

FREE MARTINGALE POLYNOMIALS FOR STATIONARY JACOBI PROCESSES

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ABSTRACT. We generalize a previous result concerning free martingale polynomials for the stationary free Jacobi process of parameters $\lambda \in]0,1], \theta = 1/2$. Hopelessly, apart from the case $\lambda = 1$, the polynomials we derive are no longer orthogonal with respect to the spectral measure. As a matter of fact, we use the multiplicative renormalization method to write down its corresponding orthogonal polynomials as well as the orthogonality measure associated with the martingale polynomials. We finally give a realization of the spectral measure of the free stationary Jacobi process by means of the corresponding one mode interacting Fock space.

1. PRELIMINARIES

Let (\mathcal{A}, ϕ) a W^* -non commutative probability space. Easily speaking, \mathcal{A} is a unital von Neumann algebra and ϕ is a tracial faithful linear functional (state). In a previous work ([8]), we defined, via matrix theory, and studied a two parameters-dependent self-adjoint free process, called free Jacobi process. Our focus will be on a particular case called the stationary Jacobi process since its spectral distribution does not depend on time. It is defined as $J_t := PUY_tQY_t^*U^*P$ where

- $(Y_t)_{t \geq 0}$ is a free multiplicative Brownian motion (see [7]).
- U is a Haar unitary operator in (\mathcal{A}, Φ) .
- P is a projection with $\Phi(P) = \lambda\theta \leq 1, \theta \in]0,1]$.
- Q is a projection with $\Phi(Q) = \theta$.
- $QP = PQ = \begin{cases} P & \text{if } \lambda \leq 1 \\ Q & \text{if } \lambda > 1 \end{cases}$
- $\{U, U^*\}$ and $\{P, Q\}$ are free (see [12] for freeness).

Thus the process takes values in *the compressed space* $(P\mathcal{A}P, (1/\phi(P))\phi)$. The spectral distribution has the following decomposition :

$$\mu_{\lambda, \theta}(dx) = \frac{1}{2\pi\lambda\theta} \frac{\sqrt{(x_+ - x)(x - x_-)}}{x(1-x)} \mathbf{1}_{[x_-, x_+]}(x) dx + a_0 \delta_0(dx) + a_1 \delta_1(dx)$$

where δ_y stands for the Dirac mass at y with corresponding weight $a_y, y \in \{0, 1\}$ and

$$x_{\pm} = \left(\sqrt{\theta(1-\lambda\theta)} \pm \sqrt{\lambda\theta(1-\theta)} \right)^2$$

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Its Cauchy transform writes

$$(1) \quad G_{\mu_{\lambda,\theta}}(z) = \frac{(2 - (1/\lambda\theta))z + (1/\lambda - 1) + \sqrt{Az^2 - Bz + C}}{2z(z - 1)}, \quad z \in \mathbb{C} \setminus [0, 1]$$

with $A = 1/(\lambda\theta)^2$, $B = 2((1/\lambda\theta)(1 + 1/\lambda) - 2/\lambda)$ et $C = (1 - 1/\lambda)^2$. It was shown in [8] that if $\lambda \in]0, 1]$, $1/\theta \geq \lambda + 1$ then the process is injective in $P\mathcal{A}P$, that is $a_0 = a_1 = 0$. Moreover, $\mu_{1,1/2}(dx)$ fits the Beta distribution $B(1/2, 1/2)$:

$$\mu_{1,1/2}(dx) = \frac{1}{\pi\sqrt{x(1-x)}} \mathbf{1}_{[0,1]}(x)dx$$

Recall that the Tchebycheff polynomials of the first kind are defined by

$$T_n(x) = \cos(n \arccos x), \quad n \geq 0, \quad |x| \leq 1.$$

and that they are orthogonal with respect to $\mu_{1,1/2}(dx)$. Their generating function is given by:

$$g(u, x) = \sum_{n \geq 0} T_n(x)u^n = \frac{1 - ux}{1 - 2ux + u^2}, \quad |u| < 1.$$

In [8], we proved that for $r > 0$

$$g(re^t, J_t) = ((1 + re^t)P - 2e^t J_t)((1 + re^t)^2 P - 4re^t J_t)^{-1}, \quad t < -\ln r$$

defines a free martingale with respect to the natural filtration of J , say \mathcal{J}_t , the unit of the compressed space being the projection P . It follows that $(e^{nt}T_n(2J_t - P))_{t \geq 0}$, $n \geq 1$ is a family of free martingale polynomials. Note also that

$$\begin{aligned} h(re^t, J_t) &:= 2g(re^t, J_t) - P = \frac{(1 - r^2 e^{2t})}{(1 + re^t)^2} (P - \frac{4re^t}{(1 + re^t)^2} J_t)^{-1} \\ &= \frac{1 - re^t}{1 + re^t} (P - \frac{4re^t}{(1 + re^t)^2} J_t)^{-1} \\ &= (1 - (re^t)^2)(P - 2re^t(2J_t - P) + (re^t)^2)^{-1} \end{aligned}$$

is also a free martingale. Let U_n denote the n -th Tchebycheff polynomial of the second kind defined by

$$U_n(\cos \alpha) = \frac{\sin(n+1)\alpha}{\sin \alpha}, \quad \alpha \in \mathbb{R}$$

with generating function given by

$$\sum_{n \geq 0} U_n(x)u^n = \frac{1}{1 - 2ux + u^2}, \quad |x| \leq 1, \quad |u| < 1.$$

Then, one deduces either from the above generating function or from the relation $2T_n = U_n - U_{n-2}$, $U_{-1} := 0$ that $\{M_t^n := e^{nt}(U_n - U_{n-2})(2J_t - P), n \geq 1\}_{t \geq 0}$ is a family of free martingale polynomials. The aim of this work is to extend this claim to the range $\theta = 1/2, \lambda \in]0, 1]$. The motivation originates from [10] where the author determines the family of orthogonal polynomials with respect to $\mu_{\lambda,\theta}$. Our first guess was that these will be free martingales polynomials for all $\lambda \in]0, 1]$, $\theta \leq 1/(\lambda + 1)$. Yet, things turn to be more complicated: not only the range is restricted but the martingale polynomials we derive are not orthogonal with respect to $\mu_{\lambda,1/2}$ except for $\lambda = 1$. As

a matter of fact, we will on one hand derive the orthogonal polynomials corresponding to $\mu_{\lambda,1/2}$ and compute on the other hand the appropriate orthogonality measure for our martingales polynomials. The last part of the paper is devoted to a realization of the free stationary Jacobi process using the Accardi-Bozejko isomorphism (see [1]) as well as some comments.

