On b-perfect chordal graphs

Frédéric Maffray, Meriem Mechebbek

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Abstract

The b-chromatic number of a graph G is the largest integer k such that G has a coloring of the vertices in k color classes such that every color class contains a vertex that has a neighbour in all other color classes. We characterize the class of chordal graphs for which the b-chromatic number is equal to the chromatic number for every induced subgraph.

1 Introduction

We deal here with finite undirected graphs. Given a graph G and an integer $k \geq 1$, a *coloring* of G with k colors is a mapping $c : V(G) \to \{1, \ldots, k\}$ such that any two adjacent vertices u, v in G satisfy $c(u) \neq c(v)$. For every vertex v, the integer $c(v)$ is called the color of v. The sets $c^{-1}(1), \ldots, c^{-1}(k)$ that are not empty are called the color classes of c. A *b-coloring* is a coloring such that every color class contains a vertex that has a neighbour in each color class other than its own, and we call any such vertex a *b-vertex*. The b-chromatic number $b(G)$ of a graph G is the largest integer k such that G admits a b-coloring with exactly k colors. The concept of b-coloring was introduced in [6] and has been studied among others in [2, 4, 7, 8, 9]. Let $\omega(G)$ be the maximum size of a clique in a graph G, and let $\chi(G)$ be the chromatic number of G. It is easy to see that every coloring of G with $\chi(G)$ colors is a b-coloring, and so every graph satisfies $\chi(G) \leq b(G)$. Hoàng and Kouider [4] call a graph G *b-perfect* if every induced subgraph H of G satisfies $b(H) = \chi(H)$. Also a graph G is *b-imperfect* if it is not b-perfect, and *minimally b-imperfect* if it is b-imperfect and every proper subgraph

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†CNRS, Laboratoire G-SCOP, 46 avenue Félix Viallet, 38031 Grenoble Cedex, France.

‡USTHB, Laboratoire LAID3, BP32 El Alia, Bab Ezzouar 16111, Alger, Algeria.

of G is b-perfect. Hoàng, Linhares Sales and Maffray [5] found a list $\mathcal{F}$ of twenty-two minimally b-imperfect graphs shown in Figure 1, and posed the following conjecture.

Conjecture 1 ([5]). *A graph is b-perfect if and only if it does not contain any member of $\mathcal{F}$ as an induced subgraph.*

