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## ON THE DIMENSION OF THE SHEETS OF A REDUCTIVE LIE ALGEBRA

*by*

Anne Moreau

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**Abstract.** — Let  $\mathfrak{g}$  be a complex finite dimensional Lie algebra and  $G$  its adjoint group. Following a suggestion of A. A. Kirillov, we investigate the dimension of the subsets of linear forms  $f \in \mathfrak{g}^*$  whose coadjoint orbit has dimension  $2m$ . In this paper we focus on the reductive case. If this case the problem reduces to compute the dimension of the sheets of  $\mathfrak{g}$ . These sheets are known to be parameterized by the pairs  $(\mathfrak{l}, \mathcal{O}_\mathfrak{l})$ , up to  $G$ -conjugation class, consisting of a Levi subalgebra  $\mathfrak{l}$  of  $\mathfrak{g}$  and a rigid nilpotent orbit  $\mathcal{O}_\mathfrak{l}$  in  $\mathfrak{l}$ . By using this parametrization, we provide the dimension of the above subsets for any  $m$ .

### Introduction

**0.1.** — Let  $\mathfrak{g}$  be a finite dimensional Lie algebra over a field  $K$ , and  $G$  be the adjoint group of  $\mathfrak{g}$ . The Lie algebra  $\mathfrak{g}$  acts on  $\mathfrak{g}$  and on its dual  $\mathfrak{g}^*$  via the adjoint and coadjoint actions. Coadjoint orbits play a leading part in representation theory. There are the origin of the famous “orbit method”, initiated by A. A. Kirillov 40 years ago motivated by the problem of describing the unitary dual of the group of upper triangular matrices with ones on the diagonal. For  $f \in \mathfrak{g}^*$ , denote by  $B_f$  the skew-symmetric bilinear form on  $\mathfrak{g} \times \mathfrak{g}$  defined by:

$$B_f(x, y) = f([x, y]), \quad \forall x, y \in \mathfrak{g}.$$

As the kernel of  $B_f$  is the stabilizer of  $f$  in  $\mathfrak{g}^*$  for the coadjoint action, the rank of  $B_f$  is equal to the dimension of the coadjoint orbit of  $f$ . The notion of the index was introduced by J. Dixmier for its importance in representation theory.

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By definition, the *index of  $\mathfrak{g}$* , denoted by  $\text{ind } \mathfrak{g}$ , is the minimal codimension of the coadjoint orbits in  $\mathfrak{g}^*$ . Namely,

$$\text{ind } \mathfrak{g} = \dim \mathfrak{g} - \max_{f \in \mathfrak{g}^*} \text{rank } B_f.$$

A linear form is said to be *regular* if the dimension of its stabilizer for the coadjoint action is equal to the index of  $\mathfrak{g}$ . As  $B_f$  is a skew-symmetric bilinear form, its rank is even and we set

$$\mathfrak{g}_m^* := \{f \in \mathfrak{g}^* \mid \text{rank } B_f = 2m\}.$$

Following a suggestion of A. A. Kirillov, we investigate the dimension of the sets  $\mathfrak{g}_m^*$ , for any  $m \in \mathbb{N}$ . If  $d_{\mathfrak{g}}$  is the integer  $(\dim \mathfrak{g} - \text{ind } \mathfrak{g})/2$ , the subset  $\mathfrak{g}_{d_{\mathfrak{g}}}^*$  has dimension  $\dim \mathfrak{g}^* = \dim \mathfrak{g}$ , since the subset of regular linear forms of  $\mathfrak{g}$  is a dense open subset of  $\mathfrak{g}^*$ . Moreover  $\mathfrak{g}_m^*$  is an empty set for any  $m$  strictly bigger than  $d_{\mathfrak{g}}$ , and it is a nonempty set for  $m = d_{\mathfrak{g}}$ . What about the smaller integers? The problem is closely related to the problem of computing the index, which is a very hard problem in general. So far, the index was studied only in some particular cases (see for instance [5], [20], [9], [27], [16] and [17]). Consequently, it would be too ambitious to get such integers, as well as the dimension of the  $\mathfrak{g}_m^*$ , for any Lie algebra. In [13], A. A. Kirillov approaches the case where  $\mathfrak{g}$  is the Lie algebra of the Lie group of upper triangular matrices with ones on the diagonal over a field  $\mathbb{F}_q$  of characteristic  $q$ . In this note, we provide the dimension of the subsets  $\mathfrak{g}_m^*$ , for all  $m \in \mathbb{N}$ , in the case where  $\mathfrak{g}$  is a complex finite dimensional reductive Lie algebra.

**0.2.** — From now on  $K = \mathbb{C}$  and  $\mathfrak{g}$  is supposed to be reductive. We denote by  $\langle \cdot, \cdot \rangle$  a non-degenerate  $G$ -invariant bilinear form on  $\mathfrak{g}$  which extends the Killing form of the semisimple part of  $\mathfrak{g}$ . The Lie algebra  $\mathfrak{g}$  can be identified with  $\mathfrak{g}^*$  through  $\langle \cdot, \cdot \rangle$ . In particular the index of  $\mathfrak{g}$  is nothing but the rank  $\text{rk } \mathfrak{g}$  of  $\mathfrak{g}$ . Moreover,  $\mathfrak{g}_m^*$  can be identified with the subset  $\mathfrak{g}^{(m)}$  of elements  $x \in \mathfrak{g}$  such that  $\dim G.x = 2m$ , where  $G.x$  is the  $G$ -orbit of  $x$  under the adjoint action of  $G$  on  $\mathfrak{g}$ . The subset  $\mathfrak{g}_{\leq m}^* := \{f \in \mathfrak{g}^* \mid \text{rank } B_f \leq 2m\}$  is a  $G$ -invariant closed subset of  $\mathfrak{g}^*$  as a finite union of nullvarieties of regular functions on  $\mathfrak{g}^*$ . Therefore the subset  $\mathfrak{g}_m^* = \mathfrak{g}_{\leq m}^* \setminus \mathfrak{g}_{\leq m-1}^*$  is a  $G$ -invariant locally closed subset of  $\mathfrak{g}^*$ , and so  $\mathfrak{g}^{(m)}$  is a  $G$ -invariant locally closed subset of  $\mathfrak{g}$ . The irreducible components of  $\mathfrak{g}^{(m)}$  are called *the sheets of  $\mathfrak{g}$*  (see [6]). Obviously,  $\mathfrak{g}$  is the finite union of its sheets. Thus, our problem reduces to compute the dimension of the sheets of  $\mathfrak{g}$ . In addition, the reductive case easily deduces from the case where  $\mathfrak{g}$  is simple. For  $x \in \mathfrak{g}$ , we denote by  $\mathfrak{g}^x$  its centralizer in  $\mathfrak{g}$  in such a way that  $\dim G.x = \dim \mathfrak{g} - \dim \mathfrak{g}^x$ . An element is said to be *regular* if its centralizer has minimal dimension  $\text{rk } \mathfrak{g} = \text{ind } \mathfrak{g}$ . We have already noticed the equality:

$\dim \mathfrak{g}^{(d_{\mathfrak{g}})} = \dim \mathfrak{g}$ . Furthermore, according to [29], the codimension of the complement of the set of regular elements in  $\mathfrak{g}$  equals 3. As a by-product of this work, we will precise this result by showing that  $\mathfrak{g}^{(d_{\mathfrak{g}}-1)}$  is equidimensional of dimension  $\dim \mathfrak{g} - 3$  (Theorem 1.16). For the smaller integers, it seems that there is no visible general rules, as we will see from our explicit computations. In particular, the subsets  $\mathfrak{g}^{(m)}$  are not always equidimensional.

**0.3.** — In Section 1, we recall some known results about  $G$ -Jordan classes, sheets, induced nilpotent orbits and rigid nilpotent orbits. We recall the links between these different notions. In particular, the subset  $\mathfrak{g}^{(m)}$  is a nonempty set if and only if  $2m$  is the dimension of a nilpotent orbit of  $\mathfrak{g}$  (Proposition 1.14) and the sheets of  $\mathfrak{g}$  are parameterized by the pairs  $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$ , up to  $G$ -conjugation class, consisting of a Levi subalgebra  $\mathfrak{l}$  of  $\mathfrak{g}$  and a rigid nilpotent orbit  $\mathcal{O}_{\mathfrak{l}}$  of  $\mathfrak{l}$  (Theorem 1.12).

Section 2 deals with the classical Lie algebras. We recall the characterization of rigid nilpotent orbits in the classical case in term of the corresponding partitions. Then, we establish an explicit formula for the dimension of  $\mathfrak{g}^{(m)}$ , for all  $m \in \mathbb{N}$  (Theorems 2.2 and 2.11). Tables 1, 3, 4 and 5 provide these dimensions for  $\mathfrak{sl}_6$ ,  $\mathfrak{so}_7$ ,  $\mathfrak{sp}_6$  and  $\mathfrak{so}_{12}$ .

Section 3 concerns the exceptional case. Using the description of the sheets of  $\mathfrak{g}$  obtained Section 1, we give for each exceptional simple Lie algebra, the dimension of the sets  $\mathfrak{g}^{(m)}$ , for all  $m \in \mathbb{N}$  (cf Tables 7, 9, 11, 15, 16).

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### 1. $G$ -Jordan Classes, sheets and induced nilpotent orbits

**1.1.** — If  $\mathfrak{a}$  is a Lie subalgebra of  $\mathfrak{g}$ , we denote by  $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$  the centralizer in  $\mathfrak{g}$  of  $\mathfrak{a}$ . As a rule, we respectively denote by  $x_s$  and  $x_n$  the semisimple and nilpotent components in  $\mathfrak{g}$  of an element  $x \in \mathfrak{g}$ . The results of this subsection are mostly due to W. Borho and H. Kraft [2] and they are also presented in [28].

**Definition 1.1.** — *Let  $x, y \in \mathfrak{g}$ . We say that  $x$  and  $y$  are  $G$ -Jordan equivalent if there exists  $g \in G$  such that:*

$$\mathfrak{g}^{y_s} = \mathfrak{g}^{g(x_s)} = g(\mathfrak{g}^{x_s}), \quad y_n = g(x_n).$$

*This defines an equivalence relation on  $\mathfrak{g}$ . The equivalence class of  $x$ , that we shall denote by  $J_G(x)$ , is called the  $G$ -Jordan class of  $x$  in  $\mathfrak{g}$ .*

For any  $G$ -invariant subset  $Y$  of  $\mathfrak{g}$  we denote by  $Y^{\text{reg}}$  the set of regular elements of  $Y$ , that is to say those of maximal  $G$ -orbit dimension. Recall that a Levi subalgebra of  $\mathfrak{g}$  is the reductive part of a parabolic subalgebra of  $\mathfrak{g}$ . The group  $G$  acts on the set of the pairs  $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$  consisting of a Levi subalgebra  $\mathfrak{l}$  of  $\mathfrak{g}$  and a nilpotent orbit  $\mathcal{O}_{\mathfrak{l}}$  of  $\mathfrak{l}$  in a obvious way.

The following lemma, proven for instance in [28], doesn't present any difficulty:

**Lemma 1.2** ([28], **Propositions 39.1.5 and 39.2.9**)

*Let  $x = x_s + x_n$  be in  $\mathfrak{g}$  and  $J_G(x)$  its  $G$ -Jordan class. Then:*

- (i)  $J_G(x) = G \cdot (\mathfrak{z}(\mathfrak{g}^{x_s})^{\text{reg}} + x_n)$ ,
- (ii)  $\dim J_G(x) = \dim(G \cdot x) + \dim \mathfrak{z}(\mathfrak{g}^{x_s})$ .

The following lemma, proven for instance in [28], doesn't present any difficulty:

**Lemma 1.3** ([28], **Propositions 39.1.5 and 39.2.9**)

*Let  $x = x_s + x_n$  be in  $\mathfrak{g}$  and  $J_G(x)$  its  $G$ -Jordan class. Then:*

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- (ii)  $\dim J_G(x) = \dim(G \cdot x) + \dim \mathfrak{z}(\mathfrak{g}^{x_s})$ .

