Bi-capacities – Part II: the Choquet integral

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Bi-capacities
Part II: The Choquet integral[1]

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Abstract

Bi-capacities arise as a natural generalization of capacities (or fuzzy measures) in a context of decision making where underlying scales are bipolar. They are able to capture a wide variety of decision behaviours, encompassing models such as Cumulative Prospect Theory (CPT). The aim of this paper in two parts is to present the machinery behind bi-capacities, and thus remains on a rather theoretical level, although some parts are firmly rooted in decision theory, notably cooperative game theory. The present second part focuses on the definition of Choquet integral. We give several expressions of it, including an expression w.r.t. the Möbius transform. This permits to express the Choquet integral for 2-additive bi-capacities w.r.t. the interaction index.

Keywords: fuzzy measure, capacity, bi-capacity, Choquet integral

1 Introduction

Bi-capacities have been introduced to model in a flexible way decision behaviours when the underlying scales are bipolar (see part I of this paper). Specifically, let us consider a finite referential set $N = \{1, \ldots, n\}$, which can be thought as the set of criteria (or states of nature, etc.), and alternatives or acts, i.e., real-valued functions on N. Then, for $i \in N$, $f(i)$ is the utility or score of f w.r.t. to criterion i (or state of nature, etc.), which we consider to lie on a *bipolar* scale, i.e., a scale with a neutral value (usually 0) such that values above this neutral reference point are considered as being good for the decision maker, and values below it are considered to be bad. Assume for simplicity that the scale used for all utilities is $[-1, 1]$, with 0 as neutral value. For any disjoint subsets A, B of N, we denote by $v(A, B)$ the overall score or utility of the ternary alternative $(1_A, -1_B, 0_{(A \cup B)^c})$, i.e., the alternative whose utility is equal to 1 for all criteria in A, to -1 for all criteria in B, and 0 otherwise. Assuming that $v(A, B)$ is non decreasing when A increases or B decreases (in the sense of inclusion), and that $v(\emptyset, \emptyset) = 0$, v viewed as a function on $\mathcal{Q}(N) := \{(A, B) \in \mathcal{P}(N) \times \mathcal{P}(N) \mid A \cap B = \emptyset\}$ is called a *bi-capacity*. Its properties, as well as the Möbius transform, the Shapley value and the interaction have been studied at length in Part I.

Let us consider any alternative $f \in [-1, 1]^N$. The problem we address here is to compute the overall utility of f by a "Choquet-like" integral, which would be defined with respect to the bicapacity v. Let us denote this new Choquet integral as $\mathcal{C}_v$. To achieve this properly, we should satisfy some basic requirements. The first one is that the Choquet integral of ternary alternatives should coincide with v in the following sense:

$$\mathcal{C}_v(1_A, -1_B, 0_{(A \cup B)^c}) = v(A, B)$$

since both expressions represent the overall utility of ternary alternatives.

The second requirement is that we should recover classical ways of computing the overall utility of real-valued acts which are based on the (usual) Choquet integral. These are, defining $f^+ := f \vee 0$ and $f^- := (-f)^+$:

- the symmetric Choquet integral $\check{\mathcal{C}}_\nu(f) := \mathcal{C}_\nu(f^+) - \mathcal{C}_\nu(f^-)$

- the asymmetric Choquet integral $\mathcal{C}_\nu(f) := \mathcal{C}_\nu(f^+) - \mathcal{C}_{\overline{\nu}}(f^-)$, where $\overline{\nu}$ indicates the conjugate capacity

- the Cumulative Prospect Theory (CPT) model of Tversky and Kahnemann [17] $\text{CPT}_{\nu_1, \nu_2}(f) := \mathcal{C}_{\nu_1}(f^+) - \mathcal{C}_{\nu_2}(f^-)$ where ν_1, ν_2 are capacities, which encompasses the symmetric and asymmetric Choquet integral.

Up to now, the CPT model was the most powerful model for decision making on bipolar scales.

We will show that it is possible to propose a definition of the Choquet integral w.r.t. bi-capacities satisfying the previous requirements. In particular, the above mentioned models will be recovered by particular bi-capacities, called respectively symmetric, asymmetric, bi-capacities of the CPT type, and have already been introduced and studied in Part I of this paper.

Section 2 introduces necessary concepts on capacities and the Choquet integral, and on bi-capacities and bipolar capacities, the latter being a related concept proposed by Greco *et al.* [11]. Section 3 gives the definition and properties of the Choquet integral w.r.t. bi-capacities, and compares this approach with the definition of Greco *et al.* Section 4 provides the expression of Choquet integral in terms of the Möbius transform, finally Section 5 focuses on 2-additive bi-capacities, and gives in this case the expression in terms of interaction.

Throughout the paper, $N := \{1, \ldots, n\}$ denotes the finite referential set. To avoid heavy notations, we will often omit braces and commas to denote sets. For example, $\{i\}, \{i, j\}, \{1, 2, 3\}$ are denoted by $i, ij, 123$ respectively. The cardinality of sets will often be denoted by the corresponding lower case, e.g., n for $|N|$, k for $|K|$, etc.

2 Preliminaries

We provide in this section definitions and results necessary for the sequel. For the sake of brevity, we will refer the reader to Part I of the paper, so as to avoid repetitions of definitions and results. However, for the convenience of the reader we will sometimes make exceptions to this rule.

2.1 Capacities and various transformations

Let us denote by ν a capacity or a game (real-valued set function such that $\nu(\emptyset) = 0$) on N. We recall from Part I that the *Möbius transform* of ν is given by

$$m^\nu(A) := \sum_{B \subseteq A} (-1)^{a-b} \nu(B), \tag{1}$$

with inverse relation

$$\nu(A) = \sum_{B \subseteq A} m^\nu(B). \tag{2}$$

If ν is a capacity, it is said to be *k-additive* if $m^\nu(A) = 0$ for all $A \subseteq N$ such that $|A| > k$.

A convenient related transformation is the *co-Möbius transform*, defined by

$$\check{m}^\nu(A) := \sum_{B \supseteq N \setminus A} (-1)^{n-b} \nu(B),$$

which is also known in Dempster-Shafer theory under the name of "commonality function" [16], and in possibility theory [4] under the name of "guaranteed possibility measure" (see, e.g., [5]). It is closely related with the Möbius transform of the conjugate capacity:

$$\check{m}^{\bar{\nu}}(A) = (-1)^{|A|+1} m^\nu(A), \quad \forall A \subseteq N, A \neq \emptyset. \tag{3}$$

The *interaction transform* of ν is given by:

$$I^\nu(A) := \sum_{B \subseteq N \setminus A} \frac{(n-b-a)! \, b!}{(n-a+1)!} \Delta_A \nu(B),$$

2

with $\Delta_A \nu(K) = \sum_{L \subseteq A} (-1)^{a-l} \nu(L \cup K)$, for any $A \subseteq N$ and $K \subseteq N \setminus A$. Observe that if ν is k-additive, then $I^\nu(A) = 0$ for all $A \subseteq N$ such that $|A| > k$. The following property will be used in the sequel:

$$I^\nu(A) = (-1)^{|A|+1} I^{\overline{\nu}}(A). \tag{4}$$

2.2 Choquet integral

Let $f : N \longrightarrow \mathbb{R}^+$ and ν be a capacity or a game. We denote $f_i := f(i)$ for simplicity. The *Choquet integral* [2] of f w.r.t. ν is defined by:

$$\mathcal{C}_\nu(f) := \int_0^\infty \nu(\{i \mid f_i \geq \alpha\}) d\alpha = \sum_{i=1}^n f_{\sigma(i)} [\nu(A_{\sigma(i)}) - \nu(A_{\sigma(i+1)})] \tag{5}$$

$$= \sum_{i=1}^n [f_{\sigma(i)} - f_{\sigma(i-1)}] \nu(A_{\sigma(i)}), \tag{6}$$

where σ is a permutation on N such that $f_{\sigma(1)} \leq \cdots \leq f_{\sigma(n)}$, $A_{\sigma(i)} := \{\sigma(i), \ldots, \sigma(n)\}$, and $A_{\sigma(n+1)} := \emptyset$, $f_{\sigma(0)} := 0$. A fundamental property is that $\mathcal{C}_\nu(1_A, 0_{A^c}) = \nu(A)$, $\forall A \subseteq N$, where $(1_A, 0_{A^c})$ is the function f defined by $f_i = 1$ if $i \in A$ and 0 else (vertices of the hypercube $[0, 1]^n$). It is known that the Choquet integral is the simplest linear interpolation between vertices of the hypercube (see [6]). It is also continuous w.r.t. f_i, for all $i \in N$.

We consider now real valued functions f, and we denote by $f_i^+ := f_i \vee 0$ and $f_i^- := (-f_i)^+$, for all $i \in N$. The *asymmetric* Choquet integral of f w.r.t. ν is defined by

$$\mathcal{C}_\nu(f) := \mathcal{C}_\nu(f^+) - \mathcal{C}_{\overline{\nu}}(f^-), \tag{7}$$

while the *symmetric* Choquet integral (or *Šipoš integral*) is defined by:

$$\check{\mathcal{C}}_\nu(f) := \mathcal{C}_\nu(f^+) - \mathcal{C}_\nu(f^-) \tag{8}$$

(see, e.g., Denneberg [3]). These two integrals are particular cases of the Cumulative Prospect Theory model:

$$\text{CPT}_{\nu_+, \nu_-}(f) := \mathcal{C}_{\nu_+}(f^+) - \mathcal{C}_{\nu_-}(f^-), \tag{9}$$

where ν_+, ν_- are two capacities [17].