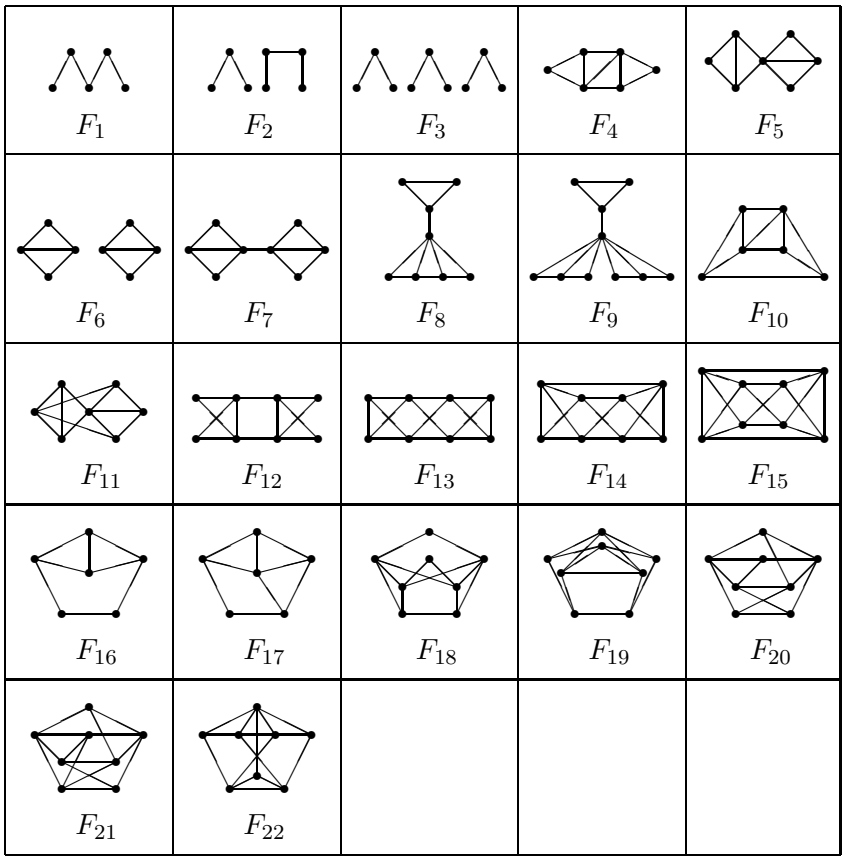

Figure 1: Class $\mathcal{F} = \{F_1, \ldots, F_{22}\}$

Given a collection $\mathcal{H}$ of graphs, a graph G is usually called $\mathcal{H}$-free if no induced subgraph of G is a member of $\mathcal{H}$. When $\mathcal{H}$ consists of only one graph H, we may write H-free instead of $\{H\}$-free. We let P_k and C_k respectively denote the graph that consists of a path (resp. cycle) on k vertices. We use $+$ to denote the disjoint union of graphs, and nF is the graph which has n components all isomorphic to F. For example, $2K_2$ is the graph with two

components of size 2, and the first three graphs in $\mathcal{F}$ are P_5, $P_4 + P_3$ and $3P_3$. We say that two vertices x, y in a graph G are *twins* if every vertex of $G \setminus \{x, y\}$ that is adjacent to any of x, y is adjacent to both. Note that two twins may be adjacent or not.

It is a routine matter to check that the graphs in class $\mathcal{F}$ are b-imperfect and minimally so. More precisely, for $i = 1, 2, 3$, we have $\chi(F_i) = 2$ and $b(F_i) = 3$, and F_i admits a b-coloring with 3 colors in which its three vertices of degree 3 have color $1, 2, 3$ respectively; and for $i = 4, \ldots, 22$, we have $\chi(F_i) = 3$ and $b(F_i) = 4$.

We will prove the conjecture in the case of chordal graphs. Recall that a graph G is *chordal* [3, 10] if every cycle of length at least four in G has a chord (an edge between non-consecutive vertices of the cycle). We call *hole* any chordless cycle of length at least four. In these terms, a graph is chordal if and only if it is hole-free.

Theorem 1. *Every $\mathcal{F}$-free chordal graph is b-perfect.*

Proof of Theorem 1. Suppose that the theorem is false, and let G be a counterexample to the theorem for which $|V(G)|+|E(G)|$ is minimal. Recall that, since G is chordal, it satisfies $\chi(G) = \omega(G)$ (see [1, 3]). Since G is a counterexample to the theorem, it admits a b-coloring c with $k \geq \chi(G)+1 = \omega(G) + 1$ colors. For $i = 1, \ldots, k$, let u_i be any b-vertex of color i, that is, a vertex that has a neighbour of each color other than i. Let $U = \{u_1, \ldots, u_k\}$. Note that, since $k > \omega(G)$, the set U does not induce a clique. As usual, we say that a vertex is *simplicial* if its neighbourhood induces a clique.

1.1. *For $i = 1, \ldots, k$, vertex u_i is not simplicial.*

Proof. Suppose on the contrary and up to symmetry that u_1 is simplicial. Since u_1 is a b-vertex, it has a neighbour v_i of each color $i = 2, \ldots, k$. Then the set $\{u_1, v_2, \ldots, v_k\}$ induces a clique of size $k > \omega(G)$, a contradiction. So Claim 1.1 holds.

1.2. *G contains a $2K_2$.*

Proof. Suppose that G contains no $2K_2$. Since U is not a clique, we may assume up to symmetry that u_1, u_2 are not adjacent. By Claim 1.1, vertex u_1 has two neighbours v, v' that are not adjacent, and vertex u_2 has two neighbours w, w' that are not adjacent. Suppose that u_1 is adjacent to w. Then u_1 is not adjacent to w', for otherwise u_1, w, u_2, w' induce a hole. One of v, v' is not equal to w, say $v \neq w$. Also $v \neq w'$ since u_1 is adjacent to v and not to w'. If v is not adjacent to u_2, then v is adjacent to w', for

otherwise $\{u_1, v, u_2, w'\}$ induces a $2K_2$; but then either $\{u_1, v, w', u_2, w\}$ or $\{u_1, v, u_2, w\}$ induce a hole. So v is adjacent to u_2. Then u_2 is not adjacent to v', for otherwise $\{u_1, v, v', u_2\}$ induces a hole. Then v' is adjacent to w', for otherwise $\{u_1, v', u_2, w'\}$ induces a $2K_2$. But then either $\{u_1, v', u_2, w, w'\}$ (if v', w are not adjacent) or $\{v', u_2, w, w'\}$ (if v', w are adjacent) induces a hole. Therefore u_1 is not adjacent to w. Similarly, u_1 is not adjacent to w', and u_2 is not adjacent to any of v, v'. Now v must be adjacent to w, for otherwise $\{u_1, v, u_2, w\}$ induces a $2K_2$, and by symmetry, to w' as well. But then $\{v, u_2, w, w'\}$ induces a hole, a contradiction. So Claim 1.2 holds.

