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# A Measurable-Group-Theoretic Solution to von Neumann's Problem

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## Abstract

We give a positive answer, in the measurable-group-theory context, to von Neumann's problem of knowing whether a non-amenable countable discrete group contains a non-cyclic free subgroup. We also get an embedding result of the free-group von Neumann factor into restricted wreath product factors.

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Amenability of groups is a concept introduced by J. von Neumann in his seminal article [vN29] to explain the so-called Banach-Tarski paradox. He proved that a discrete group containing the free group  $\mathbf{F}_2$  on two generators as a subgroup is non-amenable. Knowing whether this was a characterization of non-amenable became known as von Neumann's Problem and was solved by the negative by A. Ol'shanskii [Ol'80]. Still, this characterization could become true after relaxing the notion of "containing a subgroup". K. Whyte gave a very satisfying geometric group-theoretic solution: *A finitely generated group  $\Gamma$  is non-amenable iff it admits a partition with pieces uniformly bilipschitz equivalent to the regular 4-valent tree [Why99].* Geometric group theory admits a measurable counterpart, namely, measurable group theory. The main goal of our note is to provide a solution to von Neumann's problem in this context. We show that any countable non-amenable group admits a measure-preserving free action on some probability space, such that the orbits may be measurably partitioned into pieces given by an  $\mathbf{F}_2$ -action.

To be more precise, we use the following notation. For a finite or countable set  $M$ , let  $\mu$  denote the product  $\otimes_M Leb$  on  $[0, 1]^M$  of the Lebesgue measures  $Leb$  on  $[0, 1]$ , and for  $p \in [0, 1]$ , let  $\mu_p$  denote the product of the discrete measures  $(1-p)\delta_{\{0\}} + p\delta_{\{1\}}$  on  $\{0, 1\}^M$ . Thus, the meaning of  $\mu$  may vary from use to use as  $M$  varies. Usually  $M$  will be a countable group  $\Lambda$  or the set  $E$  of edges of a Cayley graph.

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**Theorem 1** *For any countable discrete non-amenable group  $\Lambda$ , there is a measurable ergodic essentially free action of  $\mathbf{F}_2$  on  $([0, 1]^\Lambda, \mu)$  such that almost every  $\Lambda$ -orbit of the Bernoulli shift decomposes into  $\mathbf{F}_2$ -orbits.*

In other words, the orbit equivalence relation of the  $\mathbf{F}_2$ -action is contained in that of the  $\Lambda$ -action. We give two proofs of this theorem, each with its own advantages.

For some purposes, it is useful to get a Bernoulli shift action with a discrete base space. We show:

**Theorem 2** *For any finitely generated non-amenable group  $\Gamma$ , there is  $n \in \mathbb{N}$  and a non-empty interval  $(p_1, p_2)$  of parameters  $p$  for which there is an ergodic essentially free action of  $\mathbf{F}_2$  on  $\prod_1^n(\{0, 1\}^\Gamma, \mu_p)$  such that almost every  $\Gamma$ -orbit of the diagonal Bernoulli shift decomposes into  $\mathbf{F}_2$ -orbits.*

These results have operator-algebra counterparts:

**Corollary 3** *Let  $\Lambda$  be a countable discrete non-amenable group and  $H$  be an infinite group. Then the von Neumann factor  $L(H \wr \Lambda)$  of the restricted wreath product contains a copy of the von Neumann factor  $L(\mathbf{F}_2)$  of the free group.*

**Corollary 4** *Let  $\Gamma$  be a finitely generated discrete non-amenable group. Let  $n, p_1, p_2$  be as in Theorem 2 and let  $p = \frac{\alpha}{\beta} \in (p_1, p_2)$ , with  $\alpha, \beta \in \mathbb{N}$ . Assume that  $H$  contains an abelian subgroup  $K$  of order  $k = \beta^n$ . Then the von Neumann factor  $L(H \wr \Lambda)$  of the restricted wreath product contains a copy of the von Neumann factor  $L(\mathbf{F}_2)$  of the free group.*

For this paper, we assume a certain familiarity with the results and notation of [Gab05], [Gab00] and [LS99].

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A **(countable standard) equivalence relation** on the standard Borel space  $(X, \nu)$  is an equivalence relation  $\mathcal{R}$  with countable classes that is a Borel subset of  $X \times X$  for the product  $\sigma$ -algebra.

