{1-\varepsilon_1} \frac{1}{S_\infty} \leq \frac{\beta^{y_1} - \beta^{-1}}{1-\beta^{-1}} \mid \frac{\beta^H - \beta^{-1}}{1-\beta^{-1}} S_\infty \geq \frac{1}{1-\varepsilon_1}n \right] \\
\leq &\frac{\bar{\mathbf{P}} \left[ S_\infty \geq \frac{n(1-\beta^{-1})}{(1-\varepsilon_1)(\beta^{y_1} - \beta^{-1})} \right]}{\bar{\mathbf{P}} \left[ S_\infty(\beta^H - \beta^{-1}) \geq \frac{1-\beta^{-1}}{1-\varepsilon_1}n \right]} \\
\leq &\frac{\bar{\mathbf{P}} \left[ S_\infty \geq \frac{n(1-\beta^{-1})}{(1-\varepsilon_1)(\beta^{y_1} - \beta^{-1})} \right]}{\bar{\mathbf{P}} \left[ \beta^H \geq \frac{1-\beta^{-1}}{1-\varepsilon_1}n \right]} \\
\leq &\frac{(1-\varepsilon_1)(\beta^{y_1} - \beta^{-1})}{n(1-\beta^{-1})} \frac{\bar{\mathbf{E}}[S_\infty]}{c(\varepsilon_1)n^{-\gamma}} \\
\leq &C'(\varepsilon_1)n^{\gamma-1},
\end{aligned}$$

where we used Chebyshev's inequality and the asymptotics on  $H$  given by Lemma 4.3.

This quantity goes to 0 as  $n$  goes to infinity because  $\gamma < 1$ . So for  $n$  large enough

$$\bar{\mathbf{P}} \left[ E_\delta^\omega[T_1] \geq (1 - \varepsilon_1)S_\infty \mid \frac{1-p}{p}S_\infty \geq \frac{1}{1-\varepsilon_1}n \right] \geq 1 - 2\varepsilon_2,$$

which is what we wanted at (10.9).

Recalling (10.8) and using this last equation with  $\varepsilon_2 = \frac{\varepsilon_1}{2}$  yields for  $n$  large enough

$$\bar{\mathbf{P}} \left[ \frac{1-p}{p} E_\delta^\omega[T_1] \geq n \right] \geq (1 - \varepsilon_1) \bar{\mathbf{P}} \left[ \frac{1-p}{p} S_\infty \geq \frac{1}{1-\varepsilon_1}n \right],$$

proving the second inequality and Lemma 10.1.  $\square$

Now Theorem 10.1 will follow from bounds on  $\overline{\mathbf{P}} \left[ \frac{1-p}{p} S_\infty \geq n \right]$ . Using the independence of  $H$  and  $S_\infty$  and the fact that  $p$  depends only on  $H$

$$\overline{\mathbf{P}} \left[ \frac{1-p}{p} S_\infty \geq n \right] = \int \overline{\mathbf{P}} \left[ \frac{1-p}{p} \geq \frac{n}{t} \right] \overline{\mathbf{P}}[S_\infty \in dt].$$

Then by Chebyshev's inequality

$$\overline{\mathbf{P}} \left[ S_\infty \geq n^{\frac{\gamma+1}{2}} \right] \leq \frac{1}{n^{\frac{1+\gamma}{2}}} \overline{\mathbf{E}}[S_\infty] = o(n^{-\gamma}),$$

so

$$\int \overline{\mathbf{P}} \left[ \frac{1-p}{p} \geq \frac{n}{t} \right] \overline{\mathbf{P}}[S_\infty \in dt] = \int_0^{n^{\frac{\gamma+1}{2}}} \overline{\mathbf{P}} \left[ \frac{1-p}{p} \geq \frac{n}{t} \right] \overline{\mathbf{P}}[S_\infty \in dt] + o(n^{-\gamma}).$$

We can compute the asymptotics of this integral using (10.12) to get

$$\overline{\mathbf{P}} \left[ \frac{1-p}{p} S_\infty \geq n \right] \sim \frac{\alpha}{(1-\beta^{-1})^\gamma} n^{-\gamma} \int t^\gamma \overline{\mathbf{P}}[S_\infty \in dt].$$

Theorem 10.1 follows from this asymptotic and by letting  $\varepsilon_1$  go to 0 in Lemma 10.1.

