Holomorphic dynamics, Painlevé VI equation and Character Varieties.

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HOLOMORPHIC DYNAMICS, PAINLEVÉ VI EQUATION AND CHARACTER VARIETIES.

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1. INTRODUCTION

This is the first part of a series of two papers (see [10]), the aim of which is to describe the dynamics of a polynomial action of the group

$$\Gamma_2^* = \{M \in \mathsf{PGL}(2,\mathbf{Z}) \mid \quad M = \mathrm{Id} \bmod(2)\} \tag{1.1}$$

on the family of affine cubic surfaces

$$x^2+y^2+z^2+xyz = Ax+By+Cz+D, \tag{1.2}$$

where A, B, C, and D are complex parameters. This dynamical system appears in several different mathematical areas, like the monodromy of the sixth Painlevé differential equation, the geometry of hyperbolic threefolds, and the spectral properties of certain discrete Schrödinger operators. One of our main goals here is to classify parameters (A,B,C,D) for which Γ_2^* preserves a holomorphic geometric structure, and to apply this classification to provide a galoisian proof of the irreducibility of the sixth Painlevé equation.

1.1. **Character variety.** Let $\mathbb{S}^2_4$ be the four punctured sphere. Its fundamental group is isomorphic to a free group of rank 3; if α, β, γ and δ are the four loops which are depicted on figure 1.1, then

$$\pi_1(\mathbb{S}^2_4) = \langle \alpha, \beta, \gamma, \delta \,|\, \alpha\beta\gamma\delta = 1 \rangle.$$

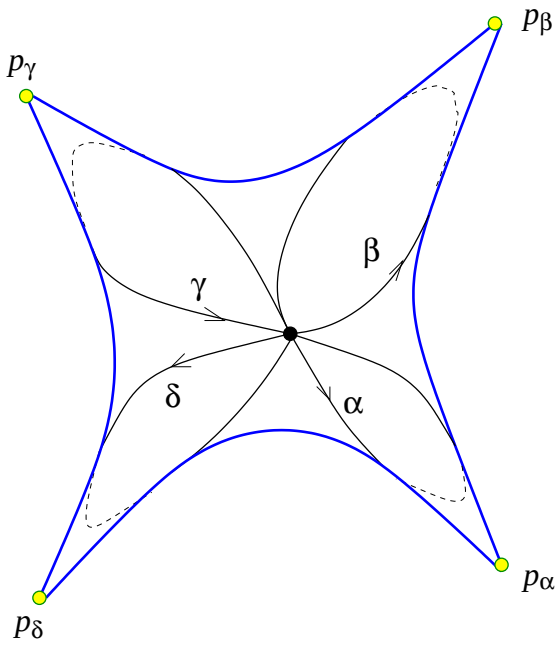

FIGURE 1. The four punctured sphere.

Let $\mathsf{Rep}(\mathbb{S}^2_4)$ be the set of representations of $\pi_1(\mathbb{S}^2_4)$ into $\mathsf{SL}(2,\mathbf{C})$. Such a representation ρ is uniquely determined by the 3 matrices $\rho(\alpha)$, $\rho(\beta)$, and $\rho(\gamma)$, so that $\mathsf{Rep}(\mathbb{S}^2_4)$ can be identified with the affine algebraic variety $(\mathsf{SL}(2,\mathbf{C}))^3$. Let us associate the 7 following traces to any element ρ of $\mathsf{Rep}(\mathbb{S}^2_4)$:

$$\begin{array}{cccccccc} a = \mathsf{tr}(\rho(\alpha)) & ; & b = \mathsf{tr}(\rho(\beta)) & ; & c = \mathsf{tr}(\rho(\gamma)) & ; & d = \mathsf{tr}(\rho(\delta)) \\ x = \mathsf{tr}(\rho(\alpha\beta)) & ; & y = \mathsf{tr}(\rho(\beta\gamma)) & ; & z = \mathsf{tr}(\rho(\gamma\alpha)). & & \end{array}$$

The polynomial map $\chi : \mathsf{Rep}(\mathbb{S}^2_4) \to \mathbf{C}^7$ defined by

$$\chi(\rho) = (a,b,c,d,x,y,z) \tag{1.3}$$

is invariant under conjugation, by which we mean that $\chi(\rho') = \chi(\rho)$ if ρ' is conjugate to ρ by an element of $\mathsf{SL}(2,\mathbf{C})$. Moreover,

(1) the algebra of polynomial functions on $\mathsf{Rep}(\mathbb{S}^2_4)$ which are invariant under conjugation is generated by the components of χ;

(2) the components of χ satisfy the quartic equation

$$x^2+y^2+z^2+xyz = Ax+By+Cz+D, \tag{1.4}$$

in which the variables A, B, C, and D are given by

$$\begin{array}{c} A = ab+cd, \quad B = bc+ad, \quad C = ac+bd, \\ \text{and} \quad D = 4-a^2-b^2-c^2-d^2-abcd. \end{array} \tag{1.5}$$

(3) the algebraic quotient $\mathsf{Rep}(\mathbb{S}^2_4)//\mathsf{SL}(2,\mathbf{C})$ of $\mathsf{Rep}(\mathbb{S}^2_4)$ by the action of $\mathsf{SL}(2,\mathbf{C})$ by conjugation is isomorphic to the six-dimensional quartic hypersurface of $\mathbf{C}^7$ defined by equation (1.4).

The affine algebraic variety $\mathsf{Rep}(\mathbb{S}^2_4)//\mathsf{SL}(2,\mathbf{C})$ will be denoted $\chi(\mathbb{S}^2_4)$ and called the *character variety of* $\mathbb{S}^2_4$. For each choice of four complex parameters A, B, C, and D, $S_{(A,B,C,D)}$ (or S if there is no obvious possible confusion) will denote the cubic surface of $\mathbf{C}^3$ defined by the equation (1.4). The family of these surfaces $S_{(A,B,C,D)}$ will be denoted Fam.

Remark 1.1. The map $\mathbf{C}^4 \to \mathbf{C}^4; (a,b,c,d) \mapsto (A,B,C,D)$ defined by (1.5) is a non Galois ramified cover of degree 24. Fibers are studied in Appendix B. It is important to notice that a point $m \in S_{(A,B,C,D)}$ will give rise to representations of very different nature depending on the choice of (a,b,c,d) in the fiber, e.g. reducible or irreducible, finite or infinite image.

Remark 1.2. As we shall see in section 2.4, if we replace the four punctured sphere by the once punctured torus, the character variety is naturally fibered by the family of cubic surfaces $S_{(0,0,0,D)}$.

1.2. **Automorphisms and modular groups.** The group of automorphisms $\mathsf{Aut}(\pi_1(\mathbb{S}^2_4))$ acts on $\mathsf{Rep}(\mathbb{S}^2_4)$ by composition: $(\Phi,\rho) \mapsto \rho\circ\Phi^{-1}$. Since inner automorphisms act trivially on $\chi(\mathbb{S}^2_4)$, we get a morphism from the group of outer automorphisms $\mathsf{Out}(\pi_1(\mathbb{S}^2_4))$ into the group of polynomial diffeomorphisms of $\chi(\mathbb{S}^2_4)$:

$$\left\{ \begin{array}{ccc} \mathsf{Out}(\pi_1(\mathbb{S}^2_4)) & \to & \mathsf{Aut}[\chi(\mathbb{S}^2_4)] \\ \Phi & \mapsto & f_\Phi \end{array} \right. \tag{1.6}$$

such that $f_\Phi(\chi(\rho)) = \chi(\rho\circ\Phi^{-1})$ for any representation ρ.

The group $\mathsf{Out}(\pi_1(\mathbb{S}^2_4))$ is isomorphic to the extended mapping class group $\mathsf{MCG}^*(\mathbb{S}^2_4)$, *i.e.* to the group of isotopy classes of homeomorphisms of $\mathbb{S}^2_4$, that preserve or reverse the orientation. It contains a copy of $\mathsf{PGL}(2,\mathbf{Z})$ which is obtained as follows. Let $\mathbb{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ be a torus of dimension 2 and σ be the involution of $\mathbb{T}^2$ defined by $\sigma(x,y) = (-x,-y)$. The fixed point set of σ is the 2-torsion subgroup $H \subset \mathbb{T}^2$, isomorphic to $\mathbf{Z}/2\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}$:

$$H = \{(0,0);(0,1/2);(1/2,0);(1/2,1/2)\}. \tag{1.7}$$

The quotient $\mathbb{T}^2/\sigma$ is homeomorphic to the sphere, $\mathbb{S}^2$, and the quotient map $\pi : \mathbb{T}^2 \to \mathbb{T}^2/\sigma = \mathbb{S}^2$ has four ramification points, corresponding to the four fixed points of σ. The affine group $\mathsf{GL}(2,\mathbf{Z}) \ltimes H$ acts linearly on $\mathbb{T}^2$ and commutes with σ. This yields an action of $\mathsf{PGL}(2,\mathbf{Z}) \ltimes H$ on the sphere $\mathbb{S}^2$, that permutes ramification points of π. Taking these four ramification points as the punctures of $\mathbb{S}^2_4$, we get a morphism

$$\mathsf{PGL}(2,\mathbf{Z}) \ltimes H \to \mathsf{MCG}^*(\mathbb{S}^2_4), \tag{1.8}$$

which, in fact, is an isomorphism (see [5], section 4.4). The image of $\mathsf{PGL}(2,\mathbf{Z})$ is the stabilizer of $\pi(0,0)$, freely permuting the three other points. As a consequence, $\mathsf{PGL}(2,\mathbf{Z})$ acts by polynomial transformations on $\chi(\mathbb{S}^2_4)$. The image of H permutes the 4 punctures by products of disjoint transpositions and acts trivially on $\chi(\mathbb{S}^2_4)$, so that the action of the whole mapping class group $\mathsf{MCG}^*(\mathbb{S}^2_4)$ on $\chi(\mathbb{S}^2_4)$ actually reduces to that of $\mathsf{PGL}(2,\mathbf{Z})$ (see section 2.2).

Let Γ_2^* be the subgroup of $\mathsf{PGL}(2,\mathbf{Z})$ whose elements coincide with the identity modulo 2. This group coincides with the (image in $\mathsf{PGL}(2,\mathbf{Z})$ of the) stabilizer of the fixed points of σ, so that Γ_2^* acts on $\mathbb{S}^2_4$ and fixes its four punctures. Consequently, Γ_2^* acts polynomially on $\chi(\mathbb{S}^2_4)$ and preserves the fibers of the projection

$$(a,b,c,d,x,y,z) \mapsto (a,b,c,d).$$

From this we obtain, for any choice of four complex parameters (A,B,C,D), a morphism from Γ_2^* to the group of polynomial diffeomorphisms of the surface $S_{(A,B,C,D)}$. The following result is essentially due to Èl'-Huti (see [19], and §3.1).

Theorem A. *For any choice of the parameters A, B, C, and D, the morphism*

$$\Gamma_2^* \to \mathsf{Aut}[S_{(A,B,C,D)}]$$

is injective and the index of its image is bounded by 24*. For a generic choice of the parameters, this morphism is an isomorphism.*

As a consequence of this result, it suffices to understand the action of Γ_2^* on the surfaces $S_{(A,B,C,D)}$ in order to get a full understanding of the action of $\mathsf{MCG}^*(\mathbb{S}^2_4)$ on $\chi(\mathbb{S}^2_4)$. (see also remark 2.4 for the case of orientation preserving transformations and an action of the pure braid group on three strings).

Remark 1.3. If the parameters A, B, C, and D belong to a ring $\mathbb{K}$, the group Γ_2^* acts on $S_{(A,B,C,D)}(\mathbb{K})$, i.e. on the set of points of the surface with coordinates in $\mathbb{K}$. In particular, when the parameters are real numbers, Γ_2^* acts on the real surface $S_{(A,B,C,D)}(\mathbf{R})$.

There are useful symmetries of the parameter space, as well as covering between distinct surfaces $S_{(A,B,C,D)}$ and $S_{(A',B',C',D')}$, that can be used to relate dynamical properties of Γ_2^* on different surfaces of our family. These symmetries and covering will be described in section 2 and appendix B.

1.3. **Projective structures.** Once $\mathbb{S}_4^2$ is endowed with a complex projective structure, which means that we have an atlas on $\mathbb{S}_4^2$ made of charts into $\mathbb{P}^1(\mathbf{C})$ with transition functions in the group of homographic transformations of $\mathbb{P}^1(\mathbf{C})$, the holonomy defines a morphism from $\pi_1(\mathbb{S}_4^2)$ to $\mathsf{PSL}(2,\mathbf{C})$. Since $\pi_1(\mathbb{S}_4^2)$ is a free group, the holonomy can be lifted to a morphism

$$\rho : \pi_1(\mathbb{S}_4^2) \to \mathsf{SL}(2,\mathbf{C}).$$

Properties of the holonomy such as discreteness, finiteness, or the presence of parabolic elements in $\rho(\pi_1(\mathbb{S}_4^2))$, are invariant by conjugation and by the action of the mapping class group $\mathsf{MCG}^*(\mathbb{S}_4^2)$. This kind of properties may be used to construct invariant subsets of $S_{(A,B,C,D)}$ for the action of Γ_2^*, and the dynamics of this action may be used to understand those invariant sets and the space of projective structures. This approach has been popularized by Goldman (see [21], [23] for example).

1.4. **Painlevé VI equation.** The dynamics of Γ_2^* on the varieties $S_{(A,B,C,D)}$ is also related to the monodromy of a famous ordinary differential equation. The sixth Painlevé equation $P_{VI} = P_{VI}(\theta_\alpha, \theta_\beta, \theta_\gamma, \theta_\delta)$ is the second order non linear O.D.E.

$$P_{VI} \quad \begin{cases} \frac{d^2q}{dt^2} = \frac{1}{2}\left(\frac{1}{q}+\frac{1}{q-1}+\frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{q-t}\right)\left(\frac{dq}{dt}\right) \\ \qquad + \frac{q(q-1)(q-t)}{t^2(t-1)^2}\left(\frac{(\theta_\delta-1)^2}{2} - \frac{\theta_\alpha^2}{2}\frac{t}{q^2} + \frac{\theta_\beta^2}{2}\frac{t-1}{(q-1)^2} + \frac{1-\theta_\gamma^2}{2}\frac{t(t-1)}{(q-t)^2}\right). \end{cases}$$

the coefficients of which depend on complex parameters

$$\theta = (\theta_\alpha, \theta_\beta, \theta_\gamma, \theta_\delta)$$

. The main property of this equation is the absence of movable singular points, the so-called Painlevé property: All essential singularities of all solutions $q(t)$ of the equation only appear when $t \in \{0, 1, \infty\}$; in other words, any solution $q(t)$ extends analytically as a meromorphic function on the universal cover of $\mathbb{P}^1(\mathbf{C}) \setminus \{0, 1, \infty\}$.

Another important property, expected by Painlevé himself, is the irreducibility. Roughly speaking, the general solution is more transcendental than solutions of linear, or first order non linear, ordinary differential equations with polynomial coefficients. Painlevé proved that any irreducible second order polynomial differential equation without movable singular point

falls after reduction into the 4-parameter family P_{VI} or one of its degenerations $P_I,\dots,P_V$. The fact that Painlevé equations are actually irreducible was proved by Nishioka and Umemura for P_I (see [35, 44]) and by Watanabe in [45] for P_{VI}. Another notion of irreducibility, related with transcendence of first integrals, was developped by Malgrange and Casale in [33, 11] and then applied to the first of Painlevé equations (see 7 for more details).

A third important property, discovered by R. Fuchs, is that solutions of P_{VI} parametrize isomonodromic deformations of rank 2 meromorphic connections over the Riemann sphere having simple poles at $\{0,t,1,\infty\}$, with respective set of local exponents $(\pm\frac{\theta_\alpha}{2},\pm\frac{\theta_\beta}{2},\pm\frac{\theta_\gamma}{2},\pm\frac{\theta_\delta}{2})$. From this point of view, the good space of initial conditions at, say, t_0, is the moduli space $\mathcal{M}_{t_0}(\theta)$ of those connections for $t=t_0$ (see [29]); it turns to be a convenient semi-compactification of the naive space of initial conditions $\mathbf{C}^2\ni(q(t_0),q'(t_0))$ (compare [37]). Via the Riemann-Hilbert correspondence, $\mathcal{M}_{t_0}(\theta)$ is analytically isomorphic to the moduli space of corresponding monodromy representations, namely to (a desingularization of) $S_{(A,B,C,D)}$ with parameters

$$a=2\cos(\pi\theta_\alpha),\ b=2\cos(\pi\theta_\beta),\ c=2\cos(\pi\theta_\gamma),\ d=2\cos(\pi\theta_\delta). \tag{1.9}$$

The (non linear) monodromy of P_{VI}, obtained after analytic continuation around 0 and 1 of local P_{VI} solutions at $t=t_0$, induces a representation

$$\pi_1(\mathbb{P}^1(\mathbf{C})\setminus\{0,1,\infty\},t_0)\to\mathsf{Aut}[S_{(A,B,C,D)}]$$

whose image coincides with the action of $\Gamma_2\subset\mathsf{PSL}(2,\mathbf{Z})$ (see [17, 29]).

1.5. **The Cayley cubic.** One very specific choice of the parameters will play a central role in this paper. The parameters are $(0,0,0,4)$, and the surface $S_{(0,0,0,4)}$ is the unique surface in our family with four singularities. Four is the maximal possible number of isolated singularities for a cubic surface, and $S_{(0,0,0,4)}$ is therefore isomorphic to the well known Cayley cubic. From the point of view of character varieties, this surface appears in the very special case $(a,b,c,d)=(0,0,0,0)$ consisting only of solvable representations (dihedral or reducible).

I. The Cayley cubic S_C ; II. $S_{(-0.2,-0.2,-0.2,4.39)}$; III. $S_{(0,0,0,3)}$; IV. $S_{(0,0,0,4.1)}$.

From the Painlevé point of view, it corresponds to the Picard parameter $(\theta_1,\theta_2,\theta_3,\theta_4)=(0,0,0,1)$. The singular foliation which is defined by the corresponding Painlevé equation $P_{VI}(0,0,0,1)$ is transversely affine (see [11]) and, as was shown by Picard himself, admits explicit first integrals by means of elliptic functions (see 7). Moreover, this specific equation has countably many agebraic solutions, that are given by finite order points on the Legendre family of elliptic curves (see 7).

The Cayley cubic has also the "maximal number of automorphisms": The whole group $\mathsf{PGL}(2,\mathbf{Z})$, in which Γ_2^* has index 6, stabilizes the Cayley cubic,

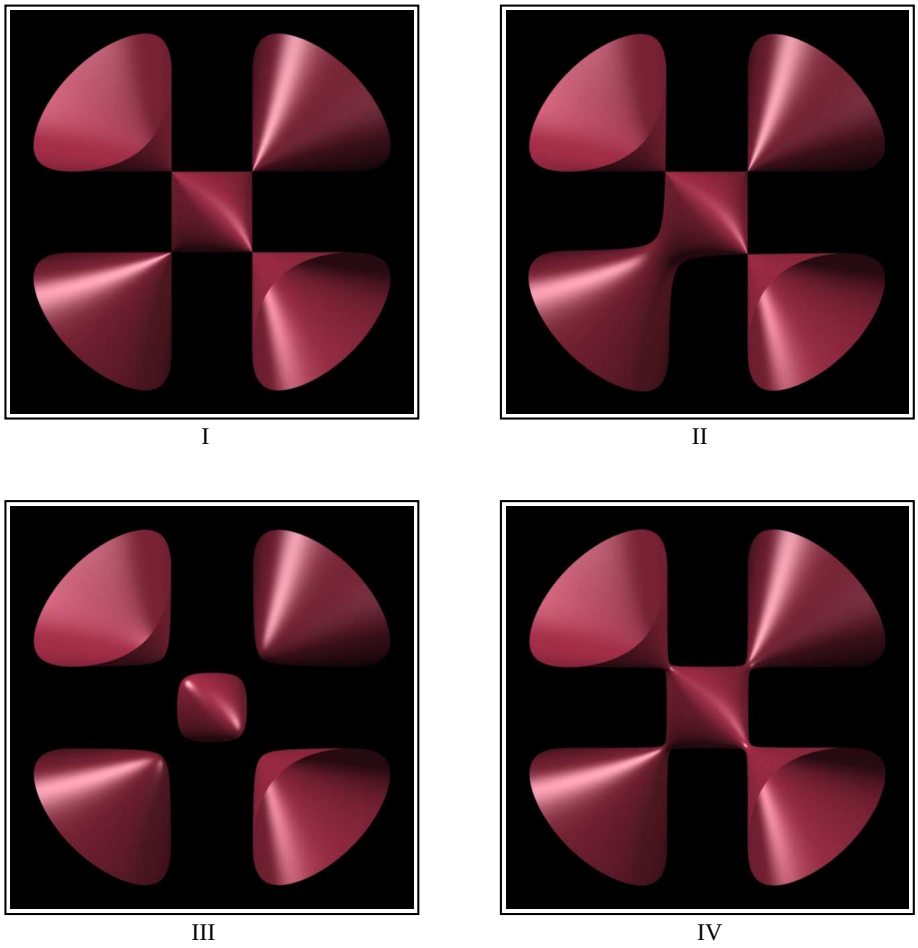

FIGURE 2. Four examples. I. The Cayley cubic S_C ; II. $S_{(-0.2,-0.2,-0.2,4.39)}$; III. $S_{(0,0,0,3)}$; IV. $S_{(0,0,0,4.1)}$.

and there are extra symmetries coming from the permutation of coordinates (see section 3.1), so that the maximal index 24 of theorem A is obtained in the case of the Cayley cubic.

Moreover, the degree 2 orbifold cover

$$\pi_C : \mathbf{C}^* \times \mathbf{C}^* \to S_{(0,0,0,4)} \tag{1.10}$$

semi-conjugates the action of $\mathsf{PGL}(2,\mathbf{Z})$ on the character surface $S_{(0,0,0,4)}$ to the monomial action of $\mathsf{GL}(2,\mathbf{Z})$ on $\mathbf{C}^* \times \mathbf{C}^*$, which is defined by

$$M\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u^{m_{11}} v^{m_{12}} \\ u^{m_{21}} v^{m_{22}} \end{pmatrix}, \tag{1.11}$$

for any element M of $\mathsf{GL}(2,\mathbf{Z})$. On the universal cover $\mathbf{C}\times\mathbf{C}\to\mathbf{C}^*\times\mathbf{C}^*$, the lifted dynamics is the usual affine action of the group $\mathsf{GL}(2,\mathbf{Z})\ltimes\mathbf{Z}^2$ on the complex plane $\mathbf{C}^2$.

1.6. **Compactification and entropy.** Our first goal is to classify automorphisms of surfaces $S_{(A,B,C,D)}$ in three types, elliptic, parabolic and hyperbolic, and to describe the main properties of the dynamics of each type of automorphisms. This classification is compatible with the description of mapping classes, Dehn twists corresponding to parabolic transformations, and pseudo-Anosov mappings to hyperbolic automorphisms. The most striking result in that direction is summarized in the following theorem.

Theorem B. *Let A, B, C, and D be four complex numbers. Let M be an element of Γ_2^*, and f_M be the automorphism of $S_{(A,B,C,D)}$ which is determined by M. The topological entropy of $f_M : S_{(A,B,C,D)}(\mathbf{C})\to S_{(A,B,C,D)}(\mathbf{C})$ is equal to the logarithm of the spectral radius of M.*

The proof is obtained by a deformation argument: We shall show that the topological entropy does not depend on the parameters (A,B,C,D), and then compute it in the case of the Cayley cubic. To do so, we first describe the geometry of surfaces $S\in\mathsf{Fam}$ (section 2), their groups of automorphisms (section 3), and the action of automorphisms by birational transformations on the Zariski closure $\overline{S}$ of S in $\mathbb{P}^3(\mathbf{C})$ (section 4).

Another algorithm to compute the topological entropy has been obtained by Iwasaki and Uehara for non singular cubics S in Fam (see [32]). The case of singular cubics is crucial for the study of the set of quasi-fuchsian deformations of fuchsian representations, in connection with Bers embedding of Teichmüller spaces (see [23] and [10]).