A **(measure-preserving oriented) graphing** on  $(X, \nu)$  is a denumerable family  $\Phi = (\varphi_i)_{i \in I}$  of partial measure-preserving isomorphisms  $\varphi_i : A_i \rightarrow B_i$  between Borel subsets  $A_i, B_i \subset X$ .

A graphing  $\Phi$  **generates** an equivalence relation  $\mathcal{R}_\Phi$ : the smallest equivalence relation that contains all pairs  $(x, \varphi_i(x))$ . The **cost** of a graphing  $\Phi = (\varphi_i)_{i \in I}$  is the sum of the measures of the domains  $\sum_{i \in I} \nu(A_i)$ . The **cost**,  $\text{cost}(\mathcal{R}, \nu)$ , of  $(\mathcal{R}, \nu)$  is the infimum of the costs of the graphings that generate  $\mathcal{R}$ . The **graph (structure)**  $\Phi[x]$  of a graphing  $\Phi$  at a point  $x \in X$  is the graph whose vertex set is the equivalence class  $\mathcal{R}_\Phi[x]$  of  $x$  and whose edges are the pairs  $(y, z) \in \mathcal{R}_\Phi[x] \times \mathcal{R}_\Phi[x]$  such that for some  $i \in I$ , either  $\varphi_i(y) = z$  or  $\varphi_i(z) = y$ . For more on cost, see [Gab00] or [KM04].

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## Proofs

Since the union of an increasing sequence of amenable groups is still amenable,  $\Lambda$  contains a non-amenable finitely generated subgroup. Let  $\Gamma$  be such a subgroup.

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If  $S$  is a finite generating set of  $\Gamma$  (maybe with repetitions),  $\mathcal{G} = (\mathbf{V}, \mathbf{E})$  denotes the associated right Cayley graph (with vertex set  $\mathbf{V}$ ): The set  $\mathbf{E}$  of edges is indexed by  $S$  and  $\Gamma$ . Given  $s \in S$  and  $\gamma \in \Gamma$ , the corresponding edge is oriented from the vertex  $\gamma$  to  $\gamma s$ . Note that  $\Gamma$  acts freely on  $\mathcal{G}$  by multiplication on the left. Let  $\rho := \text{id}$ , the identity of the group  $\Gamma$ , chosen as base vertex for  $\mathcal{G}$ .

The set of the subgraphs of  $\mathcal{G}$  (with the same set of vertices  $\mathbf{V}$ ) is naturally identified with  $\Omega := \{0, 1\}^{\mathbf{E}}$ . The connected components of  $\omega \in \Omega$  are called its **clusters**.

Consider a probability-measure-preserving essentially free (left)  $\Gamma$ -action on some standard Borel space  $(X, \nu)$  together with a  $\Gamma$ -equivariant Borel map  $\pi : X \rightarrow \{0, 1\}^{\mathbf{E}}$ .

The **full** equivalence relation  $\mathcal{R}_\Gamma$  generated by the  $\Gamma$ -action  $X$  is graphed by the graphing  $\Phi = (\varphi_s)_{s \in S}$ , where  $\varphi_s$  denotes the action by  $s^{-1}$ .

We define the following equivalence subrelation on  $X$  (see [Gab05, Sect. 1]): the **cluster equivalence subrelation**  $\mathcal{R}^{\text{cl}}$ , graphed by the graphing  $\Phi^{\text{cl}} := (\varphi_s^{\text{cl}})_{s \in S}$  of partial isomorphisms, where  $\varphi_s^{\text{cl}}$  is the restriction  $\varphi_s^{\text{cl}} := \varphi_s|_{A_s}$  of  $\varphi_s$  to the Borel subset  $A_s$  of  $x \in X$  for which the edge  $e$  labelled  $s$  from  $\rho$  to  $\rho s$  lies in  $\pi(x)$ , i.e.,  $\pi(x)(e) = 1$ . Consequently, *two points  $x, y \in X$  are  $\mathcal{R}^{\text{cl}}$ -equivalent if and only if there is some  $\gamma \in \Gamma$  such that  $\gamma^{-1}x = y$  and the vertices  $\rho, \gamma\rho$  are in the same cluster of  $\pi(x)$ .*

The graph structure  $\Phi^{\text{cl}}[x]$  given by the graphing  $\Phi^{\text{cl}}$  to the  $\mathcal{R}^{\text{cl}}$ -class of any  $x \in X$  is naturally isomorphic with the cluster  $\pi(x)_\rho$  of  $\pi(x)$  that contains the base vertex. Denote by  $U^\infty \subset X$  the Borel set of  $x \in X$  whose  $\mathcal{R}^{\text{cl}}$ -class is infinite and by  $\mathcal{R}_{\Gamma|U^\infty}$  (resp.  $\mathcal{R}_{\Gamma|U^\infty}^{\text{cl}}$ ) the restriction of  $\mathcal{R}_\Gamma$  (resp.  $\mathcal{R}^{\text{cl}}$ ) to  $U^\infty$ .