We say that a subgraph of G is *big* if it contains at least two vertices. Since G contains a $2K_2$, it contains a set S that induces a subgraph with at least two big components and is maximal with this property. Let $R = V(G) \setminus S$.

1.3. *Every vertex of R has a neighbour in every big component of S.*

Proof. Suppose on the contrary that some vertex x of R has no neighbour in some big component C of S. Then $S \cup \{x\}$ induces a subgraph with at least two big components (of which C is one), which contradicts the maximality of S. So Claim 1.3 holds.

1.4. *R is a clique.*

Proof. Suppose on the contrary that there are two non-adjacent vertices u, v in R. Consider two big components Z_1, Z_2 of S. By Claim 1.3, for each $i = 1, 2$, u has a neighbour u_i in Z_i and v has a neighbour v_i in Z_i. Since Z_i is connected, we may choose u_i, v_i and a path u_i-$\cdots$-v_i in Z_i such that this path is as short as possible (possibly $u_i = v_i$). So no interior vertex of this path is adjacent to u or v. But then the union of the two paths u_1-$\cdots$-v_1, u_2-$\cdots$-v_2, plus u and v, forms a hole in G, a contradiction. So Claim 1.4 holds.

1.5. *There is a big component Z of S such that every vertex of R is adjacent to every vertex of every big component of $S \setminus Z$.*

Proof. Suppose the contrary, that is, there are two big components Z_1, Z_2 of S and vertices x_1, x_2 of R such that x_1 has a non-neighbour in Z_1 and x_2 has a no-neighbour in Z_2. For each $i = 1, 2$, since Z_i is connected and by Claim 1.3, there are adjacent vertices y_i, z_i in Z_i such that x_i is adjacent to y_i and not to z_i. If $x_1 = x_2$, then z_1-y_1-x_1-y_2-z_2 is a P_5 in G, which contradicts that G is $\mathcal{F}$-free. So $x_1 \neq x_2$, and by the same argument we

may assume that x_1 is adjacent to all of Z_2 and that x_2 is adjacent to all of Z_1. By Claim 1.4, vertices x_1, x_2 are adjacent. Then $\{x_1, x_2, y_1, y_2, z_1, z_2\}$ induces an F_4, which contradicts that G is $\mathcal{F}$-free. So Claim 1.5 holds.

Let Z be a big component of S as described in Claim 1.5. Let $T = S \setminus Z$. So T contains a big component of S. Put $U_Z = U \cap Z$ and $U_T = U \cap T$.

1.6. *For every vertex $a \in R$ and every set $Y \subset Z$ that induces a connected subgraph and contains no neighbour of a, there exists a vertex of Z that is adjacent to all of $Y \cup \{a\}$.*

Proof. Pick any vertex y in Y. Since Z is connected, and a has a neighbour in Z by Claim 1.3, there is a shortest path z_0-z_1-$\cdots$-z_p in Z such that z_0 is adjacent to a and $z_p = y$. Let t be any vertex in a big component of T. By Claim 1.5, vertices a, t are adjacent. Then $p = 1$, for otherwise z_2-z_1-z_0-a-t is a P_5. Thus z_0 is adjacent to both a, y. We show that z_0 is adjacent to all of Y. In the opposite case, since Y is connected there are adjacent vertices y', y'' such that z_0 is adjacent to y' and not to y''; but then y''-y'-z_0-a-t is a P_5, a contradiction. So Claim 1.6 holds.

1.7. $|R| \leq \omega(G) - 2$.

Proof. By the definition of S, the set T contains two adjacent vertices a, b. By Claim 1.4, $R \cup \{a, b\}$ is a clique. So Claim 1.7 holds.

1.8. $U_Z \neq \emptyset$.

