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# CORRIGENDUM TO "ON THE DIMENSION OF THE SHEETS OF A REDUCTIVE LIE ALGEBRA"

ANNE MOREAU

ABSTRACT. This note is a corrigendum to [M08]. As it has been recently pointed out to me by Alexander Premet, [M08, Remark 3.12] is incorrect. We explain in this note the impacts of that error in [M08], and amend certain of its statements. In particular, we verify that the statement of [M08, Theorem 3.13] remains correct in spite of this error.

## 1. INTRODUCTION

Let  $\mathfrak{g}$  be a complex simple Lie algebra and  $G$  its adjoint group. We investigate in [M08] the dimension of the subsets, for  $m \in \mathbb{N}$ ,

$$\mathfrak{g}^{(m)} := \{x \in \mathfrak{g} \mid \dim(Gx) = 2m\},$$

where  $Gx$  denotes the adjoint orbit of  $x \in \mathfrak{g}$ . The irreducible components of the subsets  $\mathfrak{g}^{(m)}$  are called the *sheets* of  $\mathfrak{g}$ , [BK79, B81]. Thus, for any  $m \in \mathbb{N}$ ,

$$(1) \quad \dim \mathfrak{g}^{(m)} = \max\{\dim \mathcal{S} ; \mathcal{S} \subset \mathfrak{g}^{(m)}\},$$

where  $\mathcal{S}$  runs through all sheets contained in  $\mathfrak{g}^{(m)}$ . The sheets are known to be parameterized by the pairs  $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$ , up to  $G$ -conjugacy class, consisting of a Levi subalgebra  $\mathfrak{l}$  of  $\mathfrak{g}$  and a rigid nilpotent orbit  $\mathcal{O}_{\mathfrak{l}}$  in  $\mathfrak{l}$ , cf. [B81]. This parametrization enables to write the dimension of a sheet  $\mathcal{S}$  associated with a pair  $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$  as the sum of the dimension of the center of  $\mathfrak{l}$  and the dimension of the unique nilpotent orbit contained in  $\mathcal{S}$ , see e.g. [M08, Proposition 2.11].

In the classical case, formulas for  $\mathfrak{g}^{(m)}$  are given in [M08, Theorems 3.3 and 3.13] in term of partitions associated with nilpotent elements of  $\mathfrak{g}$ . As it has been recently pointed out by Alexander Premet, Remark 3.12 in [M08] which claims that "*in the classical case, the dimension of a sheet containing a given nilpotent orbit does not depend on the choice of a sheet containing it*" is incorrect. We give here some counter-examples (cf. Examples 3.1 and 3.2; see also [PT12, Remark 4]). This is true only for the type **A** where each nilpotent element belongs to only one sheet. The error stems from the proof of [M08, Proposition 3.11]; see Section 3 for explanations. As a consequence, the proof of [M08, Theorems 3.13], partly based on [M08, Proposition 3.11], is incorrect too. However its statement remains true. This can be shown through a recent work of Premet and Topley, [PT12]. In more details, another formula for  $\mathfrak{g}^{(m)}$  in term of partitions can be traced out from [PT12, Corollary 9] and the

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equality (1). In this note, we verify (cf. Theorems 2.10) that the Premet-Topley formula for  $\mathfrak{g}^{(m)}$  coincides with the one of [M08, Theorem 3.13].

The note is organized as follows.

In Section 2, we recall some definitions and results of [PT12] and show that the statement of [M08, Theorem 3.13] is correct in spite of the error in [M08, Proposition 3.11], see Theorem 2.10(ii). In Section 3, we precisely pin down the error in the proof [M08, Proposition 3.11] and describe the impacts of that error in [M08]. As a conclusion, we list in Section 4 all corrections which have to be taken into account in [M08].

Since the corrections in [M08] only concern the types **B**, **C** and **D**, we assume for the remaining of the note that  $\mathfrak{g}$  is either  $\mathfrak{so}(N)$  or  $\mathfrak{sp}(N)$ , with  $N \geq 2$ , and  $\varepsilon$  is 1 or  $-1$  depending on whether  $\mathfrak{g} = \mathfrak{so}(N)$  or  $\mathfrak{sp}(N)$ . Following the notations of [M08] (or [PT12]), we denote by  $\mathcal{P}_\varepsilon(N)$  the set of partitions of  $N$  associated with the nilpotent elements of  $\mathfrak{g}$ . For  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathcal{P}_\varepsilon(N)$ , we denote by  $e(\lambda)$  the corresponding nilpotent element of  $\mathfrak{g}$  whose Jordan block sizes are  $\lambda_1, \dots, \lambda_n$ . We will always assume that  $\lambda_1 \geq \dots \geq \lambda_n$ .