We deduce from Lemma 1.3 the following known proposition, whose we recall the proof:

**Proposition 1.4.** — *There is a one-to-one correspondence between  $G$ -Jordan classes and the set of pairs  $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$ , up to  $G$ -conjugation, where  $\mathfrak{l}$  is a Levi subalgebra of  $\mathfrak{g}$  and  $\mathcal{O}_{\mathfrak{l}}$  a nilpotent orbit of  $\mathfrak{l}$ .*

*Proof.* — Let  $J_G(x)$  be a  $G$ -Jordan class of  $\mathfrak{g}$ . Set  $\mathfrak{l} = \mathfrak{g}^{x_s}$ . Since  $x_s$  is semisimple,  $\mathfrak{l}$  is a Levi subalgebra of  $\mathfrak{g}$  which contains  $x_n$ , because  $[x_s, x_n] = 0$ . Denote by  $\mathcal{O}$  the nilpotent orbit of  $x_n$  in  $\mathfrak{l}$ . The  $G$ -orbit of the pair  $(\mathfrak{l}, \mathcal{O})$  only

depends on  $J_G(x)$  and not on the choice of a representative in  $J_G(x)$ . This defines the first map.

Conversely, let  $\mathfrak{l}$  be a Levi subalgebra of  $\mathfrak{g}$  and  $\mathcal{O}_\mathfrak{l}$  a nilpotent orbit in  $\mathfrak{l}$ . Fix  $z \in \mathfrak{z}_\mathfrak{g}(\mathfrak{l})^{\text{reg}}$ ,  $y \in \mathcal{O}_\mathfrak{l}$  and set  $x = y + z$ . As  $z$  belongs to the center of  $\mathfrak{l}$ ,  $[z, y] = 0$ , whence  $x_s = z$  and  $x_n = y$ . Then  $\mathfrak{g}^{x_s} = \mathfrak{g}^z = \mathfrak{l}$ , because  $z \in \mathfrak{z}_\mathfrak{g}(\mathfrak{l})^{\text{reg}}$ . Since  $J_G(x) = G \cdot (\mathfrak{z}(\mathfrak{g}^{x_s})^{\text{reg}} + x_n)$  by Lemma 1.3 (i), the  $G$ -Jordan class of  $x$  doesn't depend on the choices of  $z \in \mathfrak{z}(\mathfrak{g}^{x_s})^{\text{reg}}$  nor of  $y \in \mathcal{O}_\mathfrak{l}$ . Furthermore, it is clear that  $J_G(x)$  only depends on the  $G$ -orbit of  $(\mathfrak{l}, \mathcal{O}_\mathfrak{l})$  and not on the choice of one of its representatives, whence the other map.

These two maps are clearly inverse each other, whence the one-to-one correspondence.  $\square$

We shall say that the  $G$ -Jordan class associated via the previous correspondence, to a pair  $(\mathfrak{l}, \mathcal{O}_\mathfrak{l})$ , where  $\mathfrak{l}$  is a Levi subalgebra of  $\mathfrak{g}$  and  $\mathcal{O}_\mathfrak{l}$  a nilpotent orbit of  $\mathfrak{l}$ , *has data*  $(\mathfrak{l}, \mathcal{O}_\mathfrak{l})$ . It follows from the classification of Levi subalgebras of  $\mathfrak{g}$  as well as the one of nilpotent orbits in reductive Lie algebras that there are only finitely many  $G$ -Jordan classes. Therefore,  $\mathfrak{g}$  is a finite disjoint union of its  $G$ -Jordan classes. By Lemma 1.3, a  $G$ -Jordan class is  $G$ -stable, irreducible, and it is contained in  $\mathfrak{g}^{(m)}$ , for some  $m \in \mathbb{N}$ . As a consequence, we deduce the following proposition:

**Proposition 1.5** ([28], Proposition 39.3.3). — *Let  $\mathfrak{S}$  be a sheet of  $\mathfrak{g}$ . Then, there is an unique  $G$ -Jordan class  $\mathfrak{J}$  such that  $\mathfrak{J} \subset \mathfrak{S}$  and  $\overline{\mathfrak{S}} = \overline{\mathfrak{J}}$ . Moreover,  $\mathfrak{S} = (\overline{\mathfrak{J}})^{\text{reg}}$ .*

**1.2.** — In order to investigate the dimension of the sheets of  $\mathfrak{g}$ , we have to determinate which  $G$ -Jordan classes are dense in a sheet. Theorem 1.12 ([1], 4.4) provides an answer. Because of the importance of this result in the remainders of this paper, we recall the main steps of the proof of this theorem. In addition, we will need of some of these intermediate results. We start by recalling some results about induced nilpotent orbits (see [4] for more details). Then, we use many results due to Borho [1] presented in the thesis of Andreas im Hof [11].

**Theorem-Definition 1.6** ([4], Theorem 7.1.1). — *Let  $\mathfrak{l}$  be the reductive part of a parabolic Lie algebra  $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$  of  $\mathfrak{g}$  with nilradical  $\mathfrak{n}$ . Let  $P$  be the connected subgroup of  $G$  with Lie algebra  $\mathfrak{p}$ . Then there is a unique nilpotent orbit  $\mathcal{O}_\mathfrak{g}$  in  $\mathfrak{g}$  meeting  $\mathcal{O}_\mathfrak{l} + \mathfrak{n}$  in an open dense subset. We have  $\dim \mathcal{O}_\mathfrak{g} = \dim \mathcal{O}_\mathfrak{l} + 2 \dim \mathfrak{n}$ . The orbit  $\mathcal{O}_\mathfrak{g}$  is the unique nilpotent orbit in  $\mathfrak{g}$  of this dimension which meets  $\mathcal{O}_\mathfrak{l} + \mathfrak{n}$ . The intersection  $\mathcal{O}_\mathfrak{g} \cap (\mathcal{O}_\mathfrak{l} + \mathfrak{n})$  consists in a single  $P$ -orbit.*

*We say that the orbit  $\mathcal{O}_\mathfrak{g}$  is induced from  $\mathcal{O}_\mathfrak{l}$  and we denote it by  $\text{Ind}_\mathfrak{p}^\mathfrak{g}(\mathcal{O}_\mathfrak{l})$ . If  $\mathcal{O}_\mathfrak{l} = 0$ , then we say that  $\mathcal{O}_\mathfrak{g}$  is a Richardson orbit.*

**Lemma 1.7** ([4], **Theorem 7.1.3**). — Let  $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$  and  $\mathfrak{p}' = \mathfrak{l} \oplus \mathfrak{n}'$  be two parabolic subalgebras of  $\mathfrak{g}$  with the same Levi subalgebra  $\mathfrak{l}$ , and let  $\mathcal{O}_\mathfrak{l}$  be a nilpotent orbit in  $\mathfrak{l}$ . Then,  $\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\mathcal{O}_\mathfrak{l}) = \text{Ind}_{\mathfrak{p}'}^{\mathfrak{g}}(\mathcal{O}_\mathfrak{l})$ .

As a consequence of this result, we can use the notation  $\text{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_\mathfrak{l})$  instead of  $\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\mathcal{O}_\mathfrak{l})$ .

**Proposition 1.8** ([4], **Proposition 7.1.4**). — Let  $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$  be the Levi decomposition of a parabolic subalgebra in  $\mathfrak{g}$  and  $\mathcal{O}_\mathfrak{l}$  a nilpotent orbit in  $\mathfrak{l}$ .

- (i)  $\text{codim}_{\mathfrak{l}}(\mathcal{O}_\mathfrak{l}) = \text{codim}_{\mathfrak{g}}(\text{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_\mathfrak{l}))$ , i.e.:  $\dim \text{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_\mathfrak{l}) = \dim \mathfrak{g} - \dim \mathfrak{l} + \dim \mathcal{O}_\mathfrak{l}$ .
- (ii) Let  $\mathfrak{l}_1$  and  $\mathfrak{l}_2$  be two Levi subalgebras of  $\mathfrak{g}$  with  $\mathfrak{l}_1 \subset \mathfrak{l}_2$ . Then  $\text{Ind}_{\mathfrak{l}_2}^{\mathfrak{g}}(\text{Ind}_{\mathfrak{l}_1}^{\mathfrak{l}_2}(\mathcal{O}_{\mathfrak{l}_1})) = \text{Ind}_{\mathfrak{l}_1}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}_1})$ .

**Proposition 1.9** ([1], **3.1, Satz a**). — Let  $\mathfrak{J}$  be a  $G$ -Jordan class with data  $(\mathfrak{l}, \mathcal{O}_\mathfrak{l})$ . Then  $(\mathfrak{J})^{\text{reg}}$  contains a unique nilpotent orbit. It is equal to  $\text{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_\mathfrak{l})$ .

**Corollary 1.10** ([1], **3.6**). — Let  $\mathfrak{J}$  and  $\mathfrak{J}'$  be two  $G$ -Jordan classes with data  $(\mathfrak{l}, \mathcal{O}_\mathfrak{l})$  and  $(\mathfrak{l}', \mathcal{O}_{\mathfrak{l}'})$  respectively such that  $\mathfrak{l}$  contains  $\mathfrak{l}'$ . Then  $\mathfrak{J}$  is contained in  $\overline{\mathfrak{J}'}$  if and only if  $\mathcal{O}_\mathfrak{l}$  is  $\text{Ind}_{\mathfrak{l}'}^{\mathfrak{l}}(\mathcal{O}_{\mathfrak{l}'})$  up to  $G$ -conjugation class.

Proposition 1.8, (i) implies that not every nilpotent orbit is induced from another. A nilpotent orbit in  $\mathfrak{g}$  which is not induced from any proper parabolic subalgebra is called *rigid*.

**Corollary 1.11** ([1], **4.2**). — A  $G$ -Jordan class with data  $(\mathfrak{l}, \mathcal{O}_\mathfrak{l})$  is dense in a sheet if and only if  $\mathcal{O}_\mathfrak{l}$  is rigid in  $\mathfrak{l}$ .

We conclude this paragraph with the expected classification of sheets of  $\mathfrak{g}$ , which is a consequence of Corollary 1.11 and Proposition 1.5:

**Theorem 1.12** ([1], **4.4**). — There is a one-to-one correspondence between the set of pairs  $(\mathfrak{l}, \mathcal{O}_\mathfrak{l})$ , up to  $G$ -conjugation class, where  $\mathfrak{l}$  is a Levi subalgebra of  $\mathfrak{g}$  and  $\mathcal{O}_\mathfrak{l}$  a rigid nilpotent orbit in  $\mathfrak{l}$ , and the set of sheets of  $\mathfrak{g}$ .

