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CORRIGENDUM TO "ON THE DIMENSION OF THE SHEETS OF A REDUCTIVE LIE ALGEBRA"

ANNE MOREAU

ABSTRACT. This note is a corrigendum to [M08]. As it has been recently pointed out to me by Alexander Premet, [M08, Remark 3.12] is incorrect. We explain in this note the impacts of that error in [M08], and amend certain of its statements. In particular, we verify that the statement of [M08, Theorem 3.13] remains correct in spite of this error.

1. INTRODUCTION

Let \mathfrak{g} be a complex simple Lie algebra and G its adjoint group. We investigate in [M08] the dimension of the subsets, for $m \in \mathbb{N}$,

$$\mathfrak{g}^{(m)} := \{x \in \mathfrak{g} \mid \dim(Gx) = 2m\},$$

where Gx denotes the adjoint orbit of $x \in \mathfrak{g}$. The irreducible components of the subsets $\mathfrak{g}^{(m)}$ are called the *sheets* of \mathfrak{g} , [BK79, B81]. Thus, for any $m \in \mathbb{N}$,

$$(1) \quad \dim \mathfrak{g}^{(m)} = \max\{\dim \mathcal{S} ; \mathcal{S} \subset \mathfrak{g}^{(m)}\},$$

where \mathcal{S} runs through all sheets contained in $\mathfrak{g}^{(m)}$. The sheets are known to be parameterized by the pairs $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$, up to G -conjugacy class, consisting of a Levi subalgebra \mathfrak{l} of \mathfrak{g} and a rigid nilpotent orbit $\mathcal{O}_{\mathfrak{l}}$ in \mathfrak{l} , cf. [B81]. This parametrization enables to write the dimension of a sheet \mathcal{S} associated with a pair $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$ as the sum of the dimension of the center of \mathfrak{l} and the dimension of the unique nilpotent orbit contained in \mathcal{S} , see e.g. [M08, Proposition 2.11].

In the classical case, formulas for $\mathfrak{g}^{(m)}$ are given in [M08, Theorems 3.3 and 3.13] in term of partitions associated with nilpotent elements of \mathfrak{g} . As it has been recently pointed out by Alexander Premet, Remark 3.12 in [M08] which claims that "*in the classical case, the dimension of a sheet containing a given nilpotent orbit does not depend on the choice of a sheet containing it*" is incorrect. We give here some counter-examples (cf. Examples 3.1 and 3.2; see also [PT12, Remark 4]). This is true only for the type **A** where each nilpotent element belongs to only one sheet. The error stems from the proof of [M08, Proposition 3.11]; see Section 3 for explanations. As a consequence, the proof of [M08, Theorems 3.13], partly based on [M08, Proposition 3.11], is incorrect too. However its statement remains true. This can be shown through a recent work of Premet and Topley, [PT12]. In more details, another formula for $\mathfrak{g}^{(m)}$ in term of partitions can be traced out from [PT12, Corollary 9] and the

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equality (1). In this note, we verify (cf. Theorems 2.10) that the Premet-Topley formula for $\mathfrak{g}^{(m)}$ coincides with the one of [M08, Theorem 3.13].

The note is organized as follows.

In Section 2, we recall some definitions and results of [PT12] and show that the statement of [M08, Theorem 3.13] is correct in spite of the error in [M08, Proposition 3.11], see Theorem 2.10(ii). In Section 3, we precisely pin down the error in the proof [M08, Proposition 3.11] and describe the impacts of that error in [M08]. As a conclusion, we list in Section 4 all corrections which have to be taken into account in [M08].

Since the corrections in [M08] only concern the types **B**, **C** and **D**, we assume for the remaining of the note that \mathfrak{g} is either $\mathfrak{so}(N)$ or $\mathfrak{sp}(N)$, with $N \geq 2$, and ε is 1 or -1 depending on whether $\mathfrak{g} = \mathfrak{so}(N)$ or $\mathfrak{sp}(N)$. Following the notations of [M08] (or [PT12]), we denote by $\mathcal{P}_\varepsilon(N)$ the set of partitions of N associated with the nilpotent elements of \mathfrak{g} . For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathcal{P}_\varepsilon(N)$, we denote by $e(\lambda)$ the corresponding nilpotent element of \mathfrak{g} whose Jordan block sizes are $\lambda_1, \dots, \lambda_n$. We will always assume that $\lambda_1 \geq \dots \geq \lambda_n$.