Proof. Suppose on the contrary that Z contains no vertex of U. Consider the graph $G' = G \setminus Z$. Clearly, G' is a chordal and $\mathcal{F}$-free graph, and $|V(G')| + |E(G')| < |V(G)| + |E(G)|$. We show that c is a b-coloring of G'. To establish this, consider vertex u_i for each $i = 1, \ldots, k$ and consider any color $j \neq i$. If u_i is not in R, then u_i has the same neighbours in G and in G', so u_i is a b-vertex in G'. Now suppose that u_i is in R. If u_j is in a component of S of cardinality 1, then $N(u_j) \subseteq R$, so u_j is a simplicial vertex by Claim 1.4, which contradicts Claim 1.1. Thus u_j is in a big component of T. Then u_j is a neighbour of u_i by Claim 1.5 and the definition of Z. Thus every u_i is a b-vertex for c in G'. But then G' is a counterexample to the theorem, which contradicts the minimality of G. So Claim 1.8 holds.

1.9. *T contains no P_4 and no $2P_3$.*

Proof. Suppose on the contrary that T contains a set Q of vertices that induces a P_4 or a $2P_3$. Therefore Z contains no P_3, for otherwise taking

a P_3 in Z plus Q would give an induced F_2 or F_3. Since Z is connected and contains no P_3, it is a clique. By Claim 1.8, we may assume that u_1 is in Z. For $j = 2, \ldots, k$, let v_j be a neighbour of u_1 of color j. Since $\{u_1, v_2, \ldots, v_k\}$ is not a clique, we may assume that v_2, v_3 are not adjacent. Since $N(u_1) \subset R \cup Z$ and both R, Z are cliques, we may assume that $v_2 \in R$ and $v_3 \in Z$. By Claim 1.7, R contains at most $k-3$ of the v_j's; so we may assume that $v_4 \in Z$. Now, if v_2 is not adjacent to v_4, then $W \cup \{v_1, v_2, v_3, v_4\}$ induces an F_8 or F_9; while if v_2 is adjacent to v_4 then the same set contains an induced F_5. So Claim 1.9 holds.

1.10. *$U_T \neq \emptyset$.*

Proof. Suppose on the contrary that T contains no vertex of U. Let G' be the graph obtained from G by removing all edges whose two endvertices are in T. Graph G' satisfies $|V(G')| + |E(G')| < |V(G)| + |E(G)|$ since we have removed at least one edge because T contains a big component of S. We will show that (a) c is a b-coloring of G', (b) G' is a chordal graph, and (c) G' is $\mathcal{F}$-free. These facts will imply that G' is a counterexample to the theorem, which will contradict the minimality of G and complete the proof of the claim.

To prove (a), it suffices to observe that every vertex of U is a b-vertex for c in G', because the edges we have removed from G to obtain G' are not incident with any vertex of U.

To prove (b), observe that in G' all vertices of T are simplicial (because their neighbourhood is in R) and thus cannot lie in a hole of G'. Moreover, $G' \setminus T = G \setminus T$. So G' contains no hole and is chordal.

Now we prove (c). Suppose on the contrary that G' contains a member F of $\mathcal{F}$. Note that G' does not contain F_i for $i = 10, \ldots, 22$, because every such F_i contains a hole of length 4 or 5, while G' is chordal. Thus F must be one of $F_1, \ldots, F_9$. Graph F must contain two vertices of T that are adjacent in G, for otherwise F would be an induced subgraph of G. Let x, y be two vertices of T in F that are adjacent in G. So x, y lie in the same big component of T, and it follows from Claim 1.5 that the neighbourhood of each of them in G' is R. In particular, in F they are non-adjacent twins. This immediately implies that F cannot be F_1, F_4 or F_8 since such graphs do not have twins. Thus F must be one of F_2, F_3, F_5, F_6, F_7, F_9. Note that, in each of these six cases, there is up to symmetry only one pair of non-adjacent twins.

Suppose that F is either F_2 or F_3. So F has vertices $x, y, a, z_1, \ldots, z_p$, edges xa, ya, and either (if F is F_2) $p = 4$ and $\{z_1, \ldots, z_4\}$ induces a P_4, or (if F is F_3) $p = 6$ and $\{z_1, \ldots, z_6\}$ induces a $2P_3$ with edges $z_1z_2, z_2z_3, z_4z_5, z_5z_6$.