Write  $\mathcal{P}(Y)$  for the power set of  $Y$ . The map  $X \times \mathbf{V} \rightarrow X$  defined by  $(x, \gamma\rho) \mapsto \gamma^{-1}.x$  induces a map  $\Psi : X \times \mathcal{P}(\mathbf{V}) \rightarrow \mathcal{P}(X)$  that is invariant under the (left) diagonal  $\Gamma$ -action (i.e.,  $\Psi(\gamma.x, \gamma.C) = \Psi(x, C)$  for all  $\gamma \in \Gamma$ ,  $x \in X$ , and  $C \subset \mathbf{V}$ ) and such that  $\Psi(x, \mathbf{V})$  is the whole  $\mathcal{R}_\Gamma$ -class of  $x$ . The restriction of  $\Psi$  to the  $\Gamma$ -invariant subset  $\mathfrak{C}_\infty^{\text{cl}} := \{(x, C) : x \in X, C \in \mathcal{P}(\mathbf{V}), C \text{ is an infinite cluster of } \pi(x)\}$  sends each  $(x, C)$  (and its  $\Gamma$ -orbit) to a whole infinite  $\mathcal{R}^{\text{cl}}$ -class, namely, the  $\mathcal{R}^{\text{cl}}$ -class of  $\gamma^{-1}.x$  for any  $\gamma$  such that  $\gamma\rho \in C$ . Moreover, for each  $x \in U^\infty$ , its  $\mathcal{R}_{\Gamma|U^\infty}$ -class decomposes into infinite  $\mathcal{R}^{\text{cl}}$ -sub-classes that are in one-to-one correspondence with the elements of  $\mathfrak{C}_\infty^{\text{cl}}$  that have  $x$  as first coordinate. Note that the set  $\{(x, y, C) \in X \times X \times \mathcal{P}(\mathbf{V}) : x \in \Psi(y, C)\}$  is Borel, whence for a Borel set  $\mathcal{A} \subset \mathfrak{C}_\infty^{\text{cl}}$ , the set  $\overline{\Psi}(\mathcal{A}) := \bigcup \Psi(\mathcal{A})$  is measurable, being the projection onto the first coordinate of the Borel set  $\{(x, y, C) : x \in \Psi(y, C)\} \cap (X \times \mathcal{A})$ .

We say that  $(\nu, \pi)$  has **indistinguishable infinite clusters** if for every  $\Gamma$ -invariant Borel subset  $\mathcal{A} \subset \mathfrak{C}_\infty^{\text{cl}}$ , the set of  $x \in X$  for which some  $(x, C) \in \mathcal{A}$  and some  $(x, C) \in \mathfrak{C}_\infty^{\text{cl}} \setminus \mathcal{A}$  has  $\nu$ -measure 0. In other words, the  $\mathcal{R}^{\text{cl}}$ -invariant partition  $U^\infty = \overline{\Psi}(\mathcal{A}) \cup \mathfrak{C} \setminus \overline{\Psi}(\mathcal{A})$  is not allowed to split any  $\mathcal{R}_{\Gamma|U^\infty}$ -class (up to a union of measure 0 of such classes). The following proposition, using this refined notion of indistinguishability, corrects [Gab05, Rem. 2.3].