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2. THE MAIN RESULT

For the convenience of the reader, we recall here all the necessary definitions and results of [PT12]. Given a partition $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathcal{P}_\varepsilon(N)$ we set,

$$\Delta(\lambda) := \{1 \leq i < n ; \varepsilon(-1)^{\lambda_1} = \varepsilon(-1)^{\lambda_{i+1}} = -1, \lambda_{i-1} \neq \lambda_i \geq \lambda_{i+1} \neq \lambda_{i+2}\}.$$

Our convention is that $\lambda_0 = 0$ and $\lambda_i = 0$ for all $i > n$. Recall the following result of Kempken and Spaltenstein (also recalled in [M08] and [PT12]):

Theorem 2.1 ([K83, S82]). *Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathcal{P}_\varepsilon(N)$. Then $e(\lambda)$ is rigid if and only if*

- $\lambda_i - \lambda_{i+1} \in \{0, 1\}$ for all $1 \leq i \leq n$;
- the set $\{i \in \Delta(\lambda) ; \lambda_i = \lambda_{i+1}\}$ is empty.

Denote by $\mathcal{P}_\varepsilon^*(N)$ the set of $\lambda \in \mathcal{P}_\varepsilon(N)$ such that $e(\lambda)$ is rigid. We call the elements of $\mathcal{P}_\varepsilon^*(N)$ the *rigid partitions*. We first introduce the notion of *admissible sequences*, see [PT12, §3.1]. This is an extended version of the algorithm described in [M08] which takes $\lambda \in \mathcal{P}_\varepsilon(N)$ and returns an element of $\mathcal{P}_\varepsilon^*(N)$ compatible for the induction process of nilpotent orbits.

Let \mathbf{i} be a finite sequence of integers between 1 and n . The procedure of [PT12] is as follows: the algorithm commences with input $\lambda = \lambda^{\mathbf{i}} \in \mathcal{P}_\varepsilon(N)$ where $\mathbf{i} = \emptyset$ is the empty sequence. At the l^{th} iteration, the algorithm takes $\lambda^{\mathbf{i}} \in \mathcal{P}_\varepsilon(N - 2 \sum_{j=1}^{l-1} i_j)$ where $\mathbf{i} = (i_1, \dots, i_{l-1})$ and returns $\lambda^{\mathbf{i}'} \in \mathcal{P}_\varepsilon(N - 2 \sum_{j=1}^l i_j)$ where $\mathbf{i}' = (i_1, \dots, i_{l-1}, i_l)$ for some i_l . If the output $\lambda^{\mathbf{i}'}$ is a rigid partition then the algorithm terminates after the l^{th} iteration with output $\lambda^{\mathbf{i}'}$. We shall now explicitly describe the l^{th} iteration of the algorithm. If after the $(l-1)^{\text{th}}$ iteration

the input $\lambda^{\mathbf{i}}$ is not rigid then the algorithm behaves as follows. Let i_l denote any index in the range $1 \leq i \leq n$ such that either of the following case occur:

Case 1: $\lambda_{i_l}^{\mathbf{i}} \geq \lambda_{i_l+1}^{\mathbf{i}} + 2$;

Case 2: $i_l \in \Delta(\lambda^{\mathbf{i}})$ and $\lambda_{i_l}^{\mathbf{i}} = \lambda_{i_l+1}^{\mathbf{i}}$.