As observed above, we may assume that $x, y \in T$ and consequently $a \in R$; then vertices $z_1, \ldots, z_p$ are in a big component of S, and, by Claim 1.5, they cannot be in T, so they are in Z. Let $p = 4$. By Claim 1.6, Z contains a vertex z that is adjacent in G to $a, z_1, \ldots, z_4$. Then $\{z, z_1, \ldots, z_4, a, x, y\}$ induces an F_8 in G, a contradiction. Now let $p = 6$. By Claim 1.6, Z contains a vertex z that is adjacent in G to a, z_1, z_2, z_3 and a vertex z' that is adjacent in G to a, z_4, z_5, z_6. If $z \neq z'$, then $\{z, z', z_1, \ldots, z_6\}$ induces an F_6 or F_7 in G, a contradiction. So $z = z'$. But then $\{z, z_1, \ldots, z_6, a, x, y\}$ induces an F_9 in G, a contradiction.

Suppose that F is either F_5 or F_9. So F has vertices $x, y, a, b, z_1, \ldots, z_p$, edges xa, xb, ya, yb, ab, az_1, z_1z_2, z_1z_3, z_2z_3 and either (if F is F_5) $p = 3$ and az_2 is an edge, or (if F is F_9) $p = 6$ and vertices z_4, z_5, z_6 induce a P_3 and are adjacent to a. As observed above, we may assume that $x, y \in T$ and consequently $a, b \in R$, and so $z_1, \ldots, z_p \in Z$. By Claim 1.6, Z contains a vertex z that is adjacent in G to b, z_1, z_2, z_3. Then z is adjacent to a, for otherwise $\{z, a, b, z_1\}$ induces a hole in G. But then $\{z, a, b, z_1, z_3, x\}$ induces an F_4 in G, a contradiction.

Finally suppose that F is either F_6 or F_7. So F has vertices x, y, a, b, $z_1, \ldots,$ z_4 and edges $xa, xb, ya, yb, ab, z_1z_2, z_1z_3, z_1z_4, z_2z_3, z_2z_4$ and possibly (if F is F_7) the edge az_1. As observed above, we may assume that $x, y \in T$ and consequently $a, b \in R$ and $z_1, \ldots, z_4 \in Z$. By Claim 1.6, Z contains a vertex z that is adjacent in G to a, z_2, z_3, z_4. Vertex z is also adjacent to z_1, for otherwise $\{z, z_1, z_3, z_4\}$ induces a hole. By Claim 1.6, Z contains a vertex z' that is adjacent in G to $b, z_1, \ldots, z_4$. If none of z, z' is adjacent to both a, b, then either $\{z, z', a, b\}$ or $\{z, z', a, b, z_2\}$ induces a hole. So we may assume, up to symmetry, that z is adjacent to both a, b. But then $\{z, a, b, x, z_2, z_3, z_4\}$ induces an F_5 in G, a contradiction. Thus Claim 1.10 holds.

1.11. *U_T is a clique.*

Proof. Suppose on the contrary that u_1, u_2 are non adjacent vertices of U_T. By Claim 1.1, vertex u_1 has two neighbours v, v' that are not adjacent, and vertex u_2 has two neighbours w, w' that are not adjacent. By Claims 1.4 and 1.5 we have $v, v', w, w' \in T$. If u_1 is adjacent to w, then $\{u_1, w, u_2, w'\}$ induces a P_4 or a hole, which contradicts Claim 1.9 or the chordality of G. So u_1 is not adjacent to w, and by symmetry it is not adjacent to w', and u_2 is not adjacent to any of v, v'. If v is adjacent to w, then $\{v, u_1, v', w\}$ induce a P_4 or a hole, a contradiction. So v is not adjacent to w, and by symmetry it is not adjacent to w', and v' is not adjacent to any of w, w'.

But now $\{u_1, v, v', u_2, w, w'\}$ induces a $2P_3$, which contradicts Claim 1.9. So Claim 1.11 holds.

By Claim 1.10, there is a vertex u of U in T. By Claim 1.1, vertex u has two neighbours t, t' that are not adjacent. By Claims 1.4 and 1.5, we have $t, t' \in T$. In other words, there is a P_3 t-u-t' in T.

1.12. *Z contains no P_4 and no $2P_3$.*

Proof. In the opposite case, a P_4 or $2P_3$ from Z plus the P_3 t-u-t' from T form an induced F_2 or F_3 in G, a contradiction. So Claim 1.12 holds.