**Proposition 5** *Let  $\Gamma$  act ergodically on  $(X, \nu)$  and  $\pi : X \rightarrow \{0, 1\}^{\mathbf{E}}$  be a  $\Gamma$ -equivariant Borel map such that  $\nu(U^\infty) \neq 0$ . Then  $\mathcal{R}_{|\infty}^{\text{cl}}$ , the cluster equivalence relation restricted to its infinite locus  $U^\infty$ , is ergodic if and only if  $(\nu, \pi)$  has indistinguishable infinite clusters.*

*Proof.* Suppose that  $\mathcal{R}_{|\infty}^{\text{cl}}$  is ergodic. Then for every  $\Gamma$ -invariant Borel subset  $\mathcal{A} \subset \mathfrak{C}_{|\infty}^{\text{cl}}$ , its image  $\overline{\Psi}(\mathcal{A})$  is a union of  $\mathcal{R}_{|\infty}^{\text{cl}}$ -classes, whence  $\overline{\Psi}(\mathcal{A})$  or its complement  $\mathfrak{C}\overline{\Psi}(\mathcal{A})$  in  $U^\infty$  has measure 0. In particular, the partition  $U^\infty = \overline{\Psi}(\mathcal{A}) \cup \mathfrak{C}\overline{\Psi}(\mathcal{A})$  is trivial, whence  $\nu$  has indistinguishable infinite clusters.

Conversely, suppose that  $(\nu, \pi)$  has indistinguishable infinite clusters. An  $\mathcal{R}_{|\infty}^{\text{cl}}$ -invariant partition  $U^\infty = \mathcal{U} \cup \mathfrak{C}\mathcal{U}$  defines a partition  $\mathfrak{C}_{|\infty}^{\text{cl}} = \mathcal{A} \cup \mathfrak{C}\mathcal{A}$  according to whether  $\Psi(x, C) \in \mathcal{U}$  or  $\mathfrak{C}\mathcal{U}$ . Then for  $\nu$ -almost every  $x \in U^\infty$ , all  $\Psi(x, C)$  are in  $\mathcal{U}$  or all are in its complement, i.e., the  $\mathcal{R}_{|\infty}^{\text{cl}}$ -subclasses into which the  $\mathcal{R}_{|\infty}^{\text{cl}}$ -class of  $x$  splits all belong to one side. Since  $\mathcal{R}_{|\infty}^{\text{cl}}$  is  $\nu$ -ergodic, this side has to be the same for almost every  $x$ . This means that the other side is a null set. This holds for any partition  $\mathcal{U} \cup \mathfrak{C}\mathcal{U}$ , whence  $\mathcal{R}_{|\infty}^{\text{cl}}$  is ergodic.  $\blacksquare$

If  $X$  has the form  $X = \Omega \times Y$ , then  $\nu$  is called *insertion tolerant* (see [LS99]) if for each edge  $e \in \mathbf{E}$ , the map  $\Pi_e : X \rightarrow X$  defined by  $(\omega, y) \mapsto (\omega \cup \{e\}, y)$  quasi-preserved the measure, i.e.,  $\nu(A) > 0$  implies  $\nu(\Pi_e(A)) > 0$  for every measurable subset  $A \subseteq X$ . Call a map  $\pi : X \rightarrow \Omega$  *increasing* if  $\pi(\omega, y) \supseteq \omega$  for all  $\omega \in \Omega$ . An action of  $\Gamma$  on  $\Omega \times Y$  is always assumed to act on the first coordinate in the usual way. A slight extension of [LS99, Th. 3.3, Rem. 3.4], proved in the same way, is the following:

**Proposition 6** *Assume that  $\Gamma$  acts on  $(\Omega \times Y, \nu)$  preserving the measure and  $\pi : \Omega \times Y \rightarrow \Omega$  is an increasing  $\Gamma$ -equivariant Borel map with  $\nu(U^\infty) \neq 0$ . If  $\nu$  is insertion tolerant, then  $(\nu, \pi)$  has indistinguishable infinite clusters.*

**Proposition 7** *If  $\Gamma < \Lambda$ , then there are  $\Gamma$ -equivariant isomorphisms  $([0, 1]^{\mathbf{E}}, \mu) \simeq ([0, 1]^\Gamma, \mu) \simeq ([0, 1]^\Lambda, \mu)$  between the Bernoulli shift actions of  $\Gamma$ . In particular, the orbits of the Bernoulli shift  $\Lambda$ -action on  $[0, 1]^\Lambda$  are partitioned into subsets that are identified with the orbits of the standard Bernoulli shift  $\Gamma$ -action on  $[0, 1]^\Gamma$ .*

*Proof.* A countable set  $\mathbf{E}$  on which  $\Gamma$  acts freely may be decomposed by choosing a representative in each orbit so as to be identified with a disjoint union of  $\Gamma$ -copies,  $\mathbf{E} \simeq \coprod_J \Gamma$ , and to give  $\Gamma$ -equivariant identifications  $[0, 1]^{\mathbf{E}} = [0, 1]^{\coprod_J \Gamma} = ([0, 1]^J)^\Gamma$ .