**1.3.** — Let  $\mathfrak{S}$  be a sheet of  $\mathfrak{g}^{(m)}$ , for  $m \in \mathbb{N}$ . Then  $\mathfrak{S}$  corresponds to a pair  $(\mathfrak{l}, \mathcal{O}_\mathfrak{l})$  via the correspondence established in Theorem 1.12. According to Propositions 1.5 and 1.9,  $\text{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_\mathfrak{l})$  is the unique nilpotent orbit contained in  $\mathfrak{S}$ . Denote by  $J_G(x)$  the  $G$ -Jordan class whose data is  $(\mathfrak{l}, \mathcal{O}_\mathfrak{l})$ . Necessarily,  $\mathfrak{g}^{x_s} = \mathfrak{l}$ . So, Lemma 1.3 (ii) implies:

$$\dim J_G(x) = \dim G.x + \dim \mathfrak{z}_{\mathfrak{g}}(\mathfrak{l}).$$

In addition, since  $\text{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_\mathfrak{l})$  is contained in  $\mathfrak{S}$ ,  $\dim \text{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_\mathfrak{l}) = \dim G.x$ , because these two  $G$ -orbits belong to  $\mathfrak{g}^{(m)}$ . In conclusion, we obtain the following proposition:

**Proposition 1.13.** — *Let  $\mathfrak{S}$  be the sheet corresponding to the pair  $(\mathfrak{l}, \mathcal{O}_\mathfrak{l})$  via the correspondence established in Theorem 1.12. Then  $\text{Ind}_\mathfrak{l}^{\mathfrak{g}}(\mathcal{O}_\mathfrak{l})$  is the unique nilpotent orbit contained in  $\mathfrak{S}$ . Moreover:*

$$\begin{aligned} \dim \mathfrak{S} &= \dim \text{Ind}_\mathfrak{l}^{\mathfrak{g}}(\mathcal{O}_\mathfrak{l}) + \dim \mathfrak{z}_\mathfrak{g}(\mathfrak{l}) \\ &= \dim \mathfrak{g} - \dim \mathfrak{l} + \dim \mathcal{O}_\mathfrak{l} + \dim \mathfrak{z}_\mathfrak{g}(\mathfrak{l}). \end{aligned}$$

Denote by  $\mathbf{N}_\mathfrak{g}$  the set of all the dimensions of the nilpotent orbits in  $\mathfrak{g}$  and by  $\mathbf{N}_\mathfrak{g}^{\text{rigid}}$  those of all the dimensions of the rigid nilpotent orbits of  $\mathfrak{g}$ . As a consequence of Proposition 1.13, we get the following proposition:

**Proposition 1.14** ([28], Remarks 39.3.2). — *The set  $\mathfrak{g}^{(m)}$  is non-empty if and only if  $2m \in \mathbf{N}_\mathfrak{g}$ .*

Fix a Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$ . Let  $\Pi$  be a simple root system of the root system  $\Delta$  associated to the couple  $(\mathfrak{g}, \mathfrak{h})$ . Denote by  $\Delta_+$  the positive root system corresponding to  $\Pi$ . For  $\alpha \in \Delta$ , denote by  $\mathfrak{g}^\alpha$  the root subspace associated to  $\alpha$ . For  $S$  any subset of  $\Pi$ , we denote by  $\Delta^S$  the root subsystem of  $\Delta$  generated by  $S$ , and by  $\Delta_+^S$  the intersection  $\Delta_+ \cap \Delta^S$ . Set

$$\mathfrak{l}_S = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^S} \mathfrak{g}^\alpha.$$

Then  $\mathfrak{l}_S$  is a Levi subalgebra of  $\mathfrak{g}$  and it is well-known (see [4], Lemma 3.8.1) that every Levi subalgebra of  $\mathfrak{g}$  is  $G$ -conjugated to  $\mathfrak{l}_S$ , for some  $S$  in  $\Pi$ . For the sake of the simplicity, we shall denote by  $\mathbf{N}_S$  and  $\mathbf{N}_S^{\text{rigid}}$  respectively the sets  $\mathbf{N}_{\mathfrak{l}_S}$  and  $\mathbf{N}_{\mathfrak{l}_S}^{\text{rigid}}$ .

**Lemma 1.15.** — *Suppose that the semisimple part of  $\mathfrak{g}$  has dimension strictly bigger than 3. Then the subregular nilpotent orbit  $\mathcal{O}_\mathfrak{g}^{\text{subreg}}$  of  $\mathfrak{g}$  is not rigid.*

*Proof.* — Let  $\alpha \in \Pi$  and set  $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha}$ . Then the nilpotent orbit of  $\mathfrak{g}$  induced from the zero orbit of  $\mathfrak{l}$  has dimension,

$$\dim \mathfrak{g} - \dim \mathfrak{l} = \dim \mathfrak{g} - \text{rk} \mathfrak{g} - 2,$$

by Proposition 1.8, (i). As a consequence  $\mathcal{O}_\mathfrak{g}^{\text{subreg}} = \text{Ind}_\mathfrak{l}^{\mathfrak{g}}(\mathcal{O}_\mathfrak{l})$ , because  $\mathcal{O}_\mathfrak{g}^{\text{subreg}}$  is the only nilpotent orbit of  $\mathfrak{g}$  of dimension  $\dim \mathfrak{g} - \text{rk} \mathfrak{g} - 2$ . By hypothesis,  $\mathfrak{g}$  is different from  $\mathfrak{l}$ , whence  $\mathcal{O}_\mathfrak{g}^{\text{subreg}}$  is not rigid.  $\square$

Recall that  $2d_\mathfrak{g} = \dim \mathfrak{g} - \text{ind} \mathfrak{g}$ .

**Theorem 1.16.** — (i) *If  $m > d_\mathfrak{g}$ , then  $\mathfrak{g}^{(m)}$  is an empty set of  $\mathfrak{g}$ .*

(ii) *The subset  $\mathfrak{g}^{(d_\mathfrak{g})}$  is irreducible of dimension  $\dim \mathfrak{g}$ .*

(iii) *The subset  $\mathfrak{g}^{(d_\mathfrak{g}-1)}$  is equidimensional of dimension  $\dim \mathfrak{g} - 3$ .*



*Proof.* — (i) is an obvious consequence of the definition of  $d_{\mathfrak{g}}$ .

(ii) is true because  $\mathfrak{g}^{(d_{\mathfrak{g}})}$  is the subset of regular elements of  $\mathfrak{g}$ , which is a dense open subset of  $\mathfrak{g}$ .

(iii) Let  $\mathfrak{S}$  be a sheet of  $\mathfrak{g}^{(d_{\mathfrak{g}}-1)}$  associated to the  $G$ -Jordan class with data  $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$ , via the correspondence established in Theorem 1.12. According to Proposition 1.8, (i), the codimension in  $\mathfrak{l}$  of  $\mathcal{O}_{\mathfrak{l}}$  is  $\mathrm{rk}\mathfrak{l} + 2 = \mathrm{rk}\mathfrak{g} + 2$ . As a consequence,  $\mathcal{O}_{\mathfrak{l}}$  is the subregular nilpotent orbit of  $\mathfrak{l}$ . As  $\mathcal{O}_{\mathfrak{l}}$  is a rigid nilpotent orbit of  $\mathfrak{l}$ , Lemma 1.15 implies that the semisimple part of  $\mathfrak{l}$  has dimension 3. Then  $\mathcal{O}_{\mathfrak{l}}$  is the zero orbit of  $\mathfrak{l}$  and  $\mathfrak{S}$  has dimension

$$\begin{aligned} \dim \mathfrak{g} - \dim \mathfrak{l} + 0 + \dim \mathfrak{z}_{\mathfrak{g}}(\mathfrak{l}) &= (\dim \mathfrak{g} - \mathrm{rk}\mathfrak{g} - 2) + (\mathrm{rk}\mathfrak{g} - 1) \\ &= \dim \mathfrak{g} - 3, \end{aligned}$$

according to 1.13, because the center of  $\mathfrak{l}$  has dimension  $\mathrm{rk}\mathfrak{g} - 1$ , whence (ii).  $\square$

*Remark 1.* — *We cannot expect similar results for the smaller integers. Indeed, we will see from our explicit computations that  $\mathfrak{g}^{(m)}$  is not always equidimensional. This phenomenon mainly results from two problems. Firstly, a nilpotent orbit can be induced in two different ways. Secondly, two different orbits can have the same dimension.*

## 2. Computations in the classical cases

In this section, we study the dimensions of the sets  $\mathfrak{g}^{(n)}$ , for  $n \in \mathbb{N}$  such that  $2n \in \mathbf{N}_{\mathfrak{g}}$  in the case where  $\mathfrak{g}$  is a classical simple Lie algebra. Because of the one-to-one correspondence established in Theorem 1.12, we first need a precise description of rigid nilpotent orbits. These results will also be useful Section 3 for the exceptional case, since Levi subalgebras of simple exceptional Lie algebras may have simple factors of classical type.

**2.1. Type  $A_l$ .** — In this paragraph, we suppose that  $\mathfrak{g} = \mathfrak{sl}_N$ , with  $N \geq 2$ . First of all, recall that nilpotent orbits of  $\mathfrak{g}$  are in one-to-one correspondence with the set  $\mathcal{P}(N)$  of all partitions of  $n$ . Here, *partition* means a sequence  $\mathbf{d} = [d_1, \dots, d_N]$  of non-negative integers, possibly zero, with  $d_1 \geq \dots \geq d_N$  and  $d_1 + \dots + d_N = N$ . If  $\mathbf{d} = [d_1, \dots, d_N] \in \mathcal{P}(N)$ , we denote by  $\mathcal{O}_{\mathbf{d}}$  the corresponding nilpotent orbit of  $\mathfrak{g}$ .

Let  $\mathbf{d} = [d_1, \dots, d_N]$  be in  $\mathcal{P}(N)$ . Set  $s_i = \#\{j \mid d_j \geq i\}$  the *dual partition* of  $\mathbf{d}$ . Notice that  $d_1$  is the biggest integer  $j$  such that  $s_j$  is different from zero. Define a subset  $S^{\mathbf{d}}$  of  $\Pi$  as follows: the connected components  $S_1^{\mathbf{d}}, \dots, S_{d_1}^{\mathbf{d}}$  of

$S^{\mathbf{d}}$  have the cardinalities  $s_1 - 1, \dots, s_{d_1} - 1$  respectively. Then we denote by  $\mathfrak{l}_{\mathbf{d}}$  the Levi subalgebra  $\mathfrak{l}_{S^{\mathbf{d}}}$  with the notations of 1.3.

**Proposition 2.1** ([4], **Theorem 7.2.3**). — *The partition associated to  $\text{Ind}_{\mathfrak{l}_{\mathbf{d}}}^{\mathfrak{g}}(\mathcal{O}_0)$  is  $\mathbf{d}$ . In particular, every non-zero nilpotent orbit in  $\mathfrak{g}$  is Richardson and the unique rigid nilpotent orbit is the zero orbit.*

Let  $\mathbf{d}$  be a partition of  $N$ . The dimension of  $\mathcal{O}_{\mathbf{d}}$  is  $2m(\mathbf{d})$ , where  $m(\mathbf{d})$  is defined by ([4], Corollary 6.1.4):

$$(1) \quad m(\mathbf{d}) := \frac{1}{2}(N^2 - \sum_{i=1}^{d_1} s_i^2).$$

Let us remark that we can also obtain that formula from Propositions 2.1 and 1.8 (i), since  $\mathfrak{l}_{\mathbf{d}}$  has dimension  $\sum_{i=1}^{d_1} s_i^2 - 1$ .

**Theorem 2.2**. — *Let  $m \in \mathbb{N}$ . If  $2m \in \mathbf{N}_{\mathfrak{g}}$ , then*

$$\dim \mathfrak{g}^{(m)} = 2m + \max_{\substack{\mathbf{d} \in \mathcal{P}(N) \\ m(\mathbf{d})=m}} (d_1 - 1).$$

*Otherwise,  $\mathfrak{g}^{(m)}$  is an empty set.*

*Proof.* — Let  $m \in \mathbb{N}$  such that  $2m \in \mathbf{N}_{\mathfrak{g}}$ . Let  $\mathfrak{S}$  be a sheet of  $\mathfrak{g}^{(m)}$ . By Theorem 1.12 and proposition 2.1,  $\mathfrak{S}$  corresponds to a pair  $(\mathfrak{l}_{\mathbf{d}}, \mathcal{O}_0)$ , for some  $\mathbf{d} \in \mathcal{P}(N)$ . By Proposition 2.1 and Formula (1),  $\dim \text{Ind}_{\mathfrak{l}_{\mathbf{d}}}^{\mathfrak{g}}(\mathcal{O}_0) = 2m(\mathbf{d})$ . In particular,  $m(\mathbf{d})$  has to be equal to  $m$ . In addition, Proposition 1.13 gives:

$$\dim \mathfrak{S} = 2m(\mathbf{d}) + d_1 - 1,$$

since the center of  $\mathfrak{l}_{\mathbf{d}}$  has dimension

$$\text{rk } \mathfrak{g} - \#S^{\mathbf{d}} = N - 1 - \sum_{i=1}^{d_1} (s_i - 1) = N - 1 - (N - d_1) = d_1 - 1.$$

The first assertion of the theorem is now clear. The second assertion results from Proposition 1.14.  $\square$

As example, we present the dimension of the  $\mathfrak{g}^{(m)}$  for  $\mathfrak{g} = \mathfrak{sl}_6$  in Table 1. In this table, we provide, for each  $\mathbf{d} \in \mathcal{P}(6)$ , the dimension  $2m(\mathbf{d})$  of the corresponding nilpotent orbit and the quantity  $d_1 - 1$ . Then, we deduce the dimension of the sets  $\mathfrak{g}^{(m(\mathbf{d}))}$ , according to Theorem 2.2. Because two different orbits can have the same dimension, we need to compute  $d_1 - 1$ , for all  $\mathbf{d} \in \mathcal{P}(N)$  such that  $m(\mathbf{d}) = m$ , to get the dimension of  $\dim \mathfrak{g}^{(m)}$  (e.g. there are two nilpotent orbits of dimension 18).