## 11. BOUNDS ON THE TIME SPENT AT THE BOTTOM OF A LEAF

We use a linearization argument similar to the one used in [5].

Recalling the decomposition of  $T_0^{\ell(n)}$  given at (10.1), we get for a fixed  $\omega$

$$\begin{aligned} E_\delta^\omega[e^{-\lambda_n T_0^{\ell(n)}}] &= \sum_{k \geq 0} E_\delta^\omega[e^{-\lambda_n T_1}]^k p(1-p)^k \\ &= \frac{p}{1 - (1-p)E_\delta^\omega[e^{-\lambda_n T_1}]} \\ &= \frac{p}{1 + \frac{1-p}{p} \left(1 - E_\delta^\omega[e^{-\lambda_n T_1}]\right)}. \end{aligned}$$

We want to linearize the denominator, so we use  $1-t \leq e^{-t} \leq 1-t + \frac{t^2}{2}$  to obtain

$$E_\delta^\omega[e^{-\lambda_n T_0^{\ell(n)}}] \in \left[ \frac{1}{1 + \lambda_n \frac{1-p}{p} E_\delta^\omega[T_1]}, \frac{1}{1 + \frac{1-p}{p} (\lambda_n E_\delta^\omega[T_1] - \frac{\lambda_n^2}{2} E_\delta^\omega[T_1^2])} \right].$$

We have to bound  $E_\delta^\omega[T_1^2]$  from above. This is not possible for all  $\omega$ , but we can give bounds on the following event

$$A_5(n) = \left\{ E_\delta^\omega[T_1^2] \leq n^{\frac{1-\varepsilon}{\gamma}} \right\},$$

we have the following lemma.

**Lemma 11.1.** For  $0 < \varepsilon < \min(1/4, \gamma)$ , we have

$$\bar{\mathbf{P}}_n[A_{\delta}(n)^c] = o(n^{-\varepsilon}).$$

*Proof.* In this proof we denote for  $y$  in the leaf,  $N(y)$  the time spent at  $y$  during an excursion from  $\delta$ , which is distributed as  $\#\{0 \leq n \leq T_{\delta}^+ | X_n = y\}$  under  $P_{\delta}^{\omega}[\cdot | T_{\delta}^+ < T_{root}^+]$ .

First we compute using Cauchy-Schwarz

$$\begin{aligned} E_{\delta}^{\omega}[T_1^2] &= E_{\delta}^{\omega} \left[ \left( \sum_{y \in \text{leaf}} N(y) \right)^2 \right] \\ &\leq \sum_{y, z \in \text{leaf}} E_{\delta}^{\omega}[N(y)^2]^{1/2} E_{\delta}^{\omega}[N(z)^2]^{1/2} \\ &= \left( \sum_{y \in \text{leaf}} E_{\delta}^{\omega}[N(y)^2]^{1/2} \right)^2. \end{aligned}$$

Now fix  $y$  in the tree, denote  $q_1 = P_{\delta}^{\omega}[T_y^+ < T_{\delta}^+ | T_{\delta}^+ < T_{root}^+]$  and  $q_2 = P_y^{\omega}[T_{\delta}^+ < T_y^+ | T_{\delta}^+ < T_{root}^+]$ . Then we have

$$\forall k \geq 1, P_{\delta}^{\omega}[N(y) = k] = q_1(1 - q_2)^{k-1}q_2.$$

Hence

$$E_{\delta}^{\omega}[N(y)^2] = \sum_{n \geq 1} n^2 q_1 (1 - q_2)^{n-1} q_2 = q_1 \frac{3 - 2q_2}{q_2^2} \leq \frac{3q_1}{q_2^2}.$$