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## 2. THE MAIN RESULT

For the convenience of the reader, we recall here all the necessary definitions and results of [PT12]. Given a partition  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathcal{P}_\varepsilon(N)$  we set,

$$\Delta(\lambda) := \{1 \leq i < n ; \varepsilon(-1)^{\lambda_i} = \varepsilon(-1)^{\lambda_{i+1}} = -1, \lambda_{i-1} \neq \lambda_i \geq \lambda_{i+1} \neq \lambda_{i+2}\}.$$

Our convention is that  $\lambda_0 = 0$  and  $\lambda_i = 0$  for all  $i > n$ . Recall the following result of Kempken and Spaltenstein (also recalled in [M08] and [PT12]):

**Theorem 2.1** ([K83, S82]). *Let  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathcal{P}_\varepsilon(N)$ . Then  $e(\lambda)$  is rigid if and only if*

- $\lambda_i - \lambda_{i+1} \in \{0, 1\}$  for all  $1 \leq i \leq n$ ;
- the set  $\{i \in \Delta(\lambda) ; \lambda_i = \lambda_{i+1}\}$  is empty.

Denote by  $\mathcal{P}_\varepsilon^*(N)$  the set of  $\lambda \in \mathcal{P}_\varepsilon(N)$  such that  $e(\lambda)$  is rigid. We call the elements of  $\mathcal{P}_\varepsilon^*(N)$  the *rigid partitions*. We first introduce the notion of *admissible sequences*, see [PT12, §3.1]. This is an extended version of the algorithm described in [M08] which takes  $\lambda \in \mathcal{P}_\varepsilon(N)$  and returns an element of  $\mathcal{P}_\varepsilon^*(N)$  compatible for the induction process of nilpotent orbits.

Let  $\mathbf{i}$  be a finite sequence of integers between 1 and  $n$ . The procedure of [PT12] is as follows: the algorithm commences with input  $\lambda = \lambda^{\mathbf{i}} \in \mathcal{P}_\varepsilon(N)$  where  $\mathbf{i} = \emptyset$  is the empty sequence. At the  $l^{\text{th}}$  iteration, the algorithm takes  $\lambda^{\mathbf{i}} \in \mathcal{P}_\varepsilon(N - 2 \sum_{j=1}^{l-1} i_j)$  where  $\mathbf{i} = (i_1, \dots, i_{l-1})$  and returns  $\lambda^{\mathbf{i}'} \in \mathcal{P}_\varepsilon(N - 2 \sum_{j=1}^l i_j)$  where  $\mathbf{i}' = (i_1, \dots, i_{l-1}, i_l)$  for some  $i_l$ . If the output  $\lambda^{\mathbf{i}'}$  is a rigid partition then the algorithm terminates after the  $l^{\text{th}}$  iteration with output  $\lambda^{\mathbf{i}'}$ . We shall now explicitly describe the  $l^{\text{th}}$  iteration of the algorithm. If after the  $(l-1)^{\text{th}}$  iteration

the input  $\lambda^{\mathbf{i}}$  is not rigid then the algorithm behaves as follows. Let  $i_l$  denote any index in the range  $1 \leq i \leq n$  such that either of the following case occur:

**Case 1:**  $\lambda_{i_l}^{\mathbf{i}} \geq \lambda_{i_l+1}^{\mathbf{i}} + 2$ ;

**Case 2:**  $i_l \in \Delta(\lambda^{\mathbf{i}})$  and  $\lambda_{i_l}^{\mathbf{i}} = \lambda_{i_l+1}^{\mathbf{i}}$ .

Note that no integer  $i_l$  will fulfill both criteria. If  $\mathbf{i} = (i_1, \dots, i_{l-1})$  then define  $\mathbf{i}' = (i_1, \dots, i_{l-1}, i_l)$ . For Case 1 the algorithm has output

$$\lambda^{\mathbf{i}'} = (\lambda_1^{\mathbf{i}} - 2, \lambda_2^{\mathbf{i}} - 2, \dots, \lambda_{i_l}^{\mathbf{i}} - 2, \lambda_{i_l+1}^{\mathbf{i}}, \dots, \lambda_n^{\mathbf{i}})$$

whilst for Case 2 the algorithm has output

$$\lambda^{\mathbf{i}'} = (\lambda_1^{\mathbf{i}} - 2, \lambda_2^{\mathbf{i}} - 2, \dots, \lambda_{i_l-1}^{\mathbf{i}} - 2, \lambda_{i_l}^{\mathbf{i}} - 1, \lambda_{i_l+1}^{\mathbf{i}} - 1, \lambda_{i_l+2}^{\mathbf{i}}, \dots, \lambda_n^{\mathbf{i}}).$$

Due to its definition and the classification of rigid partitions the above algorithm certainly terminates after a finite number of steps.

