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We present an approach of the kicked rotor quantum resonances in position-space, based on its analogy with the optical Talbot effect. This approach leads to a very simple picture of the physical mechanism underlying the dynamics and to analytical expressions for relevant physical quantities, such as mean momentum or kinetic energy. The ballistic behavior, which is closely associated to quantum resonances, is analyzed and shown to emerge from a coherent adding of successive kicks applied to the rotor thanks to a periodic reconstruction of the *spatial* wavepacket.

I. INTRODUCTION

The kicked rotor is a simple system that plays a central role in studies of classical and "quantum chaos". The latter is defined as the quantum behavior of a system whose classical counterpart is chaotic. In its simpler form, a kicked rotor (KR) is formed by a particle orbiting a fixed circular orbit to which an instantaneous force (a *kick*) is applied periodically. The corresponding classical dynamics is found to be regular (periodic) for weak kick intensities, for intermediate kick intensities chaotic regions develop in limited zones of the phase space, and for strong enough forcing an ergodic diffusion appears [1]. In 1995, Raizen and co-workers [2] established this system as a privileged ground for studies of quantum chaos by realizing experimentally a kicked rotor in the quantum regime with laser-cooled atoms placed in a kicked (i.e. rapidly turned on and off) laser standing wave.

Despite its apparent simplicity, the quantum kicked rotor (QKR) has very remarkable dynamical properties. One of the most studied is the so-called "dynamical localization": in contrast to the classical case, the ergodic diffusion does not last forever in the quantum case. After a characteristic "localization time", the diffusion is stopped by destructive quantum interferences [2, 3, 4]. Another feature of the QKR that has been the object of a recent burst of theoretical and experimental activity is the existence of *quantum resonances* (QRs), whose most dramatic manifestation is the appearance, for specific values of the parameters, of a *ballistic* motion, instead of a diffusive or a localized behavior. Quantum resonances are the main subject of the present paper.

The experimental realization of the QKR has triggered in the last decade an impressive number of studies involving dynamical localization [5, 6, 7, 8, 9, 10, 11], quantum transport [12, 13, 14, 15], ratchets [16, 17, 18], chaos-assisted tunneling [19, 20], classical and quantum resonances [21, 22, 23, 24]. Quantum resonances have been used in studies of fundamental aspects of quantum chaos such as quantum stabilization [25] or measurements of the gravitation [26]. "High-order" quantum resonances were also observed recently [27, 28, 29] both with a Bose-Einstein condensate and laser-cooled atoms.

In the present work, we show that the kicked rotor quantum resonances have a very simple and intuitive interpretation in *position* space, as opposed to the more common momentum space representation (see [30] and references therein). QRs in position space have been previously considered by Izrailev and Shepelyansky [4, 31]. Here, we extend their approach and build a simple physical picture that enlightens the underlying physics and allows the calculation of quantities of experimental relevance, such as average momentum or the kinetic energy.

II. PHYSICAL ORIGIN OF QUANTUM RESONANCES

In this section we describe the fundamental properties of the kicked rotor dynamics and discuss the physical origin of quantum resonances. We aim at a simple presentation that puts into evidence the physical mechanisms even for the reader unfamiliar with quantum resonances. Some aspects of the following discussion may be considered as trivial, but the discussion is nevertheless necessary to introduce ideas and notation non-ambiguously.

The atom-optics realization of a (quantum) kicked rotor consists in placing laser-cooled atoms in a far-detuned laser standing wave. In such conditions, the atoms feel the light intensity as a mechanical potential affecting their center-of mass degree of freedom [32, 33, 34]. The standing wave is pulsed periodically, being on for a time interval so short that the motion of the atoms can be neglected; in such conditions, the pulses can be considered as delta functions (kicks). The standing wave sinusoidal modulation of intensity generates a spatial potential in $\sin(2k_L x)$, where $k_L = 2\pi/\lambda_L$ is the wavenumber of the radiation forming the standing wave. As the de Broglie wavelength of laser-cooled atoms is about $\lambda_L/3$, it is comparable to the periodicity of the potential, $\lambda_L/2$, and the system is in the quantum regime, provided that decoherence is negligible during the experiment. In practice, this implies using a far-detuned radiation to reduce spontaneous emission to acceptable levels.

We shall use a normalized spatial coordinate $X = 2k_L x = 2x/\lambda_L$ which plays in fact the role of a cyclic variable: the spatial periodicity of the potential implies that the physics is the same if translations by a multiple

of $\lambda_L/2$ are performed. We can thus use either a "linear" ("unfolded") representation of the KR, using the linear variable X or a "cyclic" or "folded" representation using the angular variable $\theta = X[\mathrm{mod}\ 2\pi]$ ($x[\mathrm{mod}\ \lambda_L/2]$ in usual units) [35].

Choosing, moreover, units such that the mass of the particle is 1 and the time-period of the forcing is $T = 1$, the Hamiltonian of the KR reads

$$H = \frac{P^2}{2} + K \cos X \sum_{n=0}^{N-1} \delta(t - n) \qquad (1)$$

where P is the momentum (scaled by a factor $M/2k_L T$) X the position in the periodic potential, K (usually called "stochasticity parameter") the intensity of the kicks, and n a discrete time corresponding to the nth kick (n is an integer in normalized units).

Labeling X_t, P_t the position and momentum immediately *after* the tth kick, and integrating the classical Hamilton equations of motion corresponding to Eq. (1) produces the so-called Chirikov's "standard map" [1]:

$$\begin{aligned} X_{t+1} &= X_t + P_t \\ P_{t+1} &= P_t + K \sin X_{t+1}. \end{aligned} \qquad (2)$$

The classical dynamics is found to be periodic below a critical value of the kick intensity $K_c \approx 0.9716$. Chaotic regions appear in phase-space for $K > K_c$, and progressively grow as K increases. For $K \gtrsim 5$ the islands of stable dynamics are barely visible, and the classical dynamics becomes an ergodic diffusion in phase space. The average kinetic energy then increases linearly with time (or kick number t): $\langle P^2 \rangle /2 = D_c t$, where $D_c \approx K^2/2$ is a diffusion coefficient that can be explicitly calculated [36].

The standard map presents well-known *classical* resonances, also called *accelerator modes*. For example, set $K = 2\pi$, and consider a particle with initial conditions $X_0 = \pi/2$ and $P_0 = 0$. Iteration of Eqs. (2) shows that $P_t = 2\pi t$ and $X_t = 2\pi(t^2 - t) + \pi/2$. The momentum increases linearly with time and the kinetic energy $P^2/2$ increases *quadratically* with time, which is a signature of a *ballistic* dynamics, in contrast with the linear increase in the diffusive case. The origin of the ballisticity is easily seen: for this particular choice of K and of the initial conditions, the particle is always kicked at the *same* position (modulus 2π), that is $\sin X_t = \sin(2\pi(t^2 - t) + \pi/2) = 1$, and thus receives the same amount of linear momentum per kick. For arbitrary values of K or of the initial conditions, the particle is kicked in different positions, and the effect of some kicks compensate the effect of other kicks, leading to a *slower* increase of the energy. One can easily convince oneself that, in general, there is a classical resonance for $K = 2\pi p$ with integer p and adequate initial conditions. The classical resonance was experimentally observed in the atom-optics realization of the KR [21]. There are also classical *antiresonances*: e.g. for the above initial condition and $K = 3\pi/2$, successive

kicks have *opposite* directions, and the momentum jumps endlessly between two values, $P_t = 0$ and $P_{t+1} = 3\pi/2$.