Note that no integer i_l will fulfill both criteria. If $\mathbf{i} = (i_1, \dots, i_{l-1})$ then define $\mathbf{i}' = (i_1, \dots, i_{l-1}, i_l)$. For Case 1 the algorithm has output

$$\lambda^{\mathbf{i}'} = (\lambda_1^{\mathbf{i}} - 2, \lambda_2^{\mathbf{i}} - 2, \dots, \lambda_{i_l}^{\mathbf{i}} - 2, \lambda_{i_l+1}^{\mathbf{i}}, \dots, \lambda_n^{\mathbf{i}})$$

whilst for Case 2 the algorithm has output

$$\lambda^{\mathbf{i}'} = (\lambda_1^{\mathbf{i}} - 2, \lambda_2^{\mathbf{i}} - 2, \dots, \lambda_{i_l-1}^{\mathbf{i}} - 2, \lambda_{i_l}^{\mathbf{i}} - 1, \lambda_{i_l+1}^{\mathbf{i}} - 1, \lambda_{i_l+2}^{\mathbf{i}}, \dots, \lambda_n^{\mathbf{i}}).$$

Due to its definition and the classification of rigid partitions the above algorithm certainly terminates after a finite number of steps.

Definition 2.2 ([PT12, §3.1]). We say that a sequence $\mathbf{i} = (i_1, \dots, i_l)$ is an *admissible sequence* for λ if Case 1 or Case 2 occurs at the point i_k for the partition $\lambda^{(i_1, \dots, i_{k-1})}$ for each $k = 1, \dots, l$. An admissible sequence \mathbf{i} for λ is be called a *maximal admissible sequence* for λ if neither Case 1 nor Case 2 occurs for any index i between 1 and n for the partition $\lambda^{\mathbf{i}}$. By convention the empty sequence is admissible for any $\lambda \in \mathcal{P}_\varepsilon(N)$.

As observed in [PT12, Lemma 6], if \mathbf{i} is an admissible sequence for λ , then \mathbf{i} is maximal admissible if and only if $\lambda^{\mathbf{i}}$ is a rigid partition. We will denote by $|\mathbf{i}| := l$ the length of an admissible sequence for λ .

Definition 2.3. The algorithm as described in [M08] corresponds to the special case where in the above algorithm, we define at each step i_l to be the smallest integer which fulfills one the Case 1 or Case 2 criteria, and $\lambda^{\mathbf{i}}$ is rigid. In the sequel, we will refer the so obtained maximal admissible sequence for λ to as the *canonical maximal admissible sequence* for λ and we denote it by \mathbf{i}^0 . Then we set

$$z_M(\lambda) := |\mathbf{i}^0|.$$

Remark. The integer $z_M(\lambda)$ corresponds to the integer $z(\lambda)$ of [M08].

Definition 2.4 ([PT12, Definition 1]). If $i \in \Delta(\lambda)$ then the pair $(i, i+1)$ is called a *2-step* of λ . If $i > 1$ and $(i, i+1)$ is a 2-step of λ then λ_{i-1} and λ_{i+2} are referred to as the *boundary* of $(i, i+1)$. If $1 \in \Delta(\lambda)$ then λ_3 is referred to as the boundary of $(1, 2)$ (if $n = 2$ then $\lambda_3 = 0$ by convention).

We observe that $\Delta(\lambda)$ is the set of 2-steps of λ , and by $|\Delta(\lambda)|$ its cardinality.

Definition 2.5 ([PT12, §3.2]). If $i \in \Delta(\lambda)$ then we say that the 2-step $(i, i+1)$ has a *good boundary* if λ_1 and the boundary of $(i, i+1)$ have the opposite parity. If the boundary of a 2-step $(i, i+1)$ of λ is not good then we say that it is *bad* and we refer to $(i, i+1)$ as a *bad 2-step*. Note that $(i, i+1)$ is a bad 2-step of λ if and only if either $i > 1$ and $\lambda_{i-1} - \lambda_i \in 2\mathbb{N}$, or $\lambda_{i+1} - \lambda_{i+2} \in 2\mathbb{N}$.

We denote by $\Delta_{\text{bad}}(\lambda)$ the set of bad 2-steps of λ , and by $|\Delta_{\text{bad}}(\lambda)|$ its cardinality.

Definition 2.6 ([PT12, Definition 2]). A sequence $1 \leq i_1 < \dots < i_k < n$ with $k \geq 2$ is called a *2-cluster* of λ whenever $i_j \in \Delta(\lambda)$ and $i_{j+1} = i_j + 2$ for all j . We say that a 2-cluster i_1, \dots, i_k has a *bad boundary* if either of the following conditions holds:

- $\lambda_{i_1-1} - \lambda_{i_1} \in 2\mathbb{N}$;
- $\lambda_{i_k+1} - \lambda_{i_k+2} \in 2\mathbb{N}$.