Then by reversibility of the walk, if  $\hat{\pi}$  is the invariant measure associated with the conductances  $\hat{c}$ , we get  $q_1 = \hat{\pi}(\delta)q_1 = \hat{\pi}(y)q_2$ . This yields

$$E_{\delta}^{\omega}[N(y)^2] \leq \frac{3\hat{\pi}(y)}{q_2}.$$

Furthermore we have  $q_2 \geq p_{\infty} \beta^{-d(y, y \wedge \delta)}/2$ . Indeed starting from  $y$ , the walker has probability at least  $\beta^{-d(y, y \wedge \delta)}$  to reach  $y \wedge \delta$  before  $y$ , then he has probability at least 1/2 to go to  $y \wedge \delta$  before going to  $z$ , where  $z$  is the first vertex on the geodesic path from  $y \wedge \delta$  to  $y$ . Finally from  $y \wedge \delta$ , the walker has probability at least  $p_{\infty}$  to go to  $\delta$  before coming back to  $y \wedge \delta$ .

Finally we denote by  $\pi$  the invariant measure associated with the  $\beta$ -biased random walk (i.e. not conditioned on  $T_{\delta}^+ < T_{root}^+$ ), normalized so as to have  $\pi(\delta) = 1$ . Then we have

- (i) For any  $y$  in the leaf,  $\hat{\pi}(y) \leq \pi(y)$  because of (10.4) and (10.5),
- (ii)  $q_2^{-1} \leq C\beta^{d(\delta, \delta \wedge y)}\pi(y)$ .

Finally we get the following bound

$$E^{\omega}[N(y)^2] \leq C\beta^{d(\delta, \delta \wedge y)}\pi(y)^2,$$

and

$$E^\omega[T_1^2]^{1/2} \leq C \sum_{y \in \text{leaf}} \beta^{d(\delta, \delta \wedge y)/2} \pi(y).$$

As a consequence, with  $A(n)$  as in Lemma 9.1 we get

$$\begin{aligned} \overline{\mathbf{E}}_n[\mathbf{1}_{A(n)} E^\omega[T_1^2]^{1/2}] &\leq C \overline{\mathbf{E}}_n \left[ \mathbf{1}_{A(n)} \sum_{i=0}^{h'_n} \beta^{-i/2} \Lambda_i \right] \\ &\leq C \sum_{i=0}^{h'_n} (\beta^{1/2} \mathbf{f}'(q))^i \\ &\leq C \max(1, (\beta^{1/2} \mathbf{f}'(q))^{h'_n}), \end{aligned}$$

where we used Lemma 4.1 and Lemma 4.4 for the first inequality.

Since  $(\beta^{1/2} \mathbf{f}'(q))^{h'_n} = n^{(1+\varepsilon)(1/2\gamma-1)}$ , we get using Chebyshev's inequality that

$$\begin{aligned} \overline{\mathbf{P}}_n[\mathbf{1}_{A(n)} E^\omega[T_1^2]^{1/2} \geq n^{\frac{1-\varepsilon}{2\gamma}}] &\leq \frac{1}{n^{\frac{1-\varepsilon}{2\gamma}}} \overline{\mathbf{E}}_n[\mathbf{1}_{A(n)} E^\omega[T_1^2]^{1/2}] \\ &\leq C \max(n^{-\frac{1-\varepsilon}{2\gamma}}, n^{\varepsilon/\gamma-1-\varepsilon}). \end{aligned}$$