1.13. *U_Z is a clique.*

Proof. Suppose on the contrary that u_1, u_2 are non adjacent vertices of U_Z. Since Z is connected, it contains a path from u_1 to u_2, and since, by Claim 1.12, Z contains no P_4, such a path has length 2, that is, Z contains a vertex x adjacent to both u_1, u_2. Suppose that some neighbour $y \neq x$ of u_1 is not adjacent to x. Then y is also not adjacent to u_2, for otherwise $\{y, u_1, x, u_2\}$ would induce a hole; and so y-u_1-x-u_2 is a P_4. If $y \in Z$ this contradicts Claim 1.12, and if $y \in R$ then u_2-x-u_1-y-t is a P_5, another contradiction. Therefore, x is adjacent to every neighbour of u_1 different from x, and similarly it is adjacent to every neighbour of u_2 different from x. By Claim 1.1, u_1 has neighbours v, v' that are not adjacent. Suppose that one of v, v', say v, is in R. Then, since R is a clique, v' is in Z, and, by the preceding argument, we have $x \neq v'$ and x is adjacent to v, v'. But then $\{v, u_1, v', x, t, u, t'\}$ induces an F_5, a contradiction. Thus v, v' are both in Z. Likewise, u_2 has neighbours w, w' that are not adjacent, and they are both in Z. If u_2 is adjacent to v, then u_2, v, u_1, v' induce either a P_4 or a hole, a contradiction. Thus u_2 is not adjacent to v, and similarly not to v', and u_1 is not adjacent to any of w, w'. Then v is not adjacent to w, for otherwise u_1-v-w-u_2 is a P_4. Similarly, v is not adjacent to w', and v' is not adjacent to any of w, w'. But now $\{u, t, t', u_1, v, v', u_2, w, w'\}$ induces a $3P_3$ in G, a contradiction. So Claim 1.13 holds.

Let C_T be the set of colors that appear in U_T. By Claim 1.10, we have $|C_T| = |U_T| \geq 1$. Let C_Z be the set of colors that do not appear in $R \cup U_T$. By Claim 1.1, a member of U must be in a big component of T, and so, by Claims 1.4, 1.5 and 1.11, $R \cup U_T$ is a clique; thus $|C_Z| \geq 1$. Consider any color $j \in C_Z$. By the definition of U, every member of U_T must have a neighbour of color j, and by the definition of C_Z, any such neighbour

must be in T. Let w_j be one vertex of color j that is adjacent to the most members of U_T. So $w_j \in T$. Suppose that w_j has a non-neighbour u' in U_T. Let w'_j be a neighbour of u' of color j. So $w'_j \in T$. Since u' is adjacent to w'_j and not to w_j, the choice of w_j implies the existence of a vertex u'' of U_T that is adjacent to w_j and not to w'_j. But then w_j-u''-u'-w'_j is a P_4, which contradicts Claim 1.9. Thus w_j is adjacent to all of U_T. Now $R \cup U_T \cup \{w_j\}$ is a clique, which implies $|C_Z| \geq 2$. Let $W = \{w_j \mid j \in C_Z\}$. Note that W is not a clique, for otherwise $R \cup U_T \cup W$ would be a clique of size k (because it contains a vertex of each color).

For each color $j \in C_Z$, the definition of C_Z implies that u_j is in Z. So

$$|U_Z| \geq |C_Z| \geq 2.$$

Consider any color $h \in C_T$. By the definition of U, every member of U_Z must have a neighbour of color h, and by the definition of C_T and by Claim 1.5, any such neighbour must be in Z. Let y_h be one vertex of color h that is adjacent to the most members of U_Z. So $y_h \in Z$. Suppose that y_h has a non-neighbour u' in U_Z. Let y'_h be a neighbour of u' of color h. So $y'_h \in Z$. Since u' is adjacent to y'_h and not to y_h, the choice of y_h implies the existence of a vertex u'' of U_Z that is adjacent to y_h and not to y'_h. But then y_h-u''-u'-y'_h is a P_4, which contradicts Claim 1.12. Thus y_h is adjacent to all of U_Z. Let $Y = \{y_h \mid h \in C_T\}$. So $|Y| = |C_T|$. Suppose that Y is not a clique. So there are non-adjacent vertices y_g, y_h in Y. Thus $|C_T| \geq 2$, and we have $u_g, u_h \in U_T$. Recall that W is not a clique, so it contains two non-adjacent vertices w_i, w_j, and by the definition of W we have $u_i, u_j \in U_T$. But then $\{y_g, y_h, u_i, u_j, w_i, w_j, u_g, u_h\}$ induces an F_6, a contradiction. Thus Y is a clique, and so