The edge set  $\mathbf{E} \simeq \coprod_S \Gamma$  of the Cayley graph of  $\Gamma$ , as well as  $\Lambda \simeq \coprod_I \Gamma$  are such countable  $\Gamma$ -sets. Then isomorphisms of standard Borel probability spaces  $([0, 1], \text{Leb}) \simeq ([0, 1]^S, \otimes_S \text{Leb}) \simeq ([0, 1]^I, \otimes_I \text{Leb})$  induce  $\Gamma$ -equivariant isomorphisms of the Bernoulli shifts:

$$\begin{array}{ccccc} [0, 1]^\Gamma & \simeq & ([0, 1]^S)^\Gamma & \simeq & ([0, 1]^I)^\Gamma \\ \parallel & & \parallel & & \parallel \\ [0, 1]^\Gamma & \simeq & [0, 1]^{\mathbf{E}} & \simeq & [0, 1]^\Lambda. \end{array}$$

 $\blacksquare$

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A subgraph  $(V', E')$  of a graph  $(V, E)$  is called **spanning** if  $V' = V$ . A vertex  $a$  in a graph is called a **cutvertex** if there are two other vertices in its component with the property that every path joining them passes through  $a$ . A **block** of a graph is a maximal connected subgraph that has no cutvertex. Every simple cycle of a graph is contained within one of its blocks.

**Lemma 8** *If all vertices of a block have finite degree and for each pair of vertices  $(a, b)$  there are only finitely many distinct paths joining  $a$  to  $b$ , then the block is finite.*

*Proof.* Suppose for a contradiction that the block is infinite. Then it contains a simple infinite path  $P$  of vertices  $a_1, a_2, \dots$ . By Menger's theorem,  $a_1$  and  $a_n$  belong to a simple cycle  $C_n$  for each  $n > 1$ . But this implies that there are infinitely many distinct paths joining  $a_1$  to  $a_2$ : Fix  $n$  and let  $a_j$  ( $2 \leq j \leq n$ ) be the vertex on  $P \cap C_n$  with minimal index  $j$ . We may assume that  $C_n$  is oriented so that it visits  $a_n$  before it visits  $a_j$ . Then simply follow  $C_n$  from  $a_1$  until  $a_j$ , and then follow  $P$  to  $a_2$ . ■

**Proposition 9** *(For any Cayley graph) Let  $X := \Omega \times [0, 1]^\Gamma$  and  $\epsilon > 0$ . Let  $\nu := \mu_\epsilon \times \mu$ . There is a  $\Gamma$ -equivariant Borel map  $f : X \rightarrow \Omega$  such that  $(\nu, f)$  has indistinguishable infinite clusters and for all sufficiently small  $\epsilon$ , the push-forward measure  $f_*(\nu)$  of  $\nu$  is supported on the set of spanning subgraphs of  $\mathcal{G}$  each of whose components is a tree with infinitely many ends.*

*Proof.* We may equivariantly identify  $([0, 1]^\Gamma, \mu)$  with  $([0, 1]^{\mathbb{N} \times \Gamma} \times [0, 1]^E, \mu \times \mu)$ , so we identify  $(X, \nu)$  with  $(\Omega \times [0, 1]^{\mathbb{N} \times \Gamma} \times [0, 1]^E, \mu_\epsilon \times \mu \times \mu)$ . Fix an ordering of  $S \amalg S^{-1}$ ; this determines an ordering of the edges incident to each vertex in  $\mathcal{G}$ , where we ignore edge orientations for the rest of this proof. With  $d$  denoting the degree of  $\mathcal{G}$ , define the function  $h(t) := \lceil dt \rceil$  for  $t \in [0, 1]$ . Given a point  $x = (\omega, (r(n, \gamma))_{n \in \mathbb{N}, \gamma \in \Gamma}, u) \in X$ , construct the wired spanning forest  $\mathcal{F}_1$  of  $\mathcal{G}$  by using the cycle-popping algorithm of D. Wilson [Wil96, Sect. 3] as adapted in [BLPS01, Th. 5.1], also called there “Wilson’s algorithm rooted at infinity”, applied to the stacks where the  $n$ th edge in the stack under  $\gamma$  is defined as the  $h(r(n, \gamma))$ th edge incident to  $\gamma$ . The measure  $\nu$  is insertion tolerant and the map  $\pi : x \mapsto \omega \cup \mathcal{F}_1$  is increasing, whence by Proposition 6, the pair  $(\nu, \pi)$  has indistinguishable infinite clusters. Notice that all clusters are infinite. Now use  $u$  to construct the free minimal spanning forest  $\mathcal{F}_2$  in each cluster of  $\pi(x)$ , that is, for every cycle  $\Delta \subset \pi(x)$ , delete the edge  $e \in \Delta$  with maximum  $u(e)$  in that cycle. The map  $f$  is  $f(x) := \mathcal{F}_2$ .