The conditions on  $\varepsilon$  ensures that this last term is a  $o(n^{-\varepsilon})$ . Hence

$$\mathbf{P}[A(n) \cap A_5(n)^c] = o(n^{-\varepsilon}),$$

and the result follows using Lemma 9.1.  $\square$

If we fix  $\xi > 0$ , we have  $A_5(n) \subset \{\lambda_n E_\delta^\omega[T_1^2] \leq 2(1 - e^{-\xi})\}$ . Hence on  $A_5(n)$

$$E_\delta^\omega[e^{-\lambda_n T_0^{\ell(n)}}] \in \left[ \frac{1}{1 + \lambda_n \frac{1-p}{p} E_\delta^\omega[T_1]}, \frac{1}{1 + e^{-\xi} \lambda_n \frac{1-p}{p} E_\delta^\omega[T_1]} \right].$$

Introducing

$$R_n(\lambda) = \overline{\mathbf{E}}_n \left[ \left( 1 + \frac{(1 - \beta^{-1}) W_n}{1 - W_n} \left( 1 - \frac{1}{1 + \lambda_n \frac{1-p}{p} E_\delta^\omega[T_1]} \right) \right)^{-1} \right],$$

we get by Proposition 9.1 and Lemma 11.1 that

$$(11.1) \quad R_n(\lambda) \leq \mathbb{E}[e^{-\lambda_n \chi_1(n)} | 0\text{-SR}] \leq R_n(e^{-\xi} \lambda) + o(n^{-\varepsilon}).$$

## 12. A TAUBERIAN RESULT

The next result and its proof are very similar to a Tauberian Theorem proven in [4]. We will prove

**Theorem 12.1.** Let  $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be such that

$$\lim_{t \rightarrow 0} t\beta^{h(t)} = 0 \text{ and } \lim_{t \rightarrow 0} h(t) = \infty.$$

Then we have the following asymptotics as  $t$  goes to 0

$$\begin{aligned} & \overline{\mathbf{E}}_{[g(t)]} \left[ 1 - \left( 1 + \frac{(1 - \beta^{-1})W_{[g(t)]}}{1 - W_{[g(t)]}} \left( 1 - \frac{1}{1 + tZ} \right) \right)^{-1} \right] \\ & \sim \frac{1}{\overline{\mathbf{P}}[H \geq h(t)]} \frac{\pi\gamma}{\sin(\pi\gamma)} \frac{\alpha \overline{\mathbf{E}}[S_\infty^\gamma]}{(1 - \beta^{-1})^\gamma} \mathbb{E} \left[ \frac{(1 - \beta^{-1})W_\infty}{1 - \beta^{-1}W_\infty} \left( \frac{1 - \beta^{-1}W_\infty}{1 - W_\infty} \right)^\gamma \right] t^\gamma, \end{aligned}$$

where  $Z := \frac{1-p}{p} E_\delta^\omega[T_1]$  and  $g(t)$  is such that  $h(t) = \frac{1-\varepsilon}{-\ln f'(q)} \ln g(t)$ .

*Proof.* We introduce

$$\tilde{t} = \frac{1 - \beta^{-1}W_{[g(t)]}}{1 - W_{[g(t)]}} t;$$

and

$$I(Z, t) = \frac{(1 - \beta^{-1})W_{[g(t)]}}{1 - \beta^{-1}W_{[g(t)]}} \frac{\tilde{t}Z}{1 + \tilde{t}Z}.$$

Note that because of Lemma 7.4 we get

$$(12.1) \quad \tilde{t} \leq Ct \text{ and } I(Z, t) \leq CtZ.$$

Now denoting  $H$  the height of  $\ell(g(t))$  the first  $h_{[g(t)]}$ -leaf met, a computation yields

$$\overline{\mathbf{E}}_{[g(t)]} \left[ 1 - \left( 1 + \frac{(1 - \beta^{-1})W_{[g(t)]}}{1 - W_{[g(t)]}} \left( 1 - \frac{1}{1 + tZ} \right) \right)^{-1} \right] = \frac{\overline{\mathbf{E}}[\mathbf{1}_{H \geq h(t)} I(Z, t)]}{\overline{\mathbf{P}}[H \geq h(t)]}.$$