In order to study the quantum dynamics of the KR, one considers the one-period evolution operator, also called Floquet operator:

$$U = e^{-iH/\hbar} = \exp\left(-i\frac{K}{\hbar} \cos X\right) \exp\left(-i\frac{P^2}{2\hbar}\right) \qquad (3)$$

where $\hbar = 4\hbar k_L^2 T/M$ is the normalized Planck's constant resulting from the definition of normalized variables satisfying the commutation relation $[X, P] = i\hbar$; it thus describes the "quanticity" of the system ($\hbar \to 0$ is the classical limit). The above operator relates the quantum state after the tth kick to the quantum state after the $(t-1)^{th}$ kick [37]. It is a remarkable fact that this operator factorizes into the product of two exponentials, a free evolution followed by the kick effect, despite the fact that X and P do not commute. This is a consequence of the δ-function time dependence: during the instantaneous kick, the evolution related to the kinetic energy term is negligible. If one starts e.g. in the P-representation, the free evolution (from t^+ to $(t+1)^-$) corresponds to simply adding a phase. One can then convert to the X-representation where the kick operator is diagonal and also simply adds a phase; one then goes back to the P-representation. The calculation of the quantum evolution over a period thus "costs" only two Fourier transforms and two multiplications. It is this formal and numerical simplicity that made the QKR so popular. Despite this simplicity, the cross-action of these two operators makes the QKR dynamics very rich: the free evolution operator mixes space components and the kick operator adds new momentum components, generating a complex quantum-interference pattern.

An important property of the kick operator is that it is periodic in space. This means that its eigenstates have a Bloch-wave (BW) structure. This is formally seen by developing it in terms of Bessel functions (noting that e^{inX} is the momentum translation operator $e^{inX}|P\rangle = |P + n\hbar\rangle$)

$$\exp\left(-i\kappa \cos X\right) = \sum_{m=-\infty}^{+\infty} (-i)^m J_m(\kappa) e^{imX}$$

where we introduced the quantity

$$\kappa \equiv \frac{K}{\hbar}. \qquad (4)$$

Therefore, if one starts from an initial state of well-defined momentum $|P_0\rangle$, only states of the form $|P_0 + m\hbar\rangle$ (m *integer*) will appear in the dynamics. We can write any arbitrary momentum P_0 in the form

$$P_0 = (m + \beta)\hbar$$

with m integer and $\beta \in [-1/2, 1/2]$ (i.e. β is in the "first Brillouin zone"). This introduces the *quasimomentum* $\hbar\beta$

which, as just demonstrated, is a constant of motion. The particle wavefunction $\psi(X) = \langle X | \psi \rangle$ can be written as

$$\psi(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \widetilde{\psi}(k) dk,$$

where $k = P/\hbar$, and can thus be decomposed in:

$$\begin{aligned}\psi(X) &= \frac{1}{\sqrt{2\pi}} \int_{-1/2}^{+1/2} d\beta \sum_{m=-\infty}^{+\infty} \widetilde{\psi}_\beta(m) e^{i(m+\beta)X} \\ &= \int_{-1/2}^{+1/2} d\beta \psi_\beta(X), \quad (5)\end{aligned}$$

where we introduced the quasimomentum component $\psi_\beta(X)$:

$$\psi_\beta(X) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \widetilde{\psi}_\beta(n) e^{i(m+\beta)X} \quad (6)$$

with

$$\psi_\beta(X) = e^{i\beta X} u_\beta(X). \quad (7)$$

The function

$$u_\beta(X) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \widetilde{\psi}_\beta(m) e^{imX} \quad (8)$$

is of a 2π-periodic function. Eq. (6) is a direct manifestation of the Bloch theorem: the particle is described by a Bloch wave, $\psi_\beta(X)$, which is the product of a plane-wave of well-defined quasimomentum β and a periodic function $u_\beta(X)$.

In most experimental realizations of the kicked rotor with laser-coled atoms the initial velocity distribution is larger than the Brillouin zone of the system and all quasimomenta are present; it may thus be necessary to average observable quantities over the quasimomentum. It is shown in the App. A that these averages values can be simply expressed in terms of the BW decomposition:

$$\langle P \rangle(t) = \int_{-1/2}^{1/2} d\beta \langle P \rangle_\beta \quad (9a)$$

$$\langle E \rangle(t) = \int_{-1/2}^{1/2} d\beta \langle E \rangle_\beta \quad (9b)$$

where the subscript β indicates an average over ψ_β.

In order to unveil the physical origin of quantum resonances, let us take, for simplicity, an initial state of quasimomentum $\beta = 0$ and apply to it the free evolution $\exp(-i\hbar P^2/2)$:

$$\psi_0(X, t = 1^-) = \frac{1}{\sqrt{2\pi}} \sum_m \widetilde{\psi}_0(m, t = 0) \exp\left(-i\hbar \frac{m^2}{2}\right) \exp(imX). \quad (10)$$

If we set in the above expression

$$\frac{\hbar}{2} = 2\pi\ell,$$

with ℓ arbitrary positive integer, we see that the argument of the exponential phase factor in Eq. (10) is always an integer multiple of 2π. This means that *the free evolution over a period leaves the wavepacket invariant.* Fig. 1 shows an initial BW that is spatially localized [38] in the first Brillouin zone and which evolves freely with time. The BW first becomes completely delocalized but, as seen above, it "focalizes" back to its initial shape just before the kick! Our using of an optics language is more than fortuitous: a monochromatic beam diffracted by a grating also reforms after a certain propagation length, an effect known in optics as the "Talbot effect". The quantum resonance is the atom-optics analog of the optical Talbot effect [22].

The only possible evolution comes from the kicks, whose effect is to add new momentum components ("sidebands") separated by a multiple of $\hbar$. As the effect of

the kick is diagonal in position representation; it adds a position-dependent phase to the wavefunction:

$$\psi_0(X, t = 1) = \exp(-i\kappa \cos X) \psi_0(X, t = 0).$$

As the wavepacket has always the same shape when the kick is applied, it acquires the *same* phase from each kick, and the kick effects add *coherently.* The quantum resonance condition is thus a constructive interference effect (in contrast with dynamical localization for instance, which results from destructive interferences that "freezes" the wavefunction evolution).

The above discussion puts into evidence an analogy (despite its very different nature) between quantum and classical resonances: in both cases, for particular values of the parameters, the dynamics is such that the particle (the wavepacket in the quantum case) is always kicked at the same position and the effect of the kicks adds (constructively interferes) to produce a linear increase of the momentum. An important difference, however, is that in the classical case the effect is related to the intensity of the kicks – the resonance condition thus depends on the
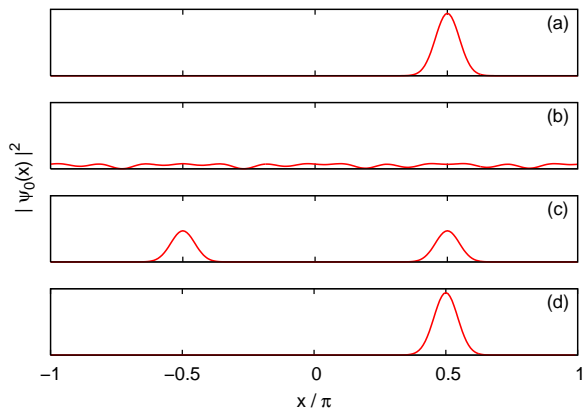
Figure 1: Free evolution of a localized wavepacket ($\beta = 0$) between two successive kicks for $\hbar = 4\pi$. (a) Initial wavepacket centered at $X = \pi/2$ shown in the interval $X \in [-\pi, \pi]$, (b) wavepacket at $t = 0.115$, showing complete delocalization, (c) wavepacket at $t = 0.25$, showing two replicas of the initial shape, (d) "reconstructed" wavepacket at $t = 1^-$, identical to the initial one.

parameter K – whereas in the quantum case it is related to the constructive accumulation of quantum phases and the resonance condition thus depends on the value of $\hbar$.

In the following sections we shall develop an approach to calculate and interpret the behaviors of the system for different types of QRs. In section III, the so-called "sim-

ple" resonances" corresponding to $\hbar = 2\pi\ell$ (ℓ positive integer) are analyzed, in which the initial wavepacket "refocuses" in a single replica of the initial wavepacket. In section IV, we focus on the so-called "high-order resonances" ($\hbar = 4\pi r/s$, r, s integers), where the free evolution of an initial wavepacket generates various spatially-separated replicas of the initial wavepacket.