1.7. **Bounded orbits.** Section 5 is devoted to the study of parabolic elements (or Dehn twists), and bounded or periodic (i.e. finite) orbits of Γ_2^*. For instance, given a representation $\rho:\mathbb{S}_4^2\to\mathsf{SU}(2)\subset\mathsf{SL}(2,\mathbf{C})$, the Γ_2^*-orbit of the correponding point $\chi(\rho)$ will be bounded, contained in the cube $[-2,2]^3$. If moreover the image of ρ is finite, then so will be the corresponding orbit. Though, there are periodic orbits with complex coordinates.

First of all, fixed points of Γ_2^* are precisely the singular points of $S_{(A,B,C,D)}$ and have been extensively studied (see [29]). Singular points arise from semi-stable points of $\mathsf{Rep}(\mathbb{S}_4^2)$, that is to say either from reducible representations, or from those representations for which one of the matrices $\rho(\alpha)$, $\rho(\beta)$, $\rho(\gamma)$ or $\rho(\delta)$ is $\pm I$. Both type of degeneracy occur at each singular point of $S_{(A,B,C,D)}$ depending on the choice of parameters (a,b,c,d) fitting to (A,B,C,D). The Riemann-Hilbert correspondance $\mathcal{M}_{t_0}(\theta)\to S_{(A,B,C,D)}$

is a minimal resolution of singularities and P_{VI} equation restricts to the exceptional divisor as a Riccati equation : this is the locus of Riccati-type solutions. We note that any point (x,y,z) is the singular point of one member $S_{(A,B,C,D)}$.

Periodic orbits of length ≥ 2 correspond to algebraic solutions of P_{VI} equation (see [31]). In Proposition 5.4, we classify orbits of length ≤ 4 : we find one 2-parameter family of length 2 orbits and two 1-parameter families of length 3 and 4 orbits. They correspond to well-known[1] algebraic solutions of P_{VI} equation (see [3]). For instance, the length 2 orbit arise when $A = C = 0$; the corresponding P_{VI}-solution is $q(t) = \sqrt{t}$.

The following result shows that infinite bounded orbits are real and contained in the cube $[-2,2]^3$.

Theorem C. *Let m be a point of $S_{(A,B,C,D)}$ with a bounded Γ_2^*-orbit of length > 4. Then, the parameters (A,B,C,D) are real numbers and the orbit is contained in the real part $S_{(A,B,C,D)}(\mathbf{R})$ of the surface.*

If the orbit of m is finite, then both the surface and the orbit are actually defined over a (real) number field.

If the orbit of m is infinite, then it corresponds to a $\mathsf{SU}(2)$-representation for a convenient choice of parameters (a,b,c,d), and the orbit is contained and dense in the unique bounded connected component of the smooth part of $S_{(A,B,C,D)}(\mathbf{R})$.

As a corollary, periodic orbits of length > 4 are rigid and we recover the main result of [3]. Recall that Cayley cubic contains infinitely many periodic orbits, of arbitrary large order. It is conjectured that there are finitely many periodic orbits apart from the Cayley member, but this is still an open problem. A classification of known periodic orbits can be found in [7].

About infinite orbits, Theorem C should be compare with results of Goldman and Previte and Xia, concerning the dynamics on the character variety for representations into $\mathsf{SU}(2)$ [40]. We note that an infinite bounded orbit may also correspond to $\mathsf{SL}(2,\mathbf{R})$-representation for an alternate choice of parameters (a,b,c,d).

This theorem stresses the particular role played by the real case, when all the parameters A, B, C, and D are real numbers; in that case, Γ_2^* preserves the real part of the surface and we have two different, but closely related, dynamical systems: The action on the complex surface $S_{(A,B,C,D)}(\mathbf{C})$ and the action on the real surface $S_{(A,B,C,D)}(\mathbf{R})$. The link between those two dynamical systems will be studied in [10].

[1] Although we usually find 4 families of algebraic solutions of P_{VI} in the litterature (see [7, 3]), there are actually 3 up to Okamoto symmetries : degree 4 solutions 3B and 4C in [3] are conjugated by the symmetry $s_1s_2s_1$ (with notations of [36]).

1.8. **Dynamics, affine structures, and the irreducibility of P_{VI}.** The last main result that we shall prove concerns the classification of parameters (A,B,C,D) for which $S_{(A,B,C,D)}$ admits a Γ_2^*-invariant holomorphic geometric structure.

Theorem D. *The group Γ_2^* does not preserve any holomorphic curve of finite type, any singular holomorphic foliation, or any singular holomorphic web. The group Γ_2^* does not preserve any meromorphic affine structure, except in the case of the Cayley cubic, i.e. when $(A,B,C,D)=(0,0,0,4)$, or equivalently when*

$$(a,b,c,d)=(0,0,0,0) \text{ or } (2,2,2,-2),$$

up to multiplication by -1 and permutation of the parameters.

Following [12], the same strategy shows that the Galois groupoid is the whole symplectic pseudo-group except in the Cayley case (see section 7), and we get

Theorem E. *The sixth Painlevé equation is irreducible in the sense of Malgrange and Casale except when $(A,B,C,D)=(0,0,0,4)$, i.e. except in one of the following cases:*

- $\theta_\omega \in \frac{1}{2}+\mathbf{Z}$, $\forall\,\omega=\alpha,\beta,\gamma,\delta$,
- $\theta_\omega \in \mathbf{Z}$, $\forall\,\omega=\alpha,\beta,\gamma,\delta$, *and* $\sum_\omega \theta_\omega$ *is even.*

Following [13], Malgrange-Casale irreducibility also implies Nishioka--Umemura irreducibility, so that theorem 1.8 indeed provides a galoisian proof of the irreducibility in the spirit of Drach and Painlevé.

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2. THE FAMILY OF SURFACES

As explained in 1.1, we shall consider the family Fam of complex affine surfaces which are defined by the following type of cubic equations

$$x^2+y^2+z^2+xyz = Ax+By+Cz+D,$$

in which A, B, C, and D are four complex parameters. Each choice of (A,B,C,D) gives rise to one surface S in our family; if necessary, S will also be denoted $S_{(A,B,C,D)}$. When the parameters are real numbers, $S(\mathbf{R})$ will denote the real part of S. Figure 1.5 presents a few pictures of $S(\mathbf{R})$ for various choices of the parameters.

This section contains preliminary results on the geometry of the surfaces $S_{(A,B,C,D)}$, and the automorphisms of these surfaces. Most of these results are well known to algebraic geometers and specialists of Painlevé VI equations.

2.1. **The Cayley cubic.** In 1869, Cayley proved that, up to projective transformations, there is a unique cubic surface in $\mathbb{P}^3(\mathbf{C})$ with four isolated singularities. One of the nicest models of the Cayley cubic is the surface $S_{(0,0,0,4)}$, whose equation is

$$x^2+y^2+z^2+xyz = 4.$$

The four singular points of S_C are rational nodes located at

$$(-2,-2,-2),\ (-2,2,2),\ (2,-2,2) \quad \text{and} \quad (2,2,-2),$$

and can be seen on figure 1.5. This specific member of our family of surfaces will be called *the Cayley cubic* and denoted S_C. This is justified by the following theorem (see Appendix A).

Theorem 2.1 (Cayley)**.** *If S is a member of the family* Fam *with four singular points, then S coincides with the Cayley cubic S_C.*

The Cayley cubic is isomorphic to the quotient of $\mathbf{C}^*\times\mathbf{C}^*$ by the involution $\eta(u,v)=(u^{-1},v^{-1})$. The map

$$\pi_C(x,y) = \left(-u-\frac{1}{u}, -v-\frac{1}{v}, -uv-\frac{1}{uv}\right)$$

gives an explicit isomorphism between $(\mathbf{C}^*\times\mathbf{C}^*)/\eta$ and S_C. The four fixed points

$$(1,1), \quad (1,-1), \quad (-1,1) \quad \text{and} \quad (-1,-1)$$

of η respectively correspond to the singular points of S_C above.

The real surface $S_C(\mathbf{R})$ contains the four singularities of S_C, and the smooth locus $S_C(\mathbf{R})\setminus \mathrm{Sing}(S_C)$ is made of five components : A bounded one, the closure of which coincides with the image of $\mathbb{T}^2=\mathbb{S}^1\times\mathbb{S}^1\subset\mathbf{C}^*\times\mathbf{C}^*$ by π_C, and four unbounded ones, corresponding to images of $\mathbf{R}^+\times\mathbf{R}^+$, $\mathbf{R}^+\times\mathbf{R}^-$, $\mathbf{R}^-\times\mathbf{R}^+$, and $\mathbf{R}^-\times\mathbf{R}^-$ (see figure 1.5).

As explained in section 1.5, the group $\mathsf{GL}(2,\mathbf{Z})$ acts on $\mathbf{C}^*\times\mathbf{C}^*$ by monomial transformations, and this action commutes with the involution η, permuting its fixed points. As a consequence, $\mathsf{PGL}(2,\mathbf{Z})$ acts on the quotient S_C. Precisely, the generators

$$\begin{pmatrix}1&0\\-1&1\end{pmatrix},\quad \begin{pmatrix}1&1\\0&1\end{pmatrix}\quad\text{and}\quad \begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

of $\mathsf{PGL}(2,\mathbf{Z})$ respectively send the triple (x,y,z) to

$$(x,-z-xy,y),\quad (z,y,-x-yz)\quad\text{and}\quad (x,y,-z-xy).$$

As we shall see, the induced action of $\mathsf{PGL}(2,\mathbf{Z})$ on S_C coincides with the action of the extended mapping class group of $\mathbb{S}^2_4$ considered in §1.2.

The group $\mathsf{PGL}(2,\mathbf{Z})$ preserves the real part of S_C ; for example, the product $\mathbf{C}^*\times\mathbf{C}^*$ retracts by deformation on the real 2-torus $\mathbb{T}^2=\mathbb{S}^1\times\mathbb{S}^1$, and the monomial action of $\mathsf{GL}(2,\mathbf{Z})$ preserves this torus (it is the standard one under the parametrization $(s,t)\mapsto(e^{2i\pi s},e^{2i\pi t})$).

2.2. **Mapping class group action.** First, let us detail section 1.2. The extended mapping class group $\mathsf{MCG}^*(\mathbb{S}^2_4)$ is the group of isotopy classes of homeomorphisms of the four punctured sphere $\mathbb{S}^2_4$; the usual mapping class group $\mathsf{MCG}(\mathbb{S}^2_4)$ is the index 2 subgroup consisting only in orientation preserving homeomorphisms. Those groups embed in the group of outer automorphisms of $\pi_1(\mathbb{S}^2_4)$ in the following way. Fix a base point $p_0\in\mathbb{S}^2_4$. In any isotopy class, one can find a homeomorphism h fixing p_0 and thus inducing an automorphism of the fundamental group

$$h_*:\pi_1(\mathbb{S}^2_4,p_0)\to\pi_1(\mathbb{S}^2_4,p_0)\ ;\ \gamma\mapsto h\circ\gamma.$$

The class of h_* modulo inner automorphisms does not depend on the choice of the representative h in the homotopy class and we get a morphism

$$\mathsf{MCG}^*(\mathbb{S}^2_4)\to\mathsf{Out}(\pi_1(\mathbb{S}^2_4)) \tag{2.1}$$

which turns out to be an isomorphism.

Now, the action of $\mathsf{Out}(\pi_1(\mathbb{S}^2_4))$ on $\chi(\mathbb{S}^2_4)$ gives rise to a morphism

$$\begin{cases}\mathsf{MCG}^*(\mathbb{S}^2_4) & \to & \mathsf{Aut}[\chi(\mathbb{S}^2_4)]\\ [h] & \mapsto & \{\chi(\rho)\mapsto\chi(\rho\circ h^{-1})\}\end{cases} \tag{2.2}$$

into the group of polynomial diffeomorphisms of $\chi(\mathbb{S}^2_4)$. (here, we use that $\rho\circ(h_*)^{-1}=\rho\circ h^*=\rho\circ h^{-1}$). Our goal in this section is to give explicit formulae for this action of $\mathsf{MCG}^*(\mathbb{S}^2_4)$ on $\chi(\mathbb{S}^2_4)$, and to describe the subgroup of $\mathsf{MCG}(\mathbb{S}^2_4)$ which stabilizes each surface $S_{(A,B,C,D)}$.

2.2.1. *Torus cover.* Consider the two-fold ramified cover

$$\pi_T : \mathbb{T}^2 = \mathbf{R}^2/\mathbf{Z}^2 \to \mathbb{S}^2 \tag{2.3}$$

with Galois involution $\sigma : (x,y) \mapsto (-x,-y)$ sending its ramification points $(1/2,0)$, $(0,1/2)$, $(1/2,1/2)$ and $(0,0)$ respectively to the four punctures p_α, p_β, p_γ and p_δ (see figure 1.1).

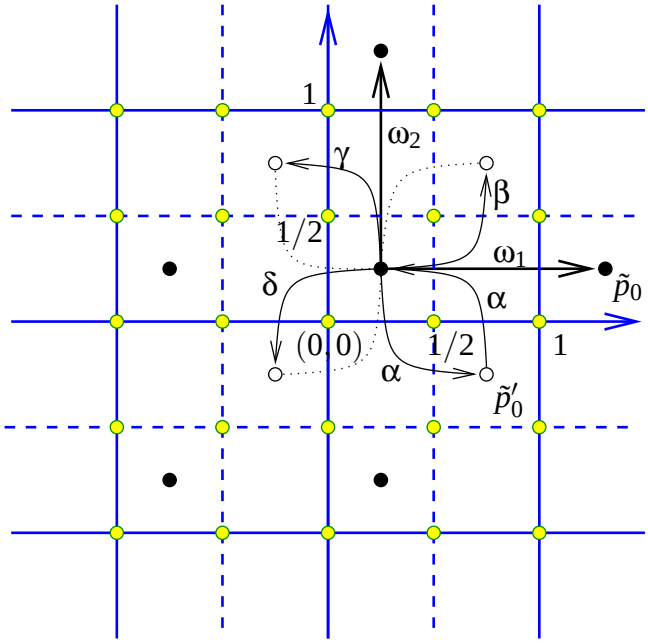

FIGURE 3. The torus cover.

The mapping class group of the torus, and also of the once punctured torus $\mathbb{T}^2_1 = \mathbb{T}^2 \setminus \{(0,0)\}$, is isomorphic to $\mathsf{GL}(2,\mathbf{Z})$. This group acts by linear homeomorphisms on the torus, fixing $(0,0)$, and permuting the other three ramification points of π_T. This action provides a section of the projection $\mathsf{Diff}(\mathbb{T}^2) \to \mathsf{MCG}^*(\mathbb{T}^2)$. Since this action commutes with the involution σ (which generates the center of $\mathsf{GL}(2,\mathbf{Z})$), we get a morphism from $\mathsf{PGL}(2,\mathbf{Z})$ to $\mathsf{MCG}^*(\mathbb{S}^2_4)$. This morphism is one to one and its image is contained in the stabilizer of p_δ in $\mathsf{MCG}^*(\mathbb{S}^2_4)$.

The subset $H \subset \mathbb{T}^2$ of ramification points of π coincides with the 2-torsion subgroup of $(\mathbb{T}^2,+)$; H acts by translation on $\mathbb{T}^2$ and commutes with the involution σ as well. This provides an isomorphism (see section 4.4 in [5])

$$\mathsf{PGL}(2,\mathbf{Z}) \ltimes H \to \mathsf{MCG}^*(\mathbb{S}^2_4). \tag{2.4}$$

Lemma 2.2. *The subgroup of* $\mathsf{Aut}(\chi(\mathbb{S}^2_4))$ *obtained by the action of the subgroup* $\mathsf{PGL}(2,\mathbf{Z})$ *of* $\mathsf{MCG}^*(\mathbb{S}^2_4)$ *is generated by the three polynomial automorphisms* B_1, B_2 *and* T_3 *of equations 2.7, 2.8, and 2.9 below. The* 4*-order translation group* H *acts trivially on parameters* (A,B,C,D,x,y,z), *permuting parameters* (a,b,c,d) *as follows*

$$P_1 = (1/2,0) : (a,b,c,d) \mapsto (d,c,b,a) \tag{2.5}$$

$$P_2 = (0,1/2) : (a,b,c,d) \mapsto (b,a,d,c) \tag{2.6}$$

Proof. Let $\tilde{p}_0$ and $\tilde{p}'_0$ be the lifts of the base point $p_0 \in \mathbb{S}^2_4$. Still denote by α, β, γ and δ the two lifts of those loops, with respective initial points $\tilde{p}_0$ and $\tilde{p}'_0$. The fundamental group of the four punctured torus $\mathbb{T}^2_4 = \mathbb{T}^2 \setminus H$ based at $\tilde{p}_0$ may be viewed as the set of even words in α, β, γ and δ, or equivalently of words in ω_1, ω_2 and δ that are even in δ where

$$\omega_1 = \beta\gamma = \alpha^{-1}\delta^{-1} \quad \text{and} \quad \omega_2 = \gamma\delta = \beta^{-1}\alpha^{-1}.$$

(see Figure 2.2.1). The action of the linear homeomorphism

$$B_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} : \mathbb{T}^2_4 \to \mathbb{T}^2_4,$$

or we should say, of a convenient isotopic homeomorphism h fixing $\tilde{p}_0$, on the fundamental groups $\pi_1(\mathbb{T}^2_4, \tilde{p}_0)$ and $\pi_1(\mathbb{S}^2_4, p_0)$ is given by :

$$h_* : \begin{cases} \omega_1 & \mapsto & \delta^{-1}\omega_1^{-1}\omega_2\delta^{-1} \\ \omega_2 & \mapsto & \omega_2 \\ \delta & \mapsto & \delta \end{cases} \quad \text{i.e.} \quad \begin{cases} \alpha & \mapsto & \alpha\beta\alpha^{-1} \\ \beta & \mapsto & \alpha \\ \gamma & \mapsto & \gamma \\ \delta & \mapsto & \delta \end{cases}$$

This automorphism of $\pi_1(\mathbb{S}^2_4, p_0)$, which depends on the choice of h in the isotopy class of B_1, induces an automorphism

$$\mathsf{Rep}(\mathbb{S}^2_4) \to \mathsf{Rep}(\mathbb{S}^2_4) \; ; \; \rho \mapsto \rho \circ (h_*)^{-1}.$$

The corresponding action on the character variety, *i.e.* on the corresponding 7-uples $(a,b,c,d,x,y,z) \in \mathbf{C}^7$, is independant of that choice. In order to compute it, note that

$$(h_*)^{-1} = h^* : \begin{cases} \alpha & \mapsto & \beta \\ \beta & \mapsto & \beta^{-1}\alpha\beta \\ \gamma & \mapsto & \gamma \\ \delta & \mapsto & \delta \end{cases}$$

We therefore obtain

$$(2.7)\quad B_1=\begin{pmatrix}1&0\\-1&1\end{pmatrix}:\ \begin{cases}a&\mapsto&b\\b&\mapsto&a\\c&\mapsto&c\\d&\mapsto&d\end{cases}\ \text{and}\ \begin{cases}x&\mapsto&x\\y&\mapsto&-z-xy+ac+bd\\z&\mapsto&y\end{cases}$$

For instance, the coordinate y' of the image is given by $y'=\mathsf{tr}(\rho\circ h^*(\beta\gamma))=\mathsf{tr}(\rho(\beta^{-1}\alpha\beta\gamma))$, and its value is easily computed using standard Fricke-Klein formulae, like

$$\mathsf{tr}(M_1)=\mathsf{tr}(M_1^{-1}),\quad \mathsf{tr}(M_1M_2)=\mathsf{tr}(M_2M_1),$$
$$\mathsf{tr}(M_1M_2^{-1})+\mathsf{tr}(M_1M_2)=\mathsf{tr}(M_1)\mathsf{tr}(M_2)$$
$$\text{and}\quad \mathsf{tr}(M_1M_2M_3)+\mathsf{tr}(M_1M_3M_2)+\mathsf{tr}(M_1)\mathsf{tr}(M_2)\mathsf{tr}(M_3)$$
$$=\mathsf{tr}(M_1)\mathsf{tr}(M_2M_3)+\mathsf{tr}(M_2)\mathsf{tr}(M_1M_3)+\mathsf{tr}(M_3)\mathsf{tr}(M_1M_2)$$

for any $M_1,M_2,M_3\in\mathsf{SL}(2,\mathbf{C})$.

A similar computation yields

$$(2.8)\quad B_2=\begin{pmatrix}1&1\\0&1\end{pmatrix}:\ \begin{cases}a&\mapsto&a\\b&\mapsto&c\\c&\mapsto&b\\d&\mapsto&d\end{cases}\ \text{and}\ \begin{cases}x&\mapsto&z\\y&\mapsto&y\\z&\mapsto&-x-yz+ab+cd\end{cases}$$

which, together with B_1, provide a system of generators for the $\mathsf{PSL}(2,\mathbf{Z})$-action. In order to generate $\mathsf{PGL}(2,\mathbf{Z})$, we have to add the involution

$$(2.9)\qquad T_3=\begin{pmatrix}0&1\\1&0\end{pmatrix}:\ \begin{cases}a&\mapsto&c\\b&\mapsto&b\\c&\mapsto&a\\d&\mapsto&d\end{cases}\ \text{and}\ \begin{cases}x&\mapsto&y\\y&\mapsto&x\\z&\mapsto&z\end{cases}$$

The formulae for the action of H are obtained in the same way. □

Remark 2.3. The formulae 2.7, 2.8, and 2.9 for B_1, B_2 and T_3 specialize to the formulae of section 2.1 when $(A,B,C,D)=(0,0,0,4)$.

Remark 2.4. The Artin Braid Group $\mathcal{B}_3=\langle\beta_1,\beta_2\mid\beta_1\beta_2\beta_1=\beta_2\beta_1\beta_2\rangle$ is isomorphic to the group of isotopy classes of the thrice punctured disk fixing its boundary. There is therefore a morphism from $\mathcal{B}$ into the subgroup of $\mathsf{MCG}(\mathbb{S}^2_4)$ that stabilizes p_δ. This morphism gives rise to the following well known exact sequence

$$I\to\langle(\beta_1\beta_2)^3\rangle\to\mathcal{B}_3\to\mathsf{PSL}(2,\mathbf{Z})\to 1,$$

where generators β_1 and β_2 are respectively sent to B_1 and B_2, and the group $\langle(\beta_1\beta_2)^3\rangle$ coincides with the center of $\mathcal{B}_3$. In particular, the action of $\mathcal{B}_3$ on $\chi(\mathbb{S}^2_4)$ coincides with the action of $\mathsf{PSL}(2,\mathbf{Z})$. We note that $\mathsf{PSL}(2,\mathbf{Z})$

is the free product of the trivolution B_1B_2 and the involution $B_1B_2B_1$. In $\mathsf{PGL}(2,\mathbf{Z})$, we also have relations $T_3^2 = I$, $T_3B_1T_3 = B_2^{-1}$ and $T_3B_2T_3 = B_1^{-1}$.

2.2.2. *The modular groups Γ_2^* and Γ_2.* Since the action of $M \in \mathsf{GL}(2,\mathbf{Z})$ on the set H of points of order 2 depends only on the equivalence class of M modulo 2, we get an exact sequence

$$I \to \Gamma_2^* \to \mathsf{PGL}(2,\mathbf{Z}) \ltimes H \to \mathsf{Sym}_4 \to 1$$

where $\Gamma_2^* \subset \mathsf{PGL}(2,\mathbf{Z})$ is the subgroup defined by those matrices $M \equiv I$ modulo 2. This group acts on the character variety, and since it preserves the punctures, it fixes a, b, c, and d. The group Γ_2^* is the free product of 3 involutions, s_x, s_y, and s_z, acting on the character variety as follows.

$$s_x = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} : \begin{cases} x & \mapsto & -x - yz + ab + cd \\ y & \mapsto & y \\ z & \mapsto & z \end{cases} \tag{2.10}$$

$$s_y = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} : \begin{cases} x & \mapsto & x \\ y & \mapsto & -y - xz + bc + ad \\ z & \mapsto & z \end{cases} \tag{2.11}$$

$$s_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} : \begin{cases} x & \mapsto & x \\ y & \mapsto & y \\ z & \mapsto & -z - xy + ac + bd \end{cases} \tag{2.12}$$

We note that $s_x = B_2B_1^{-1}B_2^{-1}T_3$, $s_y = B_2B_1B_2^{-1}T_3$ and $s_z = B_2B_1B_2T_3$. The standard modular group $\Gamma_2 \subset \mathsf{PSL}(2,\mathbf{Z})$ is generated by

$$\begin{cases} g_x = s_zs_y = B_1^2 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \\ g_y = s_xs_z = B_2^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\ g_z = s_ys_x = B_1^{-2}B_2^{-2} = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix} \end{cases}$$

(we have $g_zg_yg_x = I$); as we shall see, this corresponds to Painlevé VI monodromy (see [29] and section 7). The following proposition is now a direct consequence of lemma 2.2.