Remark. From a matrix theory point of view, the choice $\theta = 1/2$ corresponds to the ultraspherical multivariate Beta distribution (see [8]). Moreover, to our level of Knowledge, there is only one result concerning martingale polynomials for the stationary (classical) Jacobi process, which is restricted to the one dimensional case. More precisely, pick a vector (x_1, \dots, x_d) belonging to the sphere S^{d-1} , $d \geq 2$ distributed according to the uniform (Haar) measure, then form the discrete process defined by

$$s_p = \sum_{i=0}^p x_i^2, \quad 1 \leq p \leq d-1.$$

It is known that each random variable has the Beta distribution $B((d-p)/2, p/2)$. It was shown in [11] that

$$M_n^d(p) = \frac{1}{((d-p)/2)_n} P_n^{\alpha,\beta}(2s_p - 1),$$

where $P_n^{\alpha,\beta}$ denotes the n -th Jacobi polynomial of parameters $\alpha = (d-p)/2 - 1$, $\beta = (p/2) - 1$, is a martingale with respect to the natural filtration of the process. To relate this to our work, we rewrite s_p in the matrix form

$$s_p = P_1 U_d Q_p U_d^* P_1,$$

where U_d is a $d \times d$ Haar unitary matrix, P_1 is a $d \times d$ projection with only one non vanishing coefficient $(P_1)_{11} = 1$ and Q_p is a $d \times d$ projection with only p non vanishing term $(Q_p)_{11} = \dots = (Q_p)_{pp} = 1$. For $d = 2p$, we get the ultraspherical polynomials of parameter $\lambda = (p-1)/2$.

2. MAIN RESULT

One has for $\lambda \in]0, 1]$, $\theta = 1/2$

$$x_- = \left(\frac{\sqrt{2-\lambda}}{2} - \frac{\sqrt{\lambda}}{2} \right)^2 \leq x \leq x_+ = \left(\frac{\sqrt{2-\lambda}}{2} + \frac{\sqrt{\lambda}}{2} \right)^2 \Rightarrow -1 \leq \frac{2x-1}{\sqrt{\lambda(2-\lambda)}} \leq 1$$

and our main result is stated as follows:

Proposition 2.1. *Set*

$$a(\lambda) = \frac{(1-\lambda)}{\sqrt{\lambda(2-\lambda)}}, \quad x_{t,\lambda} = \frac{2J_t - P}{\sqrt{\lambda(2-\lambda)}}$$

For each $n \geq 1$, the process defined by

$$M_t^n := [U_n(x_{t,\lambda}) - 2a(\lambda)U_{n-1}(x_{t,\lambda}) - U_{n-2}(x_{t,\lambda})] \left(\frac{e^t}{\lambda(2-\lambda)} \right)^n, \quad t \geq 0$$

is a (\mathcal{I}_t) -free martingale.

3. PROOF OF THE MAIN RESULT

The proof consists of two parts: the first one consists in deriving a martingale function for all values of $\lambda \in]0, 1]$, $\theta \leq 1/2 \leq 1/(\lambda+1)$. In the second one, we specialize for $\theta = 1/2$ and show that this function corresponds to the generating function of the polynomials stated above.

First step: inspired by the above expression of $h(re^t, J_t)$, we will look for martingales of the form

$$R_t := K_t(P - Z_t J_t)^{-1} = K_t \sum_{n \geq 0} Z_t^n J_t^n := K_t H_t$$

where K, Z are differentiable functions of the variable t lying in some interval $[0, t_0[$ such that $0 < Z_t < 1$ for $t \in [0, t_0[$. The finite variation part of dR_t is given by

$$FV(dR_t) = K_t' H_t dt + K_t FV(dH_t)$$

Our main tool is the free stochastic calculus and more precisely the free stochastic differential equation already set for J_t^n , $n \geq 1$ ([8]):

$$dJ_t^n = dM_t + n(\theta P - J_t) J_t^{n-1} dt + \lambda \theta \sum_{l=1}^{n-1} l [m_{n-l}(P - J_t) J_t^{l-1} + (m_{n-l-1} - m_{n-l}) J_t^l] dt$$

where dM stands for the martingale part and m_n is the n -th moment of J_t in $P \mathcal{A} P$:

$$m_n := \tilde{\phi}(J_t^n) := \frac{1}{\phi(P)} \phi(J_t^n)$$

The finite variation part $FV(dJ_t^n)$ of J_t^n transforms to:

$$\begin{aligned} FV(dJ_t^n) &= n(\theta P - J_t) J_t^{n-1} dt + \lambda \theta \left[\sum_{l=1}^{n-1} l [m_{n-l} J_t^{l-1} + \sum_{l=1}^{n-1} l (m_{n-l-1} - 2m_{n-l}) J_t^l] \right] dt \\ &= n(\theta P - J_t) J_t^{n-1} dt + \lambda \theta \sum_{l=1}^{n-1} l m_{n-l} J_t^{l-1} + \sum_{l=1}^n (l-1)(m_{n-l} - 2m_{n-l+1}) J_t^{l-1} dt \\ &= n(\theta P - J_t) J_t^{n-1} dt + \lambda \theta \sum_{l=1}^n [l m_{n-l} + (l-1)(m_{n-l} - 2m_{n-l+1})] J_t^{l-1} dt - n \lambda \theta J_t^{n-1} dt \\ &= n\theta(1-\lambda) J_t^{n-1} dt - n J_t^n dt + \lambda \theta \sum_{l=1}^n [m_{n-l} + 2(l-1)(m_{n-l} - m_{n-l+1})] J_t^{l-1} dt \end{aligned}$$