(if $i_1 = 1$ the first condition should be omitted). A *bad 2-cluster* is one which has a bad boundary, whilst a *good 2-cluster* is one without a bad boundary.

We denote by $\Sigma(\lambda)$ the set of good 2-clusters of λ , and by $|\Sigma(\lambda)|$ its cardinality.

Lemma 2.7 ([PT12, Lemma 11]). *A good 2-cluster is maximal in the sense that it is not a proper subsequence of any 2-cluster.*

Definition 2.8 (Premet-Topley). For any $\lambda \in \mathcal{P}_\varepsilon(\lambda)$, the integer $z_{\text{PT}}(\lambda)$ is defined by the formula:

$$z_{\text{PT}}(\lambda) := s(\lambda) + |\Delta(\lambda)| - |\Delta_{\text{bad}}(\lambda)| + |\Sigma(\lambda)|$$

where

$$s(\lambda) := \sum_{i=1}^n [(\lambda_i - \lambda_{i+1})/2].$$

Remark. The integer $z_{\text{PT}}(\lambda)$ corresponds to the integer $z(\lambda)$ of [PT12].

By [PT12, Theorem 8], we have that

$$(2) \quad z_{\text{PT}}(\lambda) := \max |\mathbf{i}|$$

where the maximum is taken over all admissible sequences for λ . Hence, by [PT12, Corollary 9] and the equality (1) of the introduction, we get:

Theorem 2.9 (Premet-Topley). *For any $m \in \mathbb{N}$, we have*

$$\dim \mathfrak{g}^{(m)} = 2m + \max\{z_{\text{PT}}(\lambda) ; \lambda \in \mathcal{P}_\varepsilon(N) \text{ s.t. } \dim Ge(\lambda) = 2m\}.$$

The main result of this note is:

Theorem 2.10. (i) *For any $\lambda \in \mathcal{P}_\varepsilon(N)$, we have $z_{\text{M}}(\lambda) = z_{\text{PT}}(\lambda)$.*

(ii) *For any $m \in \mathbb{N}$, we have*

$$\dim \mathfrak{g}^{(m)} = 2m + \max\{z_{\text{M}}(\lambda) ; \lambda \in \mathcal{P}_\varepsilon(N) \text{ s.t. } \dim Ge(\lambda) = 2m\}.$$

In other words, the statement of [M08, Theorem 3.13] is correct.

Proof. (ii) is a direct consequence of (i) and Theorem 2.9.

(i) We argue by induction on N (the statement is true for small N). Let $N > 2$ and assume the statement true for any $\lambda \in \mathcal{P}_\varepsilon(N')$, with $1 \leq N' \leq N$, and let $\lambda \in \mathcal{P}_\varepsilon(N)$.

If $\lambda \in \mathcal{P}_\varepsilon^*(N)$, then $z_{\text{PT}}(\lambda) = z_{\text{M}}(\lambda) = 0$ (see Theorem 2.1, Definition 2.2 and equality (2)). So, we can assume that λ is not a rigid partition. In particular, $z_{\text{PT}}(\lambda) > 0$ and $z_{\text{M}}(\lambda) > 0$.

To ease notation, we simply denote here by $\mathbf{i} := \mathbf{i}^0$ the canonical maximal sequence for λ . Then recall that by Definition 2.3, $z_M(\lambda) = |\mathbf{i}|$. Set $\lambda' := \lambda^{(i_1)}$. Clearly, $z_M(\lambda') = z_M(\lambda) - 1$. By the induction hypothesis, we have $z_{PT}(\lambda') = z_M(\lambda')$. Hence, we have to show that:

$$z_{PT}(\lambda') = z_{PT}(\lambda) - 1.$$

Our strategy is to compare the formulas for $z_{PT}(\lambda')$ and $z_{PT}(\lambda)$ given by Definition 2.8. Recall that i_1 is the smallest integer which fulfills one of the Case 1 or Case 2 criteria for λ . First of all, we observe that if $i \in \Delta(\lambda)$ (resp. $i \in \Delta(\lambda')$), then $i \geq i_1$. Indeed, if $i \in \Delta(\lambda)$ and $i < i_1$ (if $i_1 = 1$, it is clear), then either $\lambda_i = \lambda_{i+1}$ and then i fulfills the Case 2 which contradicts the minimality of i_1 , or $\lambda_i - \lambda_{i+1} \in 2\mathbb{N} \setminus \{0\}$ and then i fulfills the Case 1 which contradicts the minimality of i_1 too.