Proposition 2.5. *Let $\mathsf{MCG}_0^*(\mathbb{S}_4^2)$ (resp. $\mathsf{MCG}_0(\mathbb{S}_4^2)$) be the subgroup of $\mathsf{MCG}^*(\mathbb{S}_4^2)$ (resp. $\mathsf{MCG}(\mathbb{S}_4^2)$) which stabilizes the four punctures of $\mathbb{S}_4^2$. This group coincides with the stabilizer of the projection $\pi : \chi(\mathbb{S}_4^2) \to \mathbf{C}^4$ which is defined by*

$$\pi(a,b,c,d,x,y,z) = (a,b,c,d).$$

Its image in $\mathsf{Aut}(\chi(\mathbb{S}_4^2))$ *coincides with the image of* Γ_2^* *(resp.* Γ_2*) and is therefore generated by the three involutions* s_x, s_y *and* s_z *(resp. the three automorphisms* g_x, g_y, g_z*).*

As we shall see in sections 3.1 and 3.2, this group is of finite index in $\mathsf{Aut}(\chi(\mathbb{S}_4^2))$.

Remark 2.6. Let us consider the exact sequence

$$I \to \Gamma_2^* \to \mathsf{PGL}(2,\mathbf{Z}) \to \mathsf{Sym}_3 \to 1,$$

where $\mathsf{Sym}_3 \subset \mathsf{Sym}_4$ is the stabilizer of p_δ, or equivalently of d, or D. A splitting $\mathsf{Sym}_3 \hookrightarrow \mathsf{PGL}(2,\mathbf{Z})$ is generated by the transpositions $T_1 = T_3B_1B_2$ and $T_2 = B_1B_2T_3$. They act as follows on the character variety.

$$T_1 = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} : \begin{cases} a & \mapsto & b \\ b & \mapsto & a \\ c & \mapsto & c \\ d & \mapsto & d \end{cases} \text{ and } \begin{cases} x & \mapsto & x \\ y & \mapsto & z \\ z & \mapsto & y \end{cases}$$

and

$$T_2 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} : \begin{cases} a & \mapsto & a \\ b & \mapsto & c \\ c & \mapsto & b \\ d & \mapsto & d \end{cases} \text{ and } \begin{cases} x & \mapsto & z \\ y & \mapsto & y \\ z & \mapsto & x \end{cases}$$

2.3. **Twists.** There are other symmetries between surfaces $S_{(A,B,C,D)}$ that do not arise from the action of the mapping class group. Indeed, given any 4-uple $\varepsilon = (\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) \in \{\pm 1\}^4$ with $\prod_{i=1}^4 \varepsilon_i = 1$, the ε-twist of a representation $\rho \in \mathsf{Rep}(\mathbb{S}_4^2)$ is the new representation $\otimes_\varepsilon \rho$ generated by

$$\begin{cases} \tilde\rho(\alpha) &=& \varepsilon_1\rho(\alpha) \\ \tilde\rho(\beta) &=& \varepsilon_2\rho(\beta) \\ \tilde\rho(\gamma) &=& \varepsilon_3\rho(\gamma) \\ \tilde\rho(\delta) &=& \varepsilon_4\rho(\delta) \end{cases}$$

This provides an action of $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ on the character variety given by

$$\otimes_\varepsilon : \begin{cases} a & \mapsto & \varepsilon_1 a \\ b & \mapsto & \varepsilon_2 c \\ c & \mapsto & \varepsilon_3 b \\ d & \mapsto & \varepsilon_4 d \end{cases} \begin{cases} A & \mapsto & \varepsilon_1\varepsilon_2 A \\ B & \mapsto & \varepsilon_2\varepsilon_3 B \\ C & \mapsto & \varepsilon_1\varepsilon_3 C \\ D & \mapsto & D \end{cases} \text{ and } \begin{cases} x & \mapsto & \varepsilon_1\varepsilon_2 x \\ y & \mapsto & \varepsilon_2\varepsilon_3 y \\ z & \mapsto & \varepsilon_1\varepsilon_3 z \end{cases}$$

The action on (A,B,C,D,x,y,z) is trivial iff $\varepsilon = \pm(1,1,1,1)$. The "Benedetto-Goldman symmetry group" of order 192 acting on (a,b,c,d,x,y,z) which is described in [4] (§3C) is precisely the group generated by ε-twists and the

symmetric group $\mathsf{Sym}_4 = \langle T_1, T_2, P_1, P_2\rangle$. The subgroup Q acting trivially on (A,B,C,D,x,y,z) is of order 8 generated by

$$Q = \langle P_1, P_2, \otimes_{(-1,-1,-1,-1)}\rangle. \tag{2.13}$$

2.4. **Character variety of the once-punctured torus.** Our family of surfaces $S_{(A,B,C,D)}$ also provides, for $(A,B,C,D) = (0,0,0,D)$, the moduli space of representations of the torus $\mathbb{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ with one puncture at $(0,0)$. Precisely, if we go back to the notations of §2.2 (see figure 2.2.1), the fundamental group $\pi_1(\mathbb{T}^2_1)$, $\mathbb{T}^2_1 = \mathbb{T}^2 \setminus \{(0,0)\}$, is the free group generated by ω_1 and ω_2. The algebraic quotient $\chi(\mathbb{T}^2_1) = \mathsf{Rep}(\mathbb{T}^2_1)//\mathsf{SL}(2,\mathbf{C})$ is given by the map

$$\begin{cases} \mathsf{Rep}(\mathbb{T}^2_1) & \to & \chi(\mathbb{T}^2_1) \simeq \mathbf{C}^3 \\ \rho & \mapsto & (X,Y,Z) = (\mathsf{tr}(\rho(\omega_1)), \mathsf{tr}(\rho(\omega_2)), -\mathsf{tr}(\rho(\omega_1\omega_2))) \end{cases}$$

(see [4]). Using that

$$\mathsf{tr}([M_1,M_2]) = \mathsf{tr}(M_1)^2 + \mathsf{tr}(M_2)^2 + \mathsf{tr}(M_1M_2)^2 - \mathsf{tr}(M_1)\mathsf{tr}(M_2)\mathsf{tr}(M_1M_2) - 2,$$

for all $M_1, M_2 \in \mathsf{SL}(2,\mathbf{C})$, we note that those representations with given trace $\tilde{d} = \mathsf{tr}(\rho([\omega_1,\omega_2]))$ are parametrized by the affine cubic

$$X^2 + Y^2 + Z^2 + XYZ = \tilde{d} + 2$$

which is precisely $S_{(0,0,0,D)}$ with $D = \tilde{d} + 2$. The reason is given by the two-fold ramified cover $\pi : \mathbb{T}^2 \to \mathbb{S}^2$ used in §2.2. Consider a representation $\rho \in \mathsf{Rep}(\mathbb{S}^2_4)$ corresponding to some point $(x,y,z) \in S_{(0,0,0,D)}$, with local traces given by $(a,b,c,d) = (0,0,0,d)$, $D = 4 - d^2$. One can lift the representation on the 4-punctured torus, where punctures are given by the set H of 2-torsion points. Since $a = b = c = 0$, we have

$$\rho(\alpha),\ \rho(\beta),\ \rho(\gamma) \sim \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

and the lifted representation $\rho \circ \pi$ has local monodromy $-I$ around the corresponding punctures $(1/2,0)$, $(0,1/2)$ and $(1/2,1/2)$. After twisting $\rho \circ \pi$ by $-I$ at each of the punctures, we finally deduce a representation $\widetilde{\rho} \in \mathsf{Rep}(\mathbb{T}^2_1)$. Since $\pi_*\omega_1 = \beta\gamma$ and $\pi_*\omega_2 = \beta^{-1}\alpha^{-1}$ (see §2.2), the character associated to the lifted representation $\widetilde{\rho}$ is given by

$$\begin{cases} X & = & \mathsf{tr}(\widetilde{\rho}(\omega_1)) & = & y \\ Y & = & \mathsf{tr}(\widetilde{\rho}(\omega_2)) & = & x \\ Z & = & -\mathsf{tr}(\widetilde{\rho}(\omega_1\omega_2) & = & -z - xy \end{cases}$$

which satisfies $X^2 + Y^2 + Z^2 + XYZ = 4 - d^2$. Note that the local monodromy of $\widetilde{\rho}$ at $(0,0)$ is $-\rho(\delta^2)$ and we indeed find $\tilde{d} = 2 - d^2$. We can now reverse

the formulae and deduce that any representation $(X,Y,Z) \in \chi(\mathbb{T}_1^2)$ is the lifting of a representation $(x,y,z) \in \chi(\mathbb{S}_4^2)$. This is due to the hyperelliptic nature of the once punctured torus.

3. Geometry and Automorphisms

This section is devoted to a geometric study of the family of surfaces $S_{(A,B,C,D)}$, and to the description of the groups of polynomial automorphisms $\mathsf{Aut}[S_{(A,B,C,D)}]$.

In section 3.4, we describe a special case that is famous in Teichmüller theory. Section 3.3 introduces the concept of elliptic, parabolic, and hyperbolic automorphisms of $S_{(A,B,C,D)}$.

3.1. **The triangle at infinity and automorphisms.** Let S be any member of the family Fam. The closure $\overline{S}$ of S in $\mathbb{P}^3(\mathbf{C})$ is given by a cubic homogeneous equation

$$w(x^2+y^2+z^2)+xyz = w^2(Ax+By+Cz)+Dw^3.$$

The intersection of $\overline{S}$ with the plane at infinity does not depend on the parameters and coincides with the triangle Δ given by the equation

$$\Delta \quad : \quad xyz = 0;$$

moreover, one easily checks that the surface $\overline{S}$ is smooth in a neighborhood of Δ (all the singularities of $\overline{S}$ are contained in S).

Since the equation defining S is of degree 2 with respect to the x variable, each point (x,y,z) of S gives rise to a unique second point (x',y,z). This procedure determines a holomorphic involution of S, namely

$$s_x(x,y,z) = (A-x-yz,y,z).$$

This automorphism coincides with the automorphism of S determined by the involution s_x of Γ_2^* (see equation 2.10, §2.2.2). Geometrically, the involution s_x corresponds to the following: If m is a point of $\overline{S}$, the projective line which joins m and the vertex $v_x = [1;0;0;0]$ of the triangle Δ intersects $\overline{S}$ on a third point; this point is $s_x(m)$. The same construction provides two more involutions

$$s_y(x,y,z) = (x,B-y-xz,z) \quad \text{and} \quad s_z(x,y,z) = (x,y,C-z-xy),$$

and therefore a subgroup

$$\mathcal{A} = \langle s_x, s_y, s_z \rangle$$

of the group $\mathsf{Aut}[S]$ of polynomial automorphisms of the surface S.

From section 2.2.2, we deduce that *for any member S of the family Fam, the group $\mathcal{A}$ coincides with the image of Γ_2^* into $\mathsf{Aut}[S]$, which is obtained by the action of $\Gamma_2^* \subset \mathsf{MCG}^*(\mathbb{S}_4^2)$ on $\chi(\mathbb{S}_4^2)$* (see §1.2).

Theorem 3.1. *Let $S = S_{(A,B,C,D)}$ be any member of the family of surfaces* Fam. *Then*

- *there is no non-trivial relation between the three involutions s_x, s_y and s_z, and $\mathcal{A}$ is therefore isomorphic to the free product $(\mathbf{Z}/2\mathbf{Z}) \star (\mathbf{Z}/2\mathbf{Z}) \star (\mathbf{Z}/2\mathbf{Z})$;*
- *the index of $\mathcal{A}$ in* Aut$[S]$ *is bounded by* 24 *;*
- *$\mathcal{A}$ coincides with the image of Γ_2^* in* Aut$[S]$.

Moreover, for a generic choice of the parameters (A,B,C,D), $\mathcal{A}$ coincides with Aut$[S]$.

This result is almost contained in Èl'-Huti's article [19] and is more precise than Horowitz's main theorem (see [25], [26]).

Proof. Since $\overline{S}$ is smooth in a neighborhood of the triangle at infinity and the three involutions are the reflexions with respect to the vertices of that triangle, we can apply the main theorems of Èl'-Huti's article:

- $\mathcal{A}$ is isomorphic to the free product
$$(\mathbf{Z}/2\mathbf{Z}) \star (\mathbf{Z}/2\mathbf{Z}) \star (\mathbf{Z}/2\mathbf{Z}) = \langle s_x \rangle \star \langle s_y \rangle \star \langle s_z \rangle;$$
- $\mathcal{A}$ is of finite index in Aut$[S]$;
- Aut$[S]$ is generated by $\mathcal{A}$ and the group of projective transformations of $\mathbb{P}^3(\mathbf{C})$ which preserve $\overline{S}$ and Δ (i.e., by affine transformations of $\mathbf{C}^3$ that preserve S).

We already know that $\mathcal{A}$ and the image of Γ_2^* in Aut$[S]$ coincide. We now need to study the index of $\mathcal{A}$ in Aut$[S]$. Let f be an affine invertible transformation of $\mathbf{C}^3$, that we decompose as the composition of a linear part M and a translation of vector T. Let S be any member of Fam. If f preserves S, then the equation of S is multiplied by a non zero complex number when we apply f. Looking at the cubic terms, this means that M is a diagonal matrix composed with a permutation of the coordinates. Looking at the quadratic terms, this implies that T is the nul vector, so that $f = M$ is linear. Coming back to the equation of S, we now see that M is one of the 24 linear transformations of the type $\sigma \circ \varepsilon$ where ε either is the identity or changes the sign of two coordinates, and σ permutes the coordinates. If (A,B,C,D) are generic, $S_{(A,B,C,D)}$ is not invariant by any of these linear maps. Moreover, one easily verifies that the subgroup $\mathcal{A}$ is a normal subgroup of Aut$[S]$: If such a linear transformation $M = \sigma \circ \varepsilon$ preserves S, then it normalizes $\mathcal{A}$. This shows that $\mathcal{A}$ is a normal subgroup of Aut$[S]$, the index of which is bounded by 24. □

3.2. **Consequences and notations.** As a corollary of theorem 3.1 and proposition 2.5, we get the following result: *The mapping class group* $\mathsf{MCG}_0^*(\mathbb{S}_4^2)$ *acts on the character variety* $\chi(\mathbb{S}_4^2)$, *preserving each surface* $S_{(A,B,C,D)}$, *and its*

image in $\mathsf{Aut}[S_{(A,B,C,D)}]$ *coincides with the image of* Γ_2^*, *and therefore with the finite index subgroup* $\mathcal{A}$ *of* $\mathsf{Aut}[S_{(A,B,C,D)}]$. In other words, up to finite index subgroups, describing the dynamics of $\mathsf{MCG}^*(\mathbb{S}_4^2)$ on the character variety $\chi(\mathbb{S}_4^2)$ or of the group $\mathsf{Aut}[S]$ on S for any member S of the family Fam is one and the same problem.

Let $\mathbb{H}$ be the Poincaré half plane. The group of isometries of $\mathbb{H}$ is isomorphic to $\mathsf{PGL}(2,\mathbf{R})$: If M is an element of $\mathsf{GL}(2,\mathbf{R})$, its action on $\mathbb{H}$ is defined by

$$M(z) = \frac{m_{11}z + m_{12}}{m_{21}z + m_{22}}$$

if the determinant of M is positive, and by the same formula but with z replaced by $\overline{z}$ if the determinant is negative. In particular, Γ_2^* acts isometrically on $\mathbb{H}$. Let j_x, j_y and j_z be the three points on the boundary of $\mathbb{H}$ with coordinates -1, 0, and ∞ respectively. Let r_x (resp. r_y and r_z) be the reflection of $\mathbb{H}$ around the geodesic between j_y and j_z (resp. j_z and j_x, resp. j_x, and j_y). These isometries are respectively induced by the three matrices s_x, s_y, and s_z given in section 2.2.2. As a consequence, Γ_2^* coincides with the group of symmetries of the tesselation of $\mathbb{H}$ by ideal triangles, one of which has vertices j_x, j_y and j_z (see the left part of figure 3.3).

In the following, we shall identify the subgroup Γ_2^* of $\mathsf{PGL}(2,\mathbf{Z})$ and the subgroup $\mathcal{A}$ of $\mathsf{Aut}[S_{(A,B,C,D)}]$: If f is an element of $\mathcal{A}$, M_f will denote the associated element of Γ_2^* (either viewed as a matrix or an isometry of $\mathbb{H}$), and if M is an element of Γ_2^*, f_M will denote the automorphism associated to M (for any surface S of the family Fam). If f is one of the three involutions s_x, s_y, or s_z (resp. the three elements g_x, g_y, or g_z), we shall use exactly the same letters to denote the element f of Γ_2^* or the corresponding automorphism $f \in \mathcal{A}$. The only place where this rule is not followed is when we study the action of Γ_2^* on the Poincar disk: We then use the notation r_x, r_y, and r_z to denote the involutive isometrys induced by s_x, s_y, and s_z.

3.3. **Elliptic, Parabolic, Hyperbolic.** Non trivial isometries of $\mathbb{H}$ are classified into three different species. Let M be an element of $\mathsf{PGL}(2,\mathbf{R}) \setminus \{\mathrm{Id}\}$, viewed as an isometry of $\mathbb{H}$. Then,

- M is elliptic if M has a fixed point in the interior of $\mathbb{H}$. Ellipticity is equivalent to $\det(M) = 1$ and $|\mathsf{tr}(M)| < 2$ (in which case M is a rotation around a unique fixed point) or $\det(M) = -1$ and $\mathsf{tr}(M) = 0$ (in which case M is a reflexion around a geodesic of fixed points).
- M is parabolic if M has a unique fixed point, which is located on the boundary of $\mathbb{H}$; M is parabolic if and only if $\det(M) = 1$ and $\mathsf{tr}(M) = 2$ or -2;

- M is hyperbolic if it has exactly two fixed points which are on the boundary of $\mathbb{H}$; this occurs if and only if $\det(M) = 1$ and $|\mathsf{tr}(M)| > 2$, or $\det(M) = -1$ and $\mathsf{tr}(M) \neq 0$.

An element f of $\mathcal{A} \setminus \{\mathrm{Id}\}$ will be termed elliptic, parabolic, or hyperbolic, according to the type of M_f. Examples of elliptic elements are given by the three involutions s_x, s_y and s_z. Examples of parabolic elements are given by the three automorphisms g_x, g_y and g_z (see section 2.2.2). The dynamics of these automorphisms will be described in details in §5.1. Let us just mention the fact that g_x (resp. g_y, g_z) preserves the conic fibration $\{x = c^{ste}\}$ (resp. $\{y = c^{ste}\}$, $\{z = c^{ste}\}$) of any member S of Fam.

Proposition 3.2. *Let S be one of the surfaces in the family Fam (S may be singular). An element f of $\mathcal{A}$ is*

- *elliptic if and only if f is conjugate to one of the involutions s_x, s_y or s_z, if and only if f is periodic;*
- *parabolic if and only if f is conjugate to a non trivial power of one of the automorphisms g_x, g_y or g_z;*
- *hyperbolic if and only if f is conjugate to a cyclically reduced composition which involves the three involutions s_x, s_y, and s_z.*

Proof. Since Γ_2^* and the image of $\mathcal{A}$ in $\mathsf{Aut}[S]$ are isomorphic for any S in Fam, we just need to prove the same statement for Γ_2^*. The group Γ_2^* is a subgroup of $\mathsf{PGL}(2,\mathbf{Z})$. As a consequence, any elliptic element of Γ_2^* is periodic. Since

$$\Gamma_2^* = (\mathbf{Z}/2\mathbf{Z}) \star (\mathbf{Z}/2\mathbf{Z}) \star (\mathbf{Z}/2\mathbf{Z}),$$

any periodic element of Γ_2^* is conjugate to one of the involutions r_x, r_y, r_z (see for example [43]), and the first property is proved.

If M is a parabolic element of Γ_2^*, its unique fixed point on the boundary $\mathbf{R} \cup \{\infty\}$ of $\mathbb{H}$ is a rational number. The action of Γ_2^* on the set $\mathbf{Q} \cup \{\infty\}$ of rational numbers has three distinct orbits: The orbits of $j_x = -1$, $j_y = 0$ and $j_z = \infty$. This implies that there exists an element F of Γ_2^* such the FMF^{-1} is parabolic and fixes one of these three points, say j_z. Any parabolic element G of Γ_2^* that fixes ∞ is of the type

$$\pm \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix}$$

where k is an integer. This fact shows that M is conjugate to a power of g_z (see section 2.2.2) and concludes the proof of the second point.

Let M be a hyperbolic element of Γ_2^*. After conjugation, we can write M as a cyclically reduced word in the involutive generators r_x, r_y and r_z. If the number of involutions that appear in this composition is equal to 1 or 2, then

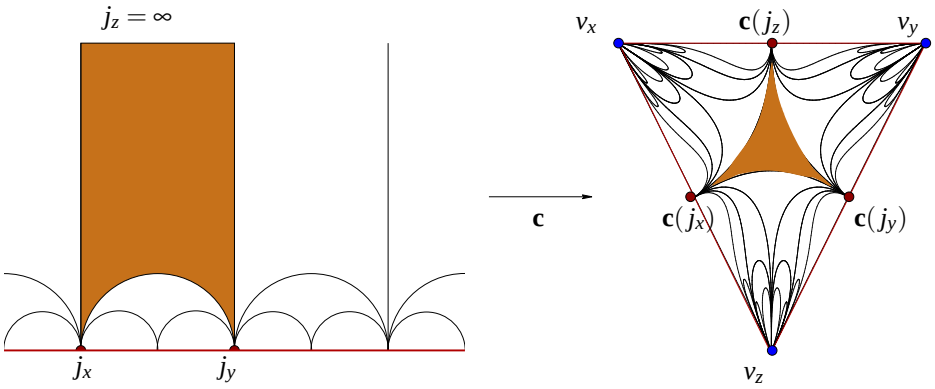

FIGURE 4. Conjugation for the Markov example. The right hand part of this figure depicts the dynamics of Γ_2^* on $S_{M+}(\mathbf{R})$, but viewed in $\mathbb{P}^2(\mathbf{R})$ after the birational change of variables $[x:y:z:w] = [XQ:YQ:ZQ:XYZ]$, with $Q = X^2+Y^2+Z^2$. This change of variables sends the interior of the triangle $\{X \geq 0, Y \geq 0, Z \geq 0\}$ onto $S_{M+}(\mathbf{R})$.

M is an involution or a power of g_x, g_y or g_z. The third property follows from this remark. □

Remark 3.3. The three vertices j_x, j_y and j_z disconnect $\partial\mathbb{H}$ in three segments $[j_y, j_z]$, $[j_z, j_x]$ and $[j_x, j_y]$. Let M be a hyperbolic element of Γ_2^*. Let α_M be the repulsive and attrating fixed points of M on the boundary of $\mathbb{H}$. The Fricke-Klein ping-pong lemma, as described in [15], page 25, shows that M is a cyclically reduced composition of r_x, r_y, and r_z if and only if the fixed points of M are contained in two distinct connected components of $\partial\mathbb{H} \setminus \{j_x, j_y, j_z\}$.

3.4. **The Markov surface.** Let S_M be the element of Fam corresponding to the parameter $(A,B,C,D) = (0,0,0,0)$. After a simultaneous multiplication of each coordinate by -3, the equation of S_M is

$$x^2 + y^2 + z^2 = 3xyz.$$

This surface has been studied by Markov in 1880 in his papers concerning diophantine approximation. The real part $S_M(\mathbf{R})$ of the Markov surface has an isolated singular point at the origin and four other connected components, each of which is homeomorphic to a disk. One of these components is

$$S_{M+}(\mathbf{R}) = S_M(\mathbf{R}) \cap (\mathbf{R}^+)^3.$$

Proposition 3.4 (Markov, [14])**.** *The action of $\mathcal{A} = \Gamma_2^*$ on the Markov surface S_M preserves each connected component of $S_M(\mathbf{R})$. There exists a diffeomorphism $\mathbf{c} : \mathbb{H} \to S_{M+}(\mathbf{R})$ such that (i) the image of the (closed) ideal triangle with vertices j_x, j_y and j_z is the subset of $S_{M+}(\mathbf{R})$ defined by the three inequalities*

$$xy \leq 2z, \quad yz \leq 2x, \quad \text{and} \quad zx \leq 2y,$$

and (ii) $\mathbf{c}$ *conjugates the action of Γ_2^* on $\mathbb{H}$ with the action of Γ_2^* on $S_{M+}(\mathbf{R})$ in such a way that*

$$\mathbf{c} \circ r_x = s_x \circ \mathbf{c}, \quad \mathbf{c} \circ r_y = s_y \circ \mathbf{c}, \quad \text{and} \quad \mathbf{c} \circ r_z = s_z \circ \mathbf{c}.$$

Remark 3.5. We refer the reader to [14] or [22] for a proof (see figure 3.3 for a visual argument). This result is not surprising if one notices that $S_{M+}(\mathbf{R})$ is a model of the Teichmüller space of the once punctured torus with a cusp at the puncture, and finite area 2π.