Now the  $\nu$ -expected number of distinct simple paths in  $\pi(x)$  that join any two vertices is finite (equation (13.7) of [BLPS01]) for all sufficiently small  $\epsilon$ . In particular, the number of such paths is finite  $\nu$ -a.s. By Lemma 8, this means that all blocks of  $\pi(x)$  are finite, so that  $\mathcal{F}_2$  is a spanning tree in each block. Therefore each component of  $\mathcal{F}_2$  spans a component of  $\pi(x)$ . Thus,  $\mathcal{F}_2$  determines the same cluster relation and so  $(\nu, f)$  also has indistinguishable (infinite) clusters. Finally, the fact that the clusters of  $\pi(x)$ , and hence those of  $\mathcal{F}_2$ , have infinitely many ends follows, e.g., from [BLPS01, Th. 13.7]. ■

The cluster relation determined by  $f$  of Proposition 9 is treeable and has cost larger than 1 by [Gab00, Cor. IV.24 (2)], has finite cost (since the degree is bounded), and is ergodic by Proposition 5. Since we may equivariantly identify  $(\Omega \times [0, 1]^\Gamma, \mu_\epsilon \times \mu)$  with  $([0, 1]^\Gamma, \mu)$ , we proved:

**Proposition 10** *For any Cayley graph of  $\Gamma$ , the Bernoulli action on  $([0, 1]^\Gamma, \mu)$  contains a treeable subrelation that is ergodic and has cost in the interval  $(1, \infty)$ .*

At this point, we already have a reasonable answer to the analogue of von Neumann’s problem, since “treeable relation” is the analogue of “free group” and cost  $\mathcal{C} > 1$  is, in the context of treeable relations, equivalent to non-amenability.

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An alternative approach begins with a more explicit  $f$  and a more common measure  $f_*(\mu)$ , namely, the Bernoulli measure  $\mu_p$  on  $\{0, 1\}^\mathbb{E}$  for a certain parameter  $p$ , but requires us to choose a particular Cayley graph for  $\Gamma$ . It also requires us to obtain a treeable subrelation in a less explicit way. This is accomplished as follows.

Results of Häggström-Peres [HP99] imply that there are two critical values  $0 < p_c \leq p_u \leq 1$  such that

- (finite phase,  $p \in [0, p_c)$ )  $\mu_p$ -a.s., the subgraph has only finite clusters;
- (**non-uniqueness phase**,  $p \in (p_c, p_u)$ )  $\mu_p$ -a.s., infinitely many of the clusters of the subgraph are infinite, each one with infinitely many ends;
- (the uniqueness phase,  $p \in (p_u, 1]$ )  $\mu_p$ -a.s., the subgraph has only one cluster that is infinite.

The situation for the critical values  $p_c$  and  $p_u$  themselves is far from clear. Benjamini and Schramm [BS96] conjectured that  $p_c \neq p_u$  for every Cayley graph of a f.g. non-amenable group. The main result of [PSN00] (Th. 1, p. 498) asserts that given a f.g. non-amenable group  $\Gamma$ , there is a finite set of generators such that the associated Cayley graph admits a non-trivial interval of non-uniqueness. Thus:

**Proposition 11** *(For particular Cayley graphs) There exists a Cayley graph of  $\Gamma$  and a non-empty interval  $(p_c, p_u)$  such that, for any  $p \in (p_c, p_u)$ , the Bernoulli measure  $\mu_p$  on  $\{0, 1\}^\mathbb{E}$  is supported on the set of subgraphs admitting infinite components, each one with infinitely many ends.*

Let  $\pi : (X, \nu) \rightarrow \{0, 1\}^\mathbb{E}$  denote either

- (i)  $f_p : ([0, 1]^\mathbb{E}, \mu) \rightarrow \{0, 1\}^\mathbb{E}$  induced by the characteristic function  $\chi_{[0, p]} : [0, 1] \rightarrow \{0, 1\}$  of  $[0, p]$ , or
- (ii) the identity map  $(\{0, 1\}^\mathbb{E}, \mu_p) \rightarrow \{0, 1\}^\mathbb{E}$ ,

both with the natural Bernoulli  $\Gamma$ -action. Notice that the action is essentially free when  $0 < p < 1$ .