We now consider the two situations Case 1 and Case 2 separately.

Case 1: $\lambda_{i_1} \geq \lambda_{i_1+1} + 2$.

We have,

$$\lambda' = (\lambda_1 - 2, \dots, \lambda_{i_1-1} - 2, \lambda_{i_1} - 2, \lambda_{i_1+1}, \dots, \lambda_n),$$

and

$$\begin{aligned} s(\lambda') &= \sum_{i=1}^{i_1-1} [(\lambda_i - \lambda_{i+1})/2] + [(\lambda_{i_1} - 2 - \lambda_{i_1+1})/2] + \sum_{i=i_1+1}^n [(\lambda_i - \lambda_{i+1})/2] \\ &= s(\lambda) - 1. \end{aligned}$$

Compare now the other terms appearing in Definition 2.8. Note that $i_1 \in \Delta(\lambda)$ (resp. $i_1 \in \Delta_{\text{bad}}(\lambda)$) if and only if $i_1 \in \Delta(\lambda')$ (resp. $i_1 \in \Delta_{\text{bad}}(\lambda')$) since the passing from λ to λ' preserves the parities. For the same reason, i_1 belongs to a good 2-cluster of λ if and only if i_1 belongs to a good 2-cluster of λ' .

Then we discuss two cases depending on whether $i_1 + 1$ is in $\Delta(\lambda)$ or not:

- $i_1 + 1 \in \Delta(\lambda)$.

Once again, we consider two cases:

- * $\lambda_{i_1} - 2 \neq \lambda_{i_1+1}$.

Then $i_1 + 1 \in \Delta(\lambda')$ too. Moreover, $i_1 + 1 \in \Delta_{\text{bad}}(\lambda')$ if and only if $i_1 + 1 \in \Delta_{\text{bad}}(\lambda)$. Hence, we conclude that $|\Delta(\lambda')| = |\Delta(\lambda)|$, $|\Delta_{\text{bad}}(\lambda')| = |\Delta_{\text{bad}}(\lambda)|$ and $|\Sigma(\lambda')| = |\Sigma(\lambda)|$.

- * $\lambda_{i_1} - 2 = \lambda_{i_1+1}$.

Then $i_1 + 1 \in \Delta_{\text{bad}}(\lambda)$ since $\lambda_{i_1} - \lambda_{i_1+1} = 2 \in 2\mathbb{N}$. But $i_1 + 1 \notin \Delta(\lambda')$. Therefore, $|\Delta(\lambda')| = |\Delta(\lambda)| - 1$ and $|\Delta_{\text{bad}}(\lambda')| = |\Delta_{\text{bad}}(\lambda)| - 1$. Moreover, if $i_1 + 1$ belongs to a 2-cluster of λ , then it is bad because $\lambda_{i_1} - \lambda_{i_1+1} \in 2\mathbb{N}$. Hence, we have $|\Sigma(\lambda')| = |\Sigma(\lambda)|$.

- $i_1 + 1 \notin \Delta(\lambda)$.

In this case, note that $i_1 + 1 \notin \Delta(\lambda')$. Hence, we conclude that $|\Delta(\lambda')| = |\Delta(\lambda)|$, $|\Delta_{\text{bad}}(\lambda')| = |\Delta_{\text{bad}}(\lambda)|$ and $|\Sigma(\lambda')| = |\Sigma(\lambda)|$.

Case 2: $i_1 \in \Delta(\lambda)$ and $\lambda_{i_1} = \lambda_{i_1+1}$.