**Definition 2.2** ([PT12, §3.1]). We say that a sequence  $\mathbf{i} = (i_1, \dots, i_l)$  is an *admissible sequence* for  $\lambda$  if Case 1 or Case 2 occurs at the point  $i_k$  for the partition  $\lambda^{(i_1, \dots, i_{k-1})}$  for each  $k = 1, \dots, l$ . An admissible sequence  $\mathbf{i}$  for  $\lambda$  is be called a *maximal admissible sequence* for  $\lambda$  if neither Case 1 nor Case 2 occurs for any index  $i$  between 1 and  $n$  for the partition  $\lambda^{\mathbf{i}}$ . By convention the empty sequence is admissible for any  $\lambda \in \mathcal{P}_\varepsilon(N)$ .

As observed in [PT12, Lemma 6], if  $\mathbf{i}$  is an admissible sequence for  $\lambda$ , then  $\mathbf{i}$  is maximal admissible if and only if  $\lambda^{\mathbf{i}}$  is a rigid partition. We will denote by  $|\mathbf{i}| := l$  the length of an admissible sequence for  $\lambda$ .

**Definition 2.3.** The algorithm as described in [M08] corresponds to the special case where in the above algorithm, we define at each step  $i_l$  to be the smallest integer which fulfills one the Case 1 or Case 2 criteria, and  $\lambda^{\mathbf{i}}$  is rigid. In the sequel, we will refer the so obtained maximal admissible sequence for  $\lambda$  to as the *canonical maximal admissible sequence* for  $\lambda$  and we denote it by  $\mathbf{i}^0$ . Then we set

$$z_M(\lambda) := |\mathbf{i}^0|.$$

*Remark.* The integer  $z_M(\lambda)$  corresponds to the integer  $z(\lambda)$  of [M08].

**Definition 2.4** ([PT12, Definition 1]). If  $i \in \Delta(\lambda)$  then the pair  $(i, i+1)$  is called a *2-step* of  $\lambda$ . If  $i > 1$  and  $(i, i+1)$  is a 2-step of  $\lambda$  then  $\lambda_{i-1}$  and  $\lambda_{i+2}$  are referred to as the *boundary* of  $(i, i+1)$ . If  $1 \in \Delta(\lambda)$  then  $\lambda_3$  is referred to as the boundary of  $(1, 2)$  (if  $n = 2$  then  $\lambda_3 = 0$  by convention).

We observe that  $\Delta(\lambda)$  is the set of 2-steps of  $\lambda$ , and by  $|\Delta(\lambda)|$  its cardinality.

**Definition 2.5** ([PT12, §3.2]). If  $i \in \Delta(\lambda)$  then we say that the 2-step  $(i, i+1)$  has a *good boundary* if  $\lambda_1$  and the boundary of  $(i, i+1)$  have the opposite parity. If the boundary of a 2-step  $(i, i+1)$  of  $\lambda$  is not good then we say that it is *bad* and we refer to  $(i, i+1)$  as a *bad 2-step*. Note that  $(i, i+1)$  is a bad 2-step of  $\lambda$  if and only if either  $i > 1$  and  $\lambda_{i-1} - \lambda_i \in 2\mathbb{N}$ , or  $\lambda_{i+1} - \lambda_{i+2} \in 2\mathbb{N}$ .

We denote by  $\Delta_{\text{bad}}(\lambda)$  the set of bad 2-steps of  $\lambda$ , and by  $|\Delta_{\text{bad}}(\lambda)|$  its cardinality.