It is easy to get the following explicit expression of the CPT model:

$$
\begin{aligned}
\text{CPT}_{\nu_+, \nu_-}(f) = \sum_{i=1}^{p-1} &\big[f_{\sigma(i)} - f_{\sigma(i-1)}\big] \nu_-(\{\sigma(1), \ldots, \sigma(i)\}) \\
&+ f_{\sigma(p)} \nu_-(\{\sigma(1), \ldots, \sigma(p)\}) + f_{\sigma(p+1)} \nu_+(\{\sigma(p+1), \ldots, \sigma(n)\}) \\
&\qquad + \sum_{i=p+2}^n \big[f_{\sigma(i)} - f_{\sigma(i-1)}\big] \nu_+(\{\sigma(i), \ldots, \sigma(n)\}), \quad (10)
\end{aligned}
$$

with σ a permutation on N such that $f_{\sigma(1)} \leq f_{\sigma(p)} < 0 \leq f_{\sigma(p+1)} \leq f_{\sigma(n)}$.

The expression of the CPT model with respect to the Möbius transforms m^+, m^- and co-Möbius transforms $\check{m}^+, \check{m}^-$ of ν_+, ν_- is given by [9]:

$$\mathrm{CPT}_{\nu_+,\nu_-}(f) = \sum_{A\subseteq N^+} m^+(A) \bigwedge_{i\in A} f_i + \sum_{A\subseteq N^-} m^-(A) \bigvee_{i\in A} f_i \tag{11}$$

$$= \sum_{A\cap N^+\neq\emptyset} (-1)^{|A|+1}\check{m}^+(A) \bigvee_{i\in A} f_i + \sum_{A\cap N^-\neq\emptyset} (-1)^{|A|+1}\check{m}^-(A) \bigwedge_{i\in A} f_i \tag{12}$$

with $N^+ := \{i \in N \mid f_i \geq 0\}$, and $N^- := N \setminus N^+$.

We turn now to the interaction transform. As the general expression is rather complicated, we limit ourself to the 2-additive case. The result for the symmetric and asymmetric integrals is the following [8] (I denotes the interaction transform of ν).

$$\mathcal{C}_\nu(f) = \sum_{I(ij)>0} (f_i \wedge f_j)I(ij) + \sum_{I(ij)<0} (f_i \vee f_j)|I(ij)| + \sum_{i=1}^n f_i\Big(I(i) - \frac{1}{2}\sum_{j\neq i}|I(ij)|\Big) \tag{13}$$

$$\check{\mathcal{C}}_\nu(f) = \sum_{i,j\in N^+, I(ij)>0} (f_i \wedge f_j)I(ij) + \sum_{i,j\in N^-, I(ij)>0} (f_i \vee f_j)I(ij)$$

$$+ \sum_{i,j\in N^+, I(ij)<0} (f_i \vee f_j)|I(ij)| + \sum_{i,j\in N^-, I(ij)<0} (f_i \wedge f_j)|I(ij)|$$

$$+ \sum_{i\in N^+} f_i\Big(\sum_{j\in N^-, I(ij)<0} |I(ij)|\Big) + \sum_{i\in N^-} f_i\Big(\sum_{j\in N^+, I(ij)<0} |I(ij)|\Big)$$

$$+ \sum_{i=1}^n f_i\Big(I(i) - \frac{1}{2}\sum_{j\neq i}|I(ij)|\Big). \tag{14}$$

Lastly, we give an important property of the (asymmetric) Choquet integral. Two real-valued functions f, g are *comonotonic* if $f_i < f'_i$ for some $i, i' \in N$ implies $g_i \leq g'_i$ (equivalently if f, g can be made non decreasing by the same permutation). If f, g are comonotonic, then the asymmetric Choquet integral is additive, for any game ν:

$$\mathcal{C}_\nu(f + g) = \mathcal{C}_\nu(f) + \mathcal{C}_\nu(g). \tag{15}$$

For other properties of the Choquet integral, see [14].

2.3 Bi-capacities and bipolar capacities

We denote by $\mathcal{Q}(N) := \{(A, B) \in \mathcal{P}(N) \times \mathcal{P}(N) \mid A \cap B = \emptyset\}$ the set of all pairs of disjoint sets, which we endow with the partial order defined by $(A, B) \sqsubseteq (C, D)$ iff $A \subseteq C$ and $B \supseteq D$. Recall that a bi-capacity is an isotone mapping $v : \mathcal{Q}(N) \longrightarrow \mathbb{R}$, with $v(\emptyset, \emptyset) = 0$. Bi-capacities of the CPT type fulfill $v(A, B) = \nu_+(A) - \nu_-(B)$, where ν_+, ν_- are capacities.

When $\nu_+ = \nu_-$, the bi-capacity is said to be *symmetric*, and *asymmetric* when $\nu_- = \overline{\nu_+}$.

We recall that the Möbius transform (Part I, Section 5) of v is given by

$$m^v(A, A') = \sum_{\substack{(B,B')\sqsubseteq(A,A')\\B'\cap A=\emptyset}} (-1)^{|A\setminus B|+|B'\setminus A'|} v(B, B') = \sum_{\substack{B\subseteq A\\A'\subseteq B'\subseteq A^c}} (-1)^{|A\setminus B|+|B'\setminus A'|} v(B, B'),$$

$$\tag{16}$$

and the inverse transform is

$$v(A, A') = \sum_{(B,B') \sqsubseteq (A,A')} m^v(B, B').$$
(17)

Equation (17) can be rewritten using bi-unanimity games $u_{(A,B)}$ defined by $u_{(A,B)}(C, D) := 1$ iff $(C, D) \sqsupseteq (A, B)$, and 0 otherwise (see (25) in Part I). This shows that, as in the classical case, the set of all bi-unanimity games is a basis for bi-capacities:

$$v(A, A') = \sum_{(B,B') \in \mathcal{Q}(N)} m^v(B, B') u_{(B,B')}(A, A').$$
(18)

A bi-capacity is k-additive if its Möbius transform vanishes for all elements (A, B) of $\mathcal{Q}(N)$ such that $|B| < n - k$.

The interaction transform of v is defined by

$$I^v_{S,T} = \sum_{K \subseteq N \setminus (S \cup T)} \frac{(n - s - t - k)! k!}{(n - s - t + 1)!} \Delta_{S,T} v(K, N \setminus (K \cup S)),$$
(19)

for all $(S, T) \in \mathcal{Q}(N)$, and

$$\Delta_{S,T} v(K, L) = \sum_{\substack{S' \subseteq S \\ T' \subseteq T}} (-1)^{(s-s')+(t-t')} v(K \cup S', L \setminus T'), \quad \forall (K, L) \in \mathcal{Q}(N \setminus S), L \supseteq T. \quad (20)$$

The following formula will be useful.

$$I^v_{S,T} = \sum_{(S',T') \in [(S,N \setminus (S \cup T)),(N \setminus T, \emptyset)]} \frac{1}{n - s - t - t' + 1} m(S', T').$$
(21)

In the sequel, we omit superscript v for m, I if no ambiguity occurs.

On the other hand, Greco *et al.* presented the notion of bipolar capacity in [11], modelling the importance of the positive and negative parts of the bipolar scale by pairs of numbers, and providing a different view of bipolarity. A *bipolar capacity* is a function $\zeta : \mathcal{Q}(N) \to [0, 1] \times [0, 1]$ with $\zeta(A, B) =: (\zeta^+(A, B), \zeta^-(A, B))$ such that

- If $A \supseteq A'$ and $B \subseteq B'$ then $\zeta^+(A, B) \geq \zeta^+(A', B')$ and $\zeta^-(A, B) \leq \zeta^-(A', B')$.

- $\zeta^-(A, \emptyset) = 0$, $\zeta^+(\emptyset, A) = 0$ for any $A \subseteq N$.

- $\zeta(N, \emptyset) = (1, 0)$ and $\zeta(\emptyset, N) = (0, 1)$.

In multicriteria decision making, $\zeta^+(A, B)$ can be interpreted as the importance of coalition A of criteria in the presence of B for the positive part. $\zeta^-(A, B)$ can be interpreted as the importance of coalition B of criteria in the presence of A for the negative part.

This notion bears a different view of bipolarity, and can be related to the works of Cacioppo *et al.* [1] on the so-called *bivariate unipolar model*. The classical view of bipolar scales (see, e.g., Osgood *et al.* [15]) amounts to consider a single axis with a central element (the neutral value), demarcating "good" values from "bad" ones. This

is called the *univariate bipolar* model. Bi-capacities are based on this view of bipolarity, since they are valued on $[-1, 1]$, which is clearly a univariate bipolar scale with central value 0. By contrast, Cacioppo *et al.* consider a bipolar scale as two independent unipolar scales, one for the positive part of the affect, and the other for the negative part of the affect (bivariate unipolar model). The reason for such a view of bipolarity is that we may have for an alternative both a positive and a negative feeling, and we cannot mix them into a single resulting feeling. Clearly, bipolar capacities rely on this view of bipolarity, since they are valued on $[0, 1]^2$.

3 The Choquet integral w.r.t. bi-capacities

3.1 Definition and properties

The expression of the Choquet integral w.r.t a bi-capacity has been introduced axiomatically in [12], see also a presentation based on symmetry considerations in [7]. In this section we elaborate on properties, and relate to similar approaches.

Definition 1 *Let v be a bi-capacity and f be a real-valued function on N. The* (general) *Choquet integral of f w.r.t v is given by*

$$\mathcal{C}_v(f) := \mathcal{C}_{\nu_{N^+}}(|f|)$$

where ν_{N^+} is a game on N defined by

$$\nu_{N^+}(C) := v(C \cap N^+, C \cap N^-),$$

and $N^+ := \{i \in N | f_i \geq 0\}$, $N^- = N \setminus N^+$.