By the minimality condition of i_1 , we have $\lambda_{i_1-1} = \lambda_{i_1} + 1$ (except for $i_1 = 1$, in which case $\lambda_{i_1-1} = 0$ by convention), and so $\lambda_{i_1-2} = \lambda_{i_1-1}$ because $\varepsilon(-1)^{\lambda_{i_1-1}} = 1$. We have

$$\lambda' = (\lambda_1 - 2, \dots, \lambda_{i_1-1} - 2, \lambda_{i_1} - 1, \lambda_{i_1+1} - 1, \lambda_{i_1+2}, \dots, \lambda_n),$$

and

$$\begin{aligned} s(\lambda') &= \sum_{i=1}^{i_1-2} [(\lambda_i - \lambda_{i+1})/2] + \underbrace{[(\lambda_{i_1-1} - \lambda_{i_1} - 1)/2]}_{=0 \text{ since } \lambda_{i_1-1} = \lambda_{i_1} + 1} \\ &\quad + [(\lambda_{i_1} - \lambda_{i_1+1})/2] + [\lambda_{i_1+1} - 1 - \lambda_{i_1+2})/2] + \sum_{i=i_1+1}^n [(\lambda_i - \lambda_{i+1})/2] \\ &= \begin{cases} s(\lambda) - 1 & \text{if } \lambda_{i_1+1} - \lambda_{i_1+2} \in 2\mathbb{N}; \\ s(\lambda) & \text{if } \lambda_{i_1+1} - \lambda_{i_1+2} \notin 2\mathbb{N}. \end{cases} \end{aligned}$$

(If $i_1 = 0$, we start at the second line and we get the same conclusion.) Also, observe that in Case 2, we have

$$|\Delta(\lambda')| = |\Delta(\lambda)| - 1.$$

Indeed, $i_1 \in \Delta(\lambda)$ but $i_1 \notin \Delta(\lambda')$ and for the indexes $i \neq i_1$ we have here the equivalence: $i \in \Delta(\lambda) \iff i \in \Delta(\lambda')$.

We discuss two cases depending on the parity of $\lambda_{i_1+1} - \lambda_{i_1+2}$.

- $\lambda_{i_1+1} - \lambda_{i_1+2} \in 2\mathbb{N}$.

Then $i_1 \in \Delta_{\text{bad}}(\lambda)$. There are two sub-cases depending on whether $i_1 + 2$ is in $\Delta(\lambda)$ or not:

- * $i_1 + 2 \in \Delta(\lambda)$.

Then, $i_1 + 2 \in \Delta_{\text{bad}}(\lambda)$ (since $\lambda_{i_1+1} - \lambda_{i_1+2} \in 2\mathbb{N}$) and $i_1 + 2 \in \Delta(\lambda')$. Once again, there are two sub-cases:

- 1) $i_1 + 2 \notin \Delta_{\text{bad}}(\lambda')$.

Then $|\Delta_{\text{bad}}(\lambda')| = |\Delta_{\text{bad}}(\lambda)| - 2$. Moreover, $(i_1, i_1 + 2)$ is a good 2-cluster of λ . Indeed, $i_1 + 2 \notin \Delta_{\text{bad}}(\lambda')$ implies that $\lambda_{i_1+3} - \lambda_{i_1+4} \notin 2\mathbb{N}$. On the other hand, $\lambda_{i_1-1} - \lambda_{i_1} = 1 \notin 2\mathbb{N}$ (if $i_1 = 1$ the first condition in Definition 2.6 should be omitted). But $(i_1, i_1 + 2)$ is not a 2-cluster of λ' since $i_1 \notin \Delta(\lambda')$. Hence, we have $|\Sigma(\lambda')| = |\Sigma(\lambda)| - 1$ by Lemma 2.7.

- 2) $i_1 + 2 \in \Delta_{\text{bad}}(\lambda')$.

Then $|\Delta_{\text{bad}}(\lambda')| = |\Delta_{\text{bad}}(\lambda)| - 1$. Moreover, the unique 2-cluster of λ which is possibly not a 2-cluster of λ' is $(i_1, i_1 + 2)$ but it is bad in this case. Indeed, $\lambda_{i_1+3} - \lambda_{i_1+4} \in 2\mathbb{N}$ since $i_1 + 2 \in \Delta_{\text{bad}}(\lambda')$ (and $\lambda'_{i_1+1} - \lambda'_{i_1+2} \notin 2\mathbb{N}$). Hence, $|\Sigma(\lambda')| = |\Sigma(\lambda)|$.