$$Y \cup U_Z \text{ is a clique of size at least } |C_T| + |C_Z| \geq 3.$$

Let R_1 be the set of vertices of R that have at most one neighbour in $Y \cup U_Z$, and let $R_2 = R \setminus R_1$. If some vertex $a \in R_2$ has a non-neighbour v in $Y \cup U_Z$, then, since a has two neighbours z, z' in $Y \cup U_Z$, we see that $\{a, z, z', v, t, u, t'\}$ induces an F_5, a contradiction (recall that t-u-t' is a P_3 in T). Thus every vertex of R_2 is adjacent to every vertex of $Y \cup U_Z$. This implies $R_1 \neq \emptyset$, for otherwise $R \cup Y \cup U_Z$ would be a clique of size k (because it contains a vertex of each color).

Consider any color ℓ that appears in R_1, and let a_ℓ be the vertex of R_1 of color ℓ. By the definition of U and R_1, every vertex of U_Z, except

possibly one, must have a neighbour of color ℓ in Z. Let x_ℓ be one vertex of Z of color ℓ that is adjacent to the most members of U_Z. By the same argument as above concerning y_h, using the fact that Z contains no P_4, we obtain that x_ℓ is adjacent to every vertex of U_Z that has a neighbour of color ℓ in Z. Now we show that x_ℓ is adjacent to all of $Y \cup U_Z$. Suppose on the contrary that x_ℓ has a non-neighbour v in $Y \cup U_Z$. If x_ℓ has two neighbours z, z' in $Y \cup U_Z$, then either t-a_ℓ-v-z-x_ℓ is a P_5 (if a_ℓ is adjacent to v), or $\{v, z, z', x_\ell, a_\ell, t, u, t'\}$ induces an F_6 or F_7, a contradiction. So x_ℓ has only one neighbour z in $Y \cup U_Z$. By the definition of x_ℓ, this implies that $U_Z = \{z, z'\}$ where z' has no neighbour of color ℓ in T. Since z' is in U, it must have a neighbour of color ℓ, and this can only be a_ℓ. But then x_ℓ-z-z'-a_ℓ-t is a P_5, a contradiction. Thus x_ℓ is adjacent to all of $Y \cup U_Z$. Now we show that x_ℓ is adjacent to all of R_2. For suppose that x_ℓ is not adjacent to some vertex a of R_2. Let z, z' be any two vertices in $Y \cup U_Z$. Then $\{x_\ell, z, z', a, t, u, t'\}$ induces an F_5, a contradiction. In summary, x_ℓ is adjacent to all of $Y \cup U_Z \cup R_2$.

Let $X = \{x_\ell \mid \text{color } \ell \text{ appear in } R_1\}$. So $X \neq \emptyset$. Suppose that there are two non-adjacent vertices x_ℓ, x_m in X. Let a_ℓ be a vertex of color ℓ in R_1. Let z, z' be any two vertices in $Y \cup U_Z$. Then a_ℓ is adjacent to x_m, for otherwise $\{x_\ell, x_m, z, z', a_\ell, t, u, t'\}$ induces an F_6 or F_7. Then a_ℓ is adjacent to z', for otherwise x_ℓ-z'-x_m-a_ℓ-t is a P_5. But then $\{x_m, z, z', a_\ell, t, u, t'\}$ induces an F_5, a contradiction. Therefore X is a clique. But now, $X \cup Y \cup U_Z \cup R_2$ is a clique of size k (because it contains a vertex of each color), a contradiction. This completes the proof of the theorem. □

Theorem 1 can be generalized slightly as follows.