Observe that we have $\mathcal{C}_v(1_A, -1_B, 0_{(A \cup B)^c}) = v(A, B)$ for any $(A, B) \in \mathcal{Q}(N)$.
Using (5) an equivalent expression of $\mathcal{C}_v(f)$ is:

$$\mathcal{C}_v(f) = \sum_{i=1}^n |f_{\sigma(i)}| \left[v(A_{\sigma(i)} \cap N^+, A_{\sigma(i)} \cap N^-) - v(A_{\sigma(i+1)} \cap N^+, A_{\sigma(i+1)} \cap N^-) \right] \quad (22)$$

where $A_{\sigma(i)} := \{\sigma(i), \ldots, \sigma(n)\}$, and σ is a permutation on N so that $|f_{\sigma(1)}| \leq \cdots \leq |f_{\sigma(n)}|$. As it is the case for capacities, the Choquet integral w.r.t. bi-capacities is continuous w.r.t f_i for all $i \in N$.

For the sake of clarity, let us give a numerical example.

> EXAMPLE 1: We consider $N = \{1, 2, 3\}$, and the function f on N defined by $f(1) = -1$, $f(2) = 3$, and $f(3) = 2$. Then $N^+ = \{2, 3\}$, $N^- = \{1\}$, so that $\nu_{N^+}(C) = v(C \cap \{2, 3\}, C \cap \{1\})$. We obtain:

$$\begin{aligned} \mathcal{C}_v(f) &= \mathcal{C}_{\nu_{N^+}}(|f|) \\ &= |f(1)|\nu_{N^+}(N) + (|f(3)| - |f(1)|)\nu_{N^+}(\{2, 3\}) + (|f(2)| - |f(3)|)\nu_{N^+}(\{2\}) \\ &= v(\{2, 3\}, \{1\}) + v(\{2, 3\}, \emptyset) + v(\{2\}, \emptyset). \end{aligned}$$

The general Choquet integral has been axiomatized in [12] as follows. In the sequel, F_v denotes any functional on $\mathbb{R}^n$ defined w.r.t. a bi-capacity v. We introduce the following axioms (names are those used in [12]).

Monotonicity w.r.t. Bi-Capacities (MBC): For any bi-capacity v on $\mathcal{Q}(N)$, $\forall f, f' \in \mathbb{R}^n$,

$$f_i \leq f'_i, \ \forall i \in N \ \Rightarrow F_v(f) \leq F_v(f')$$

Properly Weighted w.r.t. Bi-Capacities (PWBC): For any bi-capacity v, for any ternary vector, $F_v(1_A, -1_B, 0_{(A \cup B)^c}) = v(A, B)$.

Stable under Positive Linear transformations with positive shifts for Bi-Capacities and binary acts (SPLBC$^+$): For any bi-capacity v on $\mathcal{Q}(N)$, for all $A, C \subseteq N$, $\alpha > 0$, and $\beta \geq 0$,

$$F_v \left((\alpha + \beta)_A, \beta_{A^c} \right) = \alpha F_v \left(1_A, 0_{A^c} \right) + \beta v(N, \emptyset).$$

Linearity w.r.t. Bi-Capacities (LBC): Let $A \subseteq N$, and real-valued functions $v, v_1, \ldots, v_p$ on $\mathcal{Q}(N)$ vanishing at $(\emptyset, \emptyset)$. If they satisfy $v(B, B') = \sum_{i=1}^{p} \alpha_i v_i(B, B')$ with $\alpha_1, \ldots, \alpha_p \in \mathbb{R}$, for all (B, B') such that $B \subseteq A$ and $B' \cap A = \emptyset$, then for all $f \in \Sigma_A$

$$F_v(f) = \sum_{i=1}^{p} \alpha_i \, F_{v_i}(f),$$

with $\Sigma_A := \{f \in \mathbb{R}^n \mid f(i) \geq 0 \text{ iff } i \in A\}$.

For $A \subseteq N$, consider the following application $\Pi_A : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$(\Pi_A(f))_i = \begin{cases} f_i & \text{if } i \in A \\ -f_i & \text{otherwise.} \end{cases}$$

By **(PWBC)**, $v(B, B')$ corresponds to the point $\left(1_B, -1_{B'}, 0_{(B \cup B')} \right)$. Define $\Pi_A \circ v(B, B')$ as the term of the bi-capacity associated to the point

$$\Pi_A \left(1_B, -1_{B'}, 0_{(B \cup B')^c} \right) = \left(1_{(B \cap A) \cup (B' \setminus A)}, -1_{(B \setminus A) \cup (B' \cap A)}, 0_{(B \cup B')^c} \right).$$

Hence we set

$$\Pi_A \circ v(B, B') := v \left((B \cap A) \cup (B' \setminus A), (B \setminus A) \cup (B' \cap A) \right).$$

By symmetry arguments, it is reasonable to have $F_{\Pi_A \circ v} \left(\Pi_A(f) \right)$ being equal to $F_v(f)$.

Symmetry (Sym): For any $v : \mathcal{Q}(N) \to \mathbb{R}$, we have for all $A \subseteq N$

$$F_v(f) = F_{\Pi_A \circ v} \left(\Pi_A(f) \right).$$

The following can be shown [12].

Theorem 1 $\{F_v\}_v$ *satisfies* **(LBC)**, **(MBC)**, **(PWBC)**, **(SPLBC$^+$)** *and* **(Sym)** *if and only if for any bi-capacity v, and for any $N^+ \subseteq N$, $f \in \Sigma_{N^+}$,*

$$F_v(f) = \mathcal{C}_{\nu_{N^+}}(f_{N^+}, -f_{N^-})$$

where $\nu_{N^+}(C) := v(C \cap N^+, C \cap N^-)$, and $\Sigma_{N^+} := \{f \in \mathbb{R}^n, f_{N^+} \geq 0, f_{N^-} < 0\}$.

Let us present another way to derive the Choquet integral based on interpolation considerations. As said in Section 2.2, the Choquet integral w.r.t. capacities can be seen as the simplest linear interpolator between vertices of the hypercube $[0,1]^n$ [6], it is in fact the Lovász extension of pseudo-Boolean functions [13]. The interpolation is such that for any $f \in [0,1]^n$, the vertices which are used are those of the simplex $\{x \in [0,1]^n \mid x_{\sigma(1)} \leq x_{\sigma(2)} \leq \cdots \leq x_{\sigma(n)}\}$, where σ is a permutation on N such that $f_{\sigma(1)} \leq f_{\sigma(2)} \leq \cdots \leq f_{\sigma(n)}$.

Let us apply a similar approach for the case of bi-capacities, and call F the function we obtain by interpolation. To do this, we examine in details the case $n = 2$ (Fig. 1). Let us take any point f such that $f_1 \geq 0$, $f_2 \leq 0$ and $|f_1| \leq |f_2|$. Then, for $|f|$ which is
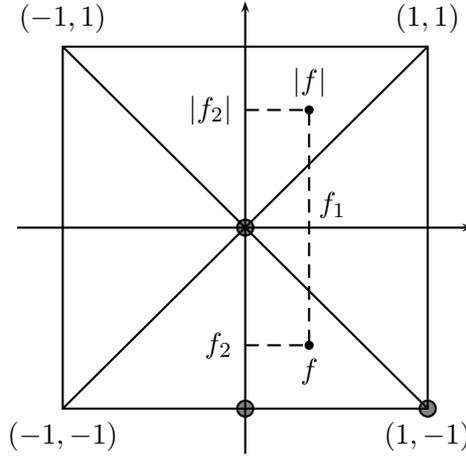

Figure 1: Interpolation for the case of bi-capacities

in the first (positive) quadrant, we already know that the best linear interpolation is the Choquet integral. It suffices to use the formula with the adequate vertices:

$$F(f_1, f_2) := |f_1|F(1, -1) + (|f_2| - |f_1|)F(0, -1)$$

This is a Choquet integral w.r.t a game ν_1 defined by:

$$\nu_1(\{1,2\}) = F(1,-1)$$
$$\nu_1(\{2\}) = F(0,-1).$$

Let us consider now the general case. Using $N^+ = \{i \in N \mid f_i \geq 0\}$, $N^- = N \setminus N^+$ as before, with similar considerations of symmetry, we obtain:

$$F(f) = |f_{\sigma(1)}|F(1_{N^+}, -1_{N^-})$$
$$+ \sum_{i=2}^{n}(|f_{\sigma(i)}| - |f_{\sigma(i-1)}|)F(1_{\{\sigma(i),...,\sigma(n)\} \cap N^+}, -1_{\{\sigma(i),...,\sigma(n)\} \cap N^-})$$

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where σ is a permutation on N such that $|f_{\sigma(1)}| \leq \cdots \leq |f_{\sigma(n)}|$. This expression is the Choquet integral of $|f|$ w.r.t. a game ν_{N+} defined by:

$$\nu_{N+}(A) := F(1_{A \cap N^+}, -1_{A \cap N^-}).$$

Recalling that $F(1_A, -1_B) =: v(A, B)$, we recover our previous Definition 1.

The following result shows that our definition encompasses the CPT model, and consequently the symmetric and asymmetric Choquet integrals.