- * $i_1 + 2 \notin \Delta(\lambda)$.

Then $|\Delta_{\text{bad}}(\lambda')| = |\Delta_{\text{bad}}(\lambda)| - 1$. Moreover, since $i_1 + 2 \notin \Delta(\lambda)$, then neither i_1 nor $i_1 + 2$ belongs to a 2-cluster for λ . Hence $|\Sigma(\lambda)| = |\Sigma(\lambda')|$.

- $\lambda_{i_1+1} - \lambda_{i_1+2} \notin 2\mathbb{N}$.

In this case, $i_1 \notin \Delta_{\text{bad}}(\lambda)$, $i_1 + 2 \notin \Delta(\lambda)$ and $i_1 + 2 \notin \Delta(\lambda')$. Hence $|\Delta_{\text{bad}}(\lambda')| = |\Delta_{\text{bad}}(\lambda)|$. Moreover, neither i_1 nor $i_1 + 2$ belongs to any 2-cluster. Hence $|\Sigma(\lambda)| = |\Sigma(\lambda')|$.

In all the cases, we can check with the formula of Definition 2.8 that $z_{\text{PT}}(\lambda') = z_{\text{PT}}(\lambda) - 1$ as desired. This concludes the proof of Theorem 2.10. \square

3. COUNTER-EXAMPLES FOR [M08, Proposition 3.11]

From now on, we shall denote by $z(\lambda)$ the integer $z_{\text{M}}(\lambda) = z_{\text{PT}}(\lambda)$ for $\lambda \in \mathcal{P}_{\varepsilon}(N)$. If \mathfrak{l} is a Levi subalgebra of \mathfrak{g} and \mathcal{O}' is a rigid nilpotent orbit of \mathfrak{l} , we denote by $\text{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}')$ the induced nilpotent orbit of \mathfrak{g} from \mathcal{O}' in \mathfrak{l} .

Proposition 3.11 of [M08] asserts that if a nilpotent element e associated with the partition $\lambda \in \mathcal{P}_{\varepsilon}(N)$ is induced from a nilpotent orbit in a Levi subalgebra \mathfrak{l} , then $z(\lambda)$ is equal to the dimension of the center of \mathfrak{l} . This result is actually incorrect. If it were true, it would imply that all the sheets containing e share the same dimension (see [M08, Remark 3.12]). But this is wrong. Below are some counter-examples (see also [PT12, Remark 4]):

Example 3.1. Assume that $\mathfrak{g} = \mathfrak{so}(8)$ and consider the nilpotent element e of \mathfrak{g} with partition $\lambda = (3, 3, 1, 1) \in \mathcal{P}_1(8) \setminus \mathcal{P}_1^*(8)$. The algorithm yields $z(\lambda) = 2$.

On the other hand, e is induced from two different ways: from the zero orbit in a Levi subalgebra \mathfrak{l}_1 of type $(3, 1; 0)$, that is $\mathfrak{l}_1 \simeq \mathfrak{gl}_3 \times \mathfrak{gl}_1 \times 0$ (see the definition after [M08, Lemma 3.2] for the meaning of *type*), and from the zero orbit in a Levi subalgebra \mathfrak{l}_2 of type $(2; 4)$, that is $\mathfrak{l}_2 \simeq \mathfrak{gl}_2 \times \mathfrak{so}_4$. The first one, \mathfrak{l}_1 , has a center of dimension 2, while the second one, \mathfrak{l}_2 , has a center of dimension 1. The nilpotent orbit of e has dimension 18 and e lies in two different sheets: one of dimension $\dim \mathfrak{z}(\mathfrak{l}_1) + \dim \text{Ind}_{\mathfrak{l}_1}^{\mathfrak{g}}(0) = 20$ and one of dimension $\dim \mathfrak{z}(\mathfrak{l}_2) + \dim \text{Ind}_{\mathfrak{l}_2}^{\mathfrak{g}}(0) = 19$ (here $\mathfrak{z}(\mathfrak{l}_i)$ denotes the center of \mathfrak{l}_i for $i = 1, 2$). This contradicts Proposition 3.11 of [M08], and also Remark 3.12 of the same paper.