Theorem 2. *Every $\mathcal{F}$-free C_4-free graph is b-perfect.*

Proof. Let G be an $\mathcal{F}$-free C_4-free graph. Since G contains no P_5, it contains no hole C_k with $k \geq 6$. We prove that $b(G) = \chi(G)$ by induction on the number of C_5's contained in G. If G contains no C_5, then it is chordal and the result follows from Theorem 1. So we may now assume that G contains a C_5. Let $z_1, \ldots, z_5$ be five vertices such that, for $i = 1, \ldots, 5$ modulo 5, vertex z_i is adjacent to z_{i+1} and not to z_{i+2}. Let $Z = \{z_1, \ldots, z_5\}$. Let x be a vertex of $G \setminus Z$ that has a neighbour in Z. If x also has a non-neighbour in Z, then it is easy to see that $Z \cup \{x\}$ contains a set that induces either a P_5, or a C_4, or an F_{16}, a contradiction. Thus x is adjacent to all of Z. Let X be the set of vertices that are adjacent to Z. Note that X is a clique, for if it contained two non-adjacent vertices x, y, then $\{x, y, z_1, z_3\}$ would induce

a C_4. Suppose that G admits a b-coloring c with $k > \chi(G)$ colors. We may assume that the colors of c that appear in Z are $1, \ldots, \ell$, with $3 \leq \ell \leq 5$. So only the colors $\ell + 1, \ldots, k$ may appear in X.

If $\ell = 3$, let G' be the graph obtained from $G \setminus Z$ by adding three new vertices a_1, a_2, a_3 that are pairwise adjacent and all adjacent to all of X. If $\ell = 4$ or 5, let G' be the graph obtained from $G \setminus Z$ by adding ℓ new vertices $a_1, \ldots, a_\ell$ that are pairwise not adjacent and all adjacent to all of X. In either case, since X is a clique the new vertices $a_1, \ldots, a_l$ are simplicial, so they cannot belong to any hole, and so G' has strictly fewer C_5's than G.

2.1. *$b(G') \geq b(G)$.*

Proof. Let c' be the coloring of the vertices of G' defined by $c'(x) = c(x)$ if x is a vertex of $G \setminus Z$ and $c'(a_i) = i$ for $i = 1, \ldots, \ell$. Clearly, c' is a coloring with k colors. For each $i = 1, \ldots, k$, let u_i be a b-vertex of color i for c in G. Suppose that u_i is in $G \setminus Z$. Consider a neighbour v_j of u_i of color j in G for any $j \neq i$. Then either v_j is in $G \setminus Z = G' \setminus Z$, and in this case v_j is a neighbour of u_i of color j in G'; or v_j is in Z, and in this case $j \in \{1, \ldots, \ell\}$ and a_j is a neighbour of u_i of color j in G'. So u_i is a b-vertex for G'. Now suppose that u_i is in Z. Then u_i must have a neighbour of every color $1, \ldots, \ell$ different from i, and since such colors do not appear in X, they must appear in Z, and so $\ell = 3$ and all colors $4, \ldots, k$ appear in X. Then a_i is a b-vertex of color i in G'. Thus c' has a b-vertex of every color $i = 1, \ldots, k$. So Claim 2.1 holds.

2.2. *$\chi(G') \leq \chi(G)$.*

Proof. Consider any coloring γ of G with $\chi(G)$. We may assume that the colors of γ that appear in Z are $1, \ldots, h$, with $3 \leq h \leq 5$. Let γ' be defined as follows. For $x \in G \setminus Z$, set $\gamma'(x) = \gamma(x)$. If $\ell = 3$, set $\gamma'(a_i) = i$ for $i = 1, 2, 3$. If $\ell = 4$ or 5, set $\gamma'(a_i) = 1$ for $i = 1, \ldots, \ell$. In either case, γ' is a coloring of G' with at most $\chi(G)$ colors. So Claim 2.2 holds.

2.3. *G' is $\mathcal{F}$-free and C_4-free.*

Proof. Suppose on the contrary that G' contains a subgraph F which is either a member of $\mathcal{F}$ or a C_4. Let $A = \{a_1, \ldots, a_\ell\}$. If F contains at most two vertices of A, then, since Z has two adjacent vertices and also two non-adjacent vertices, we can replace the vertices of $F \cap A$ by an appropriate choice of vertices of Z and we find a subgraph of G that is isomorphic to F, a contradiction. So F must contain at least three vertices of A. Note that in F, the neighbourhood of any of these vertices is equal to $F \cap X$, i.e., they

are pairwise twins. But this is impossible, because no member of $\mathcal{F} \cup \{C_4\}$ has three vertices that are pairwise twins. Thus Claim 2.3 holds.

By Claims 2.1–2.3, G' is an $\mathcal{F}$-free, C_4-free graph with $b(G') \geq b(G) > \chi(G) \geq \chi(G')$ and G' has strictly fewer C_5's than G, a contradiction. This completes the proof of Theorem 2. □

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