III. ANALYSIS OF "SIMPLE" QUANTUM RESONANCES

Quantum resonances obeying the condition $\hbar = 2\pi\ell$ (with ℓ a positive integer) are named "simple" quantum resonances (SQR). With this form for $\hbar$, it is shown that the shape of BW remains invariant in the free propagation step between two kicks and that the effects of the kicks may lead to a ballistic (linear) growth of the momentum; the kicking period is a multiple of the half-Talbot time defined in classical optics, which is the condition for the optical (integer) Talbot effect [39, 40].

A. Time-evolution

Let us consider, at some (integer) time $(t-1)$ a state $\psi_\beta(X, t-1)$ corresponding to the general form of Eq. (6), and apply to it the free-evolution operator with $\hbar = 2\pi\ell$:

$$
\begin{aligned}
\exp\left(-i\frac{P^2}{2\hbar}\right)\psi_\beta(X, t-1) &= \frac{1}{\sqrt{2\pi}} \sum_m \widetilde{\psi}_\beta(m, t-1) \exp\left(-i\pi\ell(m+\beta)^2\right) e^{i(m+\beta)X} \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-i\pi\ell\beta^2\right) \sum_m \widetilde{\psi}_\beta(m, t-1) \exp\left(-i\pi\ell m^2\right) \exp\left(-i2\pi\ell m\beta\right) e^{i(m+\beta)X}
\end{aligned}
$$

As ℓm^2 and ℓm have the same parity, the first exponential factor under the sum is equal to $\exp(-i\pi\ell m)$; it can thus be combined with the second term, yielding $\exp(-im\hbar(\beta+1/2))$. We obtain

$$
\psi_\beta(X, t^-) = \frac{1}{\sqrt{2\pi}} \exp\left(-i\frac{\hbar\beta^2}{2}\right) \sum_m \widetilde{\psi}_\beta(m, t-1) \exp(-im\hbar\beta') e^{i(m+\beta)X}
$$

with $\beta' \equiv \beta + 1/2$. This last expression can be rewritten, using Eq. (6), as

$$
\psi_\beta\left(X, t^-\right) = \exp\left(i\frac{\hbar\beta(\beta+1)}{2}\right) \psi_\beta\left(X - \hbar\beta', t-1\right). \tag{11}
$$

Applying now the kick operator (which is diagonal in the X representation), we obtain a recurrence relation linking

the wavepackets at times t and $(t-1)$:

$$
\psi_\beta(X, t) = e^{-i\kappa\cos X} \exp\left(i\frac{\hbar\beta(\beta+1)}{2}\right) \psi_\beta\left(X - \hbar\beta', t-1\right). \tag{12}
$$

The above result shows that in the conditions of a simple resonance $|\psi(X, t)|^2 = |\psi(X - v, t-1)|^2 = |\psi(X - vt, 0)|^2$: the square-modulus of the wavefunction remains invariant immediately before the kick, except for
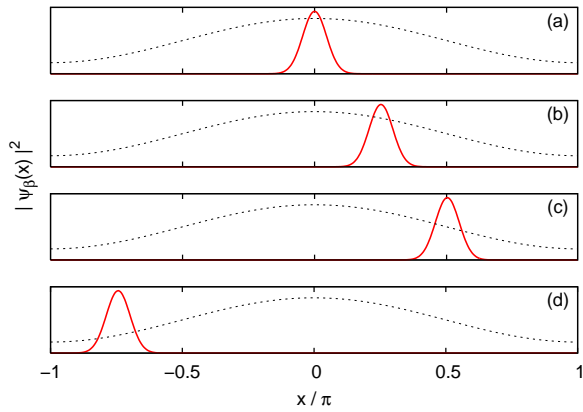
Figure 2: Wavepacket plotted at different integer times, for $\hbar = 4\pi$ and an *irrational* quasimomentum ($\beta = \frac{\pi}{50} \approx 0.063$). (a) Initial wavepacket ($|\psi_\beta|^2$) centered at $X = 0$, (b) $t = 1$, (c) $t = 2$, (d) $t = 5$. The shape of the potential is represented in dotted lines. The wavepacket interacts with the potential at different positions, and the average effect tends to zero.

a drift with a "velocity" [41]

$$v = \hbar \left(\beta + \frac{1}{2} \right). \qquad (13)$$

Moreover, the wavefunction acquires a position-dependent phase due to the kick.

By iterating Eq. (12) down to $t = 0$, we can express $\psi_\beta(x,t)$ in terms of the initial wavefunction

$$\psi_\beta(X,t) = \exp\left(-i\kappa\Phi(X,t)\right)\psi_\beta\left(X - vt, 0\right), \qquad (14)$$

where the accumulated phase is given by [42]

$$\Phi(X,t) = \sum_{s=0}^{t-1} \cos\left(X - vs\right). \qquad (15)$$

A remarkable property is obtained if

$$v = 2\pi\frac{p}{q} \qquad (16)$$

with p,q integer, e.g. for rational values of the quasimomentum. Then, the phase defined in Eq. (15) takes the values $\Phi(X, t = q) = 0$ if $q \neq 1$ and $\Phi(X, t = q) = \cos X$ if $q = 1$. Let us first consider the case $q \neq 1$. After a recurrence time $t_r = q$ the BW is given by:

$$\psi_\beta(X,t_r) = \psi_\beta\left(X - 2\pi p, 0\right) = e^{i2\pi p\beta}\psi_\beta(X, 0),$$

and the particle comes back to its initial state after t_r kicks, leading to a *periodic* evolution. To show the physical origin of this periodicity, let us take the simple case $v = \pi$, corresponding to $t_r = 2$. From Eqs. (14) and (15), the BW evolution over two successive kicks is

$$\psi_\beta(X, t = 2) = e^{-i\kappa\cos X}\psi_\beta\left(X - \pi, t = 1\right)$$

and

$$\psi_\beta(X - \pi, t = 1) = e^{i\kappa\cos X}\psi_\beta(X - 2\pi, t = 0),$$

which means that the phase added by the kicks simply cancels after two kicks. An analogous cancellation happens for any other value of the velocity v obeying Eq. (16) with $q \neq 1$: the force acting on the particle averages to zero after t_r kicks, and the motion of the wavepacket is a simple oscillation of period t_r. This behavior, that cannot lead to a ballistic behavior, is called "anti-resonance".

If $v = 2\pi p$ [or $q = 1$ in Eq. (16)], corresponding to

$$\beta = \left(\frac{p}{\ell} - \frac{1}{2} \right)$$

the evolution of the BW is given by

$$\begin{aligned} \psi_\beta(X, t_r = 1) &= e^{-i\kappa\cos X}\psi_\beta\left(X - 2\pi p, 0\right) \\ &= e^{-i2\pi p\beta}e^{-i\kappa\cos X}\psi_\beta(X, 0). \end{aligned}$$

The wavepacket exactly recovers its initial shape after each kick and the particle is subjected to an *identical* potential at *each* kick. In contrast with to the former case, the kicks will add in a *coherent* way, which, as we shall demonstrate in Sec. III B, causes a linear increase of the average momentum, or ballistic behavior. This is the quantum resonance.

If Eq. (16) is not fulfilled, our picture still allows to guess the general shape of the asymptotic evolution for large numbers of kicks ($t \gg 1$). If $v/2\pi$ is not a rational number, the phase $\Phi(X,t)$ will tend to 0 as $t \to +\infty$. Once again, the contribution of the successive kicks will average to zero and the wavepacket just "drifts" (for stroboscopic times) while keeping its initial shape (see Fig. 2).