**Definition 2.6** ([PT12, Definition 2]). A sequence  $1 \leq i_1 < \dots < i_k < n$  with  $k \geq 2$  is called a *2-cluster* of  $\lambda$  whenever  $i_j \in \Delta(\lambda)$  and  $i_{j+1} = i_j + 2$  for all  $j$ . We say that a 2-cluster  $i_1, \dots, i_k$  has a *bad boundary* if either of the following conditions holds:

- $\lambda_{i_1-1} - \lambda_{i_1} \in 2\mathbb{N}$ ;
- $\lambda_{i_k+1} - \lambda_{i_k+2} \in 2\mathbb{N}$ .

(if  $i_1 = 1$  the the first condition should be omitted). A *bad 2-cluster* is one which has a bad boundary, whilst a *good 2-cluster* is one without a bad boundary.

We denote by  $\Sigma(\lambda)$  the set of good 2-clusters of  $\lambda$ , and by  $|\Sigma(\lambda)|$  its cardinality.

**Lemma 2.7** ([PT12, Lemma 11]). *A good 2-cluster is maximal in the sense that it is not a proper subsequence of any 2-cluster.*

**Definition 2.8** (Premet-Topley). For any  $\lambda \in \mathcal{P}_\varepsilon(\lambda)$ , the integer  $z_{\text{PT}}(\lambda)$  is defined by the formula:

$$z_{\text{PT}}(\lambda) := s(\lambda) + |\Delta(\lambda)| - |\Delta_{\text{bad}}(\lambda)| + |\Sigma(\lambda)|$$

where

$$s(\lambda) := \sum_{i=1}^n [(\lambda_i - \lambda_{i+1})/2].$$

*Remark.* The integer  $z_{\text{PT}}(\lambda)$  corresponds to the integer  $z(\lambda)$  of [PT12].

By [PT12, Theorem 8], we have that

$$(2) \quad z_{\text{PT}}(\lambda) := \max |\mathbf{i}|$$

where the maximum is taken over all admissible sequences for  $\lambda$ . Hence, by [PT12, Corollary 9] and the equality (1) of the introduction, we get:

**Theorem 2.9** (Premet-Topley). *For any  $m \in \mathbb{N}$ , we have*

$$\dim \mathfrak{g}^{(m)} = 2m + \max\{z_{\text{PT}}(\lambda) ; \lambda \in \mathcal{P}_\varepsilon(N) \text{ s.t. } \dim Ge(\lambda) = 2m\}.$$

The main result of this note is:

**Theorem 2.10.** (i) *For any  $\lambda \in \mathcal{P}_\varepsilon(N)$ , we have  $z_{\text{M}}(\lambda) = z_{\text{PT}}(\lambda)$ .*

(ii) *For any  $m \in \mathbb{N}$ , we have*

$$\dim \mathfrak{g}^{(m)} = 2m + \max\{z_{\text{M}}(\lambda) ; \lambda \in \mathcal{P}_\varepsilon(N) \text{ s.t. } \dim Ge(\lambda) = 2m\}.$$

*In other words, the statement of [M08, Theorem 3.13] is correct.*

*Proof.* (ii) is a direct consequence of (i) and Theorem 2.9.

(i) We argue by induction on  $N$  (the statement is true for small  $N$ ). Let  $N > 2$  and assume the statement true for any  $\lambda \in \mathcal{P}_\varepsilon(N')$ , with  $1 \leq N' \leq N$ , and let  $\lambda \in \mathcal{P}_\varepsilon(N)$ .

If  $\lambda \in \mathcal{P}_\varepsilon^*(N)$ , then  $z_{\text{PT}}(\lambda) = z_{\text{M}}(\lambda) = 0$  (see Theorem 2.1, Definition 2.2 and equality (2)). So, we can assume that  $\lambda$  is not a rigid partition. In particular,  $z_{\text{PT}}(\lambda) > 0$  and  $z_{\text{M}}(\lambda) > 0$ .

To ease notation, we simply denote here by  $\mathbf{i} := \mathbf{i}^0$  the canonical maximal sequence for  $\lambda$ . Then recall that by Definition 2.3,  $z_M(\lambda) = |\mathbf{i}|$ . Set  $\lambda' := \lambda^{(i_1)}$ . Clearly,  $z_M(\lambda') = z_M(\lambda) - 1$ . By the induction hypothesis, we have  $z_{PT}(\lambda') = z_M(\lambda')$ . Hence, we have to show that:

$$z_{PT}(\lambda') = z_{PT}(\lambda) - 1.$$

Our strategy is to compare the formulas for  $z_{PT}(\lambda')$  and  $z_{PT}(\lambda)$  given by Definition 2.8. Recall that  $i_1$  is the smallest integer which fulfills one of the Case 1 or Case 2 criteria for  $\lambda$ . First of all, we observe that if  $i \in \Delta(\lambda)$  (resp.  $i \in \Delta(\lambda')$ ), then  $i \geq i_1$ . Indeed, if  $i \in \Delta(\lambda)$  and  $i < i_1$  (if  $i_1 = 1$ , it is clear), then either  $\lambda_i = \lambda_{i+1}$  and then  $i$  fulfills the Case 2 which contradicts the minimality of  $i_1$ , or  $\lambda_i - \lambda_{i+1} \in 2\mathbb{N} \setminus \{0\}$  and then  $i$  fulfills the Case 1 which contradicts the minimality of  $i_1$  too.

We now consider the two situations Case 1 and Case 2 separately.

**Case 1:**  $\lambda_{i_1} \geq \lambda_{i_1+1} + 2$ .

We have,

$$\lambda' = (\lambda_1 - 2, \dots, \lambda_{i_1-1} - 2, \lambda_{i_1} - 2, \lambda_{i_1+1}, \dots, \lambda_n),$$

and

$$\begin{aligned} s(\lambda') &= \sum_{i=1}^{i_1-1} [(\lambda_i - \lambda_{i+1})/2] + [(\lambda_{i_1} - 2 - \lambda_{i_1+1})/2] + \sum_{i=i_1+1}^n [(\lambda_i - \lambda_{i+1})/2] \\ &= s(\lambda) - 1. \end{aligned}$$

Compare now the other terms appearing in Definition 2.8. Note that  $i_1 \in \Delta(\lambda)$  (resp.  $i_1 \in \Delta_{\text{bad}}(\lambda)$ ) if and only if  $i_1 \in \Delta(\lambda')$  (resp.  $i_1 \in \Delta_{\text{bad}}(\lambda')$ ) since the passing from  $\lambda$  to  $\lambda'$  preserves the parities. For the same reason,  $i_1$  belongs to a good 2-cluster of  $\lambda$  if and only if  $i_1$  belongs to a good 2-cluster of  $\lambda'$ .

Then we discuss two cases depending on whether  $i_1 + 1$  is in  $\Delta(\lambda)$  or not:

- $i_1 + 1 \in \Delta(\lambda)$ .

Once again, we consider two cases:

- \*  $\lambda_{i_1} - 2 \neq \lambda_{i_1+1}$ .

Then  $i_1 + 1 \in \Delta(\lambda')$  too. Moreover,  $i_1 + 1 \in \Delta_{\text{bad}}(\lambda')$  if and only if  $i_1 + 1 \in \Delta_{\text{bad}}(\lambda)$ . Hence, we conclude that  $|\Delta(\lambda')| = |\Delta(\lambda)|$ ,  $|\Delta_{\text{bad}}(\lambda')| = |\Delta_{\text{bad}}(\lambda)|$  and  $|\Sigma(\lambda')| = |\Sigma(\lambda)|$ .

- \*  $\lambda_{i_1} - 2 = \lambda_{i_1+1}$ .

Then  $i_1 + 1 \in \Delta_{\text{bad}}(\lambda)$  since  $\lambda_{i_1} - \lambda_{i_1+1} = 2 \in 2\mathbb{N}$ . But  $i_1 + 1 \notin \Delta(\lambda')$ . Therefore,  $|\Delta(\lambda')| = |\Delta(\lambda)| - 1$  and  $|\Delta_{\text{bad}}(\lambda')| = |\Delta_{\text{bad}}(\lambda)| - 1$ . Moreover, if  $i_1 + 1$  belongs to a 2-cluster of  $\lambda$ , then it is bad because  $\lambda_{i_1} - \lambda_{i_1+1} \in 2\mathbb{N}$ . Hence, we have  $|\Sigma(\lambda')| = |\Sigma(\lambda)|$ .

- $i_1 + 1 \notin \Delta(\lambda)$ .

In this case, note that  $i_1 + 1 \notin \Delta(\lambda')$ . Hence, we conclude that  $|\Delta(\lambda')| = |\Delta(\lambda)|$ ,  $|\Delta_{\text{bad}}(\lambda')| = |\Delta_{\text{bad}}(\lambda)|$  and  $|\Sigma(\lambda')| = |\Sigma(\lambda)|$ .