Thus

$$\begin{aligned}
FV(dH_t) &= \sum_{n \geq 1} n Z_t' Z_t^{n-1} J_t^n dt + \sum_{n \geq 1} Z_t FV(J_t^n) \\
&= \sum_{n \geq 1} n Z_t' Z_t^{n-1} J_t^n dt - \sum_{n \geq 0} n Z_t^n J_t^n dt + \theta(1-\lambda) \sum_{n \geq 1} n Z_t^n J_t^{n-1} dt \\
&\quad + \lambda\theta \sum_{n \geq 1} \sum_{l=1}^n Z_t^n m_{n-l} J_t^{l-1} dt + 2\lambda\theta \sum_{n \geq 1} \sum_{l=1}^n (l-1) Z_t^n (m_{n-l} - m_{n-l+1}) J_t^{l-1} dt \\
&= \sum_{n \geq 1} n [Z_t' Z_t^{n-1} - Z_t^n] J_t^n dt + \theta(1-\lambda) \sum_{n \geq 0} (n+1) Z_t^{n+1} J_t^n dt \\
&\quad + \lambda\theta \sum_{n \geq 0} \sum_{l \geq 0} Z_t^{n+l+1} m_n J_t^l dt + 2\lambda\theta \sum_{n \geq 0} \sum_{l \geq 0} l Z_t^{n+l+1} (m_n - m_{n+1}) J_t^l dt \\
&= [Z_t'/Z_t - 1 + \theta(1-\lambda)Z_t] \sum_{n \geq 1} n Z_t^n J_t^n dt + \theta(1-\lambda) Z_t \sum_{n \geq 0} Z_t^n J_t^n dt \\
&\quad + \lambda\theta \sum_{n \geq 0} Z_t^{n+1} m_n \sum_{l \geq 0} Z_t^l J_t^l dt + 2\lambda\theta \sum_{n \geq 0} Z_t^{n+1} (m_n - m_{n+1}) \sum_{l \geq 0} l Z_t^l J_t^l dt
\end{aligned}$$

Recall that the Cauchy transform of a measure on the real line is defined by

$$G_\nu(z) = \int_{\mathbb{R}} \frac{1}{z-x} \nu(dx) = \sum_{n \geq 0} \frac{1}{z^{n+1}} \int_{\mathbb{R}} x^n \nu(dx)$$

for some values of z for which both the integral and the infinite sum make sense. Then, since $0 < Z < 1$ and $\mu_{\lambda,\theta}$ is supported in $[0, 1]$, it is easy to see that

$$\sum_{n \geq 0} Z_t^{n+1} (m_n - m_{n+1}) = \left(1 - \frac{1}{Z_t}\right) G_{\mu_{\lambda,\theta}}\left(\frac{1}{Z_t}\right) + 1$$

with $G_{\mu_{\lambda,\theta}}$ given by (1). This gives

$$2\lambda\theta(1-z)G_{\mu_{\lambda,\theta}}(z) = \frac{(1-2\lambda\theta)z - \theta(1-\lambda) - \sqrt{z^2 - (\lambda\theta)^2 Bz + (\lambda\theta)^2 C}}{z},$$

so that

$$2\lambda\theta(1-Z_t^{-1})G_{\mu_{\lambda,\theta}}(Z_t^{-1}) + 2\lambda\theta = 1 - \theta(1-\lambda)Z_t - \sqrt{1 - (\lambda\theta)^2 BZ_t + (\lambda\theta)^2 CZ_t},$$

We finally get:

$$\begin{aligned}
FV(dH_t) &= [Z_t'/Z_t - \sqrt{1 - (\lambda\theta)^2 BZ_t + (\lambda\theta)^2 CZ_t}] \sum_{n \geq 1} n Z_t^n J_t^n dt \\
&\quad + \left[\lambda\theta G_{\mu_{\lambda,\theta}}\left(\frac{1}{Z_t}\right) + \theta(1-\lambda)Z_t \right] \sum_{n \geq 0} Z_t^n J_t^n dt
\end{aligned}$$

In order to derive free martingales, we shall pick Z such that $Z_t' = Z_t \sqrt{1 - (\lambda\theta)^2 BZ_t + (\lambda\theta)^2 CZ_t}$. This shows that Z is an increasing function and one can solve the above non linear differential equation as follows: use the variables change $u = Z_t$, $t < t_0$, then integrate to

get :

$$\int_{[Z_0, Z_t]} \frac{du}{u\sqrt{1 - 2\theta(1 + \lambda - 2\lambda\theta)u + (\theta(1 - \lambda))^2 u^2}} = t$$

Remark. Let $c_1 = 2\theta(1 + \lambda - 2\lambda\theta)$, $c_2 = \theta^2(1 - \lambda)^2$. Then, the function $u \mapsto 1 - c_1u + c_2u^2$ is decreasing for $u \in]0, 1[$: in fact,

$$\begin{aligned} 2c_2u - c_1 &< 2c_2 - c_1 = 2\theta^2(1 - \lambda)^2 - 2\theta(1 + \lambda - 2\lambda\theta) \\ &= 2\theta[\theta(1 + \lambda^2) - (1 + \lambda)] \leq 2\theta\left(\frac{1 + \lambda^2}{1 + \lambda} - (1 + \lambda)\right) = -\frac{4\lambda\theta}{1 + \lambda} < 0 \end{aligned}$$

which yields $1 - c_1u + c_2u^2 > 1 - c_1 + c_2 = (1 - \theta(1 + \lambda))^2 \geq 0$.