We have

$$\overline{\mathbf{E}}[\mathbf{1}_{H \geq h(t)} I(Z, t)] = \overline{\mathbf{E}}[\mathbf{1}_{Z \geq \beta^{h(t)}} I(Z, t)] - \overline{\mathbf{E}}[\mathbf{1}_{\beta^H \geq \beta^{h(t)} \geq Z} I(Z, t)].$$

Furthermore using (12.1)

$$\begin{aligned} \overline{\mathbf{E}}[\mathbf{1}_{\beta^H \geq \beta^{h(t)} \geq Z} I(Z, t)] & \leq Ct \sum_{i=0}^{[h(t)]} \overline{\mathbf{E}}[\mathbf{1}_{H=i} \overline{\mathbf{E}}[Z|H=i]] \\ & \leq Ct \sum_{i=0}^{[h(t)]} \overline{\mathbf{E}} \left[ \mathbf{1}_{H=i} \overline{\mathbf{E}} \left[ \frac{1-p}{p} S_\infty | H=i \right] \right] \\ & \leq Ct \overline{\mathbf{E}}[S_\infty] \sum_{i=0}^{[h(t)]} \overline{\mathbf{P}}[H=i] \frac{\beta^i - 1}{1 - \beta^{-1}}, \end{aligned}$$

where we used (10.7) and (10.2). Moreover using (4.6) and  $\mathbf{f}'(q) = \beta^{-\gamma}$  we get

$$(12.2) \quad \begin{aligned} \overline{\mathbf{E}} \left[ \mathbf{1}_{\beta^H \geq e^{h(t)} \geq Z} I(Z, t) \right] &\leq Ct \sum_{i=0}^{\lfloor h(t) \rfloor} \beta^{(1-\gamma)i} \leq Ct \beta^{(1-\gamma)h(t)} \\ &\leq Ct^\gamma (t\beta^{h(t)})^{1-\gamma} = o(t^\gamma), \end{aligned}$$

since  $\gamma < 1$  and  $t\beta^{h(t)} \rightarrow 0$ .

Using the independence of  $(W_n)_{n \geq 0}$  and  $Z$ , an integration by parts with respect to  $Z$  yields

$$\begin{aligned} \overline{\mathbf{E}} \left[ \mathbf{1}_{Z \geq \beta^{h(t)}} I(Z, t) \right] &= -\overline{\mathbf{E}} \left[ \left[ \overline{\mathbf{P}}[Z \geq z] I(z, t) \right]_{\beta^{h(t)}}^\infty \right] \\ &\quad + \overline{\mathbf{E}} \left[ \frac{(1 - \beta^{-1})W_{\lfloor g(t) \rfloor}}{1 - \beta^{-1}W_{\lfloor g(t) \rfloor}} \int_{\beta^{h(t)}}^\infty \frac{\tilde{t}}{(1 + \tilde{t}z)^2} \overline{\mathbf{P}}[Z \geq z] dz \right]. \end{aligned}$$

Using Theorem 10.1 and  $I(z, t) \leq Ctz$ , we can bound the first term by

$$(12.3) \quad Ct\beta^{(1-\gamma)h(t)} = Ct^\gamma (t\beta^{h(t)})^{1-\gamma} = o(t^\gamma),$$

since  $\gamma < 1$  and  $t\beta^{h(t)} \rightarrow 0$ .