**Case 2:**  $i_1 \in \Delta(\lambda)$  and  $\lambda_{i_1} = \lambda_{i_1+1}$ .

By the minimality condition of  $i_1$ , we have  $\lambda_{i_1-1} = \lambda_{i_1} + 1$  (except for  $i_1 = 1$ , in which case  $\lambda_{i_1-1} = 0$  by convention), and so  $\lambda_{i_1-2} = \lambda_{i_1-1}$  because  $\varepsilon(-1)^{\lambda_{i_1-1}} = 1$ . We have

$$\lambda' = (\lambda_1 - 2, \dots, \lambda_{i_1-1} - 2, \lambda_{i_1} - 1, \lambda_{i_1+1} - 1, \lambda_{i_1+2}, \dots, \lambda_n),$$

and

$$\begin{aligned} s(\lambda') &= \sum_{i=1}^{i_1-2} [(\lambda_i - \lambda_{i+1})/2] + \underbrace{[(\lambda_{i_1-1} - \lambda_{i_1} - 1)/2]}_{=0 \text{ since } \lambda_{i_1-1} = \lambda_{i_1} + 1} \\ &\quad + [(\lambda_{i_1} - \lambda_{i_1+1})/2] + [\lambda_{i_1+1} - 1 - \lambda_{i_1+2}]/2 + \sum_{i=i_1+1}^n [(\lambda_i - \lambda_{i+1})/2] \\ &= \begin{cases} s(\lambda) - 1 & \text{if } \lambda_{i_1+1} - \lambda_{i_1+2} \in 2\mathbb{N}; \\ s(\lambda) & \text{if } \lambda_{i_1+1} - \lambda_{i_1+2} \notin 2\mathbb{N}. \end{cases} \end{aligned}$$

(If  $i_1 = 0$ , we start at the second line and we get the same conclusion.) Also, observe that in Case 2, we have

$$|\Delta(\lambda')| = |\Delta(\lambda)| - 1.$$

Indeed,  $i_1 \in \Delta(\lambda)$  but  $i_1 \notin \Delta(\lambda')$  and for the indexes  $i \neq i_1$  we have here the equivalence:  $i \in \Delta(\lambda) \iff i \in \Delta(\lambda')$ .

We discuss two cases depending on the parity of  $\lambda_{i_1+1} - \lambda_{i_1+2}$ .

- $\lambda_{i_1+1} - \lambda_{i_1+2} \in 2\mathbb{N}$ .

Then  $i_1 \in \Delta_{\text{bad}}(\lambda)$ . There are two sub-cases depending on whether  $i_1 + 2$  is in  $\Delta(\lambda)$  or not:

- \*  $i_1 + 2 \in \Delta(\lambda)$ .

Then,  $i_1 + 2 \in \Delta_{\text{bad}}(\lambda)$  (since  $\lambda_{i_1+1} - \lambda_{i_1+2} \in 2\mathbb{N}$ ) and  $i_1 + 2 \in \Delta(\lambda')$ . Once again, there are two sub-cases:

- 1)  $i_1 + 2 \notin \Delta_{\text{bad}}(\lambda')$ .

Then  $|\Delta_{\text{bad}}(\lambda')| = |\Delta_{\text{bad}}(\lambda)| - 2$ . Moreover,  $(i_1, i_1 + 2)$  is a good 2-cluster of  $\lambda$ . Indeed,  $i_1 + 2 \notin \Delta_{\text{bad}}(\lambda')$  implies that  $\lambda_{i_1+3} - \lambda_{i_1+4} \notin 2\mathbb{N}$ . On the other hand,  $\lambda_{i_1-1} - \lambda_{i_1} = 1 \notin 2\mathbb{N}$  (if  $i_1 = 1$  the first condition in Definition 2.6 should be omitted). But  $(i_1, i_1 + 2)$  is not a 2-cluster of  $\lambda'$  since  $i_1 \notin \Delta(\lambda')$ . Hence, we have  $|\Sigma(\lambda')| = |\Sigma(\lambda)| - 1$  by Lemma 2.7.

- 2)  $i_1 + 2 \in \Delta_{\text{bad}}(\lambda')$ .

Then  $|\Delta_{\text{bad}}(\lambda')| = |\Delta_{\text{bad}}(\lambda)| - 1$ . Moreover, the unique 2-cluster of  $\lambda$  which is possibly not a 2-cluster of  $\lambda'$  is  $(i_1, i_1 + 2)$  but it is bad in this case. Indeed,  $\lambda_{i_1+3} - \lambda_{i_1+4} \in 2\mathbb{N}$  since  $i_1 + 2 \in \Delta_{\text{bad}}(\lambda')$  (and  $\lambda'_{i_1+1} - \lambda'_{i_1+2} \notin 2\mathbb{N}$ ). Hence,  $|\Sigma(\lambda')| = |\Sigma(\lambda)|$ .