Example 3.2. We give now a counter-example in $\mathfrak{sp}(14)$. Consider the partition $\lambda = (4, 4, 2, 2, 1, 1)$ of $\mathcal{P}_{-1}(14)$. Here, the algorithm yields $z(\lambda) = 2$.

The corresponding nilpotent element is induced from the zero orbit in $\mathfrak{l}_1 \simeq \mathfrak{gl}_1 \times \mathfrak{gl}_3 \times \mathfrak{sp}(6)$, and from the rigid nilpotent orbit $0 \times \mathcal{O}'$ in $\mathfrak{l}_2 \simeq \mathfrak{gl}_2 \times \mathfrak{sp}(10)$ where \mathcal{O}' corresponds to the partition $(2, 2, 2, 2, 1, 1) \in \mathcal{P}_{-1}^*(10)$. Again the dimensions of the centers of \mathfrak{l}_1 and \mathfrak{l}_2 lead to different dimensions, 2 and 1 respectively.

The origin of the error can be pinned down in the proof of [M08, Proposition 3.11]. Let us briefly explain this. Until the end of the section, we are in the notations of [M08].

At the end of this proof, the assertion "*Consequently the smallest integer such that one of the situations (a) or (b) of Step 1 happens in $\mathbf{d}^{(p)}$ is equal to i_p* " is incorrect (here \mathbf{d} is an element of $\mathcal{P}_{\varepsilon}(N)$). And so, the main induction argument of the proof fails. We can see that is incorrect in general on an explicit example. Consider the partition $\mathbf{d} = (4, 4, 3, 3, 1, 1)$ of $\mathcal{P}_1(16)$. Then the corresponding nilpotent orbit is induced from the zero orbit in $\mathfrak{l} \simeq \mathfrak{gl}_3 \times \mathfrak{gl}_5 \times 0$ and from the rigid nilpotent orbit with partition $(2, 2, 1, 1, 1, 1)$ in $\mathfrak{l} \simeq \mathfrak{gl}(4) \times \mathfrak{so}(8)$. Consider the second induction. In the notations of the proof, we have:

$S = 1$, $i_1 = 4$, $\mathbf{d}^{(0)} = \mathbf{f} = (2, 2, 1, 1, 1, 1)$, $\mathbf{d} = \mathbf{d}^{(1)} = \widetilde{\mathbf{d}^{(0)}}$ (see [M08, Proposition 3.7] for the tilde notation). Then the smallest integer such that one of the situations (a) or (b) of Step 1 happens for $\mathbf{d} = \mathbf{d}^{(1)}$ is $3 \neq i_1$.

4. CONCLUSION

To summarize, we list below all corrections which have to be taken into account in [M08] (the numbering of [M08] is used):

- Proposition 3.11 (its proof and its statement) is incorrect.
- As a consequence Remark 3.12, the sentence "The results of this section specify that, in the classical case, the dimension of a sheet containing a given nilpotent orbit does not depend on the choice of a sheet containing it" in §1.2, and the sentence "Surprisingly, in the classical case, we will notice that if $\text{Ind}_{\mathfrak{l}_1}(\mathcal{O}_{\mathfrak{l}_1}) = \text{Ind}_{\mathfrak{l}_2}(\mathcal{O}_{\mathfrak{l}_2})$, then $\dim \mathfrak{z}_{\mathfrak{g}}(\mathfrak{l}_1) = \dim \mathfrak{z}_{\mathfrak{g}}(\mathfrak{l}_2)$ " in Remark 2.15, are also incorrect.
- The proof of Theorem 3.13 is incorrect, since it uses Proposition 3.11. Nevertheless, its statement remains valid. In particular, Tables 3, 4 and 5 are still correct.

Remark. There are some misprints in Table 5: line $2m = 48$, the partitions are $[7, 1^5]$, $[5, 3, 2^2]$, $[4^2, 3, 1]$ and not $[4^3]$, $[4^2, 3, 1]$.

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