Proposition 1 *If v is of the CPT type, with $v(A, B) = \nu_+(A) - \nu_-(B)$, the Choquet integral reduces to*

$$\mathcal{C}_v(f) = \sum_{i=1}^{n} f_{\sigma(i)}^+ \left[\nu_+(A_{\sigma(i)} \cap N^+) - \nu_+(A_{\sigma(i+1)} \cap N^+) \right]$$

$$- \sum_{i=1}^{n} f_{\sigma(i)}^- \left[\nu_-(A_{\sigma(i)} \cap N^-) - \nu_-(A_{\sigma(i+1)} \cap N^-) \right]$$

$$= \mathcal{C}_{\nu_+}(f^+) - \mathcal{C}_{\nu_-}(f^-).$$

Proof: Using the definition of v, and splitting N in N^+, N^-, Equation (22) becomes:

$$\mathcal{C}_v(f) = \sum_{\sigma(i) \in N^+} f_{\sigma(i)} \left[\nu_+(A_{\sigma(i)} \cap N^+) - \nu_-(A_{\sigma(i)} \cap N^-) - \nu_+(A_{\sigma(i+1)} \cap N^+) + \nu_-(A_{\sigma(i+1)} \cap N^-) \right]$$

$$- \sum_{\sigma(i) \in N^-} f_{\sigma(i)}^- \left[\nu_+(A_{\sigma(i)} \cap N^+) - \nu_-(A_{\sigma(i)} \cap N^-) - \nu_+(A_{\sigma(i+1)} \cap N^+) + \nu_-(A_{\sigma(i+1)} \cap N^-) \right].$$

If $\sigma(i) \in N^+$, then $A_{\sigma(i)} \cap N^- = A_{\sigma(i+1)} \cap N^-$, and if $\sigma(i) \in N^-$, we have $A_{\sigma(i)} \cap N^+ = A_{\sigma(i+1)} \cap N^+$. Substituting in the above expression leads to the first relation. Let us denote by σ^+, σ^- the permutations on N such that f^+, f^- become non decreasing. Observe that $A_{\sigma^+(i)} = A_{\sigma(i)} \cap N^+$ and $A_{\sigma^-(i)} = A_{\sigma(i)} \cap N^-$, which proves the second relation. ∎

Lastly, we give a property related to comonotonicity. Two real-valued functions are said to be *cosigned* if for all $i \in N$, $f_i > 0$ implies $g_i \geq 0$ and $f_i < 0$ implies $g_i \leq 0$.

Proposition 2 *Let f, g be real-valued functions on N. If f, g are cosigned and $|f|, |g|$ are comonotonic, then for any bi-capacity v*

$$\mathcal{C}_v(f + g) = \mathcal{C}_v(f) + \mathcal{C}_v(g).$$

Proof: Let us suppose for the moment that $f_i, g_i \neq 0$, for all $i \in N$. Let us denote $N_f := \{i \in N \mid f_i \geq 0\}$. Due to our assumptions, we have $N_{f+g} = N_f = N_g$, and $|f + g| = |f| + |g|$. Then, by comonotonic additivity of the Choquet integral:

$$\mathcal{C}_v(f + g) = \mathcal{C}_{\nu_{N_{f+g}}}(|f + g|) = \mathcal{C}_{\nu_{N_{f+g}}}(|f| + |g|)$$

$$= \mathcal{C}_{\nu_{N_f}}(|f|) + \mathcal{C}_{\nu_{N_g}}(|g|)$$

$$= \mathcal{C}_v(f) + \mathcal{C}_v(g).$$

Now suppose that f or g are zero for some $i \in N$. Define f', g' which are identical to f, g, except on those $i \in N$, where we put $f'_i = \epsilon$ or $g'_i = \epsilon$. Then we are back to our previous case, where the above result holds. Now the result holds for f, g too by continuity of $\mathcal{C}_v$ since we can obtain f, g as limits of f', g'. ∎

3.2   The Choquet integral w.r.t bipolar capacities

Formula (22) is very similar to the one proposed by Greco *et al.* [11]. For $f \in \mathbb{R}^n$, let σ be a permutation on N rearranging $|f|$ as above. Let

$$A_i^+ := \big\{ \sigma(j)\,,\; j \in \{i, \ldots, n\} \quad \text{such that } f_{\sigma(j)} \geq 0 \big\}$$
$$A_i^- := \big\{ \sigma(j)\,,\; j \in \{i, \ldots, n\} \quad \text{such that } f_{\sigma(j)} \leq 0 \big\}$$

and

$$\mathcal{C}^+(f; \zeta) = \sum_{i \in N} \left(f_{\sigma(i)}^+ - f_{\sigma(i-1)}^+ \right) \zeta^+ \left(A_i^+, A_i^- \right)$$
$$\mathcal{C}^-(f; \zeta) = \sum_{i \in N} \left(f_{\sigma(i)}^- - f_{\sigma(i-1)}^- \right) \zeta^- \left(A_i^+, A_i^- \right)$$

where $f_{\sigma(0)} := 0$, and f^+, f^- defined as above. Finally the Choquet integral w.r.t. ζ is defined by

$$\mathcal{C}(f; \zeta) := \mathcal{C}^+(f; \zeta) - \mathcal{C}^-(f; \zeta). \tag{23}$$

We illustrate the definition on the same function as in Ex. 1.

EXAMPLE 2: Let us consider again $N = \{1, 2, 3\}$, and $f(1) = -1$, $f(2) = 3$ and $f(3) = 2$. Applying the definition we get:

$$A_1^+ = \{2, 3\}, \quad A_1^- = \{1\}$$
$$A_2^+ = \{2, 3\}, \quad A_2^- = \emptyset$$
$$A_3^+ = \{2\}, \quad A_3^- = \emptyset.$$

Hence

$$\mathcal{C}^+(f; \zeta) = f^+(1)\zeta^+(\{2,3\}, \{1\}) + (f^+(3) - f^+(1))\zeta^+(\{2,3\}, \emptyset) + (f^+(2) - f^+(3))\zeta^+(\{2\}, \emptyset)$$
$$= 2\zeta^+(\{2,3\}, \emptyset) + \zeta^+(\{2\}, \emptyset).$$
$$\mathcal{C}^-(f; \zeta) = f^-(1)\zeta^-(\{2,3\}, \{1\}) + (f^-(3) - f^-(1))\zeta^-(\{2,3\}, \emptyset) + (f^-(2) - f^-(3))\zeta^-(\{2\}, \emptyset)$$
$$= \zeta^-(\{2,3\}, \{1\}) - \zeta^-(\{2,3\}, \emptyset).$$

Finally we obtain

$$\mathcal{C}(f; \zeta) = 2\zeta^+(\{2,3\}, \emptyset) + \zeta^+(\{2\}, \emptyset) - \zeta^-(\{2,3\}, \{1\}) + \zeta^-(\{2,3\}, \emptyset).$$

The two definitions (Eq. (23) and Def. 1) share similar properties. In [11], Greco *et al.* mention that cosigned comonotonic additivity (as in Prop. 2) holds, and even more, that this property together with homogeneity and idempotency is sufficient to characterize the Choquet integral w.r.t. a bipolar capacity. For further developments along this approach of the Choquet integral, see [10].

For $f \in \mathbb{R}^n$ for which several permutations σ are possible, it is easy to see that Eq. (23) depends on the choice of the permutation. This is not the case of the usual Choquet integral or the Choquet integral w.r.t. a bi-capacity. Enforcing that the results are the same for all permutations rearranging $|f|$ in increasing order, the following theorem shows that we obtain some constraints on the bipolar capacity that make it reduce to a bi-capacity.

Theorem 2 *The Choquet integral w.r.t. a bipolar capacity ζ is unambiguously defined by (23) if and only if*

$$\forall (A, B) \in \mathcal{Q}(N), \quad \zeta^+(A, B) - \zeta^-(\emptyset, B) = \zeta^+(A, \emptyset) - \zeta^-(A, B). \tag{24}$$

The bipolar capacity ζ reduces then exactly to a bi-capacity v defined by

$$\forall (A, B) \in \mathcal{Q}(N), \quad v(A, B) := \zeta^+(A, B) - \zeta^-(\emptyset, B).$$

Moreover, the Choquet integral w.r.t. ζ is equal to $\mathcal{C}_v$.

Proof: Let us show first the necessity part. Let $(A, B) \in \mathcal{Q}(N)$. Consider the ternary act $f = (1_A, -1_B, 0_{-A \cup B})$. Several permutations σ can rearrange $|f|$ in increasing order. A first one is such that $A = \{\sigma(n - a + 1), \ldots, \sigma(n)\}$, $B = \{\sigma(n - a - b + 1), \ldots, \sigma(n - a)\}$ and $N \setminus (A \cup B) = \{\sigma(1), \ldots, \sigma(n - a - b)\}$ where $a = |A|$ and $b = |B|$. It is easy to see that we obtain

$$\mathcal{C}^+(f; \zeta) = \zeta^+(A, \emptyset)$$

and

$$\mathcal{C}^-(f; \zeta) = \zeta^-(A, B) - \zeta^-(A, \emptyset) = \zeta^-(A, B)$$

so that

$$\mathcal{C}(f; \zeta) = \zeta^+(A, \emptyset) - \zeta^-(A, B) .$$

A second permutation is such that $B = \{\sigma(n - b + 1), \ldots, \sigma(n)\}$, $A = \{\sigma(n - a - b + 1), \ldots, \sigma(n - b)\}$ and $N \setminus (A \cup B) = \{\sigma(1), \ldots, \sigma(n - a - b)\}$. We get

$$\mathcal{C}^+(f; \zeta) = \zeta^+(A, B) \quad , \quad \mathcal{C}^-(f; \zeta) = \zeta^-(\emptyset, B)$$

and

$$\mathcal{C}(f; \zeta) = \zeta^+(A, B) - \zeta^-(\emptyset, B) .$$

In order to be consistent with Eq. (23), the previous two expressions of $\mathcal{C}(f; \zeta)$ shall be the same. This proves Eq. (24).