B. Averages

Let us now focus on the time evolution of average values. From Eq. (12) we can easily obtain a recurrence relation for the BW average position

$$\langle X \rangle (t) = \langle X \rangle (t - 1) + v. \qquad (17)$$

A recurrence relation for the average momentum is obtained using Eq. (12):

$$\langle P \rangle_\beta (t) = \langle P \rangle_\beta (t-1) + K \int_{-\pi}^{\pi} dX \sin X \, |\psi_\beta(X - v, t - 1)|^2$$

$$= \langle P \rangle_\beta (t - 1) + K \int_{-\pi}^{\pi} dX \sin(X + vt) \, |\psi_\beta(X, t = 0)|^2. \qquad (18)$$

This expression can be iterated down to $t = 0$, leading to:

$$\begin{aligned}
\langle P \rangle_\beta (t) &= \langle P \rangle_\beta (0) + K \left(\sum_{n=1}^{t} \int_{-\pi}^{\pi} dX \sin(X + nv) \, |\psi_\beta(X,0)|^2 \right) \\
&= \langle P \rangle_\beta (0) + K \frac{\sin(tv/2)}{\sin(v/2)} \mathrm{Im} \left(e^{i(t+1)v/2} \int_{-\pi}^{\pi} dX e^{iX} \, |\psi_\beta(X,0)|^2 \right).
\end{aligned} \tag{19}$$

The role of interferences is clearly seen in this last result: indeed, if $v = \bar{k}(\beta + \frac{1}{2}) = 2\pi \ [\mathrm{mod} \ 2\pi]$, the momentum increases linearly with the (stroboscopic) time t, i.e

$$\langle P \rangle_\beta (t) = \langle P \rangle_\beta (0) + Dt \tag{20}$$

where the slope

$$D = K \int_{-\pi}^{\pi} dx \sin x \, |\psi_\beta(x,0)|^2 \tag{21}$$

clearly appears as a force averaged over the initial spatial distribution. The observed momentum transport shows that well-controlled diffusion can be obtained by a suitable choice of initial conditions [i.e of $\psi_\beta(x,0)$].

For $v = \bar{k}(\beta + \frac{1}{2}) = \pi \ [\mathrm{mod} \ 2\pi]$, destructive interference occurs. This can be seen from Eq. (19): $\langle P \rangle_\beta (t) = \langle P \rangle_\beta (0)$ (t even) or $\langle P \rangle_\beta (t) = \langle P \rangle_\beta (0) - D$ (t odd). In the general case ($v \neq 2\pi [\mathrm{mod} \ 2\pi]$), the momentum is frozen around its initial value.

Quantum resonance effects can also be analyzed through the temporal evolutions of the average kinetic energy. Using Eq. (14), one obtains:

$$\langle E \rangle_\beta(t) = \langle E \rangle_\beta(t=0) + \frac{K^2}{2} \int_{-\pi}^{\pi} dX \left(\sum_{n=1}^{t} \sin(X + nv) \right)^2 |\psi_\beta(X, t=0)|^2 + K \int_{-\pi}^{\pi} dX \left(\sum_{n=1}^{t} \sin(X + nv) \right) J(X, t=0) \tag{22}$$

where we introduced the current

$$J(X,t) = i\frac{\bar{k}}{2} \left(\psi_\beta(X,t) \partial_X \psi_\beta^*(X,t) - c.c. \right).$$

The constructive interference case $v = 2\pi \ [\mathrm{mod} \ 2\pi]$ then leads to an average kinetic energy increasing quadratically with time:

$$\langle E \rangle_\beta (t) = \langle E \rangle_\beta (t=0) + \frac{1}{2} K^2 t^2 \int_{-\pi}^{\pi} dX \sin^2 X \, |\psi_\beta(X, t=0)|^2 + Kt \int_{-\pi}^{\pi} dX \sin X J(Xx, t=0) \tag{23}$$

which is the quantum-mechanical analog of the ballistic motion observed – in different conditions – for a classical resonance. The ballistic growth is seen to be proportional to the quantum average of the square of the force $K^2 \sin^2 X$.

C.   Map

Inspection of Eqs. (17) and Eq. (18) suggests that the dynamics of position and momentum averages of a Bloch-wave can be described by a map. In the following, we assume that the initial BW is sharply localized around its mean position $\langle X \rangle_\beta (t = 0) = X_0$. Noting $P_t = \langle P \rangle_\beta (t)$ and $X_t = \langle X \rangle_\beta (t)$ we have from Eq. (18), $P_t = P_{t-1} + K \sin(X_0 + vt)$ (with the normalization con-

dition $\int_{-\pi}^{\pi} dX \, |\psi_\beta(X, t = 0)|^2 = 1$). We then obtain

$$\begin{aligned}
P_t &= P_{t-1} + K \sin X_t \tag{24a} \\
X_t &= X_{t-1} + v, \tag{24b}
\end{aligned}$$

which evokes the classical map (see Eq. 2), with the important difference that the position in Eq. (24b) does not depend on P_t but solely on the drift velocity v. This produces after t kicks

$$\begin{aligned}
X_t &= X_0 + vt \\
P_t &= P_{t-1} + K \sin(X_0 + vt).
\end{aligned}$$

Iterating in turn this momentum equation produces

$$P_t = P_0 + K \sum_{n=1}^{t} \sin(X_0 + nv).$$

Ballisticity is found if $v = 2\pi \pmod{2\pi}$:

$$P_t = P_0 + tK \sin X_0 = P_0 + Dt$$

with

$$D = K \sin X_0. \tag{26}$$

For a localized packet, ballisticity emerges, as in the classical case, if the particle is always kicked at the same position. In the classical case, this is possible if $K = 2\pi p$ (p integer) and $X_0 = \pm\pi/2$. In the quantum case, the resonance condition depends on the quantum parameters $\bar{k}$ and β (through v), but the kick intensity K and the initial position X_0 determine only the growth rate of the average momentum.

Recalling that t is an integer (stroboscopic) variable counting the kicks, one can find also a whole class of periodic behaviors. An example is $\bar{k} = 2\pi$ and $\beta = 0$ ($v = \pi$), with a recursion time $t_r = 2$, which has already been analyzed in sect. III A. More generally, for rational quasimomentum values $\beta = p/q$ (p, q integer), after a number of kicks t_r, the average momentum and kinetic energy come back to their initial values. In such cases the dynamics is periodic, due to the effect of kick compensation discussed above.

IV. THE $\bar{k} = \pi$ HIGH-ORDER RESONANCE

"High-order" QRs are the quantum-mechanical analogs of the *fractional* optical Talbot effect [39]. In contrast with the situation found in simple resonances, after a free propagation the initial packet does not reconstruct in an identical packet, but forms two or more replicas of the original one. The action of the kick in these subpackets generates different quantum phases and produces quantum interference effects during the subsequent free propagation. This makes high-order quantum resonances fundamentally different from, and more complex than, simple ones.

High-order quantum resonances (HQRs) correspond to a dimensionless Planck's constant of the form $\bar{k} = 4\pi r/s$, with r and $s > 2$ integers. A calculation similar to that leading to Eq. (12) shows that after the free propagation the initial packet refocalizes into s uniformly spaced subpackets if s is odd, and $s/2$ subpackets if s is even. We shall consider here only the the simplest case $\bar{k} = \pi$. The method presented below can in principle be generalized to more complicated cases, but the algebra involved quickly gets very cumbersome. As in the preceding section, our approach gives a simple picture of the physical mechanism of HQRs.

Using the general expression for BW at time $(t-1)$, Eq. (6), and applying the free-evolution operator produces in the case $\bar{k} = \pi$:

$$\begin{aligned}
\psi_\beta(X, t^-) &= \frac{1}{\sqrt{2\pi}} \sum_n \widetilde{\psi}_\beta(n, t-1) \exp\left(-i\frac{\pi}{2}(n+\beta)^2\right) e^{i(n+\beta)X} \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{i\pi\beta^2}{2}\right) \sum_n \widetilde{\psi}_\beta(n, t-1) \exp\left(-\frac{i\pi n^2}{2}\right) \exp(-i\pi n\beta) \exp(i(n+\beta)X).
\end{aligned} \tag{27}$$