Next, use the variable change $1 - vu = \sqrt{1 - c_1u + c_2u^2}$. This gives

$$u = \frac{2v - c_1}{v^2 - c_2}, \quad du = -2\frac{v^2 + c_2 - c_1v}{(v^2 - c_2)^2}dv, \quad 1 - vu = -\frac{v^2 + c_2 - c_1v}{v^2 - c_2}$$

Moreover

$$u \mapsto v = \frac{1 - \sqrt{1 - c_1u + c_2u^2}}{u}, \quad 0 < u < 1$$

is an increasing function: in fact the numerator of its derivative writes

$$c_1u - 2c_2u^2 + 2(1 - c_1u + c_2u^2) - 2\sqrt{1 - c_1u + c_2u^2} = (2 - c_1u) - 2\sqrt{1 - c_1u + c_2u^2}$$

Since $2 - c_1u > 2 - c_1 = 2(1 - \theta(1 + \lambda)) + 4\lambda\theta^2 > 0$, our claim follows from the fact that $c_1^2 - 4c_2 = 16\lambda\theta^2(1 - \lambda\theta)(1 - 2\theta) \geq 0$.

Finally, the integral transforms to

$$\int_{[v_0, v_t]} \frac{2dv}{2v - c_2} = \log \left| \frac{2v_t - c_1}{2v_0 - c_1} \right| = t$$

where $1 - Z_tv_t = \sqrt{1 - c_1Z_t + c_2Z_t^2}$, $1 - Z_0v_0 = \sqrt{1 - c_1Z_0 + c_2Z_0^2}$. Note also that $c_1^2 - 4c_2 \geq 0$ implies that for all $u \in [Z_0, Z_t] \subset]0, 1[$

$$\begin{aligned} v - \frac{c_1}{2} &= \frac{1 - \sqrt{1 - c_1u + c_2u^2}}{u} - \frac{c_1}{2} = \frac{(1 - c_1u/2) - \sqrt{1 - c_1u + c_2u^2}}{u} \\ &= \frac{(1 - c_1u/2)^2 - (1 - c_1u + c_2u^2)}{u((1 - c_1u/2) + \sqrt{1 - c_1u + c_2u^2})} \geq 0 \end{aligned}$$

since $1 - c_1/2u \geq 1 - c_1/2 \geq 0$. Thus $v \geq c_1/2 \geq \sqrt{c_2}$.

$$v_t = [(2v_0 - c_1)e^t + c_1]/2 \Leftrightarrow \sqrt{1 - c_1Z_t + c_2Z_t^2} = 1 - \frac{(2v_0 - c_1)e^{\pm t} + c_1}{2}Z_t$$

We finally get

$$Z_t = \frac{4(2v_0 - c_1)e^{\pm t}}{((2v_0 - c_1)e^t + c_1)^2 - 4c_2}, \quad t \leq t_0$$

where t_0 is the first time such that $Z_{t_0} = 1 \Leftrightarrow (2v_0 - c_1)e^{t_0} + c_1)^2 - 4c_2 - 4(2v_0 - c_1)e^{t_0} = 0$. Set $r = r(\lambda, \theta) := (2v_0 - c_1)$ and $x_0 = e^{t_0} > 1$, then $r^2x_0^2 + 2(c_1 - 2)rx_0 + c_1^2 - 4c_2 = 0$. The discriminant equals to $\Delta = 16r^2(1 + c_2 - c_1) = 16r^2(1 - \theta(1 + \lambda))^2$. Thus

$$x_0 = \frac{-(c_1 - 2) - 2(1 - \theta(1 + \lambda))}{r} = \frac{2(1 - \theta(1 + \lambda)) + 4\lambda\theta^2 - 2(1 - \theta(1 + \lambda))}{r} = \frac{4\lambda\theta^2}{r} \geq 1$$

The last inequality follows from the fact that $1 - \sqrt{c_2}u \geq 1 - \theta(1 + \lambda) \geq 0$ and from

$$r - 4\lambda\theta^2 = 2v_0 - c_1 - 4\lambda\theta^2 = 2(v_0 - \theta(1 + \lambda)) = 2(v_0 - \sqrt{c_2}) \leq 0.$$

It gives $t_0 = -\ln(r/4\lambda\theta^2)$. Note also that the denominator is well defined for all $t \leq t_0$ since $c_1^2 \geq 4c_2$ and $2v_0 - c_1 \geq 0$.

For the remaining terms, we shall choose K such that

$$K'_t + K_t \left[\lambda \theta G_{\mu, \lambda, \theta} \left(\frac{1}{Z_t} \right) + \theta(1 - \lambda)Z_t \right] = 0$$

An easy computation shows that this equals to

$$K'_t + \frac{K_t}{2} \left[\theta(1 - \lambda) \frac{Z_t^2}{Z_t - 1} + (1 - 2\theta) \frac{Z_t}{Z_t - 1} - \frac{Z_t \sqrt{1 - c_1 Z_t + c_2 Z_t^2}}{Z_t - 1} \right] = 0$$

Remembering the choice of the function Z , this writes

$$K'_t - \frac{K_t}{2} \left[\frac{Z'_t}{Z_t - 1} - (1 - 2\theta) \frac{Z_t}{Z_t - 1} - \theta(1 - \lambda) \frac{Z_t^2}{Z_t - 1} \right] = 0$$

or equivalently

$$K'_t - \frac{K_t}{2} \left[\frac{Z'_t}{Z_t - 1} - (1 - \theta - \lambda\theta) \frac{Z_t}{Z_t - 1} - \theta(1 - \lambda)Z_t \right] = 0$$

If $K_t \neq 0$, then

$$\log K_t = \frac{1}{2} \log(1 - Z_t) - \frac{1 - \theta - \lambda\theta}{2} \int \frac{Z_s}{Z_s - 1} ds - \frac{\theta(1 - \lambda)}{2} \int Z_s ds + C$$