Let us show now the sufficiency part. Consider a bipolar capacity ζ satisfying (24). Consider an alternative f. The ambiguity of definition (23) lies in the fact that the two terms $\mathcal{C}^+(f; \zeta)$ and $\mathcal{C}^-(f; \zeta)$ may depend on the choice of the permutation σ if several permutations fulfill $|f_{\sigma(1)}| \leq \cdots \leq |f_{\sigma(n)}|$. Henceforth, definition (23) is fine for sure if there is a unique permutation σ, that is to say if all components of f are different in absolute value.

Consider $\alpha \in \{|f_1|, \ldots, |f_n|\}$. Let $C^+ = \{i \in N : f_i = \alpha\}$ and $C^- = \{i \in N : f_i = -\alpha\}$. Function $\alpha \mapsto \mathcal{C}(f; \zeta)$ with $|f_i| = \alpha$ is affine in α for α belonging to a small interval. Let us

show that the weight of α in this function does not depend on the choice of the permutation σ. Let $p-1$ be the number of components of f being strictly lower than α in absolute value. Let $q = p+|C^+\cup C^-|-1$. Any permutation σ such that $\{\sigma(p),\ldots,\sigma(q)\} = C^+\cup C^-$ is admissible. We have four cases

- $\sigma(p) \in C^+$ and $\sigma(q) \in C^-$. There exist $k \in N$ and two decompositions $C_1^+\cup\ldots\cup C_k^+$ and $C_1^- \cup \ldots \cup C_k^-$ of C^+ and C^- respectively, such that for any $i \in \{1,\ldots,k\}$,

$$C_i^+ = \{\sigma(p_i),\ldots,\sigma(p_i'-1)\} \quad C_i^- = \{\sigma(p_i'),\ldots,\sigma(p_{i+1}-1)\}$$

  with $p = p_1 < p_1' < \cdots < p_k' < p_{k+1} = q+1$.

- $\sigma(p) \in C^+$ and $\sigma(q) \in C^+$. There exist $k \in N$ and two decompositions $C_1^+\cup\ldots\cup C_k^+$ and $C_1^- \cup \ldots \cup C_{k-1}^-$ of C^+ and C^- respectively, such that

$$\begin{array}{ll}\forall i \in \{1,\ldots,k\}, & C_i^+ = \{\sigma(p_i),\ldots,\sigma(p_i'-1)\} \\ \forall i \in \{1,\ldots,k-1\}, & C_i^- = \{\sigma(p_i'),\ldots,\sigma(p_{i+1}-1)\}\end{array}$$

  with $p = p_1 < p_1' < \cdots < p_k < p_k' = q+1$.

- $\sigma(p) \in C^-$ and $\sigma(q) \in C^-$. There exist $k \in N$ and two decompositions $C_1^+ \cup \ldots \cup C_{k-1}^+$ and $C_1^- \cup \ldots \cup C_k^-$ of C^+ and C^- respectively, such that

$$\begin{array}{ll}\forall i \in \{1,\ldots,k-1\}, & C_i^+ = \{\sigma(p_i'),\ldots,\sigma(p_{i+1}-1)\} \\ \forall i \in \{1,\ldots,k\}, & C_i^- = \{\sigma(p_i),\ldots,\sigma(p_i'-1)\}\end{array}$$

  with $p = p_1 < p_1' < \cdots < p_k < p_k' = q+1$.

- $\sigma(p) \in C^-$ and $\sigma(q) \in C^+$. There exist $k \in N$ and two decompositions $C_1^+\cup\ldots\cup C_k^+$ and $C_1^- \cup \ldots \cup C_k^-$ of C^+ and C^- respectively, such that for any $i \in \{1,\ldots,k\}$,

$$C_i^+ = \{\sigma(p_i'),\ldots,\sigma(p_{i+1}-1)\} \quad C_i^- = \{\sigma(p_i),\ldots,\sigma(p_i'-1)\}$$

  with $p = p_1 < p_1' < \cdots < p_k' < p_{k+1} = q+1$.

These four cases are dealt with in the same way. Let us consider only the first one. For $i \in \{1,\ldots,k\}$, let $\mathcal{A}_i^+ := C_i^+ \cup \cdots \cup C_k^+ \cup A_{q+1}^+$ and $\mathcal{A}_i^- := C_i^- \cup \cdots \cup C_k^- \cup A_{q+1}^-$. From the relation

$$\mathcal{C}^+(f;\zeta) = \sum_{i\in N} f_{\sigma(i)}^+ \left(\zeta^+\left(A_i^+, A_i^-\right) - \zeta^+\left(A_{i+1}^+, A_{i+1}^-\right)\right)$$

we get that the weight of α in $\mathcal{C}^+(f;\zeta)$ is equal to

$$\begin{aligned}\sum_{j=1}^k \sum_{i=p_j}^{p_j'-1} &\Big[\zeta^+\left(\{\sigma(i),\ldots,\sigma(p_j'-1)\} \cup \mathcal{A}_{j+1}^+, \mathcal{A}_j^-\right) \\ &-\zeta^+\left(\{\sigma(i+1),\ldots,\sigma(p_j'-1)\} \cup \mathcal{A}_{j+1}^+, \mathcal{A}_j^-\right)\Big] \\ = \sum_{j=1}^k &\left(\zeta^+\left(\mathcal{A}_j^+, \mathcal{A}_j^-\right) - \zeta^+\left(\mathcal{A}_{j+1}^+, \mathcal{A}_j^-\right)\right) \ .\end{aligned}$$

where $\mathcal{A}_{k+1}^+ := A_{q+1}^+$. Similarly, the weight of α in $\mathcal{C}^-(f;\zeta)$ is equal to

$$\sum_{j=1}^k \left(\zeta^-\left(\mathcal{A}_{j+1}^+, \mathcal{A}_j^-\right) - \zeta^-\left(\mathcal{A}_{j+1}^+, \mathcal{A}_{j+1}^-\right)\right) \ ,$$

where $\mathcal{A}_{k+1}^- := A_{q+1}^-$. As a consequence, the weight of α in $\mathcal{C}(f;\zeta)$ is

$$\zeta^+\left(\mathcal{A}_1^+, \mathcal{A}_1^-\right) + \zeta^-\left(\mathcal{A}_{k+1}^+, \mathcal{A}_{k+1}^-\right) + \sum_{j=2}^k \left(\zeta^+\left(\mathcal{A}_j^+, \mathcal{A}_j^-\right) + \zeta^-\left(\mathcal{A}_j^+, \mathcal{A}_j^-\right)\right)$$
$$- \sum_{j=1}^k \left(\zeta^+\left(\mathcal{A}_{j+1}^+, \mathcal{A}_j^-\right) + \zeta^-\left(\mathcal{A}_{j+1}^+, \mathcal{A}_j^-\right)\right)$$

By (24), we have $\zeta^+(A, B) + \zeta^-(A, B) = \zeta^+(A, \emptyset) + \zeta^-(\emptyset, B)$ for any $(A, B) \in \mathcal{Q}(N)$. Hence the weight of α in $\mathcal{C}(f;\zeta)$ is

$$\zeta^+\left(A_p^+, A_p^-\right) + \zeta^-\left(A_{q+1}^+, A_{q+1}^-\right) + \sum_{j=2}^k \left(\zeta^+\left(\mathcal{A}_j^+, \emptyset\right) + \zeta^-\left(\emptyset, \mathcal{A}_j^-\right)\right)$$

$$- \sum_{j=1}^k \left(\zeta^+\left(\mathcal{A}_{j+1}^+, \emptyset\right) + \zeta^-\left(\emptyset, \mathcal{A}_j^-\right)\right)$$
$$= \left[\zeta^+\left(A_p^+, A_p^-\right) - \zeta^-\left(\emptyset, A_p^-\right)\right] - \left[\zeta^+\left(A_{q+1}^+, \emptyset\right) - \zeta^-\left(A_{q+1}^+, A_{q+1}^-\right)\right] \qquad (25)$$

Since this term depends only on p and q but not on the decomposition of C^+ and C^-, we conclude that the expression of $\mathcal{C}(f;\zeta)$ does not depend anymore on the choice of the permutation. Hence the *if* part of the proof is shown.

Suppose that ζ satisfies (24). Let v be the bi-capacity defined by

$$v(A, B) := \zeta^+(A, B) - \zeta^-(\emptyset, B) = \zeta^+(A, \emptyset) - \zeta^-(A, B), \quad \forall (A, B) \in \mathcal{Q}(N).$$

Since $\zeta^+(\emptyset, \emptyset) = 0$ and $\zeta^-(\emptyset, \emptyset) = 0$, we get

$$\zeta^+(A, \emptyset) = v(A, \emptyset) \quad, \quad \zeta^-(\emptyset, B) = -v(\emptyset, B), \quad \forall A, B \subseteq N.$$

Plugging this into previous relation, we get

$$\zeta^+(A, B) = v(A, B) - v(\emptyset, B) \quad \text{and} \quad \zeta^-(A, B) = v(A, \emptyset) - v(A, B).$$

This shows that ζ reduces exactly to v. Finally, by (25), the weight of α in $\mathcal{C}(f;\zeta)$ is

$$v\left(A_p^+, A_p^-\right) - v\left(A_{q+1}^+, A_{q+1}^-\right).$$

This is exactly the value of the weight of α in $\mathcal{C}_v(f)$. We conclude that the Choquet integral w.r.t. ζ is equal to $\mathcal{C}_v$. $\blacksquare$

The concept of bipolar capacity reduces to bi-capacities when the Choquet integral is used. In fact, it seems that considering the Choquet integral w.r.t. a bipolar capacity as the pair $(\mathcal{C}^+(f;\zeta), \mathcal{C}^-(f;\zeta))$ would be more in the spirit of the double unipolar model of Cacioppo *et al.*

4 Expressions of the Choquet integral with the Möbius transform

The following result expresses the Choquet integral in terms of the Möbius transform.