$2m \in \mathbf{N}_{\mathfrak{g}}$	$\mathbf{d} \in \mathcal{P}(6)$ such that	$2m(\mathbf{d}) = 2m$	$d_1 - 1,$	for $\mathbf{d} \in \mathcal{P}(6)$	$\dim \mathfrak{g}^{(m)} = 2m + \max_{m(\mathbf{d})=m} (d_1 - 1)$
30	[6]		5		35
28	[5, 1]		4		32
26	[4, 2]		3		29
24	[4, 1 <sup>2</sup> ]	[3 <sup>2</sup> ]	3	2	27
22	[3, 2, 1]		2		24
18	[3, 1 <sup>3</sup> ]	[2 <sup>3</sup> ]	2	1	20
16	[2 <sup>2</sup> , 1 <sup>2</sup> ]		1		17
10	[2, 1 <sup>4</sup> ]		1		11
0	[1 <sup>6</sup> ]		0		0

TABLE 1. Dimensions of the sets  $\mathfrak{g}^{(m)}$  for  $\mathfrak{sl}_6$ .

**2.2. Type  $B_l, C_l$  and  $D_l$ .** — Set  $\varepsilon = \pm 1$  and consider a non-degenerate bilinear form  $\langle \cdot, \cdot \rangle_\varepsilon$  on  $\mathbb{C}^N$  such that:

$$\langle a, b \rangle_\varepsilon = \varepsilon \langle a, b \rangle_\varepsilon, \text{ for all } b, a \in \mathbb{C}^N.$$

If  $\varepsilon = -1$  (resp. 1), then the form is  $\langle \cdot, \cdot \rangle_\varepsilon$  is symplectic (resp. symmetric). We then define:

$$\begin{aligned} I(\langle \cdot, \cdot \rangle_\varepsilon) &= \{g \in \mathrm{GL}_N \mid \langle ga, gb \rangle_\varepsilon = \langle a, b \rangle_\varepsilon, \text{ for all } a, b \in \mathbb{C}^N\}, \\ \mathfrak{g}_\varepsilon &= \{x \in \mathfrak{sl}_N \mid \langle xa, b \rangle_\varepsilon = -\langle a, xb \rangle_\varepsilon, \text{ for all } a, b \in \mathbb{C}^N\}. \end{aligned}$$

If  $\varepsilon = -1$ , then  $N = 2n$  and  $\mathfrak{g}_{-1} \simeq \mathfrak{sp}_{2n}$ . If  $\varepsilon = 1$ , then  $\mathfrak{g}_{-1} \simeq \mathfrak{so}_N$ . Thus  $I(\langle \cdot, \cdot \rangle_\varepsilon)$  is the isotropy group of the form  $\langle \cdot, \cdot \rangle_\varepsilon$  on  $\mathbb{C}^N$ , and  $\mathfrak{g}_\varepsilon$  is its Lie algebra. Now, set:

$$\mathcal{P}_\varepsilon(N) = \{[d_1, \dots, d_N] \in \mathcal{P}(N) \mid \#\{j \mid d_j = i\} \text{ is even for all } i \text{ with } (-1)^i = \varepsilon\}.$$

**Theorem 2.3** ([4], Theorem 5.1.6). — *Nilpotent orbits in  $\mathfrak{g}_\varepsilon$  are in one-to-one correspondence with the partition  $\mathcal{P}_\varepsilon(N)$  of  $N$ , except that if  $\varepsilon = 1$ , and  $N = 2n$ , then very even partitions of  $N$  (those with only even parts, each having even multiplicity) correspond to two orbits that we label with I and II.*

To make some statements more understandable, we will include some of the proofs of known results presented in [4] concerning induced nilpotent orbits in classical simple Lie algebras.

We endow the set of the partitions of  $N$  with the classical order which corresponds to the classical order on the set of nilpotent orbits of  $\mathfrak{sl}_N$ . First,

recall a result due to Gerstenhaber which generalizes the transpose operation in the set  $\mathcal{P}(N)$ ;

**Proposition-Definition 2.4 (Gerstenhaber: cf [4], Lemma 6.3.3)**

Let  $\mathbf{p} = [p_1, \dots, p_{2n+1}]$  be a partition in  $\mathcal{P}(2n+1)$ . Then there is an unique largest partition in  $\mathcal{P}_1(2n+1)$  dominated by  $\mathbf{p}$ . This partition, called the  $B$ -collapse of  $\mathbf{p}$  and denoted by  $\mathbf{p}_B$ , may be defined as follows. If  $\mathbf{p}$  is not already in  $\mathcal{P}_1(2n+1)$ , then at least one of its even parts must occur with odd multiplicity; let  $q$  be the largest such part. Replace the last occurrence of  $q$  in  $\mathbf{p}$  by  $q-1$  and the first subsequent part  $r$  strictly less than  $q-1$  by  $r+1$ ; we may have to add a 0 to  $\mathbf{p}$  to find such an  $r$ . Repeat this process until a partition in  $\mathcal{P}_1(2n+1)$  is obtained. Similarly, there are a unique largest partition  $\mathbf{q}_C$ ,  $\mathbf{q}_D$  in  $\mathcal{P}_{-1}(2n+1)$ ,  $\mathcal{P}_1(2n+1)$  dominated by any given partition  $\mathbf{q}$  of  $2n+1$ . They are called the  $C$ - and  $D$ -collapses of  $\mathbf{q}$ ; their definitions are the obvious analogues of that of  $\mathbf{p}_B$ .

From now on, we shall denote by  $X$  the type of  $\mathfrak{g}_\varepsilon$ , that is to say  $B$ ,  $C$  or  $D$ .

Let  $\mathfrak{l}$  be a Levi subalgebra of  $\mathfrak{g}$ . Then, there are integers  $i_1, \dots, i_S \geq 0$  and  $R$  such that,

$$\mathfrak{l} \simeq \mathfrak{gl}_{i_S} \times \cdots \times \mathfrak{gl}_{i_1} \times \mathfrak{g}',$$

where  $\mathfrak{g}'$  has the same type as  $\mathfrak{g}_\varepsilon$  and whose standard representation has dimension  $R$ . After a possible renumbering, we can suppose that  $[i_1, \dots, i_S]$  belongs to  $\mathcal{P}(S)$ , with  $2S + R = N$ . Then we define:

$$\mathcal{P}_\varepsilon^{\text{Levi}} = \{(\mathbf{i}, R) \in \mathcal{P}(S) \times \mathbb{N}_{\geq 0} \mid 2S + R = N, S \geq 0 \text{ and } R \neq 2, \text{ if } \varepsilon = 1\}.$$

**Lemma 2.5.** — *There is a one-to-one correspondence between  $G$ -conjugation classes of Levi subalgebras of  $\mathfrak{g}$  and elements of  $\mathcal{P}_\varepsilon^{\text{Levi}}$ .*

If  $\mathfrak{l}$  corresponds to the element  $(\mathbf{i}; R)$  of  $\mathcal{P}_\varepsilon^{\text{Levi}}$ , we shall say that  $\mathfrak{l}$  is of type  $(\mathbf{i}; R)$ .

**Proposition 2.6 ([4], Corollary 7.3.4).** — *Let  $\mathfrak{l} = \mathfrak{gl}_l \times \mathfrak{g}'$  be a maximal Levi subalgebra, where  $\mathfrak{g}'$  has the same type as  $\mathfrak{g}_\varepsilon$  (then  $2l + r = N$  if  $r$  is the dimension of the standard representation of  $\mathfrak{g}'$  and  $\mathfrak{l}$  is of type  $(l; r)$ ). Let  $\mathcal{O}_\mathfrak{l} = \mathcal{O}_0 \times \mathcal{O}_\mathfrak{f}$  be a nilpotent orbit in  $\mathfrak{l}$  whose component in the  $\mathfrak{gl}_l$  factor is the zero orbit and whose component  $\mathcal{O}_\mathfrak{f}$  in the  $\mathfrak{g}'$  factor has partition  $\mathbf{f}$ . Then the partition of  $\text{Ind}_\mathfrak{l}^{\mathfrak{g}_\varepsilon}(\mathcal{O}_\mathfrak{l})$  is  $\mathbf{p}$ , where the partition  $\mathbf{p}$  is obtained from  $\mathbf{f}$  as follows:*

*We add 2 to the first  $l$  terms of  $\mathbf{f}$ , obtaining a partition  $\tilde{\mathbf{f}}$  (extending by zero if necessary in  $\mathbf{f}$  if necessary), and then take the  $X$ -collapse of  $\tilde{\mathbf{f}}$ , where*

$X = B, C$  or  $D$  (cf Proposition-Definition 2.4). If the collapse is nontrivial (ie  $\tilde{\mathbf{f}}_X \neq \tilde{\mathbf{f}}$ ), it is obtained by subtracting 1 from the  $l^{\text{th}}$  part of  $\tilde{\mathbf{f}}$  and adding 1 to its  $(l+1)^{\text{th}}$  part. If  $\mathfrak{g}_\varepsilon = \mathfrak{so}_{4n}$ ,  $r \neq 0$  and the collapsed partition is very even, then  $\mathbf{f}$  is also very even and the induced orbit inherits the label I or II of  $\mathcal{O}_\mathbf{f}$ ; if  $r = 0$ , then the label of the induced orbit is the same as that of  $\mathcal{O}_\mathbf{f}$  if  $n$  is even but differs from it if  $n$  is odd.

Define  $\mathcal{P}_\varepsilon^*(N)$  to be the set of all the partitions  $[d_1, \dots, d_N]$  in  $\mathcal{P}_\varepsilon(N)$  such that the following two conditions hold:

- (i)  $0 \leq d_{i+1} \leq d_i \leq d_{i+1} + 1$  for all  $i$ ,
- (ii)  $\#\{j \mid d_j = i\} \neq 2$  if  $\varepsilon(-1)^i = -1$ .

We will show that  $\mathcal{P}_\varepsilon^*(N)$  encodes rigid nilpotent orbits. Let  $\mathbf{d} = [d_1, \dots, d_N]$  be in  $\mathcal{P}_\varepsilon(N)$ . We want to construct an element of  $\mathcal{P}_\varepsilon^*(N)$  from  $\mathbf{d}$ . Set  $i_0 = 0$ ,  $\mathbf{d}^{(0)} = \mathbf{d}$  and  $\mathfrak{g}'^{(0)} = \mathfrak{g}_\varepsilon$ .