If $\lambda \neq 1$, then the last term is given by

$$-\frac{\theta(1 - \lambda)}{2} \int Z_s ds = \frac{\theta(1 - \lambda)}{\sqrt{c_2}} \int \frac{(r/2\sqrt{c_2})e^t}{1 - \left(\frac{re^t + c_1}{2\sqrt{c_2}} \right)^2} = \arg \tanh \left(\frac{re^t + c_1}{2\sqrt{c_2}} \right)$$

where $\arg \tanh(u) = (1/2) \log((u + 1)/(u - 1))$, $|u| > 1$. The second term writes

$$\begin{aligned} \frac{Z_t}{Z_t - 1} &= \frac{4re^t}{4c_2 + 4re^t - (re^t + c_1)^2} = \frac{4re^t}{4c_2 - c_1^2 + (c_1 - 2)^2 - (re^t + c_1 - 2)^2} \\ &= \frac{re^t}{c_2 + 1 - c_1 - \left(\frac{re^t + c_1 - 2}{2} \right)^2} = \frac{1}{c_2 + 1 - c_1} \frac{re^t}{1 - \left(\frac{re^t + c_1 - 2}{2\sqrt{c_2 + 1 - c_1}} \right)^2} \\ &= \frac{2}{\sqrt{c_2 + 1 - c_1}} \frac{(r/2\sqrt{c_2 + 1 - c_1})e^t}{1 - \left(\frac{re^t + c_1 - 2}{2\sqrt{c_2 + 1 - c_1}} \right)^2} \end{aligned}$$

Observe that $2 - c_1 - re^t > 2 - c_1 - re^{t_0} = 2(1 - \theta(1 + \lambda)) \geq 0$. Thus, if $\theta(1 + \lambda) \neq 1$

$$\frac{1 - \theta(1 + \lambda)}{2} \int \frac{Z_s}{Z_s - 1} ds = \arg \tanh \left(\frac{2 - c_1 - re^t}{2\sqrt{c_2 + 1 - c_1}} \right)$$

Thus, if $\lambda \neq 1$ ($\theta \leq 1/2 < 1/(\lambda + 1)$),

$$K_t = C(1 - Z_t)^{1/2} \left(\frac{re^t + c_1 + 2\sqrt{c_2}}{re^t + c_1 - 2\sqrt{c_2}} \right)^{1/2} \left(\frac{2 - c_1 - 2c_3 - re^t}{2 - c_1 + 2c_3 - re^t} \right)^{1/2}$$

where $c_3 := \sqrt{c_2 + 1 - c_1} = 1 - \theta(\lambda + 1)$. Note that for $\lambda = 1, \theta = 1/2, c_1 = 1, c_2 = 0, c_3 = 0$ and

$$K_t = C \frac{1 - re^t}{1 + re^t}, \quad t < t_0 = -\ln r.$$

The case $\theta = 1/2, \lambda \neq 1$: free martingales polynomials: one has

$$\begin{aligned} c_1 = 1, c_2 &= \frac{(1 - \lambda)^2}{4}, c_3 = \sqrt{c_2} = \frac{1 - \lambda}{2}, Z_t = \frac{4re^t}{(re^t + 1)^2 - (1 - \lambda)^2} \\ c_1 + 2\sqrt{c_2} &= 2(1 + c_3) - c_1 = 2 - \lambda, c_1 - 2\sqrt{c_2} = 2(1 - c_3) - c_1 = \lambda. \\ 1 - Z_t &= \frac{(re^t - 1)^2 - (1 - \lambda)^2}{(re^t + 1)^2 - (1 - \lambda)^2} = \frac{(re^t + \lambda - 2)(re^t - \lambda)}{(re^t + 2 - \lambda)(re^t + \lambda)} \end{aligned}$$

Thus, for $t < -\ln(r/\lambda)$,

$$K_t = C \frac{\lambda - re^t}{\lambda + re^t}$$

so that

$$\begin{aligned} R_t &= C \frac{\lambda - re^t}{\lambda + re^t} (P - \frac{4re^t}{(re^t + 1)^2 - (1 - \lambda)^2} J_t)^{-1} \\ &= C(\lambda - re^t)(2 - \lambda + re^t)(\lambda(2 - \lambda)P + (re^t)^2 P - 2re^t(2J_t - P))^{-1} \\ &= \frac{C(\lambda - re^t)(2 - \lambda + re^t)}{\lambda(2 - \lambda)} \left(P - \frac{2re^t}{\sqrt{\lambda(2 - \lambda)}} \frac{(2J_t - P)}{\sqrt{\lambda(2 - \lambda)}} + \frac{(re^t)^2}{\lambda(2 - \lambda)} P \right)^{-1} \\ &= C \left(1 - 2 \frac{(1 - \lambda)}{\sqrt{\lambda(2 - \lambda)}} \frac{re^t}{\sqrt{\lambda(2 - \lambda)}} - \frac{(re^t)^2}{\lambda(2 - \lambda)} \right) \left(P - \frac{2re^t}{\sqrt{\lambda(2 - \lambda)}} \frac{(2J_t - P)}{\sqrt{\lambda(2 - \lambda)}} + \frac{(re^t)^2}{\lambda(2 - \lambda)} P \right)^{-1} \end{aligned}$$

is a free martingale with respect to the natural filtration \mathcal{J}_t . Besides, since $\lambda \in]0, 1]$, then $\lambda \leq \sqrt{\lambda(2 - \lambda)}$, hence $(re^t)/(\sqrt{\lambda(2 - \lambda)}) < 1$ for all $t < -\ln(r/\lambda)$. Now, let us consider the following generating function

$$g(u, x) = \frac{1 - 2au - u^2}{1 - 2xu + u^2}, \quad 0 < a, u < 1, |x| \leq 1.$$

It follows that

$$g(u, x) = U_0(x) + (U_1(x) - 2a)u + \sum_{n \geq 2} [U_n(x) - 2aU_{n-1}(x) - U_{n-2}(x)]u^n$$

Setting

$$u_{t,\lambda} := \frac{re^t}{\sqrt{\lambda(2 - \lambda)}}, \quad t < t_0,$$

then

$$R_t = C[P + (x_{t,\lambda} - 2a(\lambda)P)u_{t,\lambda} + \sum_{n \geq 2} [U_n(x_{t,\lambda}) - 2a(\lambda)U_{n-1}(x_{t,\lambda}) - U_{n-2}(x_{t,\lambda})]u_{t,\lambda}^n]$$

Setting $U_{-1} = U_{-2} = 0$, it can be written as

$$R_t = C \sum_{n \geq 0} [U_n(x_{t,\lambda}) - 2a(\lambda)U_{n-1}(x_{t,\lambda}) - U_{n-2}(x_{t,\lambda})] u_{t,\lambda}^n$$

Remark. The case $\lambda = 1$.