In case (ii), we have that  $(\mu_p, \pi)$  has indistinguishable infinite clusters by [LS99, Th. 3.3]. Case (i) is essentially the same, but first we must identify  $([0, 1]^\mathbb{E}, \mu)$  equivariantly as  $(\{0, 1\} \times [0, 1])^\mathbb{E} = \{0, 1\}^\mathbb{E} \times [0, 1]^\mathbb{E}$  equipped with the product measure  $\mu_p \times \mu$  in such a way that  $f_p$  becomes the identity on the first coordinate. Then we have insertion tolerance and so, by [LS99, Rem. 3.4], indistinguishable infinite clusters.

Hence, in both cases, for any  $p$  given by Prop. 11, the locus  $U^\infty$  of infinite classes of  $\mathcal{R}^{\text{cl}}$  is non-null and we have ergodicity of the restriction  $\mathcal{R}_{|\infty}^{\text{cl}}$  of  $\mathcal{R}^{\text{cl}}$  to  $U^\infty$  by Proposition 5. We claim that its normalized cost (i.e., computed with respect to the normalized probability measure  $\nu/\nu(U^\infty)$  on  $U^\infty$ ) satisfies  $1 < \mathcal{C}(\mathcal{R}_{|\infty}^{\text{cl}}) < \infty$ . The finiteness of the cost is clear since  $S$ , the index set for  $\Phi^{\text{cl}}$ , is finite. That it is strictly greater than 1 is a direct application of [Gab00, Cor. IV.24 (2)], since the graph  $\Phi^{\text{cl}}[x] \simeq \pi(x)_\rho$  associated with almost every  $x \in U^\infty$  has at least 3 ends.

In order to extend  $\mathcal{R}_{|\infty}^{\text{cl}}$  to a subrelation of  $\mathcal{R}_\Gamma$  defined on the whole of  $X$ , choose an enumeration  $\{\gamma_i\}_{i \in \mathbb{N}}$  of  $\Gamma$ . For each  $x \in X \setminus U^\infty$ , let  $\gamma_x$  be the first element  $\gamma_j \in \Gamma$  such that  $\gamma_j \cdot x \in U^\infty$ . Then the smallest equivalence relation containing  $\mathcal{R}_{|\infty}^{\text{cl}}$  and the  $(x, \gamma_x \cdot x)$ 's is a subrelation of  $\mathcal{R}_\Gamma$ , is ergodic, and has cost in  $(1, \infty)$  by the induction formula of [Gab00, Prop. II.6]. We proved:

**Proposition 12** *For a Cayley graph and a  $p$  given by Proposition 11, the Bernoulli actions on both  $([0, 1]^{\mathbb{E}}, \mu)$  and  $(\{0, 1\}^{\mathbb{E}}, \mu_p)$  contain a subrelation that is ergodic and has cost in the open interval  $(1, \infty)$ .*

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**Proposition 13** *If an equivalence relation  $\mathcal{R}$  is ergodic and has cost in  $(1, \infty)$ , then it contains a treeable subrelation  $\mathcal{S}_1$  that is ergodic and has cost in  $(1, \infty)$ .*

*Proof.* This is ensured by a result proved independently by A. Kechris and B. Miller [KM04, Lem. 28.11; 28.12] and by M. Pichot [Pic05, Cor. 40], through a process of erasing cycles from a graphing of  $\mathcal{S}_1$  with finite cost that contains an ergodic global isomorphism. ■

— O —

**Proposition 14** *If a treeable equivalence relation  $\mathcal{S}_1$  is ergodic and has cost  $> 1$ , then it contains a.e. a subrelation  $\mathcal{S}_2$  that is generated by an ergodic free action of the free group  $\mathbf{F}_2$ .*