3.5. **An (almost) invariant area form.** The monomial action of the group $\mathsf{GL}(2,\mathbf{Z})$ on $\mathbf{C}^* \times \mathbf{C}^*$ almost preserves the holomorphic 2-form

$$\Omega = \frac{dx}{x} \wedge \frac{dy}{y}.$$

More precisely, $M^*\Omega = \pm\Omega$ for any element M of $\mathsf{GL}(2,\mathbf{Z})$. This form is invariant under the action of η and determines a holomorphic volume form on the Cayley cubic, that is almost $\mathsf{Aut}[S_C]$ invariant.This property is shared by all the members of Fam (the proof is straightforward).

Proposition 3.6. *Let $S \in \mathsf{Fam}$ be the surface corresponding to the parameters (A,B,C,D). The volume form Ω, which is globally defined by the formulas*

$$\Omega = \frac{dx \wedge dy}{2z + xy - C} = \frac{dy \wedge dz}{2x + yz - A} = \frac{dz \wedge dx}{2y + zx - B}$$

on $S \setminus \mathsf{Sing}(S)$, is almost invariant under the action of $\mathsf{Aut}[S]$, by which we mean that $f^*\Omega = \pm\Omega$ for any f in $\mathsf{Aut}[S]$.

3.6. **Singularities, fixed points, and an orbifold structure.** The singularities of the elements of Fam will play an important role in this article. In this section, we collect a few results regarding these singularities.

Lemma 3.7. *Let S be a member of Fam. A point m of S is singular if and only if m is a fixed point of the group $\mathcal{A}$.*

Proof. This is a direct consequence of the fact that m is a fixed point of s_x if and only if $2x + yz = Ax$, if and only if the partial derivative of the equation of S with respect to the x-variable vanishes. □

Example 3.8. The family of surfaces with parameters $(4+2d, 4+2d, 4+2d, -(8+8d+d^2))$ with $d \in \mathbf{C}$ is a deformation of the Cayley cubic, that corresponds to $d=-2$, and any of these surfaces has 3 singular points (counted with multiplicity).

Lemma 3.9. *If m is a singular point of S, there exists a neighborhood of m which is isomorphic to the quotient of the unit ball in $\mathbf{C}^2$ by a finite subgroup of $\mathsf{SU}(2)$.*

Proof. Any singularity of a cubic surface is a quotient singularity, except when the singularity is isomorphic to $x^3+y^3+z^3+\lambda xyz=0$, for at least one parameter λ (see [8]). Since the second jet of the equation of S never vanishes when S is a member of Fam, the singularities of S are quotient singularities. Since S admits a global volume form Ω, the finite group is conjugate to a subgroup of $\mathsf{SU}(2,\mathbf{C})$. □

As a consequence, any member S of Fam is endowed with a well defined orbifold structure. If S is singular, the group $\mathcal{A}$ fixes each of the singular points and preserves the orbifold structure. We shall consider this action in the orbifold category, but we could as well extend the action of A to a smooth desingularization of S.

Lemma 3.10. *The complex affine surface S is simply connected. When S is singular, the fundamental group of the complex surface $S \setminus \mathsf{Sing}(S)$ is normally generated by the local finite fundamental groups around the singular points.*

Proof. First of all, recall that a smooth cubic surface in $\mathbb{P}^3(\mathbf{C})$ may be viewed as the blowing-up of $\mathbb{P}^2(\mathbf{C})$ at 6-points in general position. Let us be concrete. After a projective change of coordinates, one can assume that those 6 points lie on the triangle $XYZ=0$ and are labelled as follows

$$p_i=[0:1:u_i], \quad q_i=[v_i:0:1] \quad \text{et} \quad r_i=[1:w_i:0], \quad i=1,2$$

where $[X:Y:Z]$ are projective coordinates of $\mathbb{P}^2$. One can moreover assume that the three following products take the same value λ:

$$u_1u_2=v_1v_2=w_1w_2=:\lambda.$$

Now, consider the map

$$\Phi : \mathbb{P}^2 \dashrightarrow \mathbf{C}^3 \; ; \; (X:Y:Z) \mapsto \left(\frac{P}{YZ}, \frac{Q}{XZ}, \frac{R}{XY}\right)$$

where P, Q and R are degree 2 homogeneous polynomials given by

$$\begin{cases} P &= -X^2-\frac{1}{\lambda}Y^2-\lambda Z^2+(\frac{1}{w_1}+\frac{1}{w_2})XY+(v_1+v_2)XZ \\ Q &= -\lambda X^2-Y^2-\frac{1}{\lambda}Z^2+(w_1+w_2)XY+(\frac{1}{u_1}+\frac{1}{u_2})YZ \\ R &= -\frac{1}{\lambda}X^2-\lambda Y^2-Z^2+(\frac{1}{v_1}+\frac{1}{v_2})XZ+(u_1+u_2)YZ \end{cases}$$

For u_i, v_i and w_i generic, the map Φ sends the triangle $XYZ=0$ to the triangle at infinity $xyz=0$ of $\mathbb{P}^3 \supset \mathbf{C}^3$ and has simple indeterminacy points exactly at p_i, q_i, and r_i, $i=1,2$. Let $\tilde{S}$ be the surface obtained by blowing-up the 6 indeterminacy points of Φ. One can check that the image of $\Phi : \tilde{S} \to \mathbb{P}^3(\mathbf{C})$ is exactly the cubic surface $S = S_{(A,B,C,D)}$, with parameters

$$\begin{cases} A &= \left(\frac{v_1}{w_1}+\frac{v_2}{w_2}+\frac{v_1}{w_2}+\frac{v_2}{w_1}\right) - \left(u_1\lambda+\frac{1}{u_1\lambda}+u_2\lambda+\frac{1}{u_2\lambda}\right) \\ B &= \left(\frac{u_1}{w_1}+\frac{u_2}{w_2}+\frac{u_1}{w_2}+\frac{u_2}{w_1}\right) - \left(v_1\lambda+\frac{1}{v_1\lambda}+v_2\lambda+\frac{1}{v_2\lambda}\right) \\ C &= \left(\frac{u_1}{v_1}+\frac{u_2}{v_2}+\frac{u_1}{v_2}+\frac{u_2}{v_1}\right) - \left(w_1\lambda+\frac{1}{w_1\lambda}+w_2\lambda+\frac{1}{w_2\lambda}\right) \\ D &= \sum_{i,j,k\in\{1,2\}} \left(u_iv_jw_k+\frac{1}{u_iv_jw_k}\right) \\ & \quad -\left(\frac{u_1}{u_2}+\frac{u_2}{u_1}+\frac{v_1}{v_2}+\frac{v_2}{v_1}+\frac{w_1}{w_2}+\frac{w_2}{w_1}+\lambda^3+\frac{1}{\lambda^3}+4\right) \end{cases}$$

Singular cubics arise when 3 of the 6 points lie on a line, or all of them lie on a conic. In this case, the corresponding line(s) and/or conic have negative self-intersection in $\tilde{S}$, and are blown-down by Φ to singular point(s) of S. A smooth resolution of S is therefore given by $\tilde{S}$.

Our claim is that the quasi-projective surface $\tilde{S}'$ obtained by deleting the strict transform of the triangle $XYZ=0$ from $\tilde{S}$ is simply connected. Indeed, the fundamental group of $\mathbb{P}^2 - \{XYZ=0\}$ is isomorphic to $\mathbf{Z}^2$, generated by two loops, say one turning around $X=0$, and the other one around $Y=0$. After blowing-up one point lying on $X=0$, and adding the exceptional divisor (minus $X=0$), the first loop becomes homotopic to 0; after blowing-up the 6 points and adding all exceptional divisors, the two generators become trivial and the resulting surface $\tilde{S}'$ is simply connected. The affine surface S is obtained after blowing-down some rational curves in $\tilde{S}$ and is therefore simply connected as well.

The second assertion of the lemma directly follows from Van Kampen Theorem. □

4. Birational Extension and Dynamics

4.1. **Birational transformations of surfaces.** Let f be a birational transformation of a complex projective surface X and $\mathrm{Ind}(f)$ be its indeterminacy set. The critical set of f is the union of all the curves C in S such that $f(C \setminus \mathrm{Ind}(f))$ is a point (in fact a point of $\mathrm{Ind}(f^{-1})$). One says that f is not algebraically stable if there is a curve C in the critical set and a positive integer k such that $f^k(C \setminus \mathrm{Ind}(f))$ is contained in $\mathrm{Ind}(f)$. Otherwise, f is said to be algebraically stable (see [16]). Let $H^2(X,\mathbf{Z})$ be the second cohomology group of X and $f^* : H^2(X,\mathbf{Z}) \to H^2(X,\mathbf{Z})$ be the linear transformation induced by f. It turns out that f is algebraically stable if and

only if $(f^k)^* = (f^*)^k$ for any positive integer k (see [16]). More generally, $(f\circ g)^* = g^*\circ f^*$ if and only if g does not blow down any curve onto an element of $\mathrm{Ind}(f)$.

The (first) dynamical degree $\lambda(f)$ of f is the spectral radius of the sequence of linear operators $(f^k)^*$. If f is algebraically stable, $\lambda(f)$ is therefore the largest eigenvalue of f^*. It follows from Hodge theory that

$$\limsup \frac{1}{k}\log\|(f^k)^*[v]\| = \log\lambda(f).$$

for any class $[v]$ that is obtained through a hyperplane section of X. The dynamical degree of f is invariant under birational conjugation (see [16, 24]), and provides an upper bound for the topological entropy of f (see [16, 24]).

Example 4.1. If M is an element of $\mathsf{GL}(2,\mathbf{Z})$, M acts on $\mathbf{C}^*\times\mathbf{C}^*$ monomially (see equation 1.11). The dynamical degree of this monomial transformation is equal to the spectral radius $\rho(M)$ of M. If f_M is the automorphism of the Cayley cubic S_C which is induced by M, the dynamical degree of f_M coincides also with $\rho(M)$ (see [20], or the survey article [24]).

4.2. **Birational extension.** Let S be a member of the family Fam. The group $\mathcal{A}$ acts by polynomial automorphisms on S and also by birational transformations of the compactification $\overline{S}$ of S in $\mathbb{P}^3(\mathbf{C})$. Let Δ be the triangle at infinity, $\Delta = \overline{S}\setminus S$. The three sides of this triangle are the lines $D_x = \{x=0, w=0\}$, $D_y = \{y=0, w=0\}$ and $D_z = \{z=0, w=0\}$; the vertices are $v_x = [1:0:0:0]$, $v_y = [0:1:0:0]$ and $v_z = [0:0:1:0]$. The "middle points" of the sides are respectively

$$m_x = [0:1:1:0], \quad m_y = [1:0:1:0], \text{ and } m_z = [1:1:0:0]$$

(see figure 3.3 in §3.4). Let V be the subspace of $H^2(\overline{S},\mathbf{Z})$ defined by

$$V = \mathbf{Z}[D_x] + \mathbf{Z}[D_y] + \mathbf{Z}[D_z],$$

where $[D_x]$ denotes either the homology class of D_x in $H_2(\overline{S},\mathbf{Z})$ or its dual in $H^2(\overline{S},\mathbf{Z})$. Since Δ is $\mathcal{A}$-invariant, the action of any element f in $\mathcal{A}$ on $H^2(\overline{S},\mathbf{Z})$ preserves the subspace V.

Lemma 4.2 (see [19] or [32])**.** *The involution s_x acts on the triangle Δ in the following way.*

- *The image of the side D_x is the vertex v_x and the vertex v_x is blown up onto the side D_x.*
- *the sides D_y and D_z are invariant and s_x permutes the vertices and fixes the middle point of each of these sides.*

Of course, we have the same result for s_y and s_z, with the obvious required modifications. In the following, we shall denote by s_x^* (resp. s_y^* or s_z^*) the restriction of $(s_x)^*$ (resp. $(s_y)^*$ or $(s_z)^*$) on the subspace V of $H^2(\overline{S},\mathbf{Z})$.

Remark 4.3. The "action" of $\mathcal{A}$ on the triangle Δ does not depend on the choice of the parameters (A,B,C,D). Let $f = w(s_x,s_y,s_z)$ be an element of $\mathcal{A}$, given by a reduced word in the letters s_x, s_y and s_z. Since s_x (resp. s_y, s_z) does not blow down any curve on indeterminacy points of the other two involutions, the linear transformation $f^* : V \to V$ is the composition $f^* = w^t(s_x^*,s_y^*,s_z^*)$, where w^t is the transpose of w (see section 4.1). If w ends with s_x (resp. s_y or s_z), then f contracts the side D_x (resp. D_y or D_z). If w starts with s_x (resp. s_y or s_z), the image of the critical set of f is the vertex v_x (resp. v_y or v_z). In particular, $\mathrm{Ind}(f)$ and $\mathrm{Ind}(f^{-1})$ are not empty if f is different from the identity.

Example 4.4. The element $g_x = s_z \circ s_y$ preserves the coordinate variable x. Its action on Δ is the following: g_x contracts both D_y and $D_z \setminus \{v_y\}$ on v_z, and preserves D_x; its inverse contracts D_y and $D_z \setminus \{v_z\}$ on v_y. In particular $\mathrm{Ind}(g_x) = v_y$ and $\mathrm{Ind}(g_x^{-1}) = v_z$. The elements g_y and g_z act in a similar way. In particular, g_x, g_y and g_z are algebraically stable.

Let us now present a nice way of describing the "action" of $\mathcal{A}$, *i.e.* of Γ_2^*, on the triangle Δ. Since this action does not depend on the parameters, we choose $(A,B,C,D) = (0,0,0,0)$ and use what we know about the Markov surface S_M (see §3.4). The closure of $S_{M+}(\mathbf{R})$ in $\overline{S_M}$ contains a part of the triangle at infinity, namely the set $\Delta_+(\mathbf{R})$ of points $[x:y:z:0]$ such that $xyz = 0$, and $x,y,z \geq 0$. This provides a compactification of $S_{M+}(\mathbf{R})$ by the triangle $\Delta_+(\mathbf{R})$. The conjugation

$$\mathbf{c} : \mathbb{H} \to S_{M+}(\mathbf{R})$$

between the Poincaré half plane and $S_{M+}(\mathbf{R})$ described in §3.4 does not extend up to the boundary of this compactification. Nevertheless, one can "extend" the map in the following way (see figure 3.3):

- the three segments (j_y,j_z), (j_z,j_x) and (j_x,j_y) of $\partial\mathbb{H}$ are sent to the three vertices v_x, v_y and v_z of Δ;
- the three points j_x, j_y, and j_z are "sent" to the three sides D_x, D_y and D_z of $\Delta_+(\mathbf{R})$ by $\mathbf{c}$ (or equivalently to the middle points m_x, m_y and m_z);

Then, if M is a hyperbolic element of Γ_2^*, the two fixed points of M on the boundary of $\mathbb{H}$ are sent to the indeterminacy points of f_M and f_M^{-1}: If M is hyperbolic, with one attractive fixed point ω_M and one repulsive fixed point α_M, then

$$\mathrm{Ind}(f_M) = \mathbf{c}(\alpha_M), \quad \mathrm{Ind}(f_M^{-1}) = \mathbf{c}(\omega_M).$$

Remark 4.5. Let us consider the surface obtained by blowing up the vertices of the triangle Δ. This is a new compactification of the affine cubic S_M by a cycle of six rational curves. Then we blow up the six vertices of this hexagon, and so on : This defines a sequence of rational surfaces S^i. Let S^∞ be the projective limit of these surfaces. The group Γ_2^* acts continously on this space, and we can extend $\mathbf{c}^{-1}$ so as to obtain a semi-conjugation between the action on $S_M^\infty \setminus S_M$ and the action of Γ_2^* on the circle. Such a construction is presented in details in a similar context in [27], chapter 4 (see also [9] for a related approach).

The following proposition reformulates and makes more precise, section 7 of [32].

Proposition 4.6. *Let S be any member of the family* Fam *and f an element of $\mathcal{A}$.*

- *The birational transformation $f : \overline{S} \to \overline{S}$ is algebraically stable if, and only if f is a cyclically reduced composition of the three involutions s_x, s_y and s_z of length at least 2.*
- *Every hyperbolic element f of $\mathcal{A}$ is conjugate to an algebraically stable element of $\mathcal{A}$.*
- *If f is algebraically stable and hyperbolic, $Ind(f)$ and $Ind(f^{-1})$ are two distinct vertices of Δ, and f^n contracts the whole triangle $\Delta \setminus Ind(f)$ onto $Ind(f^{-1})$ as soon as n is a positive integer.*

Proof. If $\mathrm{Ind}(f) = \mathrm{Ind}(f^{-1}) \neq \emptyset$, f is not algebraically stable. This shows, for example, that an involution with a non empty indeterminacy set is not algebraically stable.

Let M be an element of $\Gamma_2^* \setminus \{Id\}$ and f_M the corresponding element of $\mathcal{A}$, viewed as a birational transformation of S. From remark 4.3, we know that $\mathrm{Ind}(f_M)$ is non empty, and from proposition 3.2 that any elliptic element of Γ_2^* is an involution. This shows that f_M is not algebraically stable if M is elliptic.

Let us now fix a non elliptic element M of Γ_2^*, which we write as a reduced word $w(r_x, r_y, r_z)$ in the generators r_x, r_y and r_z of Γ_2^* (see §3.3).

Let us first assume that M is parabolic. If f_M is a non trivial iterate of g_x (resp. g_y or g_z), we know from example 4.4 that M is algebraically stable. If not, the unique fixed point of M on ∂H is different from j_x, j_y and j_z and its image by $\mathbf{c}$ is a vertex of Δ. This vertex v coincides with $\mathrm{Ind}(f_M)$ and $\mathrm{Ind}(f_M^{-1})$, and f_M is not algebraically stable. Since M is cyclically reduced if, and only if M is an iterate of g_x, g_y, or g_z, the result is proved in the parabolic case.

Let us now suppose that M is hyperbolic : The fixed points α_M and ω_M define two distinct elements of $\partial\mathbb{H} \setminus \{j_x, j_y, j_z\}$ and the indeterminacy sets

of f_M and f_M^{-1} are the vertices $\mathrm{Ind}(f_M) = \mathbf{c}(\alpha_M)$ and $\mathrm{Ind}(f_M^{-1}) = \mathbf{c}(\omega_M)$ of Δ. These vertices are distinct if, and only if α_M and ω_M are contained in two distinct components of $\partial\mathbb{H} \setminus \{j_x, j_y, j_z\}$, if, and only if f_M is a cyclically reduced composition of the three involutions s_x, s_y, s_z (see remark 3.3). This shows that f_M is not algebraically stable if w is not cyclically reduced. In the other direction, if w is cyclically reduced, then $\mathbf{c}(\omega_M)$ is not an indeterminacy point of f_M, f_M fixes this point, and contracts the three sides of Δ on this vertex. As a consequence, the positive orbit of $\mathrm{Ind}(f_M^{-1})$ does not intersect $\mathrm{Ind}(f_M)$, and f_M is algebraically stable. □

Theorem 4.7. *Let f be an element of $\mathcal{A}$ and M_f the element of $\mathsf{PGL}(2,\mathbf{Z})$ which is associated to f. The dynamical degree $\lambda(f)$ is equal to the spectral radius of M_f.*

This result is different from, but similar to, the main theorem of [32], which provides another algorithm to compute $\lambda(f)$.

Proof. Let f be an element of $\mathcal{A}$. After conjugation inside $\mathcal{A}$ (this does not change the dynamical degree and the spectral radius of M_f), we can assume that $f = w(s_x, s_y, s_z)$ is a cyclically reduced word. If f is periodic, then f is one of the involutive generators and the theorem is proved. If f is parabolic, then f is conjugate to an iterate of g_x, g_y or g_z, f preserves a fibration of S into rational curves, and $\lambda(f) = 1$. If f is hyperbolic, proposition 4.6 shows that f is algebraically stable. Let $[v] = [D_x] + [D_y] + [D_z]$ be the class of the hyperplane section of $\overline{S}$ which is obtained by cutting $\overline{S}$ with the plane at infinity. We know that

$$\limsup_{k\to\infty} \left(\frac{1}{k} \log \|(f^k)^*[v]\| \right) = \log(\lambda(f)).$$

Since the action of f^* on the subspace V of $H^2(X,\mathbf{Z})$ does not depend on the parameters (A,B,C,D), and since $[v]$ is contained in V, $\lambda(f)$ does not depend on (A,B,C,D). Consequently, to calculate $\lambda(f)$, we can choose the parameters $(0,0,0,4)$ and work on the Cayley cubic. The conclusion now follows from example 4.1. □

4.3. **Entropy of birational transformations.** Let f be a hyperbolic element of $\mathcal{A}$ (see section 3.3). Up to conjugation, the birational transformation $f : \overline{S} \to \overline{S}$ is algebraically stable, $\mathrm{Ind}(f)$ is a fixed point of f^{-1} and $\mathrm{Ind}(f^{-1})$ is a fixed point of f. As remarked in [32], this enables us to apply the main results from [2] and [18].

Theorem 4.8. *(Bedford, Diller, Dujardin, Iwasaki, Uehara) Let f be an element of the group $\mathcal{A}$ and S be an element of Fam. The topological entropy of $f_M : S \to S$ is equal to the logarithm of the spectral radius $\lambda(f)$ of M_f, the number of periodic (saddle) points of f of period n grows like $\lambda(f)^n$ and*

these points equidistribute toward an ergodic measure of maximal entropy for f.

In [10], we shall explain how the dynamics of f is related to the dynamics of Hénon mappings, and deduce a much more precise description of the dynamics.

Example 4.9. Let M be an element of $\mathsf{GL}(2,\mathbf{Z})$. Let U be the unit circle in $\mathbf{C}^*$ and $\mathbb{T}^2$ be the subgroup $U\times U$ of $\mathbf{C}^*\times\mathbf{C}^*$. The monomial automorphism M of $\mathbf{C}^*\times\mathbf{C}^*$ preserves T and induces a "linear" automorphism on this real torus. The entropy of $M:\mathbb{T}^2\to\mathbb{T}^2$ is equal to the logarithm of the spectral radius of M. If (x,y) is a point of $\mathbf{C}^*\times\mathbf{C}^*$, the orbit $M^n(x,y)$, $n\geq 0$, converges toward $\mathbb{T}^2$ or goes to infinity. The same property remains true for the dynamics of f_M on the Cayley cubic S_C; the role played by $\mathbb{T}^2$ is now played by $\mathbb{T}^2/\eta = S_C(\mathbf{R})\cap[-2,2]^3$.

5. Bounded Orbits

5.1. **Dynamics of parabolic elements.** Parabolic elements will play an important role in the proof of theorem 1.8. In this section, we describe the dynamics of these automorphisms, on any member S of our family of cubic surfaces. Since any parabolic element is conjugate to a power of g_x, g_y or g_z, we just need to study one of these examples.

Once the parameters A, B, C, and D have been fixed, the automorphism g_z is given by

$$g_z\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}A-x-zy\\B-Az+zx+(z^2-1)y\\z\end{pmatrix}.$$

This defines a global polynomial diffeomorphism of $\mathbf{C}^3$, that preserves each horizontal plane $\Pi_{z_0}=\{(x,y,z_0),\,x\in\mathbf{C},\,y\in\mathbf{C}\}$. On each of these planes, g_z induces an affine transformation

$$\begin{pmatrix}x\\y\end{pmatrix}\mapsto\begin{pmatrix}-1&-z_0\\z_0&z_0^2-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}A\\B-Az_0\end{pmatrix},$$

which preserves the conic $S_{z_0}=S\cap\Pi_{z_0}$. The trace of the linear part of this affine transformation is z_0^2-2 while the determinant is 1.