$c_1 = 4\theta(1 - \theta)$, $c_2 = 0$ and Z_t writes

$$Z_t = \frac{4re^t}{(re^t + 4\theta(1 - \theta))^2}$$

Moreover, $c_3 = \sqrt{1 - c_1} = (1 - 2\theta)$, $2 - 2c_3 - c_1 = 4\theta^2$, $2 + 2c_3 - c_1 = 4(1 - \theta)^2$. K_t then writes

$$K_t = \frac{\sqrt{(re^t + 4\theta(1 - \theta))^2 - 4re^t}}{re^t + 4\theta(1 - \theta)} \sqrt{\frac{4\theta^2 - re^t}{4(1 - \theta)^2 - re^t}}$$

4. ONE-PARAMETER MEASURES FAMILY AND ORTHOGONAL POLYNOMIALS

Let μ be a measure on the real line which is not supported by a finite set. Assume that μ has finite moments of all orders. Applying the Gram-Schmidt orthogonalization method to the basis $(1, x, x^2, \dots)$, there exist a unique family of monic orthogonal polynomials with respect to μ , say $(P_n)_{n \geq 0}$. These polynomials satisfy the three-terms recurrence relation

$$(x - \alpha_n)P_n(x) = P_{n+1}(x) + \omega_n P_{n-1}(x), \quad n \geq 0, P_{-1} := 0.$$

where $\alpha_n \in \mathbb{R}$, $\omega_n > 0$. $(\alpha_n, \omega_n)_{n \geq 0}$ are called the Jacobi-Szegő parameters of μ . It is known that μ is symmetric if and only if $\alpha_n = 0$, $n \geq 0$. Another way to derive the family $(P_n)_n$ is the multiplicative renormalization method ([3],[4],[5], [6]) that we shall recall here : a nice function $(u, x) \mapsto \psi(u, x)$ is a generating function for the measure μ if ψ has the expansion

$$\psi(u, x) = \sum_{n \geq 0} c_n P_n(x) u^n, \quad c_n \in \mathbb{R}$$

where P_n are orthogonal with respect to μ . Of course, there is more than one generating function corresponding to a given measure and in order to claim whether a function is a generating function or not, authors in [3] provided a necessary and sufficient condition. For a particular form of ψ which fits our need, their result is formulated as follows:

Theorem 4.1. *Define*

$$\theta(u) := \int_{\mathbb{R}} \frac{1}{1 - ux} \mu(dx), \quad \theta(u, v) := \int_{\mathbb{R}} \frac{1}{(1 - ux)(1 - vx)} \mu(dx).$$

Let ρ analytic around 0 such that $\rho(0) = 0$ and $\rho'(0) \neq 0$. Then

$$(2) \quad \psi(u, x) := \frac{(1 - \rho(u)x)^{-1}}{\theta(\rho(u))}$$

is a generating function for μ if and only if

$$\Theta_\rho(u, v) := \frac{\theta(\rho(u), \rho(v))}{\theta(\rho(u))\theta(\rho(v))}$$

is a function of uv .

We will apply this result to the measures family $\nu_\lambda, \lambda \in]0, 1]$ which is the image of

$$\mu_{\lambda,1/2} = \frac{1}{\pi\lambda} \frac{\sqrt{(x_+ - x)(x - x_-)}}{x(1-x)} \mathbf{1}_{[x_-, x_+]}(x) dx, \quad x_\pm = \frac{(\sqrt{\lambda} \pm \sqrt{2-\lambda})^2}{4}$$

by the map

$$x \mapsto \frac{2x-1}{\sqrt{\lambda(2-\lambda)}}$$

Then,

$$\nu_\lambda(dx) = \frac{(2-\lambda)}{\pi} \frac{\sqrt{1-x^2}}{1-\lambda(2-\lambda)x^2} \mathbf{1}_{[-1,1]}(x) dx$$

Our scheme is the almost the same used in [9] except the computation of $\theta(u)$ which follows easily from $G_{\mu_{\lambda,1/2}}$. More precisely, authors considered the one-parameter measures family

$$\mu_a(dx) = \frac{a\sqrt{1-x^2}}{a^2 + (1-2a)x^2} \mathbf{1}_{]-1,1[}(dx), \quad a > 0.$$

It is forward that $\mu_{1/(2-\lambda)} = \nu_\lambda$ almost everywhere for $0 < \lambda \leq 1 \Leftrightarrow 1/2 < a \leq 1$.

Proposition 4.1.

$$\theta(u) = \theta_\lambda(u) = \frac{2-\lambda}{1-\lambda+\sqrt{1-u^2}}, \quad |u| < 1$$

Using

$$\frac{1}{(1-ux)(1-vx)} = \frac{1}{u-v} \left(\frac{u}{1-ux} - \frac{v}{1-vx} \right)$$

it follows that $\theta(u, v) = (u\theta(u) - v\theta(v))/(u-v)$ from which we deduce

Corollary 4.1.

$$\theta(u, v) = \theta_\lambda(u, v) = \frac{1}{2-\lambda} \left[1 - \lambda + \frac{u+v}{u\sqrt{1-v^2} + v\sqrt{1-u^2}} \right]$$

Proof: from the definition of ν_λ , one writes for $0 < u < \lambda(2-\lambda) \leq 1$:

$$\int_{\mathbb{R}} \frac{1}{1-ux} \nu_\lambda(dx) = \int_{\mathbb{R}} \frac{1}{1-u \frac{2x-1}{\sqrt{\lambda(2-\lambda)}}} \mu_{\lambda,1/2}(dx) = \frac{\sqrt{\lambda(2-\lambda)}}{2u} G_{\mu_{\lambda,1/2}} \left(\frac{\sqrt{\lambda(2-\lambda)} + u}{2u} \right)$$