We show in the Appendix B that the above expression can be written as

$$\psi_\beta(X, t) = e^{i\pi\beta^2/2} \frac{e^{-i\kappa\cos X}}{\sqrt{2}} \left[e^{-i\frac{\pi}{4}}\psi_\beta(X - \bar{k}\beta, t-1) + e^{i\frac{\pi}{4}}e^{i\beta\pi}\psi_\beta(X - \bar{k}\beta - \pi, t-1)\right] \tag{28}$$

or

$$\psi_\beta(X + wt, t) = \frac{e^{-i\kappa\phi(X,t)}}{\sqrt{2}} \left[e^{-i\pi/4}\psi_\beta(X + w(t-1), t-1) + e^{i\pi/4}e^{i\beta\pi}\psi_\beta(X + w(t-1) - \pi, t-1)\right] \tag{29}$$

where w is the packet (stroboscopic) drift velocity defined by

$$w = \bar{k}\beta \tag{30}$$

and we introduced the "local" phase [43]

$$\phi(x, t) = \kappa\cos(X + wt) \tag{31}$$

Eq. (29) shows that at any time t, $\psi_\beta(X + wt, t)$ is the superposition of two subpackets having the same shape

as the initial BW, centered at $X = 0$ and $X = \pi$. Each of these two subpackets is multiplied by a phase factor which is the sum of the accumulated phases, producing a complex interference pattern. The principle of the following calculation is to keep track of the coefficient of each subpacket, as we know that the *shape* of the subpackets is fixed. We then write:

$$\psi_\beta(X + wt, t) = c_1(X, t)\psi_\beta(X, 0) + c_2(X, t)\psi_\beta(X - \pi, 0), \quad (32)$$

where $c_1(X, t)$ and $c_2(X, t)$ are 2π-periodic complex amplitudes [44]. The above expression corresponds to a coupled two-level model where the "particle", initially in "level 1" (i.e $c_1 = 1$ at $t = 0$) is progressively "transferred" to level 2" ($c_2 \neq 0$) and then back again to "level 1", performing a kind of Rabi oscillation.

Let us define the state vector

$$\mathbf{c}_t = \begin{pmatrix} c_1(X, t) \\ c_2(X - \pi, t) \end{pmatrix}$$

It is shown in App. B 2 that these amplitudes obey a matrix recurrence relation :

$$\mathbf{c}_t = M_t \mathbf{c}_{t-1}$$

where M_t is a matrix depending on time and space having the form

$$M_t = e^{-i\pi/4}\widetilde{M_t}$$

where

$$\widetilde{M_t} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\phi} & ie^{-i\phi}e^{-i\beta\pi} \\ ie^{i\phi}e^{i\beta\pi} & e^{i\phi} \end{pmatrix}, \quad (33)$$

with $\phi(X + wt)$ given by Eq. (31). The matrix $\widetilde{M_t}$ can be recast as

$$\widetilde{M_t} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix}\begin{pmatrix} 1 & ie^{-i\beta\pi} \\ ie^{i\beta\pi} & 1 \end{pmatrix}.$$

The rightmost matrix in the above product stands for the free propagation that induces a coupling between the two subpackets; it is thus responsible of the interference effects. The leftmost one represents the effect of the kick and is obviously diagonal in x-representation.

Analytical results can be obtained in the case $\beta = 0$. As in this case $w = 0$, the matrix $\widetilde{M_t}$ is time-independent. It is then easy to write $\mathbf{c}_t$ as a function of the initial condition

$$\mathbf{c}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} :$$

$$\mathbf{c}_t = [M_t]^t \mathbf{c}_0 = e^{-it\pi/4}\left[\widetilde{M_t}\right]^t \mathbf{c}_0$$

The eigenvalues of $\widetilde{M_t}$ are:

$$\lambda = \exp(\pm i\Theta)$$

where the phase Θ depends on X and is given by

$$\begin{aligned} \cos\Theta &= \frac{\cos\phi}{\sqrt{2}} \\ &= \frac{\cos(\kappa\cos X)}{\sqrt{2}} \end{aligned}$$

(note that $\pi/4 \leq \Theta \leq 3\pi/4$). If $\mathbf{P}$ is the diagonalizing matrix, one can write (see Sec. B 3 in App. B)

$$\mathbf{c}_t = e^{-it\frac{\pi}{4}}\mathbf{P}\begin{pmatrix} e^{it\Theta} & 0 \\ 0 & e^{-it\Theta} \end{pmatrix}\mathbf{P}^{-1}\mathbf{c}_0. \quad (34)$$

After a straightforward calculation, the amplitudes are found to be

$$c_1(X, t) = e^{-it\frac{\pi}{4}}\left[\cos(t\Theta) - \frac{i\sqrt{2}}{2}\frac{\sin\phi}{\sin\Theta}\sin(t\Theta)\right] \quad (35)$$

and

$$c_2(X - \pi, t) = \frac{i\sqrt{2}}{2}\frac{e^{i\phi}}{\sin\Theta}e^{-it\frac{\pi}{4}}\sin(t\Theta). \quad (36)$$

From the above result we can calculate the average momentum, but the algebra involved is cumbersome (see App. C):

$$\langle P \rangle_\beta(t) = \langle P \rangle_\beta(t - 1) + K\int_{-\pi}^{\pi} dX \sin X\, |\psi_\beta(X, t)|^2.$$

In the general case, it is difficult to analyze the behavior of the amplitudes $c_1(X, t)$ and $c_2(X, t)$ since they depend implicitly on X in a quite complicated way. Analytic results can however be obtained in the limit which the initial BW is formed of spatially narrow wavepackets. Let us assume that $\psi_\beta(X, t = 0)$ is well localized at some position X_0 in the interval $[-\pi, \pi]$ (i.e its width $\Delta X \ll 2\pi$) and express the average momentum in terms of the coefficients c_1 and c_2

$$\langle P \rangle_\beta(t) = \langle P \rangle_\beta(t - 1) + K\int_{-\pi}^{\pi} dx \sin(X + wt)\left(|c_1(X, t)|^2 - |c_2(X - \pi, t)|^2\right)|\psi_\beta(X, 0)|^2 \quad (37)$$

This expression shows that if the two subpackets have the same weight, the momentum shift per kick is zero: the subpackets are localized around positions X_0 and $X_0 + \pi$, and subjected to opposite forces $+K \sin(X_0)/2$ and $-K \sin(X_0)/2$, an effect that is characteristic of the high-order resonances and obviously does not exist for simple resonances. This expression is valid if the two subpackets are well separated so that they do not interfere significantly.

For a wavepacket that is strongly localized at X_0, the amplitudes in the decomposition Eq. (37) can be evaluated at X_0 and depend only on time, while the phases ϕ and Θ take constant values ($\phi = \kappa \cos X_0$). Hence, the average momentum evolution is

$$\langle P \rangle_\beta (t) = \langle P \rangle_\beta (t-1) + K \sin (X_0 + wt) \left(1 - 2 \left| c_2(X_0 - \pi, t) \right|^2 \right), \tag{38}$$

which, in our simple case $\beta = 0$, takes the explicit form

$$\langle P \rangle_{\beta=0} (t) = \langle P \rangle_{\beta=0} (t-1) + K \sin X_0 \left(1 - \frac{\sin^2(t\Theta)}{\sin^2 \Theta} \right).$$

Note that the expression inside parenthesis in the above expression is characteristic of the diffraction on a grating, which puts into evidence the "wavelike" nature of the dynamics. Iterating down to $t = 0$ produces

$$\langle P \rangle_{\beta=0} (t) = \langle P \rangle_{\beta=0} (t=0) + K \sin X_0 \left[\left(\frac{\sin^2 \phi}{1 + \sin^2 \phi} \right) t + \frac{1}{1 + \sin^2 \phi} \left(\frac{\sin [(2t+1)\Theta]}{2 \sin \Theta} - \frac{1}{2} \right) \right].$$
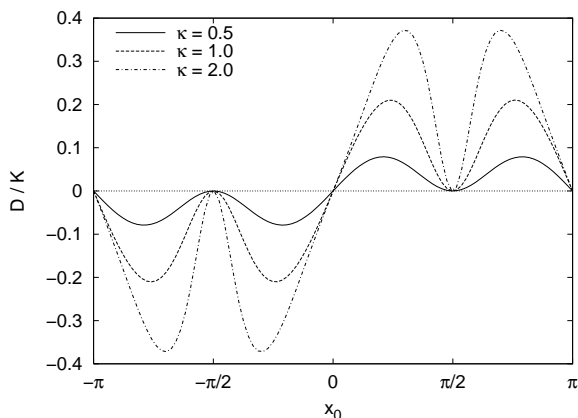
with

$$D = K \sin X_0 \left(\frac{\sin^2 \phi}{1 + \sin^2 \phi} \right). \tag{39}$$

being the rate of change of the mean momentum. The behavior of D as a function of X_0 is displayed in Fig. 3 for different values of κ. It is interesting to compare Eq. (39) with its counterpart for SQRs, which is given by Eq. (26): note in particular that in the present case $D = 0$ if $X_0 = \pi/2$, whereas for SQR, the maximum of D occurs when the force is maximum (i.e at $X_0 = \pi/2$).