As  $h(t) \rightarrow \infty$  to get the asymptotics of the second term, we can replace  $\overline{\mathbf{P}}(Z \geq z)$  by its tail estimate which is computed in Theorem (10.1). Fix for the moment  $W_{\lfloor g(t) \rfloor}$  (i.e.  $\tilde{t}$ ), we only have to compute :

$$\int_{\beta^{h(t)}}^\infty \frac{\tilde{t}}{(1 + \tilde{t}z)^2} z^{-\gamma} dz = \tilde{t}^\gamma \int_{\frac{\tilde{t}\beta^{h(t)}}{1 + \tilde{t}\beta^{h(t)}}}^1 x^{-\gamma} (1 - x)^\gamma dx,$$

using the change of variable  $x = \frac{\tilde{t}z}{1 + \tilde{t}z}$ . The r.h.s. of the previous equation can be estimated with

$$\int_0^1 x^{-\gamma} (1 - x)^\gamma dx = \Gamma(\gamma + 1)\Gamma(-\gamma + 1) = \frac{\pi\gamma}{\sin(\pi\gamma)},$$

using the relation between the  $\Gamma$  and  $\beta$  functions of Euler. More precisely, we can get the following

$$\overline{\mathbf{E}} \left[ \frac{(1 - \beta^{-1})W_{g(t)}}{1 - \beta^{-1}W_{g(t)}} \int_{\beta^{h(t)}}^\infty \frac{\tilde{t}}{(1 + \tilde{t}z)^2} z^{-\gamma} dz \right] = F(t) - G(t),$$

where

$$F(t) = \frac{\pi\gamma}{\sin(\pi\gamma)} \overline{\mathbf{E}} \left[ \frac{(1 - \beta^{-1})W_{\lfloor g(t) \rfloor}}{1 - \beta^{-1}W_{\lfloor g(t) \rfloor}} \left( \frac{1 - \beta^{-1}W_{\lfloor g(t) \rfloor}}{1 - W_{\lfloor g(t) \rfloor}} \right)^\gamma \right] t^\gamma,$$

and

$$G(t) = t^\gamma \bar{\mathbf{E}} \left[ \frac{(1 - \beta^{-1})W_{\lfloor g(t) \rfloor}}{1 - \beta^{-1}W_{\lfloor g(t) \rfloor}} \left( \frac{1 - \beta^{-1}W_{\lfloor g(t) \rfloor}}{1 - W_{\lfloor g(t) \rfloor}} \right)^\gamma \int_0^{\frac{\bar{t}\beta^{h(t)}}{1 + \bar{t}\beta^{h(t)}}} x^{-\gamma}(1-x)^\gamma dx \right].$$

Recalling (12.1), we have

$$|G(t)| \leq Ct^\gamma \int_0^{1 \wedge Ct\beta^{h(t)}} x^{-\gamma}(1-x)^\gamma dx = o(t^\gamma),$$

since  $t\beta^{h(t)} \rightarrow 0$ . Thus

$$\begin{aligned} & \bar{\mathbf{E}}[\mathbf{1}_{Z \geq \beta^{h(t)}} I(Z, t)] \\ & \sim \frac{\pi\gamma}{\sin(\pi\gamma)} \frac{\alpha \bar{\mathbf{E}}[S_\infty^\gamma]}{(1 - \beta^{-1})^\gamma} \mathbb{E} \left[ \frac{(1 - \beta^{-1})W_{\lfloor g(t) \rfloor}}{1 - \beta^{-1}W_{\lfloor g(t) \rfloor}} \left( \frac{1 - \beta^{-1}W_{\lfloor g(t) \rfloor}}{1 - W_{\lfloor g(t) \rfloor}} \right)^\gamma \right] t^\gamma, \end{aligned}$$

then using the fact that  $W_n$  is bounded away from 1 by Lemma 7.4, and using Proposition 7.1 since  $g(t) \rightarrow \infty$ , we get the expected result.  $\square$

### 13. LAPLACE TRANSFORM OF THE TIME SPENT IN A BIG LEAF

Recalling (2.3), we point out that

$$\mathbb{E} \left[ \nu \left( \frac{1}{1 - \nu} \right)^\gamma \right] = \mathbb{E} \left[ \frac{(1 - \beta^{-1})W_\infty}{1 - \beta^{-1}W_\infty} \left( \frac{1 - \beta^{-1}W_\infty}{1 - W_\infty} \right)^\gamma \right].$$