- \*  $i_1 + 2 \notin \Delta(\lambda)$ .

Then  $|\Delta_{\text{bad}}(\lambda')| = |\Delta_{\text{bad}}(\lambda)| - 1$ . Moreover, since  $i_1 + 2 \notin \Delta(\lambda)$ , then neither  $i_1$  nor  $i_1 + 2$  belongs to a 2-cluster for  $\lambda$ . Hence  $|\Sigma(\lambda)| = |\Sigma(\lambda')|$ .

- $\lambda_{i_1+1} - \lambda_{i_1+2} \notin 2\mathbb{N}$ .

In this case,  $i_1 \notin \Delta_{\text{bad}}(\lambda)$ ,  $i_1 + 2 \notin \Delta(\lambda)$  and  $i_1 + 2 \notin \Delta(\lambda')$ . Hence  $|\Delta_{\text{bad}}(\lambda')| = |\Delta_{\text{bad}}(\lambda)|$ . Moreover, neither  $i_1$  nor  $i_1 + 2$  belongs to any 2-cluster. Hence  $|\Sigma(\lambda)| = |\Sigma(\lambda')|$ .

In all the cases, we can check with the formula of Definition 2.8 that  $z_{\text{PT}}(\lambda') = z_{\text{PT}}(\lambda) - 1$  as desired. This concludes the proof of Theorem 2.10.  $\square$

### 3. COUNTER-EXAMPLES FOR [M08, Proposition 3.11]

From now on, we shall denote by  $z(\lambda)$  the integer  $z_{\text{M}}(\lambda) = z_{\text{PT}}(\lambda)$  for  $\lambda \in \mathcal{P}_\varepsilon(N)$ . If  $\mathfrak{l}$  is a Levi subalgebra of  $\mathfrak{g}$  and  $\mathcal{O}'$  is a rigid nilpotent orbit of  $\mathfrak{l}$ , we denote by  $\text{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}')$  the induced nilpotent orbit of  $\mathfrak{g}$  from  $\mathcal{O}'$  in  $\mathfrak{l}$ .

Proposition 3.11 of [M08] asserts that if a nilpotent element  $e$  associated with the partition  $\lambda \in \mathcal{P}_\varepsilon(N)$  is induced from a nilpotent orbit in a Levi subalgebra  $\mathfrak{l}$ , then  $z(\lambda)$  is equal to the dimension of the center of  $\mathfrak{l}$ . This result is actually incorrect. If it were true, it would imply that all the sheets containing  $e$  share the same dimension (see [M08, Remark 3.12]). But this is wrong. Below are some counter-examples (see also [PT12, Remark 4]):

**Example 3.1.** Assume that  $\mathfrak{g} = \mathfrak{so}(8)$  and consider the nilpotent element  $e$  of  $\mathfrak{g}$  with partition  $\lambda = (3, 3, 1, 1) \in \mathcal{P}_1(8) \setminus \mathcal{P}_1^*(8)$ . The algorithm yields  $z(\lambda) = 2$ .

On the other hand,  $e$  is induced from two different ways: from the zero orbit in a Levi subalgebra  $\mathfrak{l}_1$  of type  $(3, 1; 0)$ , that is  $\mathfrak{l}_1 \simeq \mathfrak{gl}_3 \times \mathfrak{gl}_1 \times 0$  (see the definition after [M08, Lemma 3.2] for the meaning of *type*), and from the zero orbit in a Levi subalgebra  $\mathfrak{l}_2$  of type  $(2; 4)$ , that is  $\mathfrak{l}_2 \simeq \mathfrak{gl}_2 \times \mathfrak{so}_4$ . The first one,  $\mathfrak{l}_1$ , has a center of dimension 2, while the second one,  $\mathfrak{l}_2$ , has a center of dimension 1. The nilpotent orbit of  $e$  has dimension 18 and  $e$  lies in two different sheets: one of dimension  $\dim \mathfrak{z}(\mathfrak{l}_1) + \dim \text{Ind}_{\mathfrak{l}_1}^{\mathfrak{g}}(0) = 20$  and one of dimension  $\dim \mathfrak{z}(\mathfrak{l}_2) + \dim \text{Ind}_{\mathfrak{l}_2}^{\mathfrak{g}}(0) = 19$  (here  $\mathfrak{z}(\mathfrak{l}_i)$  denotes the center of  $\mathfrak{l}_i$  for  $i = 1, 2$ ). This contradicts Proposition 3.11 of [M08], and also Remark 3.12 of the same paper.