*Proof.* If the cost of  $\mathcal{S}_1$  is  $> 2$ , this follows from a result of G. Hjorth [Hjo06] (see also [KM04, Sect. 28]). Otherwise, one first considers the restriction of the treeable  $\mathcal{S}_1$  to a small enough Borel subset  $V$ : this increases the normalized cost by the induction formula of [Gab00, Prop. II.6 (2)] to get  $\mathcal{C}(\mathcal{S}_1|V) \geq 2$ . In fact, it follows from the proof of [KM04, Th. 28.3] that *a treeable probability measure-preserving equivalence relation with cost  $\geq 2$  contains a.e. an equivalence subrelation that is given by a free action of the free group  $\mathbf{F}_2 = \langle a, b \rangle$  in such a way that the generator  $a$  acts ergodically.* By considering a subgroup of  $\mathbf{F}_2$  generated by  $a$  and some conjugates of  $a$  of the form  $b^k a b^{-k}$ , one gets an ergodic treeable subrelation of  $\mathcal{S}_1|V$  with a big enough normalized cost that, when extended to the whole of  $X$  (by using partial isomorphisms of  $\mathcal{S}_1$ ), it gets cost  $\geq 2$  (by the induction back [Gab00, Prop. II.6 (2)]) and of course remains ergodic. Another application of the above-italicized result gives the desired ergodic action of  $\mathbf{F}_2$  on  $X$ . ■



— O —

The proof of Theorem 2 is now complete as a direct consequence of Propositions 12 (for the case  $X = \{0, 1\}^{\mathbb{E}}$ ), 13 and 14. ■

— O —

In case  $X = [0, 1]^{\mathbb{E}}$  of Prop. 12, by using Prop. 7, we can see  $\mathcal{S}_2$  (with  $\mathcal{S}_2 \subset \mathcal{S}_1 \subset \mathcal{R}_\Gamma$  given by Prop. 14 and 13) as an equivalence subrelation of that given by the Bernoulli shift action of  $\Lambda$ . This finishes the proof of Theorem 1. Alternatively, we may use Prop. 10 and a similar argument to prove Theorem 1. ■

— O —

*Proof of Cor. 3.* For any diffuse abelian subalgebra  $A$  of  $L(H)$ , the von Neumann factor  $L(H \wr \Lambda) = L(\Lambda \rtimes \oplus_\Lambda H)$  contains the von Neumann algebra crossed product  $\Lambda \rtimes \otimes_\Lambda A$ , which is isomorphic with the group-measure-space factor  $\Lambda \rtimes L^\infty([0, 1]^\Lambda, \mu)$  associated with the Bernoulli shift. The corollary then follows from Th. 1. ■

— O —

*Proof of Cor. 4.* If  $\hat{K}$  is the dual group of  $K$ , then  $L(H \wr \Gamma)$  contains  $L(K \wr \Gamma)$ , which is isomorphic with the group-measure-space factor  $\Gamma \rtimes L^\infty(\hat{K}^\Gamma)$  associated with the Bernoulli shift of  $\Gamma$  on  $\hat{K}^\Gamma$ , where the finite set  $\hat{K} \simeq \{1, 2, \dots, k\}$  is equipped with the equiprobability measure  $\nu$ . The result is then obtained by taking the pull-back of the  $\mathbf{F}_2$ -action on  $\prod_1^n \{0, 1\}^\Gamma$ , given in Th. 2, by the  $\Gamma$ -equivariant Borel map  $\hat{K}^\Gamma \rightarrow (\{0, 1\}^n)^\Gamma \simeq \prod_1^n \{0, 1\}^\Gamma$ , sending  $\otimes \nu$  to  $\mu_p$ , that extends a map  $\{1, 2, \dots, k\} \rightarrow \{0, 1\}^n$  (whose existence is ensured by the form of  $k = \beta^n$ ). ■

— O —

It is likely that the free minimal spanning forest (FMSF) of a Cayley graph of  $\Gamma$  would serve as the desired ergodic subrelation  $\mathcal{S}_1$  of Prop. 13, but its indistinguishability, conjectured in [LPS06], is not known. Also, it is not known to have  $\text{cost} > 1$ , but this is equivalent to  $p_c < p_u$ , which is conjectured to hold and which we know holds for some Cayley graph. See [LPS06] for information on the FMSF and [Tim06] for a weak form of indistinguishability.

A general question remains open:

**Question:** Does every probability-measure-preserving free ergodic action of a non-amenable countable group contain an ergodic subrelation generated by a free action of a non-cyclic free group? More generally: Does every standard countable probability-measure-preserving non-amenable ergodic equivalence relation contain a treeable non-amenable ergodic equivalence subrelation?

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