Proposition 5.1. *Let S be any member of the family of cubic surfaces* Fam. *Let g_z be the automorphism of S defined by the composition $s_y\circ s_x$. On each fiber S_{z_0} of the fibration*

$$\pi_z:S\to\mathbf{C},\quad \pi_z(x,y,z)=z,$$

g_z induces a homographic transformation $\overline{g_{z_0}}$, and

- $\overline{g_{z_0}}$ *is an elliptic homography if and only if* $z_0 \in (-2,2)$*; this homography is periodic if and only if* z_0 *is of type* $\pm 2\cos(\pi\theta)$ *with* θ *rational;*
- $\overline{g_{z_0}}$ *is parabolic (or the identity) if and only if* $z_0 = \pm 2$*;*
- $\overline{g_{z_0}}$ *is loxodromic if and only if* z_0 *is not in the interval* $[-2,2]$.

If z_0 is different from 2 and -2, g_z has a unique fixed point inside Π_{z_0}, the coordinate of which are (x_0, y_0, z_0) where

$$x_0 = \frac{Bz_0 - 2A}{{z_0}^2 - 4}, \quad y_0 = \frac{Az_0 - 2B}{{z_0}^2 - 4}.$$

This fixed point is contained in the surface S if and only if z_0 satisfies the quartic equation $P_z(z_0) = 0$ where

$$P_z = z^4 - Cz^3 - (D+4)z^2 + (4C - AB)z + 4D + A^2 + B^2. \tag{5.1}$$

In that case, the union of the two g_z-invariant lines of Π_{z_0} which go through the fixed point coincides with S_{z_0}; moreover, the involutions s_x and s_y permute those two lines. If the fixed point is not contained in S, the conic S_{z_0} is smooth, and the two fixed points of the (elliptic or loxodromic) homography $\overline{g_{z_0}}$ are at infinity.

If $z_0 = 2$, the affine transformation induced by g_z on Π_{z_0} is

$$\overline{g_{z_0}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} A \\ B - 2A \end{pmatrix}.$$

Either $\overline{g_{z_0}}$ has no fixed point, or $A = B$ and there is a line of fixed points, given by $x + y = A/2$. This line of fixed points intersects the surface S if and only if S_{z_0} coincides with this (double) line. In that case the involutions s_x and s_y also fix the line pointwise. When the line does not intersect S, the conic S_{z_0} is smooth, with a unique point at infinity; this point is the unique fixed point of the parabolic transformation $\overline{g_{z_0}}$. In particular, any point of S_{z_0} goes to infinity under the action of g_z.

If $z_0 = -2$, then

$$\overline{g_{z_0}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} A \\ B + 2A \end{pmatrix}.$$

Either g_z does not have any fixed point in Π_{z_0}, or $A = -B$ and g_{z_0} has a line of fixed points given by $x - y = A/2$. This line intersects S if and only if S_{z_0} coincides with this (double) line. In that case the involutions s_x and s_y fixe the line pointwise.

Lemma 5.2. *With the notation that have just been introduced, the homographic transformation* $\overline{g_{z_0}}$ *induced by* g_z *on* S_{z_0} *has a fixed point in* S_{z_0} *if and only if* z_0 *satisfies equation (5.1). Moreover*

- *when $z_0 \neq 2, -2$, S_{z_0} is a singular conic, namely a union of two lines that are permuted by s_x and s_y, and the unique fixed point of $\overline{g_{z_0}}$ is the point of intersection of these two lines, with coordinates*
$$x_0 = \frac{Bz_0 - 2A}{{z_0}^2 - 4}, \quad y_0 = \frac{Az_0 - 2B}{{z_0}^2 - 4};$$
- *when $z_0 = 2$, then $A = B$, S_{z_0} is the double line $x + y = A/2$, and this line is pointwise fixed by $\overline{g_{z_0}}$, s_x and s_y;*
- *when $z_0 = -2$, then $A = -B$, S_{z_0} is the double line $x - y = A/2$, and this line is pointwise fixed by $\overline{g_{z_0}}$, s_x and s_y;*

The dynamics of g_z on S is now easily described. Let $p_0 = (x_0, y_0, z_0)$ be a point of S. If z_0 is in the interval $(-2, 2)$, the orbit of p_0 under g_z is bounded, and it is periodic if, and only if, either p_0 is a fixed point, or z_0 is of type $\pm 2\cos(\pi\theta)$, where θ is a rational number. If $z_0 = \pm 2$, and if p_0 is not a fixed point, $g^n(p_0)$ goes to infinity when n goes to $+\infty$ and $-\infty$. If z_0 is not contained in the interval $[-2, 2]$, for instance if the imaginary part of z_0 is not 0, either p_0 is fixed or $g^n(p_0)$ goes to infinity when n goes to $-\infty$ or $+\infty$. Of course, the same kind of results are valid for g_x and g_y, with the appropriate permutation of variables and parameters.

5.2. **Bounded Orbits.** There is a huge literature on the classification of algebraic solutions of Painlevé VI equation (see [7] and references therein). Such solutions give rise to periodic orbits for the action of $\mathcal{A}$ on the cubic surface $S_{(A,B,C,D)}$, where the parameters are defined in terms of the coefficients of the Painlevé equation (see §9). Of course, periodic orbits are bounded. Here, we study infinite bounded orbits.

Theorem 5.3. *Let $S = S_{(A,B,C,D)}$ be a surface in the family* Fam*, and p be a point with an infinite and bounded Γ_2^* orbit $Orb(p)$. Then A, B, C, and D are real numbers, the orbit is contained in $[-2, 2]^3$ and it forms a dense subset of the unique bounded component of $S(\mathbf{R}) \setminus \mathrm{Sing}(S)$.*

We fix a point p in one of the surfaces S and denote its $\Gamma^*(2)$-orbit by $\mathrm{Orb}(p)$. Let us first study orbits of small finite length. Recall that orbits of length 1 are singular points of the cubic S.

Proposition 5.4. *Modulo Benedetto-Goldman symmetries (see §2.3), Γ_2^*-orbits of length 2 are equivalent to*
$$\{(0, 0, z_1), (0, 0, z_2)\} \in S_{(0,0,C,D)}, \quad C^2 + 4D \neq 0$$
where z_1 and z_2 are the two roots of $z^2 = Cz + D$, Γ_2^*-orbits of length 3 are equivalent to
$$\{(0, 0, 1), (A, 0, 1), (0, A, 1)\} \in S_{(A,A,2,-1)},$$

and Γ_2^*-orbits of length 4 are equivalent to

$$\{(1,1,1),(A-2,1,1),(1,A-2,1),(1,1,A-2)\} \in S_{(A,A,A,4-3A)}.$$

Example 5.5. An orbit of length 2 is for instance provided by the representation ρ defined by

$$\rho \ : \ (\alpha,\beta,\gamma,\delta) \mapsto (M,N,M,-N)$$

where $M,N \in \mathsf{SL}(2,\mathbf{C})$ are any element satisfying $\mathsf{Tr}(MN)=0$ i.e. $(MN)^2 = -I$. Trace parameters are given by $(a,b,a,-b)$ where $a=\mathsf{Tr}(M)$ and $b=\mathsf{Tr}(N)$: we get $C=a^2-b^2$, $D=(a^2-2)(b^2-2)$ and $z=a^2-2$. The other representation in the orbit, given by $z'=2-b^2$, is defined by

$$\rho' \ : \ (\alpha,\beta,\gamma,\delta) \mapsto (M,M^{-1}NM,NMN^{-1},-N).$$

To this length 2 orbit corresponds a two-sheeted algebraic solution of P_{VI}-equation, namely

$$q(t) = 1+\sqrt{1-t}, \quad \text{for parameters} \quad \theta=(\theta_0,\theta_1,\theta_0,-\theta_1),$$

with $a=2\cos(\pi\theta_0)$ and $b=2\cos(\pi\theta_1)$. This representation was already considered in [39] : for convenient choice of parameters a and b, the image of the representation is a dense subgroup of $\mathsf{SU}(2)$.

Other choice of the trace parameters are provided by

$$(a',b',a',-b') \quad \text{with } a'=\sqrt{4-b^2} \text{ and } b'=\sqrt{4-a^2},$$

giving rise to a representation of the same kind, and

$$(a'',0,c'',0) \quad \text{with } a'',c''=\frac{a}{2}\sqrt{4-b^2}\pm\frac{b}{2}\sqrt{4-a^2}.$$

The later one corresponds to a dihedral representation of the form

$$(\alpha,\beta,\gamma,\delta) \mapsto \begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix},\begin{pmatrix}0 & \mu\\ -\mu^{-1} & 0\end{pmatrix},\begin{pmatrix}\tau^{-1} & 0\\ 0 & \tau\end{pmatrix},\begin{pmatrix}0 & -\nu^{-1}\\ \nu & 0\end{pmatrix})$$

with $\lambda\mu\nu\tau=1$.

Proof. Let $p=(x_0,y_0,z_0)$ be a point of $S_{(A,B,C,D)}$. Recall that p is fixed if, and only if, p is a singular point of S. On the other hand, p is periodic of order $n>1$ for g_z if, and only if,

$$z_0 = 2\cos(\pi\frac{k}{n}), \quad k\wedge n=1$$

and at least one of the equalities $P_z(z_0)=0$, $2x_0+y_0z_0=A$, $2y_0+x_0z_0=B$ does not hold. In particular, denoting by $\mathrm{Orb}_{g_z}(p)$ the orbit of p under the action of g_z, we have:

$$\#\mathrm{Orb}_{g_z}(p)=2 \ \Rightarrow \ z_0=0,$$
$$\#\mathrm{Orb}_{g_z}(p)=3 \ \Rightarrow \ z_0=\pm 1,$$

$$\#\mathrm{Orb}_{g_z}(p) = 4 \;\Rightarrow\; z_0 = \pm\sqrt{2},$$

$$\#\mathrm{Orb}_{g_z}(p) = 6 \;\Rightarrow\; z_0 = \pm\sqrt{3}.$$

Up to permutation of variables x, y and z (and correspondingly of the parameters A, B and C), an orbit of length 2 takes the form $\mathrm{Orb}(p) = \{p, s_z(p)\}$. In this case, p and $p' = s_z(p) = (x_0, y_0, z_0')$ are permuted by s_z, and thus by $g_x = s_z \circ s_y$ and $g_y = s_x \circ s_z$; this implies $x_0 = y_0 = 0$. On the other hand, p and p' are fixed by s_x, s_y, and therefore $A = B = 0$. Since $p = (0,0,z_0)$ is contained in S, we deduce that z_0 and z_0' are the roots of $z^2 = Cz + D$.

Up to permutation of the variables x, y and z, an orbit of length 3 takes the form

$$\mathrm{Orb}(p) = \{p_0, p_1 = s_x(p_0), p_2 = s_y(p_0)\}$$

with $p_0 = (x_0, y_0, z_0)$, $p_1 = (x_0', y_0, z_0)$, and $p_2 = (x_0, y_0', z_0)$. Since g_x (resp. g_y) permutes p_0 and p_2 (resp. p_0 and p_1), we get $x_0 = y_0 = 0$. On the other hand, g_z permutes cyclically $p_0 \to p_2 \to p_1$, so that $z_0 = \pm 1$. Changing signs if necessary by a twist (see 2.3), we can assume $z_0 = 1$. Now, studying the fixed points of s_x, s_y and s_z, amongst p_0, p_1 and p_2, we obtain:

$$2z_0 + x_0 y_0 = C, \quad \begin{cases} 2y_0 + x_0' z_0 = B \\ 2z_0 + x_0' y_0 = C \end{cases} \quad \text{and} \quad \begin{cases} 2x_0 + y_0' z_0 = A \\ 2z_0 + x_0 y_0' = C \end{cases}$$

and thus $C = 2$, $x_0' = B$ and $y_0' = A$. We also have

$$x_0' = A - x_0 - y_0 z_0 \quad \text{and} \quad y_0' = B - y_0 - x_0 z_0$$

(action of s_x and s_y) yielding $A = B$. Finally, the fact that p_0 is contained in S gives $1 = C + D$, whence the result.

Up to symmetry, an orbit of length 4 consists in p_0, p_1 and p_2 like before ($p_1 = s_x(p_0)$ and $p_2 = s_y(p_0)$) and there are 4 possibilities for the fourth point p_3:

(1) $p_3 = (x_0', y_0', z_0) = s_y(p_1) = s_x(p_2)$,
(2) $p_3 = (x_0', y_0'', z_0) = s_y(p_1) \neq s_x(p_2)$,
(3) $p_3 = (x_0', y_0, z_0') = s_z(p_1)$,
(4) $p_3 = (x_0, y_0, z_0') = s_z(p_0)$.

The first case is impossible: Since g_x and g_y have order 2 for each p_i, the coordinates x and y vanish for each point p_i, and therefore $p_0 = p_1 = p_2 = p_3$, a contradiction. The second case is impossible for the same reason since g_x and g_y have order 2 for p_0 and p_1, so that $p_0 = p_1$, a contradiction. The same argument applies in third case: g_x has order 2 for p_0 and p_1 implying $p_0 = p_1$, contradiction.

For the fourth case, since g_x, g_y and g_z have order 3 for p_0, we get $p_0 = (\pm 1, \pm 1, \pm 1)$. Up to symmetry, there are two subcases: $p_0 = (1,1,1)$ or

$p_0 = (-1,-1,-1)$. In the first sub-case, conditions given by the fixed points of s_x, s_y and s_z yield

$$A = B = C = 2 + x_0' = 2 + y_0' = 2 + z_0',$$

and the fact that p_i is contained in S gives $4 = 3A + D$. Proceeding in the same way with the second sub-case, conditions given by the fixed points of s_x, s_y and s_z yield

$$A = B = C = -2 - x_0' = -2 - y_0' = -2 - z_0'$$

and the fact that p_i is in S gives

$$2 = -3A + D \quad \text{and} \quad x_0' = A$$

implying $x_0' = A = -1$ and $p_1 = p_0$, a contradiction. □

Lemma 5.6. *If $Orb(p)$ is bounded and $\#Orb(p) > 4$, then A, B, C, and D are real and $p \in S(\mathbf{R})$.*

Proof. Let $p_0 = (x_0, y_0, z_0)$ be a point of the orbit. If the third coordinate $z_0 \notin (-2,2)$, the homography induced by g_z on the conic S_{z_0} is parabolic or hyperbolic. Since the orbit of p_0 is bounded, this implies that p_0 is a fixed point of g_z, s_x and s_y (see lemma 5.2). Since $\mathrm{Orb}(p_0)$ has length > 4, $s_z(p_0)$ is different from p_0, so that p_0 is not fixed by g_x, nor by g_y either ; this implies that $x_0, y_0 \in (-2,2)$. Moreover, the point $p_1 := s_z(p_0) = (x_0, y_0, z_1)$ is not fixed by g_z, otherwise the orbit would have length 2, so that $z_1 \in (-2,2)$ and $p_1 \in (-2,2)^3$. This argument shows the following: If one of the coordinates of p_0 is not contained in $(-2,2)$, then p_0 is fixed by two of the involutions s_x, s_y and s_z while the third one maps p_0 into $(-2,2)^3$.

Let now p be a point of the orbit with coordinates in $(-2,2)^3$; if the three points $s_x(p)$, $s_y(p)$ and $s_z(p)$ either escape from $(-2,2)^3$ or coincide with p, then the orbit reduces to $\{p, s_x(p), s_y(p), s_z(p)\}$, and has length ≤ 4. From this we deduce that the orbit contains at least two distinct points $p_1, p_2 \in (-2,2)^3$, which are, say, permuted by s_x. Let (x_i, y_1, z_1) be the coordinates of p_i, $i = 1,2$. Then, $A = x_1 + x_2 + y_1 z_1 \in \mathbb{R}$. If B and C are also real, then p_1 is real and satisfies the equation of S, so that D is real as well and $\mathrm{Orb}(m) = \mathrm{Orb}(p_1) \subset S(\mathbf{R})$.

Now, assume by contradiction that $B \notin \mathbf{R}$. Then, $q_i := s_y(p_i) = (x_i, B - y_1 - x_i z_1, z_1) \notin (-2,2)$ and is therefore fixed by s_x (otherwise the orbit would not be bounded): We thus have

$$2x_i + (B - y_1 - x_i z_1) z_1 = A.$$

Since B is the unique imaginary number of this equation, z_1 must vanish, and we get $x_1 = x_2 (= \frac{A}{2})$, a contradiction. A similar argument shows that C must be real as well. □

Proposition 5.7. *If $Orb(p)$ is finite and $\#Orb(p) > 4$, then A, B, C, and D are real algebraic numbers and $p \in S(\mathbf{R})$ has algebraic coordinates as well.*

The proof is exactly the same, replacing $(-2,2)$ by $(-2,2) \cap 2\cos(\pi\mathbf{Q})$ and thus $\mathbf{R}$ by $\mathbf{R} \cap \overline{\mathbf{Q}}$.

Lemma 5.8. *Let S be an element of the family* Fam *and p a point of S. There exists a positive integer N such that, if p' is a point of the orbit of p with a coordinate of the form*

$$2\cos(\pi\frac{k}{n}), \quad k \wedge n = 1,$$

then n divides N.

Proof. The point p is an element of the character variety $\chi(\mathbb{S}^2_4)$. Let us choose a representation $\rho : \pi_1(\mathbb{S}^2_4) \to \mathsf{SL}(2,\mathbf{C})$ in the conjugacy class that is determined by p. Since $\pi_1(\mathbb{S}^2_4)$ is finitely generated, Selberg's lemma (see [1]) implies the existence of a torsion free, finite index subgroup G of $\rho(\pi_1(\mathbb{S}^2_4))$. If we define N to be the cardinal of the quotient $\rho(\pi_1(\mathbb{S}^2_4))/G$, then the order of any torsion element in $\rho(\pi_1(\mathbb{S}^2_4))$ divides N.

If p' is a point of the orbit of p, the coordinates of p' are traces of elements of $\rho(\pi_1(\mathbb{S}^2_4))$. Assume that the trace of an element M in $\rho(\pi_1(\mathbb{S}^2_4))$ is of type $2\cos(\pi\theta)$. If $\theta = \frac{k}{n}$ and k and n are relatively prime integers, then M is a cyclic element of $\rho(\pi_1(\mathbb{S}^2_4))$ of order n, so that n divides N. □

The subset of $\mathsf{SU}(2)$-representations always form a connected component of $S \setminus \mathsf{Sing}(S)$ contained into $[-2,2]^3$; the corresponding orbits are bounded, generally infinite. A bounded component can also consist in $\mathsf{SL}(2,\mathbf{R})$-representations, depending on the choice of (a,b,c,d); for instance, in the Cayley case, the bounded component consists in $\mathsf{SL}(2,\mathbf{R})$-representations (resp. $\mathsf{SU}(2)$-representations) when $(a,b,c,d) = (2,2,2,-2)$ (resp. $(0,0,0,0)$).

Proposition 5.9 (Benedetto-Goldman [4])**.** *When A, B, C and D are real, then $S(\mathbf{R}) \setminus \{\mathsf{Sing}(S)\}$ has at most one bounded connected component. In this case, a, b, c and d lie in $[-2,2]$, whatever the choice of (a,b,c,d) corresponding to (A,B,C,D).*

When $S(\mathbf{R})$ is smooth, the converse is true: When a, b, c and d lie in $[-2,2]$, $S(\mathbf{R})$ has a "bounded component" maybe degenerating to a singular point, like in the Markov case. It is proved in Apendix B, §9.3, that a bounded component always correponds to $\mathsf{SU}(2)$-representations for a convenient choice of parameters (a,b,c,d).

Proof of theorem 5.3. Let $\mathrm{Orb}(p)$ be an infinite and bounded Γ^*_2-orbit in $S = S_{(A,B,C,D)}$. Following Lemma 5.6, A, B, C and D are real and $\mathrm{Orb}(p) \subset S(\mathbf{R})$.

We want to prove that the closure $\overline{\mathrm{Orb}(p)}$ is open in $S(\mathbf{R}) \setminus \{\mathsf{Sing}(S(\mathbf{R}))\}$; since $\overline{\mathrm{Orb}(p)}$ is closed, it will therefore coincide with the (unique) bounded connected component of $S \setminus \{\mathsf{Sing}(S)\}$, thus proving the theorem.

We first claim that there exists an element (actually infinitely many) $p_0 = (x_0, y_0, z_0)$ of the orbit which is contained in $(-2,2)^3$ and for which at least one of the Möbius transformations $\overline{g_{x_0}}$, $\overline{g_{y_0}}$ or $\overline{g_{z_0}}$ is (elliptic) non periodic. Indeed, if a point p_0 of the orbit is such that $\overline{g_{z_0}}$ is not of the form above, then we are in one of the following cases

- $P_z(z_0) = 0$ and p_0 is a fixed point of $\overline{g_{z_0}}$,
- $z_0 = 2\cos(\pi\frac{k}{n})$ with $k \wedge n = 1$, $n|N$ and $\overline{g_{z_0}}$ is periodic of period n

(where N is given by Lemma 5.8). This gives us finitely many possibilities for z_0; we also get finitely many possibilities for x_0 and y_0 and the claim follows.

Let p_0 be a point of $\mathrm{Orb}(p)$, with, say, $\overline{g_{x_0}}$ elliptic and non periodic, so that the closure $\overline{\mathrm{Orb}(p)}$ contains the "circle" $\overline{\mathrm{Orb}_{g_x}(p_0)} = S_{x_0}(\mathbf{R})$. Let us first prove that $\overline{\mathrm{Orb}(p)}$ contains an open neighborhood of p_0 in $S(\mathbf{R}) \setminus \{\mathsf{Sing}(S(\mathbf{R}))\}$.

Since the point p_0 is not fixed by $g_x = s_z \circ s_y$, then either s_y or s_z does not fix p_0, say s_z; this means that the point p_0 is not a critical point of the projection

$$\pi_x \times \pi_y \ : \ S(\mathbf{R}) \to \mathbf{R}^2 \ ; \ (x,y,z) \mapsto (x,y).$$

Therefore, there exists some $\varepsilon > 0$ and a neighborhood V_ε of p_0 in $S(\mathbf{R})$ such that $\pi_x \times \pi_y$ maps V_ε diffeomorphically onto the square $(x_0 - \varepsilon, x_0 + \varepsilon) \times (y_0 - \varepsilon, y_0 + \varepsilon)$. By construction, we have

$$\pi_x \times \pi_y(\overline{\mathrm{Orb}(p)}) \supset \pi_x \times \pi_y(\overline{\mathrm{Orb}_{g_x}(p_0)}) \supset \{x_0\} \times (y_0 - \varepsilon, y_0 + \varepsilon).$$

For each $y_1 \in (y_0 - \varepsilon, y_0 + \varepsilon)$ of irrational type, that is to say not of the form $2\cos(\pi\theta)$ with θ rational, there exists $p_1 = (x_0, y_1, z_1) \in \overline{\mathrm{Orb}(p)}$ (namely, the preimage of (x_0, y_1) by $\pi_x \times \pi_y$) and

$$\overline{\mathrm{Orb}(p)} \supset \overline{\mathrm{Orb}_{g_y}(p_1)} = S_{y_1}(\mathbf{R});$$

in other words, for each $y_1 \in (y_0 - \varepsilon, y_0 + \varepsilon)$ of irrational type, we have

$$\pi_x \times \pi_y(\overline{\mathrm{Orb}(p)}) \supset \pi_x \times \pi_y(\overline{\mathrm{Orb}_{g_y}(p_1)}) \supset (x_0 - \varepsilon, x_0 + \varepsilon) \times \{y_0\}.$$

Since those coordinates y_1 of irrational type are dense in $(y_0 - \varepsilon, y_0 + \varepsilon)$, we deduce that $V_\varepsilon \subset \overline{\mathrm{Orb}(p)}$, and $\overline{\mathrm{Orb}(p)}$ is open at p_0.

It remains to prove that $\overline{\mathrm{Orb}(p)}$ is open at any point $q \in \overline{\mathrm{Orb}(p)}$ which is not a singular point of $S(\mathbf{R})$. Let $q = (x_0, y_0, z_0)$ be such a point and assume that $q \notin \mathrm{Orb}(p)$ (otherwise we have already proved the assertion).