The coefficient D depends periodically on the kick intensity κ via $\phi = \kappa \cos X_0$, as shown in Fig. 4. In contrast to the simple resonance case, increasing the kick force does not necessarily increase the diffusion, an effect that persists if the momentum is averaged over the quasimomentum distribution.

Analogous results describing the ballistic behavior of an initial state which is an eigenvector of the momentum have been obtained via a quite different approach [45]. Our localized-packet approach provides a clearer picture of the underlying physics.

For $\beta \neq 0$, the occurrence of ballisticity depends on the periodicity of M_t and on the initial conditions. More precisely, ballisticity will emerge if the relation $M_{t+t_r} = M_t$ is fulfilled for some (integer) recurrence time t_r. This happens for any *rational* value of quasimomentum. For example, if $\beta = 1/2$, M_t [see Eq. (33)] has a period of 4 kicks: $M_{t+4} = M_t$. One can apply the above reasoning to $M = M_4 M_3 M_2 M_1$. To illustrate the resulting behavior, Fig. 5 shows the time-evolution of average momentum obtained by direct integration of the Schrödinger equation.
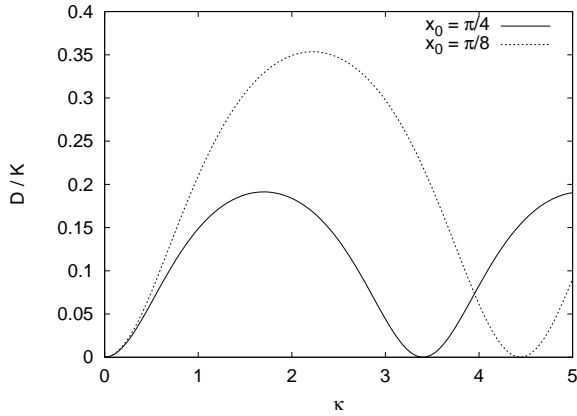

Figure 3: Momentum slope D [Eq. (39)] as a function of X_0 for 3 values of κ: 0.5 (full line), 1.0 (dots) and 2.0 (dashed).

This expression shows that average momentum evolution is, for arbitrary ϕ and Θ, a *mix* of two qualitatively different behaviors: *ballisticity*, corresponding to the first term in the brackets, and *oscillation*, described by the second term. This contrasts with SQRs, where the dynamics is *either* ballistic or oscillatory.

For long times and $\phi \neq 0$, the ballistic term dominates and one gets

$$\langle P \rangle_{\beta=0} (t) = \langle P \rangle_{\beta=0} (t=0) + Dt$$

One observes essentially the same kind of behavior.

For a general rational quasimomentum $\beta = p/q$, the ballistic diffusion rate is roughly proportional to $1/q$. Irrational quasimomenta, that may be consider as the limit $q \to +\infty$, do not produce ballistic behavior.

Figure 4: Momentum slope D [Eq. (39)] as a function of κ for two initial positions $X_0 = \pi/4$ (dots) and $X_0 = \pi/8$ (full line). One sees that increasing κ (that is, the kick intensity) does not necessarily increase the momentum slope.
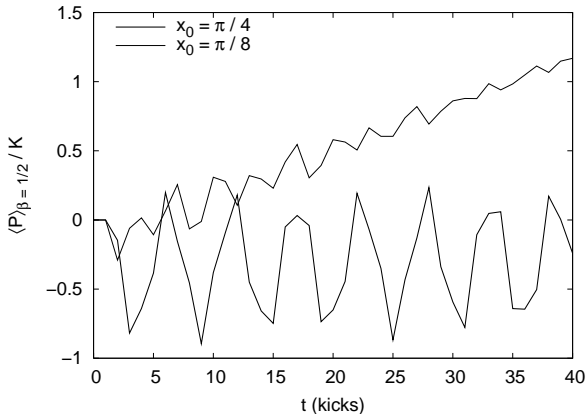

Figure 5: Evolution of the average momentum for $\beta = 1/2$ obtained by numerical integration of the Schrödinger equation. The initial initial wavefunction has the same Gaussian shape as in Fig. 1 and is centered at the positions $X_0 = \pi/4$ (dark line), which displays a dominant oscillatory behavior, or $X_0 = \pi/8$ (light line), displaying dominant ballisticity.

V.   AVERAGING OVER QUASIMOMENTUM

In experiments performed with laser-cooled atoms (not with *ultracold* atoms – BECs) the initial momentum distribution is larger than the Brillouin zone unless velocity selection is performed. In position space, this corresponds to an initial wavepacket which is localized on a single potential well, around a position X_0. In this simple case, and for $\bar{k} = 2\pi\ell$, we are able to give an analytical expression for any value of t, of the momentum and kinetic energy averaged on quasimomentum β.

Starting from Eq. (9a):

$$\langle P \rangle(t) = \langle P \rangle(t = 0) + K \sum_{n=1}^{t} \int_{-1/2}^{1/2} d\beta \int_{-\pi}^{\pi} dX \sin(X + nv) \left| \psi_\beta(X, 0) \right|^2 .$$

Assuming that $\psi_\beta(X, 0)$ is β−independent ($\psi_\beta(X, 0) = \varphi(X)$ and $\int_{-\pi}^{\pi} dx \left| \varphi(X) \right|^2 = 1$), and recalling that $v = 2\pi\ell(\beta + 1/2)$, we easily see that averaging on β lead to:

$$\langle P \rangle (t) = \langle P \rangle (t = 0).$$

A similar reasoning can be applied for kinetic energy. From Eqs. (9b) and (22), we have:

$$\langle E \rangle(t) = \langle E \rangle(t = 0) + \frac{K^2}{2} \int_{-1/2}^{1/2} d\beta \int_{-\pi}^{\pi} dX \left(\sum_{n=1}^{t} \sin(X + nv) \right)^2 \left| \varphi(X) \right|^2 + K \sum_{n=1}^{t} \int_{-1/2}^{1/2} d\beta \int_{-\pi}^{\pi} dx \sin(X + nv) J(X, t = 0)$$

When we integrate over β, the term of the last term cancels out, and the only contributions come from

$$\int_{-1/2}^{1/2} d\beta \left(\sum_{n=1}^{t} \sin(X + nv) \right)^2 = \int_{-1/2}^{1/2} d\beta \left(\sum_{n=1}^{t} \sin^2(X + \pi\ell n + 2\pi\ell n\beta) \right) = \frac{t}{2}$$

We finally get the diffusive behavior

$$\langle E \rangle(t) = \langle E \rangle(t = 0) + \frac{K^2 t}{4}. \tag{40}$$

This result evokes the classical kicked rotor in chaotic regime, whose kinetic energy grows linearly with time with the same rate. This shows that the ballistic behavior, which corresponds to a "null-measure ensemble" of rational quasimomenta, is very hard to detect by measuring quantities averaged over quasimomentum.

Experimentally, the optimum situation for observing QRs is to perform a quasimomentum selection, either by using stimulated Raman transitions [29, 46, 47] or by using a Bose-Einstein condensate [14, 27, 28]. However, it is possible to detect QRs with atoms issued of a magneto-optical trap if one can measure the full momentum distribution with enough precision to see the ballistic parts of the wavefunction separating out of the diffusive part for long enough times, as experimentally evidenced by d'Arcy *et al.* [23].