Step 1: if  $\mathbf{d} \in \mathcal{P}_\varepsilon^*(N)$ , set  $\mathbf{d}^{(1)} = \mathbf{d} = \mathbf{d}^{(0)}$ . Otherwise, there is  $j \in \{1, \dots, N\}$  such that either  $d_j \geq d_{j+1} + 2$ , or  $d_{j-1} > d_j = d_{j+1} > d_{j+2}$  with  $\varepsilon(-1)^{d_j} = -1$  (where we have set  $d_0 = 0$  and  $d_j = 0$ , for all  $j > N$ , by convention). Denote by  $i_1$  the smallest integer  $j$  such that one of these two situations happens.

a) In the first situation, set:

$$\mathbf{d}^{(1)} := [d_1 - 2, \dots, d_{i_1} - 2, d_{i_1+1}, \dots, d_N].$$

b) In the second situation, set:

$$\mathbf{d}^{(1)} := [d_1 - 2, \dots, d_{i_1-1} - 2, d_{i_1} - 1, d_{i_1+1} - 1, d_{i_1+2}, \dots, d_N].$$

In both situations,  $\mathbf{d}^{(1)}$  remains an element of  $\mathcal{P}_\varepsilon(N - 2i_1)$  so  $\mathcal{O}_{\mathbf{d}^{(1)}}$  is a nilpotent orbit of  $\mathfrak{g}'^{(1)}$ , where  $\mathfrak{g}'^{(1)}$  is a classical simple Lie algebra of the same type as  $\mathfrak{g}_\varepsilon$  whose standard representation has dimension  $N - 2i_1$ . Moreover, by Proposition 2.6,

$$\mathcal{O}_{\mathbf{d}} = \text{Ind}_{\mathfrak{l}^{(1)}}^{\mathfrak{g}_\varepsilon}(\mathcal{O}_0 \times \mathcal{O}_{\mathbf{d}^{(1)}}),$$

where  $\mathfrak{l}^{(1)}$  is a Levi subalgebra of  $\mathfrak{g}_\varepsilon$  of type  $(i_1; N - 2i_1)$ .

Step 2: suppose that  $i_0, i_1, \dots, i_{p-1}$ ,  $\mathbf{d}^{(0)}, \mathbf{d}^{(1)}, \dots, \mathbf{d}^{(p-1)}$ ,  $\mathfrak{g}'^{(0)}, \mathfrak{g}'^{(1)}, \dots, \mathfrak{g}'^{(p-1)}$  and  $\mathfrak{l}^{(1)}, \dots, \mathfrak{l}^{(p-1)}$  are defined for some  $p \in \{1, \dots, N\}$  in such way that:

(c<sub>1</sub>) for all  $k \in \{0, \dots, p-1\}$ ,  $\mathcal{O}_{\mathbf{d}^{(k)}}$  is a nilpotent orbit of  $\mathfrak{g}'^{(k)}$ , where  $\mathfrak{g}'^{(k)}$  is a classical simple Lie algebra of same type as  $\mathfrak{g}_\varepsilon$  whose corresponding

standard representation has dimension  $N - 2i_0 - 2i_1 - \dots - 2i_k$ ,

(c<sub>2</sub>) for all  $k \in \{1, \dots, p-1\}$ ,  $\mathcal{O}_{\mathbf{d}^{(k-1)}} = \text{Ind}_{\mathfrak{l}^{(k)}}^{\mathfrak{g}'^{(k-1)}}(\mathcal{O}_0 \times \mathcal{O}_{\mathbf{d}^{(k)}})$ , where  $\mathfrak{l}^{(k)}$  is a Levi subalgebra of  $\mathfrak{g}'^{(k-1)}$  of type  $(i_k; N - 2i_0 - 2i_1 - \dots - 2i_k)$ .

Then, we define  $i_p$ ,  $\mathbf{d}^{(p)}$ ,  $\mathfrak{l}^{(p)}$  and  $\mathfrak{g}'^{(p)}$  as in Step 1. More precisely, if  $\mathbf{d}^{(p-1)} \in \mathcal{P}_\varepsilon^*(N - 2i_0 - 2i_1 \dots - 2i_{p-1})$ , set  $\mathbf{d}^{(p)} = \mathbf{d}^{(p-1)}$ . Otherwise, there is  $j \in \{1, \dots, N - 2i_0 - 2i_1 \dots - 2i_{p-1}\}$  such that either  $d_j^{(p-1)} \geq d_{j+1}^{(p-1)} + 2$ , or  $d_{j-1}^{(p-1)} > d_j^{(p-1)} = d_{j+1}^{(p-1)} > d_{j+2}^{(p-1)}$  with  $d_j^{(p-1)}$  odd. Denote by  $i_p$  the smallest integer  $j$  such that one of these two situations happens.

a) In the first situation, set:

$$\mathbf{d}^{(p)} := [d_1^{(p-1)} - 2, \dots, d_{i_p}^{(p-1)} - 2, d_{i_p+1}^{(p-1)}, \dots, d_{N-2i_0-2i_1 \dots - 2i_{p-1}}^{(p-1)}].$$

b) In the second situation, set:

$$\mathbf{d}^{(p)} := [d_1^{(p-1)} - 2, \dots, d_{i_p-1}^{(p-1)} - 2, d_{i_p}^{(p-1)} - 1, d_{i_p+1}^{(p-1)} - 1, d_{i_p+2}^{(p-1)}, \dots, d_{N-2i_0-2i_1 \dots - 2i_{p-1}}^{(p-1)}].$$

As before, in both cases  $\mathbf{d}^{(p)}$  is an element of  $\mathcal{P}_\varepsilon(N - 2i_0 - 2i_1 \dots - 2i_p)$  so  $\mathcal{O}_{\mathbf{d}^{(p)}}$  is a nilpotent orbit of  $\mathfrak{g}'^{(p)}$ , where  $\mathfrak{g}'^{(p)}$  is a classical simple Lie algebra of the same type as  $\mathfrak{g}_\varepsilon$  whose standard representation has dimension  $N - 2i_0 - 2i_1 \dots - 2i_p$ . And, by Proposition 2.6,

$$\mathcal{O}_{\mathbf{d}^{(p-1)}} = \text{Ind}_{\mathfrak{l}^{(p)}}^{\mathfrak{g}'^{(p-1)}}(\mathcal{O}_0 \times \mathcal{O}_{\mathbf{d}^{(p)}}),$$

where  $\mathfrak{l}^{(p)}$  is a Levi subalgebra of  $\mathfrak{g}'^{(p-1)}$  of type  $(i_p; N - 2i_0 - 2i_1 \dots - 2i_p)$ . Then  $i_0, i_1, \dots, i_p$ ,  $\mathbf{d}^{(0)}, \mathbf{d}^{(1)}, \dots, \mathbf{d}^{(p)}$ ,  $\mathfrak{g}'^{(0)}, \mathfrak{g}'^{(1)}, \dots, \mathfrak{g}'^{(p)}$  and  $\mathfrak{l}^{(1)}, \dots, \mathfrak{l}^{(p)}$  satisfy conditions (c<sub>1</sub>) and (c<sub>2</sub>).

The process clearly ends after a finite number of steps.

**Definition 2.7.** — We denote by  $z(\mathbf{d})$  the smallest integer  $j$  such that  $\mathbf{d}^{(j)} = \mathbf{d}^{(j+1)}$ .

If a partition  $\mathbf{d}'$  is obtained from another partition  $\mathbf{d}$  by a transformation of type a) or b) as described in Steps 1 or 2 (in particular  $\mathbf{d}$  has to be different from  $\mathbf{d}'$ ), then we shall say that  $\mathbf{d}'$  is deduced from  $\mathbf{d}$  by an elementary transformation. In this case, it is clear that

$$(2) \quad z(\mathbf{d}') = z(\mathbf{d}) - 1.$$

Clearly, the previous construction proves that  $\mathcal{O}_{\mathbf{d}}$  is not rigid whenever  $z(\mathbf{d}) \neq 0$ . Using Proposition 2.6, we can easily prove that the converse also holds, whence the following corollary:

**Corollary 2.8.** — *The nilpotent orbit corresponding to a partition  $\mathbf{d} \in \mathcal{P}_\varepsilon(N)$  is rigid if and only if  $z(\mathbf{d}) = 0$ .*

As  $z(\mathbf{d})$  is clearly equal to 0 if and only if  $\mathbf{d}$  belongs to  $\mathcal{P}_\varepsilon^*(N)$ , Corollary 2.8 gives the following result, enounced in [4]:

**Corollary 2.9** ([4], **Corollary 7.3.5**). — *The orbit corresponding to a partition  $\mathbf{d} \in \mathcal{P}_\varepsilon(N)$  is rigid if and only if  $\mathbf{d} \in \mathcal{P}_\varepsilon(N)^*$ .*

Let  $\mathbf{d} = [d_1, \dots, d_N]$  be in  $\mathcal{P}_\varepsilon(N)$ . Put  $r_i = \#\{j \mid d_j = i\}$  and  $s_i = \#\{j \mid d_j \geq i\}$ . Denote by  $m(\mathbf{d})$  the half dimension of  $\mathcal{O}_{\mathbf{d}}$ . According to [4], Corollary 6.1.4,  $m(\mathbf{d})$  is given by the following formulas:

$$(3) \quad m(\mathbf{d}) = \begin{cases} \frac{1}{2}(2n^2 + n - \frac{1}{2} \sum_i s_i^2 + \frac{1}{2} \sum_{i \text{ odd}} r_i), & \text{if } \mathfrak{g} = \mathfrak{so}_{2n+1} \\ \frac{1}{2}(2n^2 + n - \frac{1}{2} \sum_i s_i^2 - \frac{1}{2} \sum_{i \text{ odd}} r_i), & \text{if } \mathfrak{g} = \mathfrak{sp}_{2n} \\ \frac{1}{2}(2n^2 - n - \frac{1}{2} \sum_i s_i^2 + \frac{1}{2} \sum_{i \text{ odd}} r_i), & \text{if } \mathfrak{g} = \mathfrak{so}_{2n}. \end{cases}$$

The construction preceding Definition 2.7 gives a process to compute the number  $z(\mathbf{d})$ , for  $\mathbf{d} \in \mathcal{P}_\varepsilon(N)$ . Then, according to Corollary 2.8, we obtain the list of rigid nilpotent orbits with their dimensions (computed with formulas (3)) in  $B_2, B_3, C_3, D_4, D_5, D_6$  and  $D_7$  in Table 2. We list these cases since they all appear as simple factors of Levi subalgebras in the exceptional Lie algebras.

$B_2$		$C_3$		$B_3$		$D_4$		$D_5$		$D_6$		$D_7$	
$[1^5]$	0	$[1^6]$	0	$[1^7]$	0	$[1^8]$	0	$[1^{10}]$	0	$[1^{12}]$	0	$[1^{14}]$	0
$[2^2, 1]$	4	$[2, 1^4]$	6	$[2^2, 1^3]$	8	$[2^2, 1^4]$	10	$[2^2, 1^6]$	14	$[2^2, 1^8]$	18	$[2^2, 1^{10}]$	22
						$[3, 2^2, 1]$	16	$[3, 2^2, 1^3]$	24	$[2^4, 1^4]$	28	$[2^4, 1^6]$	36
										$[3, 2^2, 1^5]$	32	$[3, 2^2, 1^7]$	40
										$[3, 2^4, 1]$	36	$[3, 2^4, 1^3]$	48
												$[3^3, 2^2, 1]$	58

TABLE 2. Rigid nilpotent orbits with their dimensions in  $\mathfrak{so}_5, \mathfrak{sp}_6, \mathfrak{so}_7, \mathfrak{so}_8, \mathfrak{so}_{10}, \mathfrak{so}_{12}$  and  $\mathfrak{so}_{14}$

**Proposition 2.10.** — Let  $\mathbf{d} = [d_1, \dots, d_N]$  be in  $\mathcal{P}_\varepsilon(N)$ . Suppose that  $\mathcal{O}_{\mathbf{d}} = \text{Ind}_{\mathfrak{l}}^{\mathfrak{g}_\varepsilon}(\mathcal{O}_{\mathfrak{l}})$ , where  $\mathfrak{l}$  is a Levi subalgebra of  $\mathfrak{g}_\varepsilon$  and  $\mathcal{O}_{\mathfrak{l}}$  a rigid nilpotent orbit in  $\mathfrak{l}$ . Then

$$\dim \mathfrak{z}_{\mathfrak{g}_\varepsilon}(\mathfrak{l}) = z(\mathbf{d}).$$

*Remark 2.* — A nilpotent orbit  $\mathcal{O}_{\mathbf{d}}$  can be induced in different ways by a rigid nilpotent orbit of a Levi subalgebra of  $\mathfrak{g}$ . Proposition 2.10 claims that the dimension of the center of a such Levi subalgebra only depends on  $\mathbf{d}$  and not on the choice of this Levi subalgebra, nor on the choice of a rigid nilpotent orbit in this Levi subalgebra.