The result follows from

$$G_{\mu_{\lambda,1/2}}(z) = \frac{(1-\lambda)(2z-1) - \sqrt{4z^2 - 4z + (1-\lambda)^2}}{2\lambda z(1-z)}, \quad z \in \mathbb{C} \setminus [0, 1] \quad \blacksquare$$

Let $\rho(u) = 2u/(1+u^2)$, then

$$\frac{\rho(u) + \rho(v)}{\rho(u)\sqrt{1-\rho^2(v)} + \rho(v)\sqrt{1-\rho^2(u)}} = \frac{1+uv}{1-uv}$$

so that Theorem 4.1 applies and claims that

$$\psi_\lambda(u, x) = \frac{1 - \lambda/(2-\lambda)u^2}{1 - 2ux + u^2}$$

is a generating function for ν_λ corresponding to the polynomials

$$Q_n^\lambda(x) = U_n(x) - \frac{\lambda}{2-\lambda}U_{n-2}(x), \quad n \geq 0, U_{-1} = U_{-2} := 0.$$

Using the recurrence relation

$$(3) \quad 2xU_n(x) = U_{n+1}(x) + U_{n-1}(x), \quad U_{-1} := 0,$$

These polynomials satisfy

$$\begin{aligned} 2xQ_0^\lambda(x) &= Q_1^\lambda(x) \\ 2xQ_1^\lambda(x) &= Q_2^\lambda(x) + \left(1 + \frac{\lambda}{2-\lambda}\right)Q_0^\lambda(x) \\ 2xQ_n^\lambda(x) &= Q_{n+1}^\lambda(x) + Q_{n-1}^\lambda(x), \quad n \geq 2. \end{aligned}$$

Setting $Q_{-1}^\lambda := 0$ and since the coefficient of the leading power in $Q_n^\lambda(x)$ is 2^n , then one deduces that the Jacobi-Szegő parameters are given by : $\alpha_n = 0, n \geq 0, w_1 = 1/(2(2-\lambda)), w_n = 1/4, n \geq 2$.

Remark. In [10], authors characterize the absolutely continuous measures for which the multiplicative renormalization method is applicable with the generating function given by (2). They derived a two-parameters densities family written as

$$f(x) = \frac{c\sqrt{1-x^2}}{\pi[b^2 + c^2 - 2b(1-c)x + (1-2c)x^2]} \mathbf{1}_{[-1,1]}(x), \quad |b| < 1-c, 0 < c \leq 1.$$

These densities fit the image of absolutely continuous part of $\mu_{\lambda,\theta}$ by the map

$$u = \frac{2x-s}{d} \in [-1, 1]$$

with $d = d(\lambda, \theta) = x_+ - x_- = 4\theta\sqrt{\lambda(1-\theta)(1-\lambda\theta)}$, $s = s(\lambda, \theta) = x_+ + x_- = 2\theta(1 + \lambda - 2\lambda\theta)$. One gets

$$\nu_{\lambda,\theta}(dx) = \frac{d^2}{2\pi\lambda\theta} \frac{\sqrt{1-x^2}}{s(2-s) + 2d(1-s)x - d^2x^2} dx$$

which provides the following relations

$$(4) \quad c = \frac{1}{2(1-\lambda\theta)}, \quad b = \sqrt{\frac{\lambda}{(1-\theta)(1-\lambda\theta)}}(2\theta-1)$$

As a result, one can derive the corresponding orthogonal polynomials for $\lambda \in]0, 1], \theta \leq 1/(\lambda+1)$ from the generating function ([10]):

$$(5) \quad \phi(u, x) = \frac{1 - 2bu + (1-2c)u^2}{1 - 2ux + u^2}.$$

5. MORE ORTHOGONAL POLYNOMIALS

Consider the polynomials P_n^λ defined by

$$P_n^\lambda(x) = U_n(x) - 2a(\lambda)U_{n-1}(x) - U_{n-2}(x), \quad U_{-1} = U_{-2} := 0$$

with generating function

$$g(u, x) = \frac{1 - 2a(\lambda)u - u^2}{1 - 2xu + u^2}, \quad a(\lambda) = \frac{1 - \lambda}{\lambda(2 - \lambda)}, \quad 0 < u < 1.$$

The P_n^λ 's appear in [2] as a limiting case of the q -Pollaczek polynomials. The coefficient of the highest monomial is equal to 2^n . Using (3), one deduces that

$$\begin{aligned} 2[x - a(\lambda)]P_0^\lambda(x) &= P_1^\lambda(x) \\ 2xP_1^\lambda(x) &= P_2^\lambda(x) + 2P_0^\lambda(x) \\ 2xP_n^\lambda(x) &= P_{n+1}^\lambda(x) + P_{n-1}^\lambda(x), \quad n \geq 2. \end{aligned}$$

Thus the Jacobi-Szegő parameters are given by $\alpha_0 = a(\lambda)$ and $\alpha_n = 0$ for all $n \geq 1$ and $\omega_1 = 1/2$, $\omega_n = 1/4$, $n \geq 2$ ($P_{-1}^\lambda = 0$).