VI. CONCLUSION

In the present work, we presented a description of quantum resonances of the kicked rotor in position space, both for simple and for high-order quantum resonances. We have shown that the dynamics can be understood by considering that the spatial wavepacket comes back to its

in ital form after a finite number of kicks, according to the (rational) value of the quasimomentum. For a localized wavepacket (or an finite ensemble of localized wavepackets, in the case of HQRs), one can interpret the dynamical behavior in terms of the action of a finite number of successive kicks. This picture, inspired of the atom-optics analog of the Talbot effect, proves very useful both as it providing a intuitive understanding of the underlying physics ans as it leads to analytical developments for experimentally-relevant quantities.

Appendix A: AVERAGE MOMENTUM

The average momentum reads:

$$\langle P \rangle = \hbar \sum_n \int_{-1/2}^{1/2} d\beta (n + \beta) \left| \widetilde{\psi}_\beta(n) \right|^2$$

and can be related to $\langle P \rangle_\beta$ in the following way. One has from the definition of section II:

$$\widetilde{\psi}_\beta(n) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \psi_\beta(X) e^{-i(\beta+n)X} dx$$

therefore:

$$(n+\beta)\widetilde{\psi}_\beta(n) = \frac{i}{\sqrt{2\pi}} \int_{-\pi}^{\pi} dX \psi_\beta(X) \frac{\partial}{\partial X} e^{-i(\beta+n)X} = \frac{i}{\sqrt{2\pi}} \left[\psi_\beta(X) e^{-i(\beta+n)X} \right]_{-\pi}^{\pi} - \frac{i}{\sqrt{2\pi}} \int_{-\pi}^{\pi} dX e^{-i(\beta+n)X} \frac{\partial}{\partial X} \psi_\beta(X)$$

The first term on the RHS vanishes ($u_\beta(X) = \psi_\beta(X) e^{-i\beta X}$ is $2\pi-$periodic). One then has:

$$\begin{aligned}
\langle P \rangle &= -\frac{i\hbar}{\sqrt{2\pi}} \sum_n \int d\beta \widetilde{\psi}_\beta^*(n) \int_{-\pi}^{\pi} dX e^{-i(\beta+n)X} \frac{\partial}{\partial X} \psi_\beta(X) \\
&= -\frac{i\hbar}{2\pi} \int d\beta \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} dX dX' e^{-i\beta(X-X')} \psi_\beta(X') \frac{\partial}{\partial X} \psi_\beta(X) \sum_n e^{-in(X-X')}
\end{aligned}$$

Finally, using $\sum_n e^{-in(X-X')} = \sum_k \delta(X - X' - 2k\pi)$

$$\langle P \rangle = -i\hbar \int d\beta \int_{-\pi}^{\pi} dX \psi_\beta^*(X) \frac{\partial}{\partial X} \psi_\beta(X) = \int_{-1/2}^{1/2} d\beta \langle P \rangle_\beta$$

where

$$\langle P \rangle_\beta = \int_{-\pi}^{\pi} dX \psi_\beta^*(X) [P \psi_\beta(X)]$$

For the kinetic energy, the same developments lead to:

$$\langle P^2 \rangle = \sum_n \int d\beta (n+\beta)^2 \left| \widetilde{\psi}_\beta(n) \right|^2 = \int d\beta \left\langle P^2 \right\rangle_\beta$$

where

$$\left\langle P^2 \right\rangle_\beta = \int_{-\pi}^{\pi} dX \left[P\psi_\beta(X) \right]^* \left[P\psi_\beta(X) \right]$$

(note that the same rules can be obtained for $\left\langle P^k \right\rangle = \int d\beta \left\langle P^k \right\rangle_\beta$).

Appendix B: BLOCH WAVE EVOLUTION FOR THE $k = \pi$ RESONANCE

1.   Bloch wave evolution

The free-propagation factor $\exp(-i\frac{\pi}{2}n^2)$ in Eq. (27) has two different values according to the parity of n:

$$\exp\left(-i\frac{\pi}{2}n^2\right) = 1 \quad (n \text{ even})$$
$$= -i \quad (n \text{ odd}).$$

Replacing

$$\widetilde{\psi}_\beta(n,t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} dX e^{-inX} u_\beta(X,t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} dX e^{-i(n+\beta)X} \psi_\beta(X,t)$$

in Eq. (27), one finds

$$\psi_\beta(X,t^-) = \frac{\exp\left(-i\pi\beta^2/2\right)}{2\pi} \sum_n \int_{-\pi}^{\pi} dX' e^{-i(n+\beta)(X-X')} \psi_\beta(X',t-1) e^{-i\pi n^2/2} e^{-i\pi n\beta}$$

one can separate even and odd terms ; this leads to:

$$\psi_\beta(X,t^-) = \frac{\exp\left(-i\pi\beta^2/2\right)}{2\pi} \int_{-\pi}^{\pi} dX' \psi_\beta(X',t-1) e^{-i\beta(X-X')} \left[\sum_p e^{-i2\pi p\beta} e^{-i2p(X-X')} \left(1 - ie^{-i\pi\beta} e^{-i(X-X')} \right) \right]$$

Using the relation $\sum_p e^{-i2\pi p\beta} e^{-i2p(X-X')} = (1/2)\sum_n \delta(X - X' - \pi\beta + n\pi)$ and integrating with respect to X', gives Eq. (29) (note that only $n = 0, 1$ contributes for $\beta > 0$ and $n = 0, -1$ for $\beta < 0$).

2.   "Two-level" system

The coupled equation for the amplitudes $c_{1,2}(X,t)$ are obtained in the following way. Insertion of Eq. (32) at time $(t-1)$

$$\psi_\beta(X + w(t-1), t-1) = c_1(X,t-1)\psi_\beta(X,0) + c_2(X,t-1)\psi_\beta(X-\pi,0), \tag{B1}$$

in Eq. (29) gives

$$\psi_\beta(X + wt, t) = \frac{e^{-i\phi(X,t)}}{\sqrt{2}} \Big(e^{-i\pi/4} \left[c_1(X,t-1)\psi_\beta(X,0) + c_2(X,t-1)\psi_\beta(X-\pi,0) \right] +$$
$$e^{i\pi/4} e^{i\beta\pi} \left[c_1(X-\pi,t-1)\psi_\beta(X-\pi,0) + c_2(X-\pi,t-1)\psi_\beta(X-2\pi,0) \right] \Big)$$

and can be put in a simpler form [using $\psi_\beta(X - 2\pi, 0) = e^{-i2\pi\beta}\psi_\beta(X, 0)$]:

$$\psi_\beta(X + wt, t) = \frac{e^{-i\phi(X,t)}}{\sqrt{2}}\left(\left[e^{-i\pi/4}c_1(X, t-1) + e^{i\pi/4}e^{-i\beta\pi}c_2(X-\pi, t-1)\right]\psi_\beta(X, 0)+\right.$$
$$\left.\left[e^{i\pi/4}e^{i\beta\pi}c_1(X-\pi, t-1) + e^{-i\pi/4}c_2(X, t-1)\right]\psi_\beta(X-\pi, 0)\right)$$

Comparing to Eq. (32) one obtains:

$$c_1(X, t) = \frac{e^{-i\phi(X,t)}}{\sqrt{2}}\left[e^{-i\pi/4}c_1(X, t-1) + e^{i\pi/4}e^{-i\beta\pi}c_2(X-\pi, t-1)\right]$$
$$c_2(X, t) = \frac{e^{-i\phi(X,t)}}{\sqrt{2}}\left[e^{i\pi/4}e^{i\beta\pi}c_1(X-\pi, t-1) + e^{-i\pi/4}c_2(X, t-1)\right].$$

This last expression can be put into the form

$$c_2(X - \pi, t) = \frac{e^{i\phi(X,t)}}{\sqrt{2}}\left[e^{i\pi/4}e^{i\beta\pi}c_1(X-2\pi, t-1) + e^{-i\pi/4}c_2(X-\pi, t-1)\right].$$