Proposition 3 *For any bi-capacity v, any real valued function f on N,*

$$\mathcal{C}_v(f) = \sum_{B \subseteq N} m(\emptyset, B)\left(\bigwedge_{i \in B^c \cap N^-} f_i\right) + \sum_{\substack{(A,B) \in \mathcal{Q}(N) \\ A \neq \emptyset}} m(A,B)\left[\left(\bigwedge_{i \in (A \cup B)^c \cap N^-} f_i + \bigwedge_{i \in A} f_i\right) \vee 0\right]$$

with the convention $\wedge_\emptyset f_i := 0$.

The proof is based on the following result.

Lemma 1 *For any real valued function f on N, the Choquet integral of f w.r.t. a bi-unanimity game $u_{(A,B)}$ (see Section 2.3 or (25) in Part I) is given by:*

$$\mathcal{C}_{u_{(A,B)}}(f) = \begin{cases} 0, & \text{if } (A,B) = (\emptyset, N) \\ \left(\bigwedge_{i \in B^c} f_i\right) \wedge 0, & \text{if } (A,B) = (\emptyset, B) \\ \left(\bigwedge_{i \in A} f_i\right) \vee 0, & \text{if } (A,B) = (A, A^c), A \neq \emptyset \\ \left(\bigwedge_{i \in (A \cup B)^c \cap N^-} f_i + \bigwedge_{i \in A} f_i\right) \vee 0 & , \text{ otherwise.} \end{cases} \tag{26}$$

Proof: Using (22), we get:

$$\mathcal{C}_{u_{(A,B)}}(f) = \sum_{i=1}^{n} |f_{\sigma(i)}| \left[u_{(A,B)}(A_{\sigma(i)} \cap N^+, A_{\sigma(i)} \cap N^-) - u_{(A,B)}(A_{\sigma(i+1)} \cap N^+, A_{\sigma(i+1)} \cap N^-) \right]$$

with the above defined notations. For a given $i \in N$, the difference between brackets is non zero only in the following cases: the left term is 0 and the second one is equal to 1 (case 1_i), or the converse (case 2_i). Case 1_i happens if and only if $A_{\sigma(i+1)} \cap N^+ \supset A$ and $A_{\sigma(i+1)} \cap N^- \subseteq B$, and either $A_{\sigma(i)} \cap N^+ \not\supset A$ or $A_{\sigma(i)} \cap N^- \not\subseteq B$. Observing that the third condition cannot occur if the first holds, this amounts to:

$$\text{Case } 1_i \Leftrightarrow \begin{cases} (\alpha_i) & \sigma(i) \in B^c \cap N^- \\ (\beta_i) & A_{\sigma(i)} \cap N^+ \supset A \\ (\gamma_i) & A_{\sigma(i+1)} \cap N^- \subseteq B. \end{cases}$$

Proceeding similarly with case 2_i, we get:

$$\text{Case } 2_i \Leftrightarrow \begin{cases} (\alpha'_i) & \sigma(i) \in A \cap N^+ \\ (\beta_i) & A_{\sigma(i)} \cap N^+ \supset A \\ (\gamma_i) & A_{\sigma(i+1)} \cap N^- \subseteq B. \end{cases}$$

We remark that only the first conditions differ.

Let us suppose that $(A, B) = (\emptyset, N)$. Then, for any $i \in N$, neither case 1_i nor case 2_i can happen, since conditions (α_i), (α'_i) cannot hold. Hence $\mathcal{C}_{u_{(\emptyset,N)}}(f) = 0$. This proves the first line in (26).

Let us suppose $(A, B) = (\emptyset, B)$, $B \neq N$. Then for any $i \in N$, case 2_i can never occur since condition (α'_i) is not fulfilled, and condition (β_i) is always fulfilled. If $B^c \cap N^- \neq \emptyset$, there is at least one i such that both (α_i) and (γ_i) are fulfilled, namely $\sigma(i) \in B^c \cap N^-$ such that $|f_{\sigma(i)}|$ is maximum (this forces $\sigma(i+1)$ to be either in N^+ or in B). Hence we

get $\mathcal{C}_{u_{(\emptyset,B)}}(f) = \wedge_{i \in B^c} f_i$ under $B^c \cap N^- \neq \emptyset$, and 0 otherwise. This proves the second line of (26).

Let us suppose $(A, B) = (A, A^c)$. If condition β_i holds for some i, then we have $A \subseteq N^+$. For any $i \in N$, case 1_i cannot occur since $\sigma(i) \in A \cap N^-$ by condition α_i, so that by condition β_i, $\sigma(i) \in N^+$, which contradicts condition α_i. Any $\sigma(i) \in A$ fulfills conditions α_i' and γ_i, while fulfilling also condition β_i imposes to choose $\sigma(i) \in A$ such that $f_{\sigma(i)}$ is minimum. Hence, under these conditions we get $\mathcal{C}_{u_{(A,A^c)}}(f) = \wedge_{i \in A} f_i$. This proves the third line of (26).

Let us consider the general case, with $A \neq \emptyset, B^c$. Let us suppose that case 1_i occurs for some $i \in N$. Then necessarily, $A \subseteq N^+$ by condition β_i, $\sigma(i) \in (A \cup B)^c \cap N^- \neq \emptyset$ (conditions α_i, β_i), and moreover $\sigma(i)$ is such that $|f_{\sigma(i)}|$ is maximum on $(A \cup B)^c \cap N^-$ and

$$|f_{\sigma(i)}| < \wedge_{j \in A} f_{\sigma(j)} \tag{27}$$

(conditions β_i, γ_i). Under the assumption that case 1_i holds, let us show that case 2_j must occur for some j. Since $\emptyset \neq A \subseteq N^+$, condition α_j' holds for some j. For any such j, $j > i$ by (27). Now, condition β_j imposes to choose j such that $f_{\sigma(j)}$ is minimum in $A \cap N^+$. j being such defined, let us show that γ_j holds. Suppose it is not the case. This means that there exists j' such that $\sigma(j') \in A_{\sigma(j+1)} \cap N^-$ and $\sigma(j') \notin B$. Since $A \subseteq N^+$, this means that $\sigma(j') \in (A \cup B)^c \cap N^-$. However, $i < j < j'$, so that $|f_{\sigma(j')}| > |f_{\sigma(i)}|$, which contradicts the definition of $\sigma(i)$. The reverse result can be shown as well, so that either both cases hold or both fail to hold. If both hold, $\mathcal{C}_{u_{(A,B)}}(f) = \wedge_{i \in (A \cup B)^c \cap N^-} f_i + \wedge_{i \in A} f_i$, which is ≥ 0 by (27). This proves the last line of (26). ∎

Using the above lemma, the proof of Prop. 3 is immediate considering Eq. (18) and convention $\wedge_\emptyset = 0$.

We can recover the result for the CPT model (11) and (12), using Prop. 1 of Part I, and (3). Recalling the convention $\wedge_\emptyset f_i := 0$, we get:

$$\mathcal{C}_v(f) = \sum_{B \subsetneq N} m^{\overline{\nu_-}}(B^c) \bigwedge_{B^c \cap N^-} f_i + \sum_{A \neq \emptyset} m^{\nu_+}(A) \left(\bigwedge_{i \in A} f_i \vee 0 \right)$$

$$= \sum_{B \cap N^- \neq \emptyset} (-1)^{b+1} \check{m}^{\nu_-}(B) \bigwedge_{i \in B} f_i + \sum_{A \subseteq N^+} m^{\nu_+}(A) \bigwedge_{i \in A} f_i$$

after some simplifications.

We illustrate the expression w.r.t. the Möbius transform taking again the same example.

> EXAMPLE 3: We consider as before $N = \{1, 2, 3\}$ and $f(1) = -1$, $f(2) = 3$ and $f(3) = 2$. Since $N^- = \{1\}$, among the 8 terms in $m(\emptyset, B)$, only those such that $1 \notin B$ are non zero. Among the 19 terms in $m(A, B)$, $A \neq \emptyset$, those

such that $1 \in A$ are zero. Hence it remains

$$\begin{aligned}
\mathcal{C}_v(f) = & [m(\emptyset, \emptyset) + m(\emptyset, 2) + m(\emptyset, 3) + m(\emptyset, \{2,3\})]f(1) \\
& + [m(2, \emptyset) + m(2, 3)][(f(1) + f(2)) \vee 0] + [m(3, \emptyset) + m(3, 2)][(f(1) + f(3)) \vee 0] \\
& + m(\{2,3\}, \emptyset)[(f(1) + (f(2) \wedge f(3))) \vee 0] \\
& + [m(2, 1) + m(2, \{1,3\})]f(2) + +[m(3, 1) + m(3, \{1,2\})]f(3) + m(\{2,3\}, 1)[f(2) \wedge f(3)] \\
= & -[m(\emptyset, \emptyset) + m(\emptyset, 2) + m(\emptyset, 3) + m(\emptyset, \{2,3\})] \\
& + 2[m(2, \emptyset) + m(2, 3)] + m(3, \emptyset) + m(3, 2) + m(\{2,3\}, \emptyset) + 3[m(2, 1) + m(2, \{1,3\})] \\
& + 2[m(3, 1) + m(3, \{1,2\}) + m(\{2,3\}, 1)].
\end{aligned}$$

5   The case of 2-additive bi-capacities

Let us consider a 2-additive bi-capacity v. The Möbius transform is non zero for any (A, B) such that $|B| \geq n-2$, i.e., for elements $(\emptyset, i^c)$, $(\emptyset, \{i, j\}^c)$, (i, i^c), and $(\{i, j\}, \{i, j\}^c)$ (denoted $(ij, (ij)^c)$ for short), for any $i, j \in N$.

We express it w.r.t. the interaction I. For convenience, we use the notation $I_{S,T}$.