Since q is not a singular point of $S(\mathbf{R})$, one of the projections, say $\pi_x \times \pi_y : S(\mathbf{R}) \to \mathbf{R}^2$, is regular at q and we consider a neighborhood V_ε like above, $\pi_x \times \pi_y(V_\varepsilon) = (x_0-\varepsilon, x_0+\varepsilon) \times (y_0-\varepsilon, y_0+\varepsilon)$. By assumption, $\mathrm{Orb}(p) \cap V_\varepsilon$ is infinite (accumulating q) and, applying once again Lemma 5.8, one can find one such point $p_1 = (x_1, y_1, z_1) \in \mathrm{Orb}(p) \cap V_\varepsilon$ such that either x_1 or y_1 has irrational type, say x_1. Now, reasonning with p_1 like we did above with p_0, we eventually conclude that $V_\varepsilon \supset \overline{\mathrm{Orb}(p)}$, and $\overline{\mathrm{Orb}(p)}$ is open at q. □

6. Invariant geometric structures

In this section, we study the existence of $\mathcal{A}$-invariant geometric structures on surfaces S of the family Fam. An example of such an invariant structure is given by the area form Ω, defined in Proposition 3.6. Another example occurs for the Cayley cubic: S_C is covered by $\mathbf{C}^* \times \mathbf{C}^*$ and the action of $\mathcal{A}$ on S_C is covered by the monomial action of $\mathsf{GL}(2,\mathbf{Z})$, that is also covered by the linear action of $\mathsf{GL}(2,\mathbf{Z})$ on $\mathbf{C} \times \mathbf{C}$ if we use the covering mapping

$$\pi : \mathbf{C} \times \mathbf{C} \to \mathbf{C}^* \times \mathbf{C}^*, \quad \pi(\theta, \phi) = (\exp(\theta), \exp(\phi));$$

as a consequence, there is an obvious $\mathcal{A}$-invariant affine structure on S_C.

Remark 6.1. The surface S_C is endowed with a natural orbifold structure, the analytic structure near its singular points being locally isomorphic to the quotient of $\mathbf{C}^2$ near the origin by the involution $\sigma(x,y) = (-x,-y)$. The affine structure can be understood either in the orbifold language, or as an affine structure defined only outside the singularities (see below).

6.1. Invariant curves, foliations and webs.

We start with

Lemma 6.2. *Whatever the choice of S in the family* Fam, *the group $\mathcal{A}$ does not preserve any (affine) algebraic curve on S.*

Of course, invariant curves appear if we blow up singularities. This is important for the study of special (Riccati) solutions of Painlevé VI equation (see section 7).

Proof. Let C be an algebraic curve on S. Either C is contained in a fiber of π_z, or the projection $\pi_z(C)$ covers $\mathbf{C}$ minus at most finitely many points. If C is not contained in a fiber, we can choose $m_0 = (x_0, y_0, z_0)$ in C and a neighborhood U of m_0 such that z_0 is contained in $(0,2)$ and, in U, C intersects each fiber S_z of the projection π_z in exactly one point. Let $m' = (x', y', z')$ be any element of $C \cap U$ such that z' is an element of $(0,2)$. Then $\overline{g_z}$ is an elliptic transformation of $S_{z'}$ that preserves $C \cap S_{z'}$; since the intersection of C and $S_{z'}$ contains an isolated point m', this point is g_z periodic. As a consequence, z' is of the form $2\cos(\pi p/q)$ (see proposition 5.1). Since any $z' \in (0,2)$ sufficiently close to z_0 should satisfy an equation of this type, we obtain a contradiction.

Since no curve can be simultaneously contained in fibers of π_x, π_y and π_z, the lemma is proved. □

A (singular) web on a surface X is given by a hypersurface in the projectivized tangent bundle $\mathbb{P}TX$; for each point, the web determines a finite collection of directions tangent to X through that point. The number of directions is constant on an open subset of X but it may vary along the singular locus of the web. Foliations are particular cases of webs, and any web is locally made of a finite collection of foliations in the complement of its singular locus.

Proposition 6.3. *Whatever the choice of S in the family* Fam, *the group $\mathcal{A}$ does not preserve any web on S.*

Proof. Let us suppose that there exists an invariant web W on one of the surfaces S. Let k and l be coprime positive integers and $m = (x,y,z)$ be a periodic point of g_z of period l, with

$$z = 2\cos(\pi k/l).$$

Let L_1, ..., L_d be the directions determined by W through the point m, and C_1, ..., C_d the local leaves of W which are tangent to these directions. The automorphism g_z^s, with $s = l(d!)$, fixes m, preserves the web and fixes each of the directions L_i; it therefore preserves each of the C_i. The proof of lemma 6.2 now shows that $d = 1$ and that the curves C_i are contained in the fiber of π_z through m. Since the set of points m which are g_z-periodic is Zariski dense in S, this argument shows that the web is the foliation by fibers of π_z. The same argument shows that the web should also coincide with the foliations by fibers of π_x or π_y, a contradiction. □

Corollary 6.4. *Whatever the choice of S in the family* Fam, *the group $\mathcal{A}$ does not preserve any holomorphic riemannian metric on S.*

Proof. Let g be an invariant holomorphic riemannian metric. At each point m of S, g has two isotropic lines. This determines an $\mathcal{A}$-invariant web, and we get a contradiction with the previous proposition. □

6.2. **Invariant Affine Structures.** A holomorphic affine structure on a complex surface M is given by an atlas of charts $\Phi_i : U_i \to \mathbf{C}^2$ for which the transition functions $\Phi_i \circ \Phi_j^{-1}$ are affine transformations of the plane $\mathbf{C}^2$. A local chart $\Phi : U \to \mathbf{C}^2$ is said to be affine if, for all i, $\Phi \circ \Phi_i^{-1}$ is the restriction of an affine transformation of $\mathbf{C}^2$ to $\Phi_i(U_i) \cap \Phi(U)$. A subgroup G of $\mathsf{Aut}(M)$ preserves the affine structure if elements of G are given by affine transformations in local affine charts.

Theorem 6.5. *Let S be an element of* Fam. *Let G be a finite index subgroup of* $\mathsf{Aut}(S)$. *The group G preserves an affine structure on $S \setminus \mathsf{Sing}(S)$, if, and only if S is the Cayley cubic S_C.*

In what follows, S is a cubic of the family Fam and G will be a finite index subgroup of $\mathcal{A}$ preserving an affine structure on S.

Before giving the proof of this statement, we collect a few basic results concerning affine structures. Let X be a complex surface with a holomorphic affine structure. Let $\pi : \widetilde{X} \to X$ be the universal cover of X; the group of deck transformations of this covering is isomorphic to the fundamental group $\pi_1(X)$. Gluing together the affine local charts of X, we get a developping map

$$\mathbf{dev} : \widetilde{X} \to \mathbf{C}^2,$$

and a monodromy representation $\mathbf{Mon} : \pi_1(X) \to \mathsf{Aff}(\mathbf{C}^2)$ such that

$$\mathbf{dev}(\gamma(m)) = \mathbf{Mon}(\gamma)(\mathbf{dev}(m))$$

for all γ in $\pi_1(X)$ and all m in $\widetilde{X}$. The map **dev** is a local diffeomorphism but, a priori, it is neither surjective, nor a covering onto its image.

Let f be an element of $\mathsf{Aut}(X)$ that preserves the affine structure of X. Let m_0 be a fixed point of f, let $\widetilde{m_0}$ be an element of the fiber $\pi^{-1}(m_0)$, and let $\widetilde{f} : \widetilde{X} \to \widetilde{X}$ be the lift of f that fixes $\widetilde{m_0}$. Since f is affine, there exists a unique affine automorphism $\mathbf{Aff}(f)$ of $\mathbf{C}^2$ such that

$$\mathbf{dev} \circ \widetilde{f} = \mathbf{Aff}(f) \circ \mathbf{dev}.$$

6.3. **Proof of theorem 6.5; step 1.** In this first step, we show that $S \setminus \mathsf{Sing}(S)$ cannot be simply connected, and deduce from this fact that S is singular. Then we study the singularities of S and the fundamental group of $S \setminus \mathsf{Sing}(S)$.

6.3.1. *Simple connectedness.* Assume that $S \setminus \mathsf{Sing}(S)$ is simply connected. The developping map **dev** is therefore defined on $S \setminus \mathsf{Sing}(S) \to \mathbf{C}^2$. Let N be a positive integer for which g_x^N is contained in G. Choose a fixed point m_0 of g_x as a base point. Since g_x^N preserves the affine structure, there exists an affine transformation $\mathbf{Aff}(g_x^N)$ such that

$$\mathbf{dev} \circ g_x^N = \mathbf{Aff}(g_x^N) \circ \mathbf{dev}.$$

In particular, **dev** sends periodic points of g_x^N to periodic points of $\mathbf{Aff}(g_x^N)$. Let m be a nonsingular point of S with its first coordinate in the interval $(-2,2)$, and let U be an open neighborhood of m. Section 5.1 shows that periodic points of g_x^N form a Zariski-dense subset of U, by which we mean that any holomorphic functions $\Psi : U \to \mathbf{C}$ which vanishes on the set of periodic points of g_x^N vanishes everywhere. Since **dev** is a local diffeomorphism, periodic points of $\mathbf{Aff}(g_x^N)$ are Zariski-dense in a neighborhood of $\mathbf{dev}(m)$,

and therefore $\mathbf{Aff}(g_x^N) = \mathrm{Id}$. This provides a contradiction, and shows that $S \setminus \mathsf{Sing}(S)$ is not simply connected.

Consequently, lemma 3.10 implies that S is singular and that the fundamental group of $S \setminus \mathsf{Sing}(S)$ is generated, as a normal subgroup, by the local fundamental groups around the singularities.

6.3.2. *Orbifold structure.* We already explained in section 3.6 that the singularities of S are quotient singularities. If q is a singular point of S, S is locally isomorphic to the quotient of the unit ball $\mathbb{B}$ in $\mathbf{C}^2$ by a finite subgroup H of $\mathsf{SU}(2)$.

The local affine structure around q can therefore be lifted into a H-invariant affine structure on $\mathbb{B} \setminus \{(0,0)\}$, and then extended up to the origin by Hartogs theorem. In particular, **dev** lifts to a local diffeomorphism between $\mathbb{B}$ and an open subset of $\mathbf{C}^2$. This remark shows that the affine structure is compatible with the orbifold structure of S defined in section 3.6.

Let h be an element of the local fundamental group H. Let us lift the affine structure on $\mathbb{B}$ and assume that the monodromy action of h is trivial, i.e. $\mathbf{dev} \circ h = \mathbf{dev}$. Since **dev** is a local diffeomorphism, the singularity is isomorphic to a quotient of $\mathbb{B}$ by a proper quotient of H, namely the quotient of H by the smallest normal subgroup containing h. This provides a contradiction and shows that (i) H embeds in the global fundamental group of $S \setminus \mathsf{Sing}(S)$ and (ii) the universal cover of S in the orbifold sense is smooth (it is obtained by adding points to the universal cover of $S \setminus \mathsf{Sing}(S)$ above singularities of S).

In what follows, we denote the orbifold universal cover by $\pi : \widetilde{S} \to S$, and the developing map by $\mathbf{dev} : \widetilde{S} \to \mathbf{C}^2$.

6.3.3. *Singularities.* Let q be a singular point of S. Let $\widetilde{q}$ be a point of the fiber $\pi^{-1}(q)$. Since the group $\mathcal{A}$ fixes all the singularities of S, it fixes q and one can lift the action of $\mathcal{A}$ on S to an action of $\mathcal{A}$ on $\widetilde{S}$ that fixes $\widetilde{q}$. If f is an element of $\mathcal{A}$, $\widetilde{f}$ will denote the corresponding holomorphic diffeomorphism of $\widetilde{S}$. Then we compose **dev** by a translation of the affine plane $\mathbf{C}^2$ in order to assume that

$$\mathbf{dev}(\widetilde{q}) = (0,0).$$

By assumption, $\mathbf{dev} \circ \widetilde{g} = \mathbf{Aff}(g) \circ \mathbf{dev}$ for any element g in G, from which we deduce that the affine transformation $\mathbf{Aff}(g)$ are in fact linear. Since $\mathcal{A}$ almost preserves an area form, $\mathbf{Aff}(g)$ is an element of $\mathsf{GL}(2,\mathbf{C})$ with determinant $+1$ or -1; passing to a subgroup of index 2 in G, we shall assume that the determinant is 1. Since **dev** realizes a local conjugation between the action of G near $\widetilde{q}$ and the action of $\mathbf{Aff}(G)$ near the origin, the morphism

$$\left\{ \begin{array}{ccc} G & \to & \mathsf{SL}(2,\mathbf{C}) \\ g & \mapsto & \mathbf{Aff}(g) \end{array} \right.$$

is injective.

Since G is a finite index subgroup of $\mathsf{Aut}(S)$, G contains a non abelian free group of finite index and is not virtually solvable. Let H be the finite subgroup of $\pi_1(S \setminus \mathsf{Sing}(S))$ that fixes the point $\widetilde{q}$. This group is normalized by the action of $\mathcal{A}$ on $\widetilde{S}$. Consequently, using the local affine chart determined by **dev**, the group $\mathbf{Aff}(G)$ normalizes the monodromy group $\mathbf{Mon}(H)$. If $\mathbf{Mon}(H)$ is not contained in the center of $\mathsf{SL}(2,\mathbf{C})$, the eigenlines of the elements of $\mathbf{Mon}(H)$ determine a finite, non empty, and $\mathbf{Aff}(G)$-invariant set of lines in $\mathbf{C}^2$, so that $\mathbf{Aff}(G)$ is virtually solvable. This would contradict the injectivity of $g \mapsto \mathbf{Aff}(g)$. From this we deduce that any element of $\mathbf{Mon}(H)$ is a homothety with determinant 1. Since the monodromy representation is injective on H, we conclude that H "coincides" with the subgroup $\{+\mathrm{Id}, -\mathrm{Id}\}$ of $\mathsf{SU}(2)$.

6.3.4. *Linear part of the monodromy.* By lemma 3.10, the fundamental group of $S \setminus \mathsf{Sing}(S)$ is generated, as a normal subgroup, by the finite fundamental groups around the singularities of S. Since $\pm\mathrm{Id}$ is in the center of $\mathsf{GL}(2,\mathbf{C})$, the linear part of the monodromy $\mathbf{Mon}(\gamma)$ of any element γ in $\pi_1(S \setminus \mathsf{Sing}(S))$ is equal to $+\mathrm{Id}$ or $-\mathrm{Id}$.

6.4. **Proof of theorem 6.5; step 2.** We now study the dynamics of the parabolic elements of G near the fixed point q.

6.4.1. *Linear part of automorphisms.* Let g be an element of the group G. Let m be a fixed point of g and $\widetilde{m}$ a point of the fiber $\pi^{-1}(m)$. Let $\widetilde{g}_{\widetilde{m}}$ be the unique lift of g to $\widetilde{S}$ fixing $\widetilde{m}$ (with the notation used in step 1, $\widetilde{g}_{\widetilde{q}} = \widetilde{g}$). Since g preserves the affine structure, there exists an affine transformation $\mathbf{Aff}(\widetilde{g}_{\widetilde{m}})$ such that

$$\mathbf{dev} \circ \widetilde{g}_{\widetilde{m}} = \mathbf{Aff}(\widetilde{g}_{\widetilde{m}}) \circ \mathbf{dev}.$$

Note that $\mathbf{Aff}(\widetilde{g}_{\widetilde{m}})$ depends on the choice of m and $\widetilde{m}$, but that $\mathbf{Aff}(\widetilde{g}_{\widetilde{m}})$ is uniquely determined by g up to composition by an element of the monodromy group $\mathbf{Mon}(\pi_1(S \setminus \mathsf{Sing}(S))$. Since the linear parts of the monodromy are equal to $+\mathrm{Id}$ or $-\mathrm{Id}$, we get a well defined morphism

$$\left\{ \begin{array}{ccc} G & \to & \mathsf{PSL}(2,\mathbf{C}) \\ g & \mapsto & \mathbf{Lin}(g) \end{array} \right.$$

that determines the linear part of $\mathbf{Aff}(\widetilde{g}_{\widetilde{m}})$ modulo $\pm\mathrm{Id}$ for any choice of m and $\widetilde{m}$.

6.4.2. *Parabolic elements.* Since the linear part $\mathbf{Lin}(g)$ does not depend on the fixed point m, it turns out that **Lin** preserves the type of g: We now prove and use this fact in the particular case of the parabolic elements g_x, g_y and g_z.

Let N be a positive integer such that g_x^N is contained in G. For m, we choose a regular point of S which is periodic of period l for g_x^N and which is not a critical point of the projection π_x. Then g_x^{Nl} fixes the fiber S_x of π_x through m pointwise. Since g_x is not periodic and preserves the fibers of π_x, this implies that the differential of g_x^{Nl} at m is parabolic. Let $\widetilde{m}$ be a point of $\pi^{-1}(m)$ and $(\widetilde{g_x^{Nl}})_{\widetilde{m}}$ the lift of g_x^{Nl} fixing that point. The universal cover π provides a local conjugation between g_x^{Nl} and $(\widetilde{g_x^{Nl}})_{\widetilde{m}}$ around m and $\widetilde{m}$, and the developping map provides a local conjugation between $(\widetilde{g_x^{Nl}})_{\widetilde{m}}$ and $\mathbf{Lin}(g_x^{Nl})$. As a consequence, $\mathbf{Lin}(g_x^{Nl})$ is a parabolic element of $\mathsf{PSL}(2,\mathbf{C})$.

Since a power of $\mathbf{Lin}(\widetilde{g_x^N})$ is parabolic, $\mathbf{Lin}(g_x^N)$ itself is parabolic. In particular, the dynamics of $\widetilde{g_x^N}$ near $\widetilde{q}$ is conjugate to a linear upper triangular transformation of $\mathbf{C}^2$ with diagonal entries equal to 1.

As a consequence, the lift $\widetilde{g_x}$ is locally conjugate near $\widetilde{q}$ to a linear parabolic transformation with eigenvalues ± 1. The eigenline of this transformation corresponds to the fiber S_z through q. Since the local fundamental group H coincides with $\pm Id$, this eigenline is mapped to a curve a fixed point by the covering π. In particular, the fiber S_z through q is a curve of fixed points for g_x.

Of course, a similar study holds for g_y and g_z.

6.4.3. *Fixed points and coordinates of the singular point.* The study of fixed points of g_x, g_y and g_z (see lemma 5.2) now shows that the coordinates of the singular point q are equal to ± 2. Let ε_x, ε_y and ε_z be the sign of the coordinates of q, so that

$$q = (2\varepsilon_x, 2\varepsilon_y, 2\varepsilon_z).$$

Recall from section 3.6 that the coefficients A, B, C, and D are uniquely determined by the coordinates of any singular point of S. If the product $\varepsilon_x\varepsilon_y\varepsilon_z$ is positive, then, up to symmetry, $q = (2,2,2)$ and S is the surface

$$x^2 + y^2 + z^2 + xyz = 8x + 8y + 8z - 28;$$

in this case, q is the unique singular point of S, and this singular point is not a node: The second jet of the equation near q is $(x+y+z)^2 = 0$. This contradicts the fact that q has to be a node (see section 6.3.3). From this we deduce that the product $\varepsilon_x\varepsilon_y\varepsilon_z$ is equal to -1, and that S is the Cayley cubic.

7. IRREDUCIBILITY OF PAINLEVÉ VI EQUATION.

The goal of this section is to apply the previous section to the irreducibility of Painlevé VI equation.

7.1. **Phase space and space of initial conditions.** The naive phase space of Painlevé VI equation is parametrized by coordinates $(t, q(t), q'(t)) \in (\mathbb{P}^1 \setminus \{0,1,\infty\}) \times \mathbf{C}^2$; the "good" phase space is a convenient semi-compactification still fibering over the three punctured sphere

$$\mathcal{M}(\theta) \to \mathbb{P}^1 \setminus \{0,1,\infty\}$$

whose fibre $\mathcal{M}_{t_0}(\theta)$, at any point $t_0 \in \mathbb{P}^1 \setminus \{0,1,\infty\}$, is the Hirzebruch surface $\mathbb{F}_2$ blown-up at 8-points minus some divisor, a union of 5 rational curves (see [37]). The analytic type of the fibre, namely the position of the 8 centers and the 5 rational curves, only depends on Painlevé parameters $\theta = (\theta_\alpha, \theta_\beta, \theta_\gamma, \theta_\delta) \in \mathbf{C}^4$ and t_0. This fibre bundle is analytically (but not algebraically!) locally trivial: The local trivialization is given by the Painlevé foliation (see [41]) which is transversal to the fibration. The monodromy of Painlevé equation is given by a representation

$$\pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}, t_0) \to \mathsf{Diff}\,(\mathcal{M}_{t_0}(\theta))$$

into the group of analytic diffeomorphisms of the fibre.

7.2. **The Riemann-Hilbert correspondance and P_{VI}-monodromy.** On the other hand, the space of initial conditions $\mathcal{M}_{t_0}(\theta)$ may be interpreted as the moduli space of rank 2, trace free meromorphic connections over $\mathbb{P}^1$ having simple poles at $(p_\alpha, p_\beta, p_\gamma, p_\delta) = (0, t_0, 1, \infty)$ with prescribed residual eigenvalues $\pm\frac{\theta_\alpha}{2}$, $\pm\frac{\theta_\beta}{2}$, $\pm\frac{\theta_\gamma}{2}$ and $\pm\frac{\theta_\delta}{2}$. The Riemann-Hilbert correspondance therefore provides an analytic diffeomorphism

$$\mathcal{M}_{t_0}(\theta) \to \hat{S}_{(A,B,C,D)}$$

where $\hat{S}_{(A,B,C,D)}$ is the minimal desingularization of $S = S_{(A,B,C,D)}$, the parameters (A,B,C,D) being given by formulae (1.9) and (1.5). From this point of view, the Painlevé VI foliation coincides with the isomonodromic foliation: Leaves correspond to universal isomonodromic deformations of those connections. The monodromy of Painlevé VI equation correspond to a morphism

$$\pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}, t_0) \to \mathsf{Aut}(S_{(A,B,C,D)})$$

and coincides with the Γ_2-action described in section 2.2.2. For instance, g_x (resp. g_y) is the Painlevé VI monodromy when t_0 turns around 0 (resp. 1) in the obvious simplest way. All this is described with much detail in [29].

7.3. **Riccati solutions and singular points.** When $S_{(A,B,C,D)}$ is singular, the exceptional divisor in $\hat{S}_{(A,B,C,D)}$ is a finite union of rational curves in restriction to which Γ_2 acts by Möbius transformations. To each such rational curve corresponds a rational hypersurface $\mathcal{H}$ of the phase space $\mathcal{M}(\theta)$ invariant by the Painlevé VI foliation. On $\mathcal{H}$, the projection $\mathcal{M}(\theta) \to \mathbb{P}^1 \setminus \{0,1,\infty\}$ restricts to a regular rational fibration and the Painlevé equation restricts to a Riccati equation of hypergeometric type: We get a one parameter family of Riccati solutions. See [45, 42, 29] for a classification of singular points of $S_{(A,B,C,D)}$ and their link with Riccati solutions; they occur precisely when either one of the θ-parameter is an integer, or when the sum $\sum \theta_i$ is an integer. Since $S_{(A,B,C,D)}$ is affine, there are obviously no other complete curve in $\mathcal{M}_{t_0}(\theta)$ (see section 6.1).

7.4. **Algebraic solutions and periodic orbits.** A complete list of algebraic solutions of Painlevé VI equation is still unknown. Apart from those solutions arising as special cases of Riccati solutions, that are well known, they correspond to periodic Γ_2-orbits on the smooth part of $S_{(A,B,C,D)}$ (see [31]). Following section 5.2, apart from the three well-known families of 2, 3 and 4-sheeted algebraic solutions, other algebraic solutions are countable and the cosines of the corresponding θ-parameters are real algebraic numbers. In the particular Cayley case $S_C = S_{(0,0,0,4)}$, periodic Γ_2-orbits arise from pairs of roots of unity (u,v) on the two-fold cover $(\mathbf{C}^*)^2$ (see 2.1); there are infinitely many periodic orbits in this case and they are dense in the real bounded component of $S_C \setminus \{\mathsf{Sing}(S_C)\}$. The corresponding algebraic solutions were discovered by Picard in 1889 (before Painlevé discovered the general P_{VI}-equation !); see [34] and below. All algebraic solutions (resp. periodic Γ_2-orbits) have been classified in the particular case $\theta = (0,0,0,*)$ (resp. $(A,B,C,D) = (0,0,0,*)$) in [17, 34]: Apart from Riccati and Picard algebraic solutions, there are 5 extra solutions up to symmetry (see also [6] for finite orbit coming from finite subgroups of $\mathsf{SU}(2)$).