**Example 3.2.** We give now a counter-example in  $\mathfrak{sp}(14)$ . Consider the partition  $\lambda = (4, 4, 2, 2, 1, 1)$  of  $\mathcal{P}_{-1}(14)$ . Here, the algorithm yields  $z(\lambda) = 2$ .

The corresponding nilpotent element is induced from the zero orbit in  $\mathfrak{l}_1 \simeq \mathfrak{gl}_1 \times \mathfrak{gl}_3 \times \mathfrak{sp}(6)$ , and from the rigid nilpotent orbit  $0 \times \mathcal{O}'$  in  $\mathfrak{l}_2 \simeq \mathfrak{gl}_2 \times \mathfrak{sp}(10)$  where  $\mathcal{O}'$  corresponds to the partition  $(2, 2, 2, 2, 1, 1) \in \mathcal{P}_{-1}^*(10)$ . Again the dimensions of the centers of  $\mathfrak{l}_1$  and  $\mathfrak{l}_2$  lead to different dimensions, 2 and 1 respectively.

The origin of the error can be pinned down in the proof of [M08, Proposition 3.11]. Let us briefly explain this. Until the end of the section, we are in the notations of [M08].

At the end of this proof, the assertion "Consequently the smallest integer such that one of the situations (a) or (b) of **Step 1** happens in  $\mathfrak{d}^{(p)}$  is equal to  $i_p$ " is incorrect (here  $\mathfrak{d}$  is an element of  $\mathcal{P}_\varepsilon(N)$ ). And so, the main induction argument of the proof fails. We can see that is incorrect in general on an explicit example. Consider the partition  $\mathfrak{d} = (4, 4, 3, 3, 1, 1)$  of  $\mathcal{P}_1(16)$ . Then the corresponding nilpotent orbit is induced from the zero orbit in  $\mathfrak{l} \simeq \mathfrak{gl}_3 \times \mathfrak{gl}_5 \times 0$  and from the rigid nilpotent orbit with partition  $(2, 2, 1, 1, 1, 1)$  in  $\mathfrak{l} \simeq \mathfrak{gl}(4) \times \mathfrak{so}(8)$ . Consider the second induction. In the notations of the proof, we have:

$S = 1$ ,  $i_1 = 4$ ,  $\mathbf{d}^{(0)} = \mathbf{f} = (2, 2, 1, 1, 1, 1)$ ,  $\mathbf{d} = \mathbf{d}^{(1)} = \widetilde{\mathbf{d}^{(0)}}$  (see [M08, Proposition 3.7] for the tilda notation). Then the smallest integer such that one of the situations (a) or (b) of Step 1 happens for  $\mathbf{d} = \mathbf{d}^{(1)}$  is  $3 \neq i_1$ .

#### 4. CONCLUSION

To summarize, we list below all corrections which have to be taken into account in [M08] (the numbering of [M08] is used):

- Proposition 3.11 (its proof and its statement) is incorrect.
- As a consequence Remark 3.12, the sentence "The results of this section specify that, in the classical case, the dimension of a sheet containing a given nilpotent orbit does not depend on the choice of a sheet containing it" in §1.2, and the sentence "Surprisingly, in the classical case, we will notice that if  $\text{Ind}_{\mathfrak{l}_1}(\mathcal{O}_{\mathfrak{l}_1}) = \text{Ind}_{\mathfrak{l}_2}(\mathcal{O}_{\mathfrak{l}_2})$ , then  $\dim \mathfrak{z}_{\mathfrak{g}}(\mathfrak{l}_1) = \dim \mathfrak{z}_{\mathfrak{g}}(\mathfrak{l}_2)$ " in Remark 2.15, are also incorrect.
- The proof of Theorem 3.13 is incorrect, since it uses Proposition 3.11. Nevertheless, its statement remains valid. In particular, Tables 3, 4 and 5 are still correct.

*Remark.* There are some misprints in Table 5: line  $2m = 48$ , the partitions are  $[7, 1^5]$ ,  $[5, 3, 2^2]$ ,  $[4^2, 3, 1]$  and not  $[4^3]$ ,  $[4^2, 3, 1]$ .

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