Proposition 4 *For any 2-additive bi-capacity, we have:*

$$m(ij, (ij)^c) = I_{ij, \emptyset} \tag{28}$$

$$m(\emptyset, (ij)^c) = I_{\emptyset, ij} \tag{29}$$

$$m(i, (ij)^c) = I_{i, j} \tag{30}$$

$$m(i, i^c) = I_{i, \emptyset} - \frac{1}{2} \sum_{j \neq i} [I_{i,j} + I_{ij, \emptyset}] \tag{31}$$

$$m(\emptyset, i^c) = I_{\emptyset, i} - \frac{1}{2} \sum_{j \neq i} [I_{j,i} + I_{\emptyset, ij}] \tag{32}$$

Proof: The proof comes readily from Prop. 6 in Part I for the three first ones, and Eq. (21) for the last two. We just detail the last two. We have:

$$I_{i, \emptyset} = m(i, i^c) + \sum_{j \neq i} \frac{1}{2}[m(i, (ij)^c) + m(ij, (ij)^c)].$$

Replacing $m(i, (ij)^c)$ and $m(ij, (ij)^c)$ by their expression in terms of I gives the result. Similarly, the last equation comes from

$$I_{\emptyset, i} = m(\emptyset, i^c) + \sum_{j \neq i} \frac{1}{2}[m(j, (ij)^c) + m(\emptyset, (ij)^c)].$$

$\blacksquare$

Based on this, we obtain the following result.

Proposition 5 *For any 2-additive (normalized) bi-capacity,*

(i) $\displaystyle\sum_{i\in N}[I_{i,\emptyset}+I_{\emptyset,i}]=2.$

(ii) $\displaystyle\sum_{i\in N}I_{\emptyset,i}=\frac{1}{2}\sum_{i,j\in N}I_{i,j}+1.$

(iii) $\displaystyle I_{\emptyset,\emptyset}=-\frac{1}{6}\Big[\sum_{i,j\in N}I_{i,j}+\sum_{ij\subseteq N}[I_{ij,\emptyset}+I_{\emptyset,ij}]\Big].$

Proof: (i) Since $\sum_{(A,B)\in\mathcal{Q}(N)}m(A,B)=1$, we have:

$$1=\sum_{ij\subseteq N}m(\emptyset,(ij)^c)+\sum_{ij\subseteq N}m(i,(ij)^c)+\sum_{ij\subseteq N}m(ij,(ij)^c)+\sum_{i\in N}m(\emptyset,i^c)+\sum_{i\in N}m(i,i^c)+m(\emptyset,N)$$

$$=\sum_{ij\subseteq N}I_{\emptyset,ij}+\sum_{ij\subseteq N}I_{i,j}+\sum_{ij\subseteq N}I_{ij,\emptyset}+\sum_{i\in N}I_{\emptyset,i}-\frac{1}{2}\sum_{ij\subseteq N}[I_{j,i}+2I_{\emptyset,ij}]+$$

$$\sum_{i\in N}I_{i,\emptyset}-\frac{1}{2}\sum_{ij\subseteq N}[2I_{ij,\emptyset}+I_{i,j}]-1$$

$$=\sum_{i\in N}I_{i,\emptyset}+\sum_{i\in N}I_{\emptyset,i}-1$$

hence the result.

(ii) This is due to $\sum_{B\subsetneq N}m(\emptyset,B)=1$.

(iii) Using Eq. (21), we have:

$$I_{\emptyset,\emptyset}=\sum_{(S',T')\in[(\emptyset,N),(N,\emptyset)]}\frac{1}{n-t'+1}m(S',T')$$

$$=m(\emptyset,N)+\sum_{(S',T')>(\emptyset,N)}\frac{1}{n-t'+1}m(S',T')$$

$$=-1+\frac{1}{3}\sum_{ij\subseteq N}[m(ij,(ij)^c)+m(i,(ij)^c)+m(\emptyset,(ij)^c)]$$

$$+\frac{1}{2}\sum_{i\in N}[m(i,i^c)+m(\emptyset,i^c)]$$

$$=-1+\frac{1}{3}\sum_{ij\subseteq N}[I_{ij,\emptyset}+I_{\emptyset,ij}+I_{i,j}]$$

$$+\frac{1}{2}\sum_{i\in N}\Big[I_{i,\emptyset}-\frac{1}{2}\sum_{j\neq i}[I_{i,j}+I_{ij,\emptyset}]+I_{\emptyset,i}-\frac{1}{2}\sum_{j\neq i}[I_{j,i}+I_{\emptyset,ij}]\Big]$$

$$=-1-\frac{1}{6}\sum_{ij\subseteq N}[I_{ij,\emptyset}+I_{\emptyset,ij}]-\frac{1}{6}\sum_{i,j\in N}I_{i,j}+\frac{1}{2}\sum_{i\in N}[I_{i,\emptyset}+I_{\emptyset,i}]$$

hence the result, using (i) above. ∎

We are now able to express the monotonicity conditions for m and I.

Proposition 6 *Let m be a function defined on $\mathcal{Q}(N)$. It corresponds to the Möbius transform of a normalized bi-capacity v if and only if:*

(i) $\displaystyle\sum_{(A,B)\in\mathcal{Q}(N)} m(A,B) = 1, \sum_{B\subsetneq N} m(\emptyset, B) = 1, \text{ and } m(\emptyset, N) = -1.$

(ii) For all $i \in N$, for all $(A, B) \in \mathcal{Q}(N \setminus i)$,

(ii.1) $\displaystyle m(i, i^c) + \sum_{j\in(B\cup i)^c} m(i, (ij)^c) + \sum_{j\in A} m(ij, (ij)^c) \geq 0$

(ii.2) $\displaystyle m(\emptyset, i^c) + \sum_{j\in(B\cup i)^c} m(\emptyset, (ij)^c) + \sum_{j\in A} m(j, (ij)^c) \geq 0$

Proof: Condition (i) is clear. Conditions (ii) come from Lemma 1 of Part I:

$$v(A \cup i, B) - v(A, B) = \Delta_{(i,i^c)} v(A, B) = \sum_{(S,T)\in[(i,i^c),(A\cup i,B)]} m(S, T) \geq 0$$

$$v(A, B) - v(A, B \cup i) = \Delta_{(\emptyset,i^c)} v(A, B \cup i) = \sum_{(S,T)\in[(\emptyset,i^c),(A,B)]} m(S, T) \geq 0.$$

$\blacksquare$

Proposition 7 *Let I be a function defined on $\mathcal{Q}(N)$. It corresponds to the interaction of a normalized bi-capacity v if and only if:*

(i) $\displaystyle\sum_{i\in N}[I_{i,\emptyset} + I_{\emptyset,i}] = 2, \sum_{i\in N} I_{\emptyset,i} = \frac{1}{2}\sum_{i,j\in N} I_{i,j} + 1,$
$\displaystyle I_{\emptyset,\emptyset} = -\frac{1}{6}\Big[\sum_{i,j\in N} I_{i,j} + \sum_{ij\subseteq N}[I_{ij,\emptyset} + I_{\emptyset,ij}]\Big].$

(ii) For all $i \in N$, for all $(A, B) \in \mathcal{Q}(N \setminus i)$,

(ii.1)
$$I_{i,\emptyset} + \frac{1}{2}\Big[\sum_{j\in A} I_{ij,\emptyset} - \sum_{j\in(A\cup i)^c} I_{ij,\emptyset} - \sum_{j\in B} I_{i,j} + \sum_{j\in(B\cup i)^c} I_{i,j}\Big] \geq 0$$

(ii.2)
$$I_{\emptyset,i} + \frac{1}{2}\Big[\sum_{j\in(B\cup i)^c} I_{\emptyset,ij} - \sum_{j\in B} I_{\emptyset,ij} - \sum_{j\in(A\cup i)^c} I_{j,i} + \sum_{j\in A} I_{j,i}\Big] \geq 0$$

The proof is immediate from previous propositions.

Let us express the Choquet integral in terms of the Möbius transform, applying Prop. 3. With the 2-additive case, we obtain:

$$
\mathcal{C}_v(f) = \sum_{i \in N^-} m(\emptyset, i^c) f_i + \sum_{ij \subseteq N^-} m(\emptyset, (ij)^c)(f_i \wedge f_j) + \sum_{\substack{i \in N^- \\ j \in N^+}} m(\emptyset, (ij)^c) f_i +
$$

$$
\sum_{\substack{i \in N^+ \\ j \in N^-}} m(i, (ij)^c)[(f_i + f_j) \vee 0] + \sum_{ij \subseteq N^+} m(i, (ij)^c) f_i +
$$

$$
\sum_{ij \subseteq N^+} m(ij, (ij)^c)(f_i \wedge f_j) + \sum_{i \in N^+} m(i, i^c) f_i. \tag{33}
$$

Based on this formula, we obtain the expression in terms of interaction indices. We propose two formulas.