Bounded Γ_2-orbits correspond to what Iwasaki calls "tame solutions" in [30].

7.5. **Nishioka-Umemura irreducibility.** In 1998, Watanabe proved in [45] the irreducibility of Painlevé VI equation in the sense of Nishioka-Umemura for any parameter θ: The generic solution of $P_{VI}(\theta)$ is non classical, and classical solutions are

- Riccati solutions (like above),
- algebraic solutions.

Non classical roughly means "very transcendental" with regards to the XIXth century special functions: The general solution cannot be expressed in an

algebraic way by means of solutions of linear, or first order non linear differential equations. A precise definition can be found in [13].

7.6. **Malgrange irreducibility.** Another notion of irreducibility was introduced by Malgrange in [33]: He defines the Galois groupoid of an algebraic foliation to be the smallest algebraic Lie-pseudo-group that contains the tangent pseudo-group of the foliation (hereafter referred to as the "pseudo-group"); this may be viewed as a kind of Zariski closure for the pseudo-group of the foliation. Larger Galois groupoids correspond to more complicated foliations. From this point of view, it is natural to call irreducible any foliation whose Galois groupoid is as large as possible, *i.e.* coincides with the basic pseudo-group.

For Painlevé equations, a small restriction has to be taken into account: It has been known since Malmquist that Painlevé foliations may be defined as kernels of closed meromorphic 2-forms. The pseudo-group, and the Galois groupoid, both preserve the closed 2-form. The irreducibility conjectured by Malgrange is that the Galois groupoid of Painlevé equations coincide with the algebraic Lie-pseudo-group of those transformations on the phase space preserving ω. This was proved for Painlevé I equation by Casale in [12].

For a second order polynomial differential equation $P(t,y,y',y'')=0$, like Painlevé equations, Casale proved in [13] that Malgrange-irreducibility implies Nishioka-Umemura-irreducibility; the converse is not true as we shall see.

7.7. **Invariant geometric structures.** Restricting to a transversal, e.g. the space of initial conditions $\mathcal{M}_{t_0}(\theta)$ for Painlevé VI equations, the Galois groupoid defines an algebraic geometric structure which is invariant under monodromy transformations; reducibility would imply the existence of an extra geometric structure on $\mathcal{M}_{t_0}(\theta)$, additional to the volume form ω, preserved by all monodromy transformations. In that case, a well known result of Cartan, adapted to our algebraic setting by Casale in [12], asserts that monodromy transformations

- either preserve an algebraic foliation,
- or preserve an algebraic affine structure.

Here, "algebraic" means that the object is defined over an algebraic extension of the field of rational functions, or equivalently, becomes well-defined over the field of rational functions after some finite ramified cover. For instance, "algebraic foliation" means polynomial web. As a corollary of proposition 6.3 and Theorem 6.5, we shall prove the following

Theorem 7.1. *The sixth Painlevé equation is irreducible in the sense of Malgrange, except in one of the following cases:*

- $\theta_\omega \in \frac{1}{2}+\mathbf{Z}$, $\omega=\alpha,\beta,\gamma,\delta$,

- $\theta_\omega \in \mathbf{Z}$, $\omega = \alpha, \beta, \gamma, \delta$, *and* $\sum_\omega \theta_\omega$ *is even.*

All these special parameters are equivalent, modulo Okamoto symmetries, to the case $\theta = (0,0,0,1)$. *The corresponding cubic surface is the Cayley cubic.*

Of course, in the Cayley case, the existence of an invariant affine structure shows that the Painlevé foliation is Malgrange-reducible (see [11]). This will be made more precise in section 7.9.

Before proving the theorem, we need a stronger version of Lemma 6.2

Lemma 7.2. *Let S be an element of the family* Fam. *There is no $\mathcal{A}$-invariant curve of finite type in S.*

By "curve of finite type" we mean a complex analytic curve in S with a finite number of irreducible components C_i, such that the desingularization of each C_i is a Riemann surface of finite type.

Proof. Let $C \subset S$ be a complex analytic curve of finite type. Since S is embedded in $\mathbf{C}^3$, C is not compact. In particular, C is not isomorphic to the projective line and the group of holomorphic diffeomorphisms of C is virtually solvable. Since $\mathcal{A}$ contains a non abelian free subgroup, there exists an element f in $\mathcal{A} \setminus \{\mathrm{Id}\}$ which fixes C pointwise. From this we deduce that C is contained in the algebraic curve of fixed points of f. This shows that the Zariski closure of C is an $\mathcal{A}$-invariant algebraic curve, and we conclude by Lemma 6.2. □

7.8. **Proof of theorem 7.1.** In order to prove that Painlevé VI equation, for a given parameter $\theta \in \mathbf{C}^4$ is irreducible, it suffices, due to [12] and the discussion above, to prove that the space of initial conditions $\mathcal{M}_{t_0}(\theta)$ does not admit any monodromy-invariant web or algebraic affine structure. Via the Riemann-Hilbert correspondance, such a geometric structure will induce a similar Γ_2-invariant structure on the corresponding character variety $S_{(A,B,C,D)}$. But we have to be carefull: The Riemann-Hilbert map is not algebraic but analytic. As a consequence, the geometric structures we have now to deal with on $S_{(A,B,C,D)}$ are not rational anymore, but meromorphic (on a finite ramified cover). Anyway, the proof of proposition 6.3 is still valid in this context and exclude the possibility of Γ_2-invariant analytic web.

7.8.1. *Multivalued affine structures.* We now explain more precisely what is a Γ_2-invariant multivalued meromorphic affine structure in the above sense. First of all, a meromorphic affine structure is an affine structure in the sense of section 6.2 defined on the complement of a proper analytic subset Z, having moderate growth along Z in a sense that we do not need to consider here. This structure is said to be Γ_2-invariant if both Z, and the regular affine structure induced on the complement of Z, are Γ_2-invariant. Now, a multivalued

meromorphic affine structure is a meromorphic structure (with polar locus Z') defined on a finite analytic ramified cover $\pi' : S' \to S$; the ramification locus X is an analytic set. This structure is said to be Γ_2-invariant if both X and $Z = \pi'(Z')$ are invariant and, over the complement of $X \cup Z$, Γ_2 permutes the various regular affine structures induced by the various branches of π'.

Let us prove that the multivalued meromorphic affine structure induced on S by a reduction of Painlevé VI Galois groupoid has actually no pole, and no ramification apart from singular points of S. Indeed, let C be the union of Z and R; then C is analytic in S but comes from an algebraic curve in $\mathcal{M}_{t_0}(\theta)$ (the initial geometric structure is algebraic in $\mathcal{M}_{t_0}(\theta)$), so that the 1-dimensional part of C is a curve of finite type. Lemma 7.2 then show that C is indeed a finite set. In other words, C is contained in $\mathsf{Sing}(S)$, R itself is contained in $\mathsf{Sing}(S)$ and Z is empty.

7.8.2. *Singularities of S.* Since the ramification set R is contained in $\mathsf{Sing}(S)$, the cover π' is an étale cover in the orbifold sense (singularities of S' are also quotient singularities). Changing the cover $\pi' : S' \to S$ if necessary, we may assume that π' is a Galois cover.

If S is simply connected, then of course π' is trivial, the affine structure is univalued, and theorem 6.5 provides a contradiction. We can therefore choose a singularity q of S, and a point q' in the fiber $(\pi')^{-1}(q)$. Since $\pi_1(S;q)$ is finitely generated, the number of subgroups of index $\deg(\pi')$ in $\pi_1(S;q)$ is finite. As a consequence, there is a finite index subgroup G in Γ_2 which lifts to S' and preserves the univalued affine structure defined on S'.

We now follow the proof of theorem 6.5 for G, S' and its affine structure. First, we denote $\pi : \tilde{S} \to S'$ the universal cover of S', we choose a point $\tilde{q}$ in the fiber $\pi^{-1}(q')$, and we lift the action of G to an action on the universal cover $\tilde{S}$ fixing $\tilde{q}$. Then we fix a developping map $\mathbf{dev} : \tilde{S} \to \mathbf{C}^2$ with $\mathbf{dev}(\tilde{q}) = 0$; these choices imply that $\mathbf{Aff}(g)$ is linear for any g in G. Section 6.3.3 shows that the singularities of S and S' are simple nodes.

Now comes the main difference with sections 6.3.4 and 6.4: A priori, the fundamental group of S' is not generated, as a normal subgroup, by the local fundamental groups around the singularities of $\mathsf{Sing}(S')$. It could be the case that S' is smooth, with an infinite fundamental group. So, we need a new argument to prove that g_x (*resp.* g_y and g_z) has a curve of fixed points through the singularity q.

7.8.3. *Parabolic dynamics.* Let $g = g_x^n$ be a non trivial iterate of g_x that is contained in G. The affine transformation $\mathbf{Aff}(g)$ is linear, with determinant 1 ; we want to show that this transformation is parabolic.

Let $\tilde{U}$ be an open subset of $\tilde{S}$ on which both **dev** and the universal cover $\pi' \circ \pi$ are local diffeomorphisms, and let U be the projection of $\tilde{U}$ on S by $\pi' \circ \pi$. We choose $\tilde{U}$ in such a way that U contains points $m = (x,y,z)$ with

x in the interval $[-2,2]$. The fibration of U by fibers of the projection π_x is mapped onto a fibration $\mathcal{F}$ of $\mathbf{dev}(\tilde{U})$ by the local diffeomorphism $\mathbf{dev}\circ(\pi'\circ\pi)^{-1}$. Let us prove, first, that $\mathcal{F}$ is a foliation by parallel lines.

Let m be a point of U which is g-periodic, of period l. Then, the fiber of π_x through m is a curve of fixed point for g^l. If $\tilde{m}$ is a lift of m in $\tilde{S}$, one can find a lift $\gamma\circ\tilde{g}^l$ of g to $\tilde{S}$ (γ in $\pi_1(S,q)=\mathsf{Aut}(\pi)$) that fixes pointwise the fiber through $\tilde{m}$. As a consequence, the fiber of $\mathcal{F}$ through $\mathbf{dev}(\tilde{m})$ coincides locally with the set of fixed points of the affine transformation $\mathbf{Aff}(g^l)\circ\mathbf{Mon}(\gamma)$. As such, the fiber of $\mathcal{F}$ through $\mathbf{dev}(\tilde{m})$ is an affine line.

This argument shows that an infinite number of leaves of $\mathcal{F}$ are affine lines, or more precisely coincide with the intersection of affine lines with $\mathbf{dev}(\tilde{U})$. Since g preserves each fiber of π_x, the foliation $\mathcal{F}$ is leafwise $(\mathbf{Aff}(g^l)\circ\mathbf{Mon}(\gamma))$-invariant. Assume now that L is a line which coincides with a leaf of $\mathcal{F}$ on $\mathbf{dev}(\tilde{U})$. If L is not parallel to the line of fixed points of $\mathbf{Aff}(g^l)\circ\mathbf{Mon}(\gamma)$, then the affine transformation $\mathbf{Aff}(g^l)\circ\mathbf{Mon}(\gamma)$ is a linear map (since it has a fixed point), with determinant ±1, and with two eigenlines, one of them, the line of fixed points, corresponding to the eigenvalue 1. This implies that $\mathbf{Aff}(g^l)\circ\mathbf{Mon}(\gamma)$ has finiter order. Since g is not periodic, we conclude that L is parallel to the line of fixed points of $\mathbf{Aff}(g^l)\circ\mathbf{Mon}(\gamma)$, and that the foliation $\mathcal{F}$ is a foliation by parallel lines.

By holomorphic continuation, we get that the image by $\mathbf{dev}$ of the fibration $\pi_x\circ\pi$ is a foliation of the plane by parallel lines.

Let us now study the dynamics of $\tilde{g}$ near the fixed point $\tilde{q}$. Using the local chart $\mathbf{dev}$, $\tilde{g}$ is conjugate to the linear transformation $\mathbf{Aff}(g)$. Since g preserves each fiber of π_x, $\mathbf{Aff}(g)$ preserves each leaf of the foliation $\mathcal{F}$. Since g is not periodic, $\mathbf{Aff}(g)$ is not periodic either, and $\mathbf{Aff}(g)$ is a linear parabolic transformation. As a consequence, g has a curve of fixed points through q.

7.8.4. *Conclusion.* We can now apply the arguments of section 6.4.3 word by word to conclude that S is the Cayley cubic.

7.9. **Picard parameters of Painlevé VI equation and the Cayley cubic.** Let us now explain in more details why the Cayley case is so special with respect to Painlevé equations. Consider the universal cover

$$\pi_t : \mathbf{C} \to \{y^2 = x(x-1)(x-t)\}\ ;\ z \mapsto (x(t,z),y(t,z))$$

of the Legendre elliptic curve with periods $\mathbf{Z}+\tau\mathbf{Z}$ - this makes sense at least on a neighborhood of $t_0 \in \mathbb{P}^1 \setminus \{0,1,\infty\}$. The functions $\tau=\tau(t)$ and π_t are analytic in t.

The following theorem, obtained by Picard in 1889, shows that the Painlevé equations corresponding to the Cayley cubic have (almost) classical solutions.

Theorem 7.3 (Picard, see [11] for example)**.** *The general solution of the Painlevé sixth differential equation $P_{VI}(0,0,0,1)$ is given by*

$$t \mapsto x(t, c_1 + c_2 \cdot \tau(t)), \quad c_1, c_2 \in \mathbf{C}.$$

Moreover, the solution is algebraic if, and only if c_1 and c_2 are rational numbers.

Note that $c_1, c_2 \in \mathbf{Q}$ exactly means that $\pi_t(c_1 + c_2 \cdot \tau(t))$ is a torsion point of the elliptic curve.

Finally, $P_{VI}(0,0,0,1)$-equation can actually be integrated by means of elliptic functions, but in a way that is non classical with respect to Nishioka-Umemura definition. Coming back to Malgrange's point of view, the corresponding polynomial affine structure on the phase space $\mathcal{M}(0,0,0,1)$ has been computed by Casale in [11], thus proving the reducibility of $P_{VI}(0,0,0,1)$ equation (and all its birational Okamoto symmetrics) in the sense of Malgrange.

8. Appendix A

This section is devoted to the proof of theorem 2.1, according to which the unique surface in the family Fam with four singularities is the Cayley cubic S_C.

Proof. **I.** The point $q = (x,y,z)$ is a singular point of $S_{(A,B,C,D)}$ if, and only if q is contained in $S_{(A,B,C,D)}$ and

$$2x + yz = A, \quad 2y + zx = B, \text{ and } 2z + xy = C.$$

In particular, any pair of two coordinates of q determines the third coordinate.

II. If (u,v) is a couple of complex numbers, $\kappa_{uv}(X)$ will denote the following quadratic polynomial

$$\kappa_{uv}(X) = X^2 - uvX + (u^2 + v^2 - 4).$$

This polynomial has a double root, namely $\alpha = uv/2$, if and only if $\kappa_{uv}(X) = (X - uv/2)^2$, if and only if $(u^2-4)(v^2-4) = 0$.

Let us now fix a set of (a,b,c,d) parameters that determines (A,B,C,D). It is proved in [4] that the coordinates of a singular point q satisfy the following properties:

(i) The x coordinate satisfy one of the following conditions
   - x is a double root of $\kappa_{ab}(X)$,
   - x is a double root of $\kappa_{cd}(X)$,

– x is a common root of κ_{ab} and $\kappa_{cd}(X)$;
(i) y satisfies the same kind of conditions with respect to κ_{ad} and κ_{bc};
(i) z also, with respect to κ_{ac} and κ_{bd}.

This shows that the number of possible x (resp. y, z)-coordinates for q is bounded from above by 2. Together with step **I**, this shows that $S_{(A,B,C,D)}$ has at most four singularities.

When $S_{(A,B,C,D)}$ has four singuarities, there are two possibilities for the x coordinate, and either κ_{ab} and κ_{cd} both have a double root, of κ_{ab} and κ_{cd} coincide and have two simple roots.

III. Let us assume that κ_{ab} and κ_{cd} have a double root. After a symmetry (see §2.3), we may assume that $a = c = 2$. Then, κ_{ac}, κ_{ad} and κ_{bc} all have a double root. In particular, since $S_{(A,B,C,D)}$ has four singularities, the two choices for the z-coordinate of singular points are two double roots, the root of κ_{ac}, and, necessarily, the double root of κ_{bd}. This implies that b or d is equal to ± 2. Applying a symmetry of the parameters, we may assume that $b = 2$, so that (a,b,c,d) is now of type $(2,2,2,d)$.

Under this assumption, the x, y and z coordinates of singular points are contained in $\{2,d\}$ (these are the possible double roots). If $d^2 \neq 4$, the equations of step **I** show that two of the coordinates are equal to 2, when one is equal to d. This gives at most three singularities. As a consequence, $d = 2$ or $d = -2$, and the conclusion follows from the fact that when $d = 2$, there is only one singularity, namely $(2,2,2)$.

IV. The last case that we need to consider is when all polynomials κ_{uv}, $u, v \in \{a,b,c,d\}$, coincide. In that case, up to symmetries, $a = b = c = d$. Then, a similar argument shows that $a = 0$ if S has four singularities (another way to see it is to apply the covering Quad $\circ$ Quad from section 9.4). □

9. APPENDIX B

9.1. **Painlevé VI parameters $(\theta_\alpha, \theta_\beta, \theta_\gamma, \theta_\delta)$ and Okamoto symmetries.** Many kinds of conjugacy classes of representations ρ with

$$\chi(\rho) = (a,b,c,d,x,y,z)$$

give rise to the same (A,B,C,D,x,y,z)-point ; in order to underline this phenomenon, we would like to understand the ramified cover

$$\Pi : \begin{cases} \mathbf{C}^4 & \to & \mathbf{C}^4 \\ (a,b,c,d) & \mapsto & (A,B,C,D) \end{cases}$$

defined by equation (1.5).

9.1.1. *Degree of* Π.

Lemma 9.1. *The degree of the covering map* Π*, that is the number of points* (a,b,c,d) *giving rise to a given generic* (A,B,C,D)*-point, is* 24*.*

Proof. We firstly assume $B \neq \pm C$ so that $a \neq \pm b$. Then, solving $B = bc+ad$ and $C = ac+bd$ in c and d yields

$$c = \frac{aC-bB}{a^2-b^2} \quad \text{and} \quad d = \frac{aB-bC}{a^2-b^2}.$$

Subsituting in $A = ab+cd$ and $D = 4-a^2-b^2-c^2-d^2-abcd$ gives $\{P = Q = 0\}$ with

$$\begin{aligned} P &= -ab(a^2-b^2)^2 + A(a^2-b^2)^2 + (B^2+C^2)ab - BC(a^2+b^2) \\ \text{and } Q &= (a^2+b^2)(a^2-b^2)^2 + (D-4)(a^2-b^2)^2 \\ &\quad + (B^2+C^2)(a^2-a^2b^2+b^2) + BCab(a^2+b^2-4). \end{aligned}$$

These two polynomials have both degree 6 in (a,b) and the corresponding curves must intersect in 36 points. However, one easily check that they intersect along the line at infinity with multiplicity 4 at each of the two points $(a:b) = (1:1)$ and $(1:-1)$; moreover, they also intersect along the forbidden lines $a = \pm b$ at $(a,b) = (0,0)$ with multiplicity 4 as well, provided that $BC \neq 0$. As a consequence, the number of preimages of (A,B,C,D) is $36-4-4-4 = 24$ (counted with multiplicity). □

Remark 9.2. Π is not a Galois cover: The group of deck transformations is the order 8 group $Q = \langle P_1, P_2, \otimes_{(-1,-1,-1,-1)} \rangle$ (see §2.3).

9.1.2. *Okamoto symmetries.* To understand the previous remark, it is convenient to introduce the Painlevé VI parameters, which are related to (a,b,c,d) by the map

$$\begin{array}{ccc} \mathbf{C}^4 & \to & \mathbf{C}^4 \\ (\theta_\alpha,\theta_\beta,\theta_\gamma,\theta_\delta) & \mapsto & (a,b,c,d) \end{array} \quad \text{with} \quad \begin{cases} a &= 2\cos(\pi\theta_\alpha) \\ b &= 2\cos(\pi\theta_\beta) \\ c &= 2\cos(\pi\theta_\gamma) \\ d &= 2\cos(\pi\theta_\delta) \end{cases}$$

The composite map $(\theta_\alpha,\theta_\beta,\theta_\gamma,\theta_\delta) \mapsto (A,B,C,D)$ has been studied in [28]: It is an infinite Galois ramified cover whose deck transformations coincide with the group G of so called Okamoto symmetries. Those symmetries are "birational transformations" of Painlevé VI equation; they have been computed directly on the equation by Okamoto in [38] (see [36] for a modern presentation). Let $\mathsf{Bir}(P_{VI})$ be the group of all birational symmetries of Painlevé sixth equation. The Galois group G is the subgroup of $\mathsf{Bir}(P_{VI})$ generated by the following four kind of affine transformations.

(1) Even translations by integers

$$\oplus_n \ : \ \begin{cases} \theta_\alpha & \mapsto & \theta_\alpha + n_1 \\ \theta_\beta & \mapsto & \theta_\beta + n_2 \\ \theta_\gamma & \mapsto & \theta_\gamma + n_3 \\ \theta_\delta & \mapsto & \theta_\delta + n_4 \end{cases} \quad \text{with} \quad \begin{cases} n = (n_1, n_2, n_3, n_4) \in \mathbf{Z}^4, \\ n_1 + n_2 + n_3 + n_4 \in 2\mathbf{Z}. \end{cases}$$

Those symmetries also act on the space of initial conditions of P_{VI} in a non trivial way, but the corresponding action on (x,y,z) is very simple: We recover the twist symmetries $\otimes_\varepsilon$ of section 2.3 by considering n modulo $2\mathbf{Z}^4$.

(2) An action of Sym_4 permuting $(\theta_\alpha, \theta_\beta, \theta_\gamma, \theta_\delta)$. This corresponds to the action of Sym_4 on (a,b,c,d,x,y,z) permuting (a,b,c,d) in the same way. This group is generated by the four permutations T_1, T_2, P_1 and P_2 (see sections 2.2.1 and 2.2.2).

(3) Twist symmetries on Painlevé parameters

$$\otimes_\varepsilon \ : \ \begin{cases} \theta_\alpha & \mapsto & \varepsilon_1 \theta_\alpha \\ \theta_\beta & \mapsto & \varepsilon_2 \theta_\beta \\ \theta_\gamma & \mapsto & \varepsilon_3 \theta_\gamma \\ \theta_\delta & \mapsto & \varepsilon_4 \theta_\delta \end{cases} \quad \text{with} \quad \varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \in \{\pm 1\}^4.$$

The corresponding action on (a,b,c,d,x,y,z) is trivial.

(4) The special Okamoto symmetry (called s_2 in [36])

$$\mathrm{Ok} \ : \ \begin{cases} \theta_\alpha & \mapsto & \frac{\theta_\alpha - \theta_\beta - \theta_\gamma - \theta_\delta}{2} + 1 \\ \theta_\beta & \mapsto & \frac{-\theta_\alpha + \theta_\beta - \theta_\gamma - \theta_\delta}{2} + 1 \\ \theta_\gamma & \mapsto & \frac{-\theta_\alpha - \theta_\beta + \theta_\gamma - \theta_\delta}{2} + 1 \\ \theta_\delta & \mapsto & \frac{-\theta_\alpha - \theta_\beta - \theta_\gamma + \theta_\delta}{2} + 1 \end{cases}$$

The corresponding action on (A,B,C,D,x,y,z) is trivial (see [28]), but the action on (a,b,c,d) is rather subbtle, as we shall see.