*Proof.* — 1) If  $\mathcal{O}_{\mathbf{d}}$  is rigid, then  $\mathfrak{l} = \mathfrak{g}_\varepsilon$ , and  $\dim \mathfrak{z}_{\mathfrak{g}_\varepsilon}(\mathfrak{l}) = 0$ . On the other hand,  $z(\mathbf{d}) = 0$ , according to Corollary 2.8.

2) We suppose that  $\mathbf{d} \notin \mathcal{P}_\varepsilon(N)$ . In other words,  $\mathcal{O}_{\mathbf{d}}$  is not rigid, therefore  $\mathfrak{l}$  is strictly contained in  $\mathfrak{g}_\varepsilon$ . According to Lemma 2.5,  $\mathfrak{l}$  is  $G$ -conjugated to

$$\mathfrak{gl}_{i_S} \times \cdots \times \mathfrak{gl}_{i_1} \times \mathfrak{g}',$$

with  $2i_1 + \cdots + 2i_S + R = N$ ,  $i_1 \geq \cdots \geq i_S$ , and where  $\mathfrak{g}'$  is a Lie algebra of same type as  $\mathfrak{g}_\varepsilon$  whose standard representation has dimension  $R$ . In other words,  $\mathfrak{l}$  has type  $(\mathbf{i}; R)$ , with  $\mathbf{i} = [i_1, \dots, i_S] \in \mathcal{P}(S)$ . Notice that the center of  $\mathfrak{l}$  has dimension  $S$ . By Theorem 2.1,

$$\mathcal{O}_{\mathfrak{l}} = \mathcal{O}_0 \times \cdots \times \mathcal{O}_0 \times \mathcal{O}_{\mathbf{f}},$$

where  $\mathbf{f}$  is an element of  $\mathcal{P}_\varepsilon(R)$  such that  $\mathcal{O}_{\mathbf{f}}$  is rigid in  $\mathfrak{g}'$  and where the component of  $\mathcal{O}_{\mathfrak{l}}$  on the  $\mathfrak{gl}_{i_k}$  factor is zero, for  $k = 1, \dots, S$ .

Set  $i_0 = 0$  and  $\mathfrak{g}^{(0)} = \mathfrak{l}^{(0)} = \mathfrak{g}'$ . For  $p = 1, \dots, S$ , denote by  $\mathfrak{g}^{(p)}$  a Lie algebra of the same type as  $\mathfrak{g}_\varepsilon$  whose standard representation has dimension

$$R_p := 2i_p + \cdots + 2i_1 + 2i_0 + R,$$

and by  $\mathfrak{l}^{(p)}$  a Levi subalgebra of  $\mathfrak{g}^{(p)}$  of type  $(i_p; 2i_{p-1} + \cdots + 2i_1 + 2i_0 + R)$ .

Set  $\mathfrak{l}^{(S)} = \mathfrak{g}_\varepsilon$  and, for  $p = 0, \dots, S-1$ , denote by  $\mathfrak{l}^{(p)}$  a Levi subalgebra of  $\mathfrak{g}_\varepsilon$  of type  $([i_{p+1}, \dots, i_S]; 2i_p + \cdots + 2i_1 + 2i_0 + R)$  such that:

$$\mathfrak{l} = \mathfrak{l}^{(0)} \subseteq \cdots \subseteq \mathfrak{l}^{(S)} = \mathfrak{g}_\varepsilon.$$

As defined,  $\mathfrak{l}^{(p)}$  is  $G$ -conjugated to

$$\mathfrak{gl}_{i_S} \times \cdots \times \mathfrak{gl}_{i_{p+1}} \times \mathfrak{g}^{(p)},$$

for all  $p = 0, \dots, S-1$ .

Set  $\mathbf{d}^{(0)} = \mathbf{f}$  and define  $\mathbf{d}^{(p)}$ , for  $p = 1, \dots, S$ , by induction as follows;  $\mathbf{d}^{(p)}$  is the element of  $\mathcal{P}_\varepsilon(R_p)$  such that:

$$\mathcal{O}_{\mathbf{d}^{(p)}} = \text{Ind}_{\mathfrak{l}^{(p)}}^{\mathfrak{g}^{(p)}}(\mathcal{O}_0 \times \mathcal{O}_{\mathbf{d}^{(p-1)}}),$$



where  $\mathcal{O}_0$  is the zero orbit in the  $\mathfrak{gl}_{i_p}$  factor.

Let us first check that  $\mathbf{d}^{(S)} = \mathbf{d}$ . By Lemma 1.8, (ii), we can write:

$$\mathcal{O}_{\mathbf{d}} = \text{Ind}_{\mathfrak{l}^{(S-1)}}^{\mathfrak{l}^{(S)}}(\dots(\text{Ind}_{\mathfrak{l}^{(0)}}^{\mathfrak{l}^{(1)}}(\underbrace{\mathcal{O}_0 \times \dots \times \mathcal{O}_0 \times \mathcal{O}_{\mathbf{f}}}_{S \text{ factors}})))$$

In addition, we easily see that

$$\begin{aligned} \text{Ind}_{\mathfrak{l}^{(p-1)}}^{\mathfrak{l}^{(p)}}(\underbrace{\mathcal{O}_0 \times \dots \times \mathcal{O}_0}_{S-p+1 \text{ factors}} \times \mathcal{O}_{\mathbf{d}^{(p-1)}}) &\simeq \underbrace{\mathcal{O}_0 \times \dots \times \mathcal{O}_0}_{S-p \text{ factors}} \times \text{Ind}_{\mathfrak{l}^{(p)}}^{\mathfrak{g}'^{(p)}}(\mathcal{O}_0 \times \mathcal{O}_{\mathbf{d}^{(p-1)}}) \\ &= \underbrace{\mathcal{O}_0 \times \dots \times \mathcal{O}_0}_{S-p \text{ factors}} \times \mathcal{O}_{\mathbf{d}^{(p)}}, \end{aligned}$$

for all  $p = 1, \dots, S$ , because the  $S - p$  first factors of  $\mathfrak{l}^{(p-1)}$  and  $\mathfrak{l}^{(p)}$  are the same. Then, by induction, we obtain:

$$\begin{aligned} \mathcal{O}_{\mathbf{d}} &= \text{Ind}_{\mathfrak{l}^{(S-1)}}^{\mathfrak{l}^{(S)}}(\dots(\text{Ind}_{\mathfrak{l}^{(0)}}^{\mathfrak{l}^{(1)}}(\underbrace{\mathcal{O}_0 \times \dots \times \mathcal{O}_0 \times \mathcal{O}_{\mathbf{f}}}_{S \text{ factors}}))) \\ &= \text{Ind}_{\mathfrak{l}^{(S-1)}}^{\mathfrak{l}^{(S)}}(\dots \text{Ind}_{\mathfrak{l}^{(1)}}^{\mathfrak{l}^{(2)}}(\underbrace{\mathcal{O}_0 \times \dots \times \mathcal{O}_0}_{S-1 \text{ factors}} \times \mathcal{O}_{\mathbf{d}^{(1)}})) \\ &\quad \vdots \\ &= \text{Ind}_{\mathfrak{l}^{(S-1)}}^{\mathfrak{l}^{(S)}}(\mathcal{O}_0 \times \mathcal{O}_{\mathbf{d}^{(S-1)}}) \\ &= \text{Ind}_{\mathfrak{l}^{(S)}}^{\mathfrak{g}'^{(S)}}(\mathcal{O}_0 \times \mathcal{O}_{\mathbf{d}^{(S-1)}}), \end{aligned}$$

because  $\mathfrak{l}^{(S)} \simeq \mathfrak{g}'^{(S)} \simeq \mathfrak{g}_{\varepsilon}$  and  $\tilde{\mathfrak{l}}^{(S)} \simeq \mathfrak{l}^{(S-1)}$ . The definition of  $\mathbf{d}^{(S)}$  thus implies  $\mathbf{d} = \mathbf{d}^{(S)}$ .

It remains to compute the number  $z(\mathbf{d}^{(S)})$ . Prove by induction on  $p \in \{0, \dots, S\}$ , that  $z(\mathbf{d}^{(p)}) = p$  and that, for all  $i < i_{p+1}$ :

$$\left\{ \begin{array}{l} \text{(i)'} \quad 0 \leq d_{i+1}^{(p)} \leq d_i^{(p)} \leq d_{i+1}^{(p)} + 1, \\ \text{(ii)'} \quad \#\{j \mid d_j^{(p)} = i\} \neq 2, \text{ if } \varepsilon(-1)^i = -1. \end{array} \right.$$

Then we will deduce the expected result from the  $p = S$  case, since the center of  $\mathfrak{l}$  has dimension  $S$ .

$p = 0$ : since  $\mathcal{O}_{\mathbf{f}}$  is rigid in  $\mathfrak{g}'$ , it follows from Corollary 2.8 that  $z(\mathbf{f}) = z(\mathbf{d}^{(0)}) = 0$  and that the conditions (i)' and (ii)' hold.

$(p-1) \Rightarrow p$ : suppose that, for all  $k \in \{0, \dots, p-1\}$ ,  $z(\mathbf{d}^{(k)}) = k$  and that conditions (i)' and (ii)' holds for  $i < i_{k+1}$ , for some  $p \in \{1, \dots, S\}$ . We have to prove first that  $z(\mathbf{d}^{(p)}) = p$ . According to relation (2), it is enough to prove that  $\mathbf{d}^{(p-1)}$  is deduced from  $\mathbf{d}^{(p)}$  by an elementary transformation, since  $z(\mathbf{d}^{(p-1)}) = p-1$  by the induction hypothesis. As  $\mathfrak{l}^{(p)}$  is a maximal Levi subalgebra of  $\mathfrak{g}^{(p)}$ , we can apply Proposition 2.6 to

$$\mathcal{O}_{\mathbf{d}^{(p)}} = \text{Ind}_{\mathfrak{l}^{(p)}}^{\mathfrak{g}^{(p)}}(\mathcal{O}_0 \times \mathcal{O}_{\mathbf{d}^{(p-1)}}).$$

With the notations of Proposition 2.6, we have:

$$\mathbf{d}^{(p)} = \widetilde{(\mathbf{d}^{(p-1)})_X}.$$

By the induction hypothesis, for all  $i < i_p$ , conditions (i)' and (ii)' hold for  $\mathbf{d}^{(p-1)}$ . Consequently the smallest integer  $l$  such that one of the situations a) or b) of Step 1 happens in  $\mathbf{d}^{(p)}$  is equal to  $i_p$ , because  $\mathfrak{l}^{(p)}$  is of type  $(i_p; R_p - 2i_p)$ . We distingue two cases: either  $\widetilde{(\mathbf{d}^{(p-1)})_X}$  equals to  $\widetilde{\mathbf{d}^{(p-1)}}$  or not. We easily check that in both situations,  $\mathbf{d}^{(p-1)}$  is deduced from  $\widetilde{(\mathbf{d}^{(p-1)})_X}$  by an elementary transformation. Moreover, for all  $i < i_{p+1}$ , conditions (i)' and (ii)' hold for  $\mathbf{d}^{(p)}$  because  $i_{p+1} \leq i_p$ .

By induction, for all  $p = 1, \dots, S$ ,  $z(\mathbf{d}^{(p)}) = p$  and conditions (i)' and (ii)' hold, for all  $i < i_{p+1}$ . In particular, with  $p = S$ , we have:  $z(\mathbf{d}^{(S)}) = z(\mathbf{d}) = S = \dim \mathfrak{z}_{\mathfrak{g}}(\mathfrak{l})$ .  $\square$

We are now ready to compute the dimension of the subsets  $\mathfrak{g}_{\varepsilon}^{(m)}$ . Recall that  $m(\mathbf{d})$  is given by the formulas (3).