One can use Theorem 4.1 to determine the probability measure, ξ_λ , with respect to which the P_n^λ s are orthogonal. Since $\alpha_0 \neq 0$, then ξ_λ is not symmetric. Indeed, keeping the same function ρ previously defined, then the function θ must be equal to

$$\theta(\rho(u)) = \frac{1 + u^2}{1 - 2a(\lambda) - u^2}$$

so that

$$\theta(u) = \frac{1}{\sqrt{1 - u^2 - a(\lambda)u}}$$

From the definition of θ , one deduces that

$$G_{\xi_\lambda}(u) := \int_{\mathbb{R}} \frac{1}{u - x} \xi_\lambda(dx) = \frac{1}{u} \theta\left(\frac{1}{u}\right) = \frac{\sqrt{u^2 - 1} + a(\lambda)}{u^2 - (1 + a^2(\lambda))}$$

for $|u| > 1$, $u \neq \pm\sqrt{1 + a(\lambda)^2}$. Thus, ξ_λ has two atoms a_\pm at $\pm\sqrt{a^2(\lambda) + 1}$ and an absolutely continuous part given by

$$a_\pm = - \lim_{y \rightarrow 0^+} y \Im G_{\xi_\lambda}(\pm\sqrt{a^2(\lambda) + 1} + iy), \quad g(x) = -\frac{1}{\pi} \lim_{y \rightarrow 0^+} \Im G_{\xi_\lambda}(x + iy)$$

Using that the Cauchy transform maps \mathbb{C}^+ to \mathbb{C}^- , one finally gets

$$\xi_\lambda(dx) = \frac{a(\lambda)}{\sqrt{a^2(\lambda) + 1}} \delta_{\sqrt{a^2(\lambda) + 1}}(dx) + \frac{1}{\pi} \frac{\sqrt{1 - x^2}}{a^2(\lambda) + 1 - x^2} \mathbf{1}_{|x| < 1} dx$$

Remark. To see that this defines a probability measure for $\lambda \neq 1$, it suffices to write

$$\begin{aligned} \frac{1}{\pi} \int_{-1}^1 \frac{\sqrt{1 - x^2}}{a^2(\lambda) + 1 - x^2} dx &= \frac{1}{\pi} \int_0^1 \frac{\sqrt{1 - x}}{\sqrt{x}(a^2(\lambda) + 1 - x)} dx \\ &= \frac{1}{2(a^2(\lambda) + 1)} {}_2F_1\left(1, \frac{1}{2}; 2; \frac{1}{a^2(\lambda) + 1}\right) \end{aligned}$$

where ${}_2F_1$ denotes the Gauss hypergeometric function given by

$${}_2F_1(e, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-e} dx, \quad \Re(b) \wedge \Re(c-b) > 0$$

for $|u| < 1$. Then, one uses the identity

$${}_2F_1(1, b, 2; z) = \frac{1 - (1-z)^{1-b}}{(1-b)z}$$

to get

$$\frac{1}{\pi} \int_{-1}^1 \frac{\sqrt{1-x^2}}{a^2(\lambda) + 1 - x^2} dx = 1 - \frac{a(\lambda)}{\sqrt{a^2(\lambda) + 1}}$$

6. ONE MODE INTERACTING FOCK SPACE

In the sequel, we give a realization of $\nu_{\lambda, \theta}$, image of the spectral measure $\mu_{\lambda, \theta}$ for $\lambda \in]0, 1], \theta \leq 1/(\lambda + 1)$ so that the support is $[-1, 1]$. In the quantum scope, it is known as the quantum decomposition of $\nu_{\lambda, \theta}$. We only need the Jacobi-Szegö parameters in order to apply Accardi-Bozejko Theorem ([1]). We first write down from the generating function (5) the orthogonal polynomials (see [10]) corresponding to $\nu_{\lambda, \theta}$:

$$Q_n^{\lambda, \theta} = U_n - 2bU_{n-1} + (1-2c)U_{n-2}, \quad U_{-1} = U_{-2} = 0,$$

where $b = b(\lambda, \theta), c = c(\lambda, \theta)$ are given by (4). It follows that $\alpha_0 = b, \alpha_n = 0$ for $n \geq 1$ and $\omega_1 = c/2, \omega_n = 1/4$ for $n \geq 1$. In order to use Accardi-Bozejko Theorem ([1]), we shall introduce the so-called *one-mode interacting Fock space*: let \mathcal{H} be a one dimensional separable complex Hilbert space $\sim \mathbb{C}$. Then the n -th tensor product $\mathcal{H}^{\otimes n}$ is one dimensional: indeed $z_1 \otimes \cdots \otimes z_n = (z_1 \dots z_n)1 \otimes \cdots \otimes 1 \in \mathbb{C}\Phi_n$. The one-mode interacting Fock space associated to $\nu_{\lambda, \theta}$ is defined as $\Gamma(\mathbb{C}\Phi_n, (\lambda_n))$ as the infinite orthogonal sum of $\mathbb{C}\Phi_n$ equipped with the weighted scalar product

$$(z_1\Phi_n, z_2\Phi_n) := \lambda_n \bar{z}_1 z_2, \quad z_1, z_2 \in \mathbb{C},$$

where $\lambda_n = \omega_1 \dots \omega_n$. Then $\nu_{\lambda, \theta}$ is the vacuum distribution (in the vacuum state Φ_1) of any extension of the operator $a^+ + a + \alpha_N$ where

$$\begin{aligned} a^+\Phi_n &= \Phi_{n+1} \quad (\text{creation operator}) \\ a\Phi_{n+1} &= \omega_{n+1}\Phi_n = \frac{\lambda_{n+1}}{\lambda_n}\Phi_n, \quad a\Phi_1 = 0, \quad (\text{annihilation operator}) \\ N\Phi_n &= n\Phi_n \quad (\text{Number operator}), \quad aa^+\Phi_n = \frac{\lambda_{n+1}}{\lambda_n}\Phi_n, \end{aligned}$$

and α_N is defined by the spectral Theorem, that is $\alpha_N\Phi_n = \alpha_n\Phi_n$.

Remark. The concept of one mode interacting Fock space (IFS) is purely algebraic as the reader can see from [1] and is fully characterized by both the commutation relations between creation and annihilation operators and $a\Phi_1 = 0$. The most important feature of Accardi-Bozejko Theorem is illustrated in the *canonical* isomorphism between one mode IFS and the L^2 -space of a given measure μ of all order moments. It is noteworthy that only the ω_n s are involved in the commutation relations (thus in both one mode IFS and $L^2(\mu)$) while the α_n s reflect only the symmetry of μ .

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