We are finally able to derive the following estimate

**Theorem 13.1.** *Let  $\varepsilon > 0$  be small enough, then for any  $\lambda \geq 0$  we have as  $n$  goes to infinity*

$$1 - \mathbb{E}[e^{-\lambda_n \chi_1^{(n)}} | 0\text{-SR}] \sim C_b \lambda^\gamma n^{-\varepsilon},$$

with

$$C_b = \frac{\pi\gamma}{\sin(\pi\gamma)} \frac{\bar{\mathbf{E}}[S_\infty^\gamma]}{(1 - \beta^{-1})^\gamma} \mathbb{E} \left[ \nu \left( \frac{1}{1 - \nu} \right)^\gamma \right].$$

*Proof.* Fix  $\lambda$ , we apply Theorem 12.1, with  $h(x) = \frac{1-\varepsilon}{\ln \beta} (\ln x - \ln \lambda)$  and  $g(x) = \lambda^\gamma x^{-\gamma}$  to get

$$1 - R_n(\lambda) \sim \frac{1}{\eta_n} \frac{\pi\gamma}{\sin(\pi\gamma)} \frac{\alpha \bar{\mathbf{E}}[S_\infty^\gamma]}{(1 - \beta^{-1})^\gamma} \mathbb{E} \left[ \nu \left( \frac{1}{1 - \nu} \right)^\gamma \right] \lambda_n^\gamma.$$

Inserting this into (11.1) and using (6.1), we get for any  $\xi > 0$

$$\begin{aligned} \limsup_{n \rightarrow \infty} n^\varepsilon (1 - \mathbb{E}[e^{-\lambda_n \chi_1^{(n)}} | 0\text{-SR}]) & \leq C_b \lambda^\gamma, \\ \liminf_{n \rightarrow \infty} n^\varepsilon (1 - \mathbb{E}[e^{-\lambda_n \chi_1^{(n)}} | 0\text{-SR}]) & \geq C_b (e^{-\xi} \lambda)^\gamma. \end{aligned}$$

Letting  $\xi$  go to 0 in the previous equation, we have

$$1 - \mathbb{E}[e^{-\lambda_n \chi_1(n)} | 0 - \text{SR}] \sim C_b \lambda^\gamma n^{-\varepsilon},$$

which proves Theorem 13.1. □

#### 14. PROOF OF THEOREM 2.1

We finally prove Theorem 2.1.

*Proof.* Fix  $\lambda \geq 0$  and  $\varepsilon' > 0$ , using (8.2) and Theorem 13.1, we get

$$\begin{aligned} \mathbb{E}[e^{-\lambda_n \chi(n)}] &\geq (1 - C_b \lambda^\gamma n^{-\varepsilon} + o(n^{-\varepsilon}))^{(1+\varepsilon')\rho C_a n^\varepsilon} + o(1), \\ \mathbb{E}[e^{-\lambda_n \chi(n)}] &\leq (1 - C_b \lambda^\gamma n^{-\varepsilon} + o(n^{-\varepsilon}))^{(1-\varepsilon')\rho C_a n^\varepsilon} + o(1), \end{aligned}$$

hence recalling that  $\zeta = \rho C_a C_b$ ,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathbb{E}[e^{-\lambda_n \chi(n)}] &\leq e^{-(1-\varepsilon')\zeta \lambda^\gamma} \\ \liminf_{n \rightarrow \infty} \mathbb{E}[e^{-\lambda_n \chi(n)}] &\geq e^{-(1+\varepsilon')\zeta \lambda^\gamma} \end{aligned}$$

Now letting  $\varepsilon'$  go to 0 and using Theorem 6.1, we get

$$\forall \lambda \geq 0, \quad \lim_{n \rightarrow \infty} \mathbb{E}[e^{-\lambda_n \Delta_n}] = e^{-\zeta \lambda^\gamma},$$

which proves (2.2). □

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