**Theorem 2.11.** — *Let  $m \in \mathbb{N}$ . If  $2m \in \mathbf{N}_{\mathfrak{g}_{\varepsilon}}$ , then*

$$\dim \mathfrak{g}_{\varepsilon}^{(m)} = 2m + \max_{\substack{\mathbf{d} \in \mathcal{P}_{\varepsilon}(\mathbb{N}) \\ m(\mathbf{d})=m}} z(\mathbf{d}).$$

*Otherwise,  $\mathfrak{g}_{\varepsilon}^{(m)}$  is an empty set.*

*Proof.* — Let  $m \in \mathbb{N}$  such that  $2m \in \mathbf{N}_{\mathfrak{g}_{\varepsilon}}$ . Let  $\mathfrak{S}$  be a sheet of  $\mathfrak{g}_{\varepsilon}^{(m)}$  associated to the pair  $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$ , where  $\mathfrak{l}$  is a Levi subalgebra of  $\mathfrak{g}_{\varepsilon}$  and  $\mathcal{O}_{\mathfrak{l}}$  a nilpotent rigid orbit in  $\mathfrak{l}$ . According to Proposition 1.9, the unique nilpotent orbit  $\mathcal{O}_{\mathbf{d}}$  contained in  $\mathfrak{S}$  is  $\text{Ind}_{\mathfrak{l}}^{\mathfrak{g}_{\varepsilon}}(\mathcal{O}_{\mathfrak{l}})$ . In particular,  $\text{Ind}_{\mathfrak{l}}^{\mathfrak{g}_{\varepsilon}}(\mathcal{O}_{\mathfrak{l}})$  has dimension  $2m$ , and so  $m = m(\mathbf{d})$ . Proposition 2.10 implies that the center of  $\mathfrak{l}$  has dimension  $z(\mathbf{d})$ . As a result, we deduce from Proposition 1.13 the expected relation. The second assertion results from Proposition 1.14.  $\square$

Tables 3, 3 and 5 provide, for each  $\mathbf{d} \in \mathcal{P}_\varepsilon(N)$ , the integers  $2m(\mathbf{d})$  and  $z(\mathbf{d})$ , and then the dimensions of the sets  $\mathfrak{g}_\varepsilon^{(m(\mathbf{d}))}$ , for  $N = 3$  with  $\varepsilon = \pm 1$  and for  $N = 6$  with  $\varepsilon = 1$ .

$2m \in \mathbf{N}_\mathfrak{g}$	$\mathbf{d} \in \mathcal{P}(6)$ such that $2m(\mathbf{d}) = 2m$	$z(\mathbf{d})$ , for $\mathbf{d} \in \mathcal{P}(6)$	$\dim \mathfrak{g}_\varepsilon^{(m)} = 2m + \max_{m(\mathbf{d})=m} z(\mathbf{d})$
18	[7]	3	21
16	[5, 1 <sup>2</sup> ]	2	18
14	[3 <sup>2</sup> , 1]	1	15
12	[3, 2 <sup>2</sup> ]	1	13
10	[3, 1 <sup>4</sup> ]	1	11
8	[2 <sup>2</sup> , 1 <sup>3</sup> ]	0	8
0	[1 <sup>7</sup> ]	0	0

TABLE 3. Dimensions of the sets  $\mathfrak{g}_\varepsilon^{(m)}$  for  $\mathfrak{so}_7$ .

$2m \in \mathbf{N}_\mathfrak{g}$	$\mathbf{d} \in \mathcal{P}(6)$ such that $2m(\mathbf{d}) = 2m$	$z(\mathbf{d})$ , for $\mathbf{d} \in \mathcal{P}(6)$	$\dim \mathfrak{g}_\varepsilon^{(m)} = 2m + \max_{m(\mathbf{d})=m} z(\mathbf{d})$
18	[6]	3	21
16	[4, 2]	2	18
14	[3 <sup>2</sup> ]	1	15
12	[2 <sup>3</sup> ]	1	13
10	[2 <sup>2</sup> , 1 <sup>2</sup> ]	1	11
6	[2, 1 <sup>4</sup> ]	0	6
0	[1 <sup>6</sup> ]	0	0

TABLE 4. Dimensions of the sets  $\mathfrak{g}_\varepsilon^{(m)}$  for  $\mathfrak{sp}_6$ .

### 3. Computations in the exceptional cases

We suppose in this section that  $\mathfrak{g}$  is a simple exceptional Lie algebra. We intend in this section to explicitly compute the dimensions of all the sheets of  $\mathfrak{g}$ .

$2m \in \mathbf{N}_{\mathfrak{g}}$	$\mathbf{d} \in \mathcal{P}(6)$ such that $2m(\mathbf{d}) = 2m$	$z(\mathbf{d}),$ for $\mathbf{d} \in \mathcal{P}(6)$	$\dim \mathfrak{g}_{\varepsilon}^{(m)} = 2m + \max_{m(\mathbf{d})=m} z(\mathbf{d})$
60	[11, 1]	6	66
58	[9, 3]	5	63
56	[9, 1 <sup>3</sup> ] [7, 5]	4 4	60
54	[7, 3, 1 <sup>2</sup> ] [6 <sup>2</sup> ]	4 3	58
52	[5 <sup>2</sup> , 1 <sup>2</sup> ] [7, 2 <sup>2</sup> , 1]	3 2	55
50	[5, 3, 2 <sup>2</sup> ]	2	52
48	[4 <sup>3</sup> ] [4 <sup>2</sup> , 3, 1]	2 2	50
46	[4 <sup>2</sup> , 2 <sup>2</sup> ] [5, 3, 1 <sup>4</sup> ]	2 2	48
44	[4 <sup>2</sup> , 1 <sup>4</sup> ] [5, 2 <sup>2</sup> , 1 <sup>3</sup> ] [3 <sup>4</sup> ]	1 1 1	45
42	[3 <sup>3</sup> , 1 <sup>3</sup> ]	1	43
40	[3 <sup>2</sup> , 2 <sup>2</sup> , 1 <sup>2</sup> ]	1	41
36	[5, 1 <sup>7</sup> ] [3, 2 <sup>4</sup> , 1]	2 0	38
34	[3 <sup>2</sup> , 1 <sup>6</sup> ]	1	35
32	[3, 2 <sup>2</sup> , 1 <sup>5</sup> ]	0	32
30	[2 <sup>6</sup> ]	1	31
28	[2 <sup>4</sup> , 1 <sup>4</sup> ]	0	28
18	[2 <sup>2</sup> , 1 <sup>8</sup> ]	0	18
0	[1 <sup>12</sup> ]	0	0

TABLE 5. Dimensions of the sets  $\mathfrak{g}_{\varepsilon}^{(m)}$  for  $\mathfrak{so}_{12}$ .

For each pair  $(S, p)$  consisting of a subset  $S$  of  $\Pi$  and the dimension  $p$  of a nilpotent orbit of the Levi subalgebra  $\mathfrak{l}_S = \mathfrak{h} \oplus \left( \bigoplus_{\alpha \in \Delta^S} \mathfrak{g}^{\alpha} \right)$ , set

$$d_{S,p} := \dim \mathfrak{g} - \operatorname{rk} \mathfrak{g} - 2\#\Delta_+^S + p.$$

According to Lemma 1.8 (i),  $d_{S,p}$  is the dimension of the nilpotent orbit induced by any nilpotent orbit of dimension  $p$  in  $\mathfrak{l}_S$ . Then, by Lemma 1.3, the dimension of a  $G$ -Jordan class with data  $(\mathfrak{l}_S, \mathcal{O})$ , for any nilpotent orbit  $\mathcal{O}$  in  $\mathfrak{l}_S$  of dimension  $p$ , is

$$(\dim \mathfrak{g} - \operatorname{rk} \mathfrak{g} - 2\#\Delta_+^S + p) + (\operatorname{rk} \mathfrak{g} - \#S),$$

since the center of  $\mathfrak{l}_S$  has dimension  $\operatorname{rk} \mathfrak{g} - \#S$ . In addition, by Proposition 1.12, this dimension corresponds to the dimension of a sheet of  $\mathfrak{g}^{(m)}$ , where  $2m = d_{S,p}$ , if and only if  $p$  is the dimension of a rigid nilpotent orbit in  $\mathfrak{l}_S$ .

When all the simple factors of  $\mathfrak{l}_S$  are of classical type, then the set  $\mathbf{N}_S^{\text{rigid}}$  is given by Table 2. In [24], Chapter II, the rigid nilpotent orbits of the exceptional types are listed from Elashvili's computations. We observe that the present method "almost" allows to recover in a simpler way the dimensions of the rigid nilpotent orbits in the exceptional types. More precisely, if for some  $m \in \mathbf{N}_{\mathfrak{g}}$ , there is no pair  $(S, p)$ , with  $S \subset \Pi$ ,  $S \neq \Pi$ , and  $p \in \mathbf{N}_{\mathfrak{l}_S}^{\text{rigid}}$  such that  $d_{S,p} = m$ , then  $m \in \mathbf{N}_{\mathfrak{g}}^{\text{rigid}}$ . Unfortunately, the converse is not true in general. Indeed, a non-rigid nilpotent orbit can be induced in two different ways. Nevertheless, if there is only one nilpotent orbit of dimension  $m$  then we can decide if  $m$  belongs to  $\mathbf{N}_{\mathfrak{g}}^{\text{rigid}}$  or not. Whatever the case, for what we intend to do, this approach is sufficient because computing the dimension of a  $G$ -Jordan class possibly not dense in a sheet doesn't affect the final result: in that case, the corresponding dimension will not appear as a dimension of some  $\mathfrak{g}^{(m)}$ . To precisely know the  $G$ -Jordan classes which are dense in a sheet is useful only to simplify the computations. For completeness, we present here the computations without referring to [24]. Noting that our conclusions agree with the data of [24].

In the Tables 6, 8, 10, 12, 13 and 14, we present the necessary data for each exceptional type. For each subset  $S \in \Pi$ , we give the type of  $\Delta^S$ , the cardinality of  $S$ , the cardinality of  $\Delta_+^S$ , the set  $\mathbf{N}_S^{\text{rigid}}$  when we are able to compute it, and a set possibly bigger otherwise, the set  $d_{S,p}$ , for  $p$  running through the 4<sup>th</sup> column, and at last the set of the numbers  $d_{S,p} + (\operatorname{rk} \mathfrak{g} - \#S)$ , for  $p$  running through the 4<sup>th</sup> column. In the 4<sup>th</sup> column, we add a question mark to identify the dimensions which might not belong to  $\mathbf{N}_S^{\text{rigid}}$ . The dimension of the varieties  $\mathfrak{g}^{(m)}$  are given Tables 7, 9, 11, 15 and 16.