One can show that the amplitudes are periodic functions, i.e $c_{1,2}(X, t) = c_{1,2}(X + 2\pi, t)$ by combining Eq. (32) combined with the equality $\psi_\beta(X - 2\pi, t) = e^{-i2\pi\beta}\psi_\beta(X, t)$. This property leads to the matrix expression

$$\begin{pmatrix} c_1(X, t) \\ c_2(X + \pi, t) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\phi}e^{-i\pi/4} & e^{-i\phi}e^{i\pi/4}e^{-i\beta\pi} \\ e^{i\phi}e^{i\pi/4}e^{i\beta\pi} & e^{i\phi}e^{-i\pi/4} \end{pmatrix}\begin{pmatrix} c_1(X, t-1) \\ c_2(X + \pi, t-1) \end{pmatrix}$$

3. The case $\beta = 0$

The explicit expression for the amplitudes $c_{1,2}(X, t)$ is obtained for $\beta = 0$ as follows. The diagonalization matrix $\mathbf{P}$ is formed with the eigenvectors of $\widetilde{M_t}$. For $\lambda = e^{\pm i\Theta}$ they are given by $(-ie^{-i\phi}, \; e^{-i\phi} - \sqrt{2}e^{\pm i\Theta})^T$. $\mathbf{P}$ is then obtained as

$$\mathbf{P} = \begin{bmatrix} -ie^{-i\phi} & -ie^{-i\phi} \\ e^{-i\phi} - \sqrt{2}e^{i\Theta} & e^{-i\phi} - \sqrt{2}e^{-i\Theta} \end{bmatrix}.$$

In order to obtain the amplitudes c_1 and c_2 at time t, one performs explicitly the development corresponding to Eq. (34). The algebra is simple, although rather long, and the final result is Eq. (35).

Appendix C: CALCULUS OF AVERAGE VALUES FOR THE $k = \pi$ RESONANCE

Starting from Eq. (28), recursion relation for $\psi_\beta(X, t)$, we easily obtain a recursion relation for its derivative

$$\frac{\partial}{\partial X}\psi_\beta(X, t) = i\kappa \sin X e^{-i\kappa \cos X}\psi_\beta(X, t)$$
$$+ \frac{e^{-i\kappa \cos X}}{\sqrt{2}}\left(e^{-i\frac{\pi}{4}}\frac{\partial}{\partial X}\psi_\beta(X - w, t-1) + e^{i\frac{\pi}{4}}e^{i\beta\pi}\frac{\partial}{\partial X}\psi_\beta(X - w - \pi, t-1)\right). \quad (C1)$$

Using these expressions into for calculating the average momentum produces two terms; let us call them p_1 and p_2. The first one is given by:

$$p_1 = K\int_{-\pi}^{\pi} dX \sin X \, |\psi_\beta(X, t)|^2$$

For the second term p_2, on obtains

$$
\begin{aligned}
p_2 &= -\frac{i\hbar}{2} \int_{-\pi}^{\pi} dX \left(e^{i\frac{\pi}{4}} \psi_\beta^* (X-w, t-1) + e^{-i\frac{\pi}{4}} e^{-i\beta\pi} \psi_\beta^* (X-x-\pi, t-1) \right) \\
&\quad \times \left(e^{-i\frac{\pi}{4}} \frac{\partial}{\partial X} \psi_\beta (X-w, t-1) + e^{i\frac{\pi}{4}} e^{i\beta\pi} \frac{\partial}{\partial X} \psi_\beta (X-w-\pi, t-1) \right) \\
&= -\frac{i\hbar}{2} \int_{-\pi}^{\pi} dX \left(\psi_\beta^* (X, t-1) \frac{\partial}{\partial X} \psi_\beta (X, t-1) + \psi_\beta^* (X-\pi, t-1) \frac{\partial}{\partial X} \psi_\beta (X-\pi, t-1) \right) \\
&\quad + \frac{\hbar}{2} \int_{-\pi}^{\pi} dX \left(e^{i\beta\pi} \psi_\beta^* (X, t-1) \frac{\partial}{\partial X} \psi_\beta (X-\pi, t-1) - e^{-i\beta\pi} \psi_\beta^* (X-\pi, t-1) \frac{\partial}{\partial X} \psi_\beta (X, t-1) \right)
\end{aligned}
$$

[in the last equality we replaced $(X-w)$ by X, as the integration is over one period, the integration limits can be kept the same]. One recognizes $\langle P \rangle_\beta (t-1)$ in the first term while the second term cancels out. The recursion relation for the momentum thus reads

$$
\langle P \rangle_\beta (t) = \langle P \rangle_\beta (t-1) + K \int_{-\pi}^{\pi} dX \sin X \, |\psi_\beta(X,t)|^2 . \tag{C2}
$$

We now use the decomposition of $\psi_\beta(X,t)$ proposed in Eq. (32). Eq. (C2) then transforms into:

$$
\begin{aligned}
\langle P \rangle_\beta (t) &= \langle P \rangle_\beta (t-1) + iK \cos(\beta\pi) \int_{-\pi}^{\pi} dX \sin(X+wt) \left(c_1^*(X,t)\psi_\beta^*(X,0) + c_2^*(X,t)\psi_\beta^*(X-\pi,0) \right) \\
&\quad \times \left(c_1(X,t)\psi_\beta(X,0) + c_2(X,t)\psi_\beta(X-\pi,0) \right) .
\end{aligned}
$$

If the initial wavefunction has a narrow distribution centered around position X_0, all terms involving overlaps of functions $\psi_\beta(X,0)$ and $\psi_\beta(X+\pi)$ tend to zero [48]. This expression therefore simplifies to

$$
\langle P \rangle_\beta (t) = \langle P \rangle_\beta (t-1) + K \int_{-\pi}^{\pi} dX \sin(X+wt) \left(|c_1(X,t)|^2 - |c_2(X-\pi,t)|^2 \right) |\psi_\beta(X,0)|^2 \tag{C3}
$$

Finally use the fact that $\psi_\beta(X,0)$ is much narrower then any other factor, and assuming $\int_{-\pi}^{\pi} dX \, |\psi_\beta(X,0)|^2 = 1$ [49] gives

$$
\langle P \rangle_\beta (t) = \langle P \rangle_\beta (t-1) + K \sin(X_0+wt) \left(1 - 2|c_2(X_0-\pi,t)|^2 \right)
$$

where we used the normalization $|c_1(X,t)|^2 + |c_2(X-\pi,t)|^2 = 1$.

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lowing, consider a wave packet that is "localized" around a given position – say θ_0 – in the folded representation. In the unfolded representation such a packet corresponds in fact to a "comb" (of spatial step $\lambda_L/2$) of packets periodically distributed in X, with period 2π.

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[37] Throughout this work, we shall use the convention that t indicates the time immediately after the tth kick. When necessary, we shall use the notation t^+ to stress the fact that we are considering the instant after the tth kick has been applied, and t^- to indicate the instant just before the tth kick is applied.

[38] Indeed, a wavepacket can be spatially localized in the first Brillouin zone and can also have a well-defined quasimomentum. Both in the (unfolded) real space and in the momentum space the corresponding wavefunction is a periodic distribution of packets (an infinite comb) and the Heisenberg principle is fulfilled.

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[41] Note that the wavepacket in not simply drifting. For an arbitrary time between kicks it is completely delocalized (cf. Fig. 1). It is only because we are studying the wavepacket at "stroboscopic" times t in the vicinity of the kick, together with the "refocalizing" effect of the quantum resonance at these times that gives the impression of a drift. The true evolution for arbitrary times is much more complex.

[42] The constant phase factor in Eq. (12) has been omitted in Eq. (14) for the sake of simplicity.

[43] The first constant phase factor in Eq. (28) has been omitted for simplicity.

[44] The periodicity of $c_{1,2}(x,t)$ follows from Eq. (32) and the condition $\psi_\beta(x + 2\pi, t) = \exp(2\pi i\beta)\psi_\beta(x,t)$.

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[48] Note that we cannot use the narrow wavepacket approximation directly in the calculation of the coefficients $c_{1,2}$ because in such case the derivative in Eq. (C1) is meaningless. We can however use it in the expression of the average momentum thanks to the presence of the integral of x.

[49] This normalization is not *a priori* fulfilled since the BW ψ_β is a component of the full wavefunction.