Proposition 8 *For any 2-additive v, the Choquet integral is given by:*

$$
\mathcal{C}_v(f) = \sum_{i \in N^+} f_i \Big[I_{i,\emptyset} + \frac{1}{2} \Big(\sum_{j \in N^+ \backslash i} I_{i,j} - \sum_{j \in N^-} I_{i,j} - \sum_{j \neq i} I_{ij,\emptyset} \Big) \Big]
$$

$$
+ \sum_{i \in N^-} f_i \Big[I_{\emptyset,i} + \frac{1}{2} \Big(- \sum_{j \neq i} I_{j,i} + \sum_{j \in N^+} I_{\emptyset,ij} - \sum_{j \in N^- \backslash i} I_{\emptyset,ij} \Big) \Big]
$$

$$
+ \sum_{ij \subseteq N^-} I_{\emptyset,ij}(f_i \wedge f_j) + \sum_{ij \subseteq N^+} I_{ij,\emptyset}(f_i \wedge f_j) + \sum_{\substack{i \in N^+ \\ j \in N^-}} I_{i,j}[(f_i + f_j) \vee 0]. \tag{34}
$$

Another expression is:

$$
\mathcal{C}_v(f) = \sum_{\substack{ij \subseteq N^+ \\ I_{ij,\emptyset} > 0}} I_{ij,\emptyset}(f_i \wedge f_j) + \sum_{\substack{ij \subseteq N^+ \\ I_{ij,\emptyset} < 0}} |I_{ij,\emptyset}|(f_i \vee f_j)
$$

$$
+ \sum_{\substack{ij \subseteq N^- \\ I_{\emptyset,ij} > 0}} I_{\emptyset,ij}(f_i \wedge f_j) + \sum_{\substack{ij \subseteq N^- \\ I_{\emptyset,ij} < 0}} |I_{\emptyset,ij}|(f_i \vee f_j)
$$

$$
+ \sum_{i \in N^+} f_i \Big[I_{i,\emptyset} + \frac{1}{2} \Big(\sum_{j \in N^+ \backslash i} \big(I_{i,j} - |I_{ij,\emptyset}| \big) - \sum_{j \in N^-} \big(|I_{i,j}| + I_{ij,\emptyset} \big) \Big) \Big]
$$

$$
+ \sum_{i \in N^-} f_i \Big[I_{\emptyset,i} + \frac{1}{2} \Big(\sum_{j \in N^+} \big(I_{\emptyset,ij} - |I_{j,i}| \big) - \sum_{j \in N^- \backslash i} \big(I_{j,i} + |I_{\emptyset,ij}| \big) \Big) \Big]
$$

$$
- \sum_{\substack{i \in N^+ \\ j \in N^- \\ I_{i,j} < 0}} |I_{i,j}|[(f_i + f_j) \vee 0] - \sum_{\substack{i \in N^+ \\ j \in N^- \\ I_{i,j} > 0}} I_{i,j}[(f_i + f_j) \wedge 0]. \tag{35}
$$

In addition, the coefficients in $\sum_{i \in N^+}$, $\sum_{i \in N^-}$ are non negative, and satisfy (partial

convexity)

$$\sum_{\substack{ij\subseteq N \\ I_{ij,\emptyset}>0}} I_{ij,\emptyset} + \sum_{\substack{ij\subseteq N \\ I_{ij,\emptyset}<0}} |I_{ij,\emptyset}| + \sum_{i\in N}\left[I_{i,\emptyset} - \frac{1}{2}\sum_{j\neq i}\left(I_{i,j}+|I_{ij,\emptyset}|\right)\right] = 1 \tag{36}$$

$$\sum_{\substack{ij\subseteq N \\ I_{\emptyset,ij}>0}} I_{\emptyset,ij} + \sum_{\substack{ij\subseteq N \\ I_{\emptyset,ij}<0}} |I_{\emptyset,ij}| + \sum_{i\in N}\left[I_{\emptyset,i} - \frac{1}{2}\sum_{j\neq i}\left(I_{j,i}+|I_{\emptyset,ij}|\right)\right] = 1 \tag{37}$$

Proof: Eq. (34) comes readily from (33) and Prop. 4. In order to show Eq. (35), we start with (34). Let us express the different terms in the summation. Remarking that for any $a, b \in \mathbb{R}$, $a \wedge b + a \vee b = a + b$, we have:

$$\sum_{ij\subseteq N^+} I_{ij,\emptyset}(f_i \wedge f_j) =$$

$$\sum_{\substack{ij\subseteq N^+ \\ I_{ij,\emptyset}>0}} I_{ij,\emptyset}(f_i \wedge f_j) + \sum_{\substack{ij\subseteq N^+ \\ I_{ij,\emptyset}<0}} (-I_{ij,\emptyset})(f_i \vee f_j) - \sum_{i\in N^+} f_i \sum_{\substack{j\in N^+\setminus i \\ I_{ij,\emptyset}<0}} (-I_{ij,\emptyset}).$$

Let us regroup this with the term $\sum_{i\in N^+}$. We get:

$$\sum_{\substack{ij\subseteq N^+ \\ I_{ij,\emptyset}>0}} I_{ij,\emptyset}(f_i \wedge f_j) + \sum_{\substack{ij\subseteq N^+ \\ I_{ij,\emptyset}<0}} (-I_{ij,\emptyset})(f_i \vee f_j)$$

$$+ \sum_{i\in N^+} f_i \left[I_{i,\emptyset} - \frac{1}{2}\sum_{j\in N^+} |I_{ij,\emptyset}| - \frac{1}{2}\sum_{j\in N^-} I_{ij,\emptyset} - \frac{1}{2}\sum_{j\in N^-} I_{i,j} + \frac{1}{2}\sum_{j\in N^+\setminus i} I_{i,j}\right].$$

Similarly, we have:

$$\sum_{ij\subseteq N^-} I_{\emptyset,ij}(f_i \wedge f_j) =$$

$$\sum_{\substack{ij\subseteq N^- \\ I_{\emptyset,ij}>0}} I_{\emptyset,ij}(f_i \wedge f_j) + \sum_{\substack{ij\subseteq N^- \\ I_{\emptyset,ij}<0}} (-I_{\emptyset,ij})(f_i \vee f_j) - \sum_{i\in N^-} f_i \sum_{\substack{j\in N^-\setminus i \\ I_{\emptyset,ij}<0}} (-I_{\emptyset,ij}),$$

and regrouping with the term $\sum_{i\in N^-}$, we get:

$$\sum_{\substack{ij\subseteq N^- \\ I_{\emptyset,ij}>0}} I_{\emptyset,ij}(f_i \wedge f_j) + \sum_{\substack{ij\subseteq N^- \\ I_{\emptyset,ij}<0}} (-I_{\emptyset,ij})(f_i \vee f_j)$$

$$+ \sum_{i\in N^-} f_i \left[I_{\emptyset,i} - \frac{1}{2}\sum_{j\neq i} I_{j,i} + \frac{1}{2}\sum_{j\in N^+} I_{\emptyset,ij} - \frac{1}{2}\sum_{j\in N^-\setminus i} |I_{\emptyset,ij}|\right].$$

Now, remark that for any $a, b \in \mathbb{R}$, we have $(a+b)\vee 0 + (a+b)\wedge 0 = a+b$. Applying

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this, we get:

$$\sum_{\substack{i\in N^+ \\ j\in N^-}} I_{i,j}[(f_i + f_j) \vee 0] =$$

$$\sum_{\substack{i\in N^+ \\ j\in N^- \\ I_{i,j}<0}} I_{i,j}[(f_i + f_j) \vee 0] + \sum_{\substack{i\in N^+ \\ j\in N^- \\ I_{i,j}>0}} (-I_{i,j})[(f_i + f_j) \wedge 0]$$

$$- \sum_{i\in N^+} f_i \sum_{\substack{j\in N^- \\ I_{i,j}<0}} (-I_{i,j}) - \sum_{j\in N^-} f_j \sum_{\substack{i\in N^+ \\ I_{i,j}<0}} (-I_{i,j}).$$

Regrouping all terms gives the desired result.

We study the signs of the coefficients for $\sum_{i\in N^+}$ and $\sum_{i\in N^-}$, by applying Prop. 7. For the summation on N^+ and a given $i \in N^+$, taking $A_i := N^+ \cap \{j|I_{ij,\emptyset} < 0\}$ and $B_i := N^- \cap \{j|I_{i,j} > 0\}$, and replacing into (ii.1) of Prop. 7, we recover the coefficient of f_i, which proves that it is non negative. Similarly for the summation on N^-, taking $A_i := N^+ \cap \{j|I_{j,i} < 0\}$, and $B_i := N^- \cap \{j|I_{\emptyset,ij} > 0\}$ proves the non negativity of the coefficient of f_i.

Partial convexity is obtained by letting $f_i = 1$ and $f_i = -1$ for all i. ∎

Although complicated, the second expression gives in detail all interaction phenomena occuring in a 2-additive bi-capacity model. As for (14), the interest of the expression lies in the fact that we have (almost) a convex sum, so that the importance of each term in the summation is clear. Let us write down the case of the CPT model. Using Prop. 6 of Part I and (4) and denoting I^+, I^- the interactions of ν_+, ν_-, we obtain

$$\mathrm{CPT}_{\nu^+,\nu^-}(f) = \sum_{\substack{ij\subseteq N^+ \\ I^+(ij)>0}} I^+(ij)(f_i \wedge f_j) + \sum_{\substack{ij\subseteq N^+ \\ I^+(ij)<0}} |I^+(ij)|(f_i \vee f_j)$$

$$+ \sum_{\substack{ij\subseteq N^- \\ I^-(ij)>0}} I^-(ij)(f_i \vee f_j) + \sum_{\substack{ij\subseteq N^- \\ I^-(ij)<0}} |I^-(ij)|(f_i \wedge f_j)$$

$$+ \sum_{i\in N^+} f_i \Big[I^+(i) - \frac{1}{2}\Big(\sum_{j\in N^+\setminus i} |I^+(ij)| + \sum_{j\in N^-} I^+(ij) \Big)\Big]$$

$$+ \sum_{i\in N^-} f_i \Big[I^-(i) - \frac{1}{2}\Big(\sum_{j\in N^+} I^-(ij) - \sum_{j\in N^-\setminus i} |I^-(ij)| \Big)\Big].$$

If $\nu^+ = \nu^-$, then after some rearrangements, we find the symmetric Choquet integral (14), or the asymmetric one if $\nu_+ = \overline{\nu_-}$.

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