*Remark 3.* — *A short look at the tables of [24] shows that, each time there is a question mark in Tables 6, 8, 10, 12, 13 and 14, the corresponding orbit is actually not rigid.*

$\Delta^S$	$\#S$	$\#\Delta_+^S$	$\mathbf{N}_S^{\text{rigid}}$	$d_{S,p} = \dim \mathfrak{g} - \text{rk} \mathfrak{g} - 2(\#\Delta_+^S) + p$ , for $p \in \mathbf{N}_S^{\text{rigid}}$	$d_{S,p} + (\text{rk} \mathfrak{g} - \#S)$ , for $p \in \mathbf{N}_S^{\text{rigid}}$
$\emptyset$	0	0	0	12	14
$A_1$	1	1	0	10	11
$G_2$	2	6	8 6 0	8 6 0	8 6 0

TABLE 6. Dimension of the sheets for  $\mathfrak{g} = G_2$  ( $\dim \mathfrak{g} = 14$ ,  $\text{rk} \mathfrak{g} = 2$ )

$2m \in \mathbf{N}_{\mathfrak{g}}$	12	10	8	6	0
$\dim \mathfrak{g}^{(m)}$	14	11	8	6	0

TABLE 7. Dimensions of the subsets  $\mathfrak{g}^{(m)}$  for  $G_2$

$\Delta^S$	$\#S$	$\#\Delta_+^S$	$\mathbf{N}_S^{\text{rigid}}$	$d_{S,p}$ , for $p \in \mathbf{N}_S^{\text{rigid}}$	$d_{S,p} + (\text{rk} \mathfrak{g} - \#S)$ , for $p \in \mathbf{N}_S^{\text{rigid}}$
$\emptyset$	0	0	0	48	52
$A_1$	1	1	0	46	49
$A_1 \times A_1$	2	2	0	44	46
$A_2$	2	3	0	42	44
$B_2$	2	4	4 0	44 40	46 42
$A_1 \times A_2$	3	4	0	40	41
$B_3$	3	9	8 0	38 30	39 31
$C_3$	3	9	6 0	36 30	37 31
$F_4$	4	24	36 34 30? 28 22 16 0	36 34 30? 28 22 16 0	36 34 30? 28 22 16 0

TABLE 8. Dimension of the sheets for  $\mathfrak{g} = F_4$  ( $\dim \mathfrak{g} = 52$ ,  $\text{rk} \mathfrak{g} = 4$ )

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$2m \in \mathbf{N}_{\mathfrak{g}}$	48	46	44	42	40	38	36	34	30	28	22	16	0
$\dim \mathfrak{g}^{(m)}$	52	49	46	44	42	39	37	34	31	28	22	16	0

TABLE 9. Dimensions of the subsets  $\mathfrak{g}^{(m)}$  for  $F_4$ 

$\Delta^S$	$\#S$	$\#\Delta_+^S$	$\mathbf{N}_S^{\text{rigid}}$	$d_{S,p}$ , for $p \in \mathbf{N}_S^{\text{rigid}}$	$d_{S,p} + (\text{rk} - \#S)$ , for $p \in \mathbf{N}_S^{\text{rigid}}$
$\emptyset$	0	0	0	72	78
$A_1$	1	1	0	70	75
$A_1 \times A_1$	2	2	0	68	72
$A_2$	2	3	0	66	70
$A_1 \times A_1 \times A_1$	3	3	0	66	69
$A_2 \times A_1$	3	4	0	64	67
$A_3$	3	6	0	60	63
$A_2 \times A_1 \times A_1$	4	5	0	62	64
$A_2 \times A_2$	4	6	0	60	62
$A_3 \times A_1$	4	7	0	58	60
$A_4$	4	10	0	52	54
$D_4$	4	12	16 10 0	74 68 48	76 70 50
$A_5$	5	15	0	42	43
$A_4 \times A_1$	5	11	0	54	55
$D_5$	5	20	24 14 0	56 46 32	57 47 33
$E_6$	6	56	64? 60? 54 40 22 0	64? 60? 54 40 22 0	64? 60? 54 40 22 0

TABLE 10. Dimension of the sheets for  $\mathfrak{g} = E_6$  ( $\dim \mathfrak{g} = 78$ ,  $\text{rk} \mathfrak{g} = 6$ )

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$2m \in \mathbf{N}_{\mathfrak{g}}$	72	70	68	66	64	62	60	58	56	54	52	50	48	46	42	40	32	22	0
$\dim \mathfrak{g}^{(m)}$	78	75	72	70	67	64	63	60	57	55	54	50	50	47	43	40	33	22	0

TABLE 11. Dimensions of the subsets  $\mathfrak{g}^{(m)}$  for  $E_6$ 

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$\Delta^S$	$\#S$	$\#\Delta_+^S$	$\mathbf{N}_S^{\text{rigid}}$	$d_{S,p}$ , for $p \in \mathbf{N}_S^{\text{rigid}}$	$d_{S,p} + (\text{rk} - \#S)$ , for $p \in \mathbf{N}_S^{\text{rigid}}$
$\emptyset$	0	0	0	126	133
$A_1$	1	1	0	124	130
$A_1 \times A_1$	2	2	0	122	127
$A_2$	2	3	0	120	125
$A_1 \times A_1 \times A_1$	3	3	0	120	124
$A_2 \times A_1$	3	4	0	118	122
$A_3$	3	6	0	114	118
$A_1 \times A_1 \times A_1 \times A_1$	4	4	0	118	121
$A_2 \times A_1 \times A_1$	4	5	0	116	119
$A_2 \times A_2$	4	6	0	114	117
$A_3 \times A_1$	4	7	0	112	115
$A_4$	4	10	0	106	109
$D_4$	4	12	16 10 0	118 112 102	121 115 105
$A_2 \times A_1 \times A_1 \times A_1$	5	6	0	114	116
$A_2 \times A_2 \times A_1$	5	7	0	112	114
$A_3 \times A_1 \times A_1$	5	8	0	110	112
$A_3 \times A_2$	5	9	0	108	110
$A_4 \times A_1$	5	11	0	104	106
$D_4 \times A_1$	5	13	16 10 0	116 110 100	118 112 102
$A_5$	5	15	0	96	98
$D_5$	5	20	24 14 0	110 100 86	112 102 88
$A_3 \times A_2 \times A_1$	6	10	0	106	107
$A_4 \times A_2$	6	13	0	100	101
$A_5 \times A_1$	6	16	0	94	95
$A_6$	6	21	0	84	85
$D_5 \times A_1$	6	21	24 14 0	108 98 84	109 99 85
$D_6$	6	30	36 32 28 18 0	102 98 94 84 66	103 99 95 85 67
$E_6$	6	36	64? 60? 54 40 22 0	118? 114? 108 94 76 54	119? 115? 109 95 77 55
$E_7$	7	63	120? 118? 114? 112? 110? 108? 106? 100? 96? 92 90 84? 82 70 64 52 34 0	120? 118? 114? 112? 110? 108? 106? 100 ? 96? 92 90 84? 82 70 64 52 34 0	120? 118? 114? 112? 110? 108? 106? 100 ? 96? 92 90 84? 82 70 64 52 34 0

TABLE 12. Dimension of the sheets for  $\mathfrak{g} = E_7$  ( $\dim \mathfrak{g} = 133$ ,  $\text{rk} \mathfrak{g} = 7$ )

$\Delta^S$	$\#S$	$\#\Delta_+^S$	$\mathbf{N}_S^{\text{rigid}}$	$d_{S,p}$ , for $p \in \mathbf{N}_S^{\text{rigid}}$	$d_{S,p} + (\text{rk} - \#S)$ , for $p \in \mathbf{N}_S^{\text{rigid}}$
$\emptyset$	0	0	0	240	248
$A_1$	1	1	0	238	245
$A_1 \times A_1$	2	2	0	236	242
$A_2$	2	3	0	234	240
$A_1 \times A_1 \times A_1$	3	3	0	234	239
$A_2 \times A_1$	3	4	0	232	237
$A_3$	3	6	0	228	233
$A_1 \times A_1 \times A_1 \times A_1$	4	4	0	232	236
$A_2 \times A_1 \times A_1$	4	5	0	230	234
$A_2 \times A_2$	4	6	0	228	232
$A_3 \times A_1$	4	7	0	226	230
$A_4$	4	10	0	220	224
$D_4$	4	12	16 10 0	232 226 216	236 230 220
$A_2 \times A_1 \times A_1 \times A_1$	5	6	0	228	231
$A_2 \times A_2 \times A_1$	5	7	0	226	229
$A_3 \times A_1 \times A_1$	5	8	0	224	227
$A_3 \times A_2$	5	9	0	222	225
$A_4 \times A_1$	5	11	0	218	221
$D_4 \times A_1$	5	13	16 10 0	230 224 214	233 227 217
$A_5$	5	15	0	210	213
$D_5$	5	20	24 14 0	224 214 200	227 217 203

TABLE 13. Dimension of the sheets for  $\dim \mathfrak{g} = E_8$  ( $\dim \mathfrak{g} = 248$ ,  $\text{rk} = 8$ ) I

$\Delta^S$	$\#S$	$\#\Delta_+^S$	$\mathbf{N}_S^{\text{rigid}}$	$d_{S,p}$ , for $p \in \mathbf{N}_S^{\text{rigid}}$	$d_{S,p} + (\text{rk} \mathfrak{g} - \#S)$ , for $p \in \mathbf{N}_S^{\text{rigid}}$
$A_2 \times A_2 \times A_1 \times A_1$	6	8	0	224	226
$A_3 \times A_2 \times A_1$	6	10	0	220	222
$A_3 \times A_3$	6	12	0	216	218
$A_4 \times A_2$	6	13	0	214	216
$D_4 \times A_2$	6	15	16 10 0	226 220 210	228 222 212
$A_5 \times A_1$	6	16	0	208	210
$A_6$	6	21	0	198	200
$D_5 \times A_1$	6	21	24 14 0	222 212 198	224 214 200
$D_6$	6	30	36 32 28 18 0	216 212 208 198 180	218 214 210 200 182
$E_6$	6	36	64? 60? 54 40 22 0	232? 228? 222 208 190 168	234? 230? 224 210 192 170
$D_4 \times A_3$	7	16	0	208	209
$A_6 \times A_1$	7	22	0	196	197
$D_5 \times A_2$	7	23	24 14 0	218 208 194	219 209 195
$A_7$	7	28	0	184	185
$E_6 \times A_1$	7	37	64? 60? 54 40 22 0	230? 226? 220 206 188 166	231? 227? 221 207 189 167
$D_7$	7	42	58 48 40 36 22 0	214 204 196 192 178 156	215 205 197 193 179 157
$E_7$	7	63	120? 118? 114? 112? 110? 108? 106? 100? 96? 92 90 84? 82 70 64 52 34 0	234? 232? 228? 226? 224? 222? 220? 214? 210? 206 204 198? 196 184 178 166 148 114	235? 233? 229? 227? 225? 223? 229? 215? 211? 207 205 199? 197 185 179 167 149 115
$E_8$	8	120	232? 228? 226? 224? 222? 220? 218? 216? 214? 212? 210? 208? 204? 202 200 198? 196 188 184? 182 176 172 164 162 154 146 136 128 112 92 58 0	232? 228? 226? 224? 222? 220? 218? 216? 214? 212? 210? 208? 204? 202 200 198? 196 188 184? 182 176 172 164 162 154 146 136 128 112 92 58 0	232? 228? 226? 224? 222? 220? 218? 216? 214? 212? 210? 208? 204? 202 200 198? 196 188 184? 182 176 172 164 162 154 146 136 128 112 92 58 0

TABLE 14. Dimension of the sheets for  $\dim \mathfrak{g} = E_8$  II

$2m \in \mathbf{N}_{\mathfrak{g}}$	126	124	122	120	118	116	114	112	110	108	106	104	102	100	98
$\dim \mathfrak{g}^{(m)}$	133	130	127	125	122	119	118	115	112	110	109	106	105	102	99
$2m \in \mathbf{N}_{\mathfrak{g}}$	96	94	92	90	86	84	82	76	70	66	64	54	52	34	0
$\dim \mathfrak{g}^{(m)}$	98	95	92	90	88	85	82	77	70	67	64	55	52	34	0

TABLE 15. Dimensions of the subsets  $\mathfrak{g}^{(m)}$  for  $E_7$ 

$2m \in \mathbf{N}_{\mathfrak{g}}$	240	238	236	234	232	230	228	226	224	222	220	218	216	214	212	210	208
$\dim \mathfrak{g}^{(m)}$	248	245	242	240	237	234	233	230	227	225	224	221	220	217	214	213	210
$2m \in \mathbf{N}_{\mathfrak{g}}$	206	204	202	200	198	196	194	192	190	188	184	182	180	178	176	172	168
$\dim \mathfrak{g}^{(m)}$	207	205	202	203	200	197	195	193	192	189	185	182	182	179	176	172	170
$2m \in \mathbf{N}_{\mathfrak{g}}$	166	164	162	156	154	148	146	136	128	114	112	92	58	0			
$\dim \mathfrak{g}^{(m)}$	167	164	162	157	154	149	146	136	128	115	112	92	58	0			

TABLE 16. Dimensions of the subsets  $\mathfrak{g}^{(m)}$  for  $E_8$