The ramified cover $(\theta_\alpha, \theta_\beta, \theta_\gamma, \theta_\delta) \mapsto (a,b,c,d)$ is also a Galois cover: Its Galois group K is the subgroup of G generated by those translations $\oplus_n$ with $n \in (2\mathbf{Z})^4$ and the twists $\otimes_\varepsilon$. One can check that $[G:K] = 24$ but K is not a normal subgroup of G: It is not Ok-invariant. In fact, K is normal in the subgroup $G' \subset G$ where we omit the generator Ok and $Q = G'/K$ coincides with the order 8 group of symmetries fixing (A,B,C,D). Therefore, G/K may be viewed as the disjoint union of left cosets

$$G/K = Q \cup \mathrm{Ok} \cdot Q \cup \widetilde{\mathrm{Ok}} \cdot Q$$

where $\widetilde{\mathrm{Ok}}$ is the following symmetry (called $s_1s_2s_1$ in [36])

$$\widetilde{\mathrm{Ok}}\ :\ \begin{cases} \theta_\alpha & \mapsto & \frac{\theta_\alpha-\theta_\beta-\theta_\gamma+\theta_\delta}{2} \\ \theta_\beta & \mapsto & \frac{-\theta_\alpha+\theta_\beta-\theta_\gamma+\theta_\delta}{2} \\ \theta_\gamma & \mapsto & \frac{-\theta_\alpha-\theta_\beta+\theta_\gamma+\theta_\delta}{2} \\ \theta_\delta & \mapsto & \frac{\theta_\alpha+\theta_\beta+\theta_\gamma+\theta_\delta}{2} \end{cases}$$

9.1.3. *From (A,B,C,D) to (a,b,c,d).* Now, given a (a,b,c,d)-point, we would like to describe explicitly all other parameters (a',b',c',d') in the same Π-fibre, i.e. giving rise to the same parameter (A,B,C,D). We already know that the Q-orbit

$$\left\{\begin{matrix}(a,b,c,d) & (-a,-b,-c,-d) & (d,c,b,a) & (-d,-c,-b,-a)\\ (b,a,d,c) & (-b,-a,-d,-c) & (c,d,a,b) & (-c,-d,-a,-b)\end{matrix}\right\},$$

, which generically is of length 8, is contained in the fibre. In order to describe the remaining part of the fibre, let us choose $(a_\varepsilon,b_\varepsilon,c_\varepsilon,d_\varepsilon)\in\mathbf{C}^4$, $\varepsilon=0,1$, such that

$$\begin{cases} a_0 &=& \frac{\sqrt{2+a}}{2} \\ b_0 &=& \frac{\sqrt{2+b}}{2} \\ c_0 &=& \frac{\sqrt{2+c}}{2} \\ d_0 &=& \frac{\sqrt{2+d}}{2} \end{cases} \quad\text{and}\quad \begin{cases} a_1 &=& \frac{\sqrt{2-a}}{2} \\ b_1 &=& \frac{\sqrt{2-b}}{2} \\ c_1 &=& \frac{\sqrt{2-c}}{2} \\ d_1 &=& \frac{\sqrt{2-d}}{2} \end{cases}$$

If θ_α is such that $(a_0,a_1)=(\cos(\pi\frac{\theta_\alpha}{2}),\sin(\pi\frac{\theta_\alpha}{2}))$, then $a=2\cos(\pi\theta_\alpha)$; therefore, the choice of (a_0,a_1) is equivalent to the choice of a P_{VI}-parameter θ_α modulo $2\mathbf{Z}$, i.e. of $\frac{\theta_\alpha}{2}$ modulo $\mathbf{Z}$. Then, looking at the action of the special Okamoto symmetry Ok on Painlevé parameters $(\theta_\alpha,\theta_\beta,\theta_\gamma,\theta_\delta)$, we derive the following new point (a',b',c',d') in the Π-fibre

$$\begin{cases} a' &=& -2\sum(-1)^{\frac{\sum\varepsilon_i}{2}+\varepsilon_1}a_{\varepsilon_1}b_{\varepsilon_2}c_{\varepsilon_3}d_{\varepsilon_4} \\ b' &=& -2\sum(-1)^{\frac{\sum\varepsilon_i}{2}+\varepsilon_2}a_{\varepsilon_1}b_{\varepsilon_2}c_{\varepsilon_3}d_{\varepsilon_4} \\ c' &=& -2\sum(-1)^{\frac{\sum\varepsilon_i}{2}+\varepsilon_3}a_{\varepsilon_1}b_{\varepsilon_2}c_{\varepsilon_3}d_{\varepsilon_4} \\ d' &=& -2\sum(-1)^{\frac{\sum\varepsilon_i}{2}+\varepsilon_4}a_{\varepsilon_1}b_{\varepsilon_2}c_{\varepsilon_3}d_{\varepsilon_4} \end{cases}$$

where the sum is taken over all $\varepsilon=(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)\in(\{0,1\})^4$ for which $\sum_{i=1}^4\varepsilon_i$ is even. One can check that the different choices for (a_0,b_0,c_0,d_0) and (a_1,b_1,c_1,d_1) lead to 16 distinct possible (a',b',c',d'), namely 2 distinct Q-orbits, which together with the Q-orbit of (a,b,c,d) above provide the whole Π-fibre.

Example 9.3. When $(a,b,c,d) = (0,0,0,d)$, we have $(A,B,C,D) = (0,0,0,D)$ with $D = 4-d^2$. The Π-fibre is given by the Q-orbits of the 3 points

$$(0,0,0,d) \quad \text{and} \quad (\tilde{d},\tilde{d},\tilde{d},-\tilde{d}) \quad \text{where} \quad \tilde{d} = \sqrt{2\pm\sqrt{4-d^2}}$$

(only the sign of the square root inside is relevant up to Q). The fibre has length 24 except in the Cayley case $d = 0$ where it has length 9, consisting of the two Q-orbits of

$$(0,0,0,0) \quad \text{and} \quad (2,2,2,-2)$$

(note that $(0,0,0,0)$ is Q-invariant) and in the Markov case $d = 2$ where it has length 16, consisting of the two Q-orbits of

$$(0,0,0,2) \quad \text{and} \quad (\sqrt{2},\sqrt{2},\sqrt{2},-\sqrt{2}).$$

9.2. **Reducible representations versus singularities.**

Theorem 9.4 ([4, 29])**.** *The surface $S_{(A,B,C,D)}$ is singular if, and only if, we are in one of the following cases*

- $\Delta(a,b,c,d) = 0$ *where*

$$\Delta = (2(a^2+b^2+c^2+d^2) - abcd - 16)^2 - (4-a^2)(4-b^2)(4-c^2)(4-d^2),$$

- *at least one of the parameters a, b, c or d equals ± 2.*

More precisely, a representation ρ is sent to a singular point if, and only if, we are in one of the following cases :

- *the representation ρ is reducible and then $\Delta = 0$,*
- *one of the generators $\rho(\alpha)$, $\rho(\beta)$, $\rho(\gamma)$ or $\rho(\delta)$ equals $\pm I$ (the corresponding trace parameter is then equal to ± 2).*

In fact, it is proved in [4] that the set Z of parameters (A,B,C,D) for which $S_{(A,B,C,D)}$ is singular is defined by $\delta = 0$ where δ is the discriminant of the polynomial

$$P_z = z^4 - Cz^3 - (D+4)z^2 + (4C-AB)z + 4D + A^2 + B^2$$

defined in section 5.1: P_z has a multiple root at each singular point. Now, consider the ramified cover

$$\Pi : \mathbf{C}^4 \to \mathbf{C}^4; (a,b,c,d) \mapsto (A,B,C,D)$$

defined by (1.5). One can check by direct computation that

$$\delta \circ \Pi = \frac{1}{16}(a^2-4)(b^2-4)(c^2-4)(d^2-4)\Delta^2.$$

One also easily verifies that the locus of reducible representations is also the ramification locus of Π:

$$\mathrm{Jac}(\Pi) = -\frac{1}{2}\Delta.$$

It is a well known fact (see [29]) that Okamoto symmetries permute the two kinds of degenerate representations given by Theorem 9.4. For instance, a singular point is defined by the following equations:

$$A = 2x + yz, \quad B = 2y + xz, \quad C = 2z + xy$$
$$\text{and} \quad x^2 + y^2 + z^2 + xyz = Ax + By + Cz + D.$$

Now, a compatible choice of parameters (a,b,c,d) is provided by

$$(a,b,c,d) = (y,z,x,2)$$

and one easily check that the corresponding representations satisfy $\rho(\delta) = I$.

9.3. $\mathsf{SU}(2)$**-representations versus bounded components.** When a, b, c, and d are real numbers, A, B, C, and D are real as well. In that case, the real part $S_{(A,B,C,D)}(\mathbf{R})$ stands for $\mathsf{SU}(2)$ and $\mathsf{SL}(2,\mathbf{R})$-representations; precisely, each connected component of the smooth part of $S_{(A,B,C,D)}(\mathbf{R})$ is either purely $\mathsf{SU}(2)$, or purely $\mathsf{SL}(2,\mathbf{R})$, depending on the choice of (a,b,c,d) fitting to (A,B,C,D).

Moving into the parameter space $\{(a,b,c,d)\}$, when we pass from $\mathsf{SU}(2)$ to $\mathsf{SL}(2,\mathbf{R})$-representations, we must go through a representation of the group $\mathsf{SU}(2) \cap \mathsf{SL}(2,\mathbf{R}) = \mathsf{SO}(2,\mathbf{R})$. Since representations into $\mathsf{SO}(2,\mathbf{R})$ are reducible, they correspond to singular points of the cubic surface (see §9.2). In other words, any bifurcation between $\mathsf{SU}(2)$ and $\mathsf{SL}(2,\mathbf{R})$-representations creates a real singular point of $S_{(A,B,C,D)}$.

Since $\mathsf{SU}(2)$-representations are contained in the cube $[-2,2]^3$, they always form a bounded component of the smooth part of $S_{(A,B,C,D)}(\mathbf{R})$: Unbounded components always correspond to $\mathsf{SL}(2,\mathbf{R})$-representations, whatever the choice of parameters (a,b,c,d) is.

The topology of $S_{(A,B,C,D)}(\mathbf{R})$ is studied in [4] when (a,b,c,d) are real numbers. There are at most four singular points, and the smooth part has at most one bounded and at most four unbounded components. On the other hand, if A, B, C, and D are real numbers, then a, b, c, and d are not necessarily real.

Example 9.5. If a, b, c, and d are purely imaginary numbers, then A, B, C, and D are real numbers. In this specific example, there are representations $\rho : \pi_1(\mathbb{S}^2_4) \to \mathsf{SL}(2,\mathbf{C})$ with trace parameters

$$(a,b,c,d,x,y,z) \in (i\mathbf{R})^4 \times (\mathbf{R})^3,$$

the image of which are Zariski dense in the (real) Lie group $\mathsf{SL}(2,\mathbf{C})$. Such a representation correspond to a point (x,y,z) on $\S_{(A,B,C,D)(\mathbf{R})}$ which is not realized by a representation into $\mathsf{SL}(2,\mathbf{R})$.

The goal of this section is to prove the following theorem, which partly extends the above mentionned results of Benedetto and Goldman [4].

Theorem 9.6. *Let A, B, C, and D be real numbers, for which the smooth part of $S_{(A,B,C,D)}(\mathbf{R})$ has a bounded component. Then for any choice of parameters (a,b,c,d) fitting to (A,B,C,D), the numbers a, b, c, and d are real, contained in $(-2,2)$ and the bounded component stands for $\mathsf{SU}(2)$ or $\mathsf{SL}(2,\mathbf{R})$-representations. Moreover, for any such parameter (A,B,C,D), we can choose between $\mathsf{SU}(2)$ and $\mathsf{SL}(2,\mathbf{R})$ by conveniently choosing (a,b,c,d): The two cases both occur.*

In particular, bounded components of real surfaces $S_{(A,B,C,D)}(\mathbf{R})$ always arise from $\mathsf{SU}(2)$-representations[2].

Denote by $Z \subset \mathbf{R}^4$ the subset of those parameters (A,B,C,D) for which the corresponding surface $S_{(A,B,C,D)}(\mathbf{R})$ is singular (see section 9.2). Over each connected component of $\mathbf{R}^4 \setminus Z$, the surface $S_{(A,B,C,D)}(\mathbf{R})$ is smooth and has constant topological type. Let $\mathcal{B}$ be the union of connected components of $\mathbf{R}^4 \setminus Z$ over which the smooth surface has a bounded component.

The ramified cover $\Pi : \mathbf{C}^4 \to \mathbf{C}^4; (a,b,c,d) \mapsto (A,B,C,D)$ has degree 24; Okamoto correspondences, defined in section 9.1, "act" transitively on fibers (recall that Π is not Galois). Because of their real nature, these correspondences permute real parameters (a,b,c,d): Therefore, Π restricts as a degree 24 ramified cover $\Pi|_{\mathbf{R}^4} : \mathbf{R}^4 \to \Pi(\mathbf{R}^4)$. Following [4], we have

$$\Pi^{-1}(\mathcal{B}) \cap \mathbb{R}^4 = (-2,2)^4 \setminus \{\Delta = 0\}.$$

Using again that $\mathsf{SU}(2) \cap \mathsf{SL}(2,\mathbf{R}) = \mathrm{SO}(2)$ is abelian, and therefore corresponds to reducible representations, we promptly deduce that, along each connected component of $(-2,2)^4 \setminus \{\Delta = 0\}$, the bounded component of the corresponding surface $S_{(A,B,C,D)}(\mathbf{R})$ constantly stands either for $\mathsf{SU}(2)$-representations, or for $\mathsf{SL}(2,\mathbf{R})$-representations. We shall denote by $\mathcal{B}^{\mathsf{SU}(2)}$ and $\mathcal{B}^{\mathsf{SL}(2,\mathbf{R})}$ the corresponding components of $\mathcal{B}$. Theorem 9.6 may now be rephrased as the following equalities:

$$\mathcal{B} = \mathcal{B}^{\mathsf{SU}(2)} = \mathcal{B}^{\mathsf{SL}(2,\mathbf{R})}.$$

To prove these equalities, we first note that t $\mathcal{B}^{\mathsf{SU}(2)} \cup \mathcal{B}^{\mathsf{SL}(2,\mathbf{R})} \subset \Pi([-2,2]^4)$ is obviously bounded by $-8 \leq A,B,C \leq 8$ and $-20 \leq D \leq 28$ (this bound is not sharp !).

Lemma 9.7. *The set $\mathcal{B}$ is bounded, contained into $-8 \leq A,B,C \leq 8$ and $-56 \leq D \leq 68$.*

Proof. The orbit of any point p belonging to a bounded component of $S_{(A,B,C,D)}(\mathbf{R})$ is bounded. Applying the tools involved in section 5, we deduce that the

[2]This strengthens the results of [40] where the bounded component was assumed to arise from $\mathsf{SU}(2)$-representations.

bounded component is contained into $[-2,2]^3$. Therefore, for any $p=(x,y,z)$ and $s_x(p)=(x',y,z)$ belonging to the bounded component, we get $A=x+x'+yz$ and then $-8\leq A\leq 8$. Using s_y and s_z, we get the same bounds for B and C. Since p is in the surface, we also get $D=x^2+y^2+z^2+xyz-Ax-By-Cz$. □

The order 24 group of Benedetto-Goldman symmetries act on the parameters (A,B,C,D) by freely permutting the triple (A,B,C), and freely changing sign for two of them. This group acts on the set of connected components of $\mathbf{R}^4\setminus Z$, $\mathcal{B}$, $\mathcal{B}^{\mathsf{SU}(2)}$ and $\mathcal{B}^{\mathsf{SL}(2,\mathbf{R})}$. The crucial Lemma is

Lemma 9.8. *Up to Benedetto-Goldman symmetries, $\mathbf{R}^4\setminus Z$ has only one bounded component.*

Proof. Up to Benedetto-Goldman symmetries, one can always assume $0\leq A\leq B\leq C$. This fact is easily checked by looking at the action of symmetries on the projective coordinates $[A:B:C]=[X:Y:1]$: the triangle $T=\{0\leq X\leq Y\leq 1\}$ happens to be a fundamental domain for this group action. We shall show that $\mathbf{R}^4\setminus Z$ has at most one bounded component over the cone

$$\mathcal{C}=\{(A,B,C)\,;\,0\leq A\leq B\leq C\}$$

with respect to the projection $(A,B,C,D)\mapsto(A,B,C)$.

The discriminant of δ with respect to D reads

$$\mathrm{disc}(\delta)=-65536\,(B-C)^2(B+C)^2(A-C)^2(A+C)^2(A-B)^2(A+B)^2\,\delta_1^3$$

where δ_1 is the following polynomial (with $(X,Y)=(\frac{A}{C},\frac{B}{C})$)

$$\begin{aligned}\delta_1=&-C^9X^3Y^3+\left(27Y^4+27X^4Y^4-6X^2Y^4-6X^4Y^2+27X^4-6X^2Y^2\right)C^8\\&+\left(-768X^5Y+192Y^3X-768XY+192X^3Y-768Y^5X+192X^3Y^3\right)C^7\\&+\Big(4096Y^6-1536Y^2+4096+23808X^2Y^2-1536X^4-1536X^2Y^4\\&\qquad-1536X^4Y^2+4096X^6-1536X^2-1536Y^4\Big)C^6\\&+\left(-86016X^3Y-86016Y^3X-86016XY\right)C^5+\left(712704X^2Y^2\right.\\&\left.-196608Y^4-196608-196608X^4+712704X^2+712704Y^2\right)C^4\\&-5505024C^3XY+\left(3145728X^2+3145728+3145728Y^2\right)C^2-16777216\end{aligned}$$

First, we want to show that $\mathcal{C}\setminus\{\mathrm{disc}(\delta)=0\}$ has 5 connected components, only two of which are bounded. The polynomial δ_1 has degree 9 in C in restriction to any line $L_{X,Y}=\{A=XC,B=YC\}\subset\mathcal{C}$ with $0<X<Y<1$; we claim that it has constantly 3 simple real roots (and 6 non real ones)

$$c_1(X,Y)<0<c_2(X,Y)<c_3(X,Y).$$

In order to check this, let us verify that the discriminant of δ_1 with respect to C does not vanish in the interior of the triangle T. After computations, we find

$$\mathrm{disc}(\delta_1) = k(X^2-Y^2)^8(X^2-1)^8(Y^2-1)^8(Y\delta_2)^2$$

where k is a huge constant and δ_2 is given, setting $X = tY$, by

$$\begin{aligned}\delta_2 = &\left(22272t^8+40337t^6+16384t^{10}+16384t^2+22272t^4\right)Y^{10}\\ &+\left(-59233t^4+16384t^{10}-59233t^6+40448t^8+16384+40448t^2\right)Y^8\\ &+\left(22272+22272t^8-59233t^2-59233t^6-118893t^4\right)Y^6\\ &+\left(40337t^6+40337-59233t^2-59233t^4\right)Y^4\\ &+\left(22272+22272t^4+40448t^2\right)Y^2+16384+16384t^2.\end{aligned}$$

This later polynomial has non vanishing discriminant with respect to Y for $0<t<1$ and has a non real root, for instance, when $t=1/2$: Thus $\mathrm{disc}(\delta_1)$ does not vanish in the interior of the triangle T. Therefore, the polynomial δ_1 has always the same number of real roots when (X,Y) lie inside the triangle T and one can easily check that 0 is never a root, and by specializing (X,Y), that there are indeed 3 roots, one of them being negative. The claim is proved.

The cone $\mathcal{C}$ is cutted off by $\mathrm{disc}(\delta)=0$ into 5 components, namely

$$\mathcal{C}_1=\{C<c_1(X,Y)\},\quad \mathcal{C}_2=\{c_1(X,Y)<C<0\},\quad \mathcal{C}_3=\{0<C<c_2(X,Y)\},$$
$$\mathcal{C}_4=\{c_2(X,Y)<C<c_3(X,Y)\}\quad\text{and}\quad \mathcal{C}_5=\{c_3(X,Y)<C\}.$$

But δ_1 has degree 8 when $X=0$ and one of the roots $c_i(X,Y)$ tends to infinity when $X\to 0$. One can check that $c_3\to\infty$ and only $\mathcal{C}_2$ and $\mathcal{C}_3$ are bounded.

We now study the possible bounded components of $\mathbf{R}^4\setminus Z$ over the cona $\mathcal{C}$; they necessarily project onto $\mathcal{C}_2$, $\mathcal{C}_3$ or the union (together with $(A,B,C)=0$). The polynomial δ defining Z has degree 5 in D. After several numerical specializations, we obtain the following picture:

- the polynomial δ has 5 real roots $d_1<d_2<d_3<d_4<d_5$ over $\mathcal{C}_2$ and $\mathcal{C}_3$, $d_i=d_i(A,B,C)$ for $i=1,\dots,5$,
- over $C=c_1$ or $C=c_2$, $0<A<B<C$, the 5 roots extend continuously, satisfying $d_1=d_2<d_3<d_4<d_5$
- over $(A,B,C)=0$, the 5 roots extend continuously as $0=d_1<d_2=d_3=d_4=4$.

Among the 6 connected components of $\mathbf{R}^4\setminus Z$ over $\mathcal{C}_2$ (resp. $\mathcal{C}_3$), only that one defined by $\{d_1(A,B,C)<D<d_2(A,B,C)\}$ does not "extend" over the unbounded component $\mathcal{C}_1$ (resp. $\mathcal{C}_4$). The unique bounded component of $\mathbf{R}^4\setminus Z$ over the cona $\mathcal{C}$ is therefore defined over $\mathcal{C}_2\cup\{A=B=C=0\}\cup\mathcal{C}_3$

by $\{d_1(A,B,C) < D < d_2(A,B,C)\}$. The corresponding connected component of $\mathbf{R}^4 \setminus Z$ must be bounded as well, since there is at least one bounded component, given by $\mathcal{B}^{\mathsf{SU}(2)}$, or $\mathcal{B}^{\mathsf{SL}(2,\mathbf{R})}$. □

We thus conclude that $\mathcal{B} = \mathcal{B}^{\mathsf{SU}(2)} = \mathcal{B}^{\mathsf{SL}(2,\mathbf{R})}$ and Theorem 9.6 is proved in the case the real surface $S_{(A,B,C,D)}(\mathbf{R})$ is smooth. The general case follows from the following lemma, the proof of which is left to the reader.

Lemma 9.9. *Let (A,B,C,D) be real parameters such that the smooth part of the surface $S_{(A,B,C,D)}(\mathbf{R})$ has a bounded component. Then, there exist an arbitrary small real perturbation of (A,B,C,D) such that the corresponding surface is smooth and has a bounded component.*

We would like now to show that there is actually only one bounded component in $\mathbf{R}^4 \setminus Z$ (up to nothing).

Inside $[-2,2]^4$, the equation Δ splits into the following two equations

$$2(a^2+b^2+c^2+d^2) - abcd - 16 = \pm\sqrt{(4-a^2)(4-b^2)(4-c^2)(4-d^2)}.$$

Those two equations cut-off the parameter space $[-2,2]^4$ into many connected components and we have[3]

Theorem 9.10 (Benedetto-Goldman [4])**.** *When a, b, c and d are real and $S_{(A,B,C,D)}(\mathbf{R})$ is smooth, then $S_{(A,B,C,D)}(\mathbf{R})$ has a bounded component if, and only if, a, b, c and d both lie in $(-2,2)$. In this case, the bounded component corresponds to $\mathsf{SL}(2,\mathbf{R})$-representations if, and only if,*

$$2(a^2+b^2+c^2+d^2) - abcd - 16 > \sqrt{(4-a^2)(4-b^2)(4-c^2)(4-d^2)}.$$

When we cross the boundary

$$2(a^2+b^2+c^2+d^2) - abcd - 16 = \sqrt{(4-a^2)(4-b^2)(4-c^2)(4-d^2)}$$

inside $(-2,2)^4$, we pass from $\mathsf{SL}(2,\mathbf{R})$ to $\mathsf{SU}(2)$-representations: At the boundary, the bounded component must degenerate down to a singular point.

We now prove the

Proposition 9.11. *The set $(-2,2)^4 \setminus \{\Delta = 0\}$ has 24 connected components, 8 of them corresponding to $\mathsf{SL}(2,\mathbf{R})$-representations. Okamoto correspondence permute transitively those components.*

Recall that the group of cover transformations Q has order 8 and does not change the nature of the representation: The image $\rho(\pi_1(\mathbb{S}^2_4))$ remains

[3] In [4], the connected components of $[-2,2]^4$ standing for $\mathsf{SL}(2,\mathbf{R})$-representations are equivalently defined by $\Delta > 0$ and $2(a^2+b^2+c^2+d^2) - abcd - 16 > 0$.

unchanged in $\mathsf{PGL}(2,\mathbb{C})$. Therefore, up to this tame action, Okamoto correspondence provides, to any smooth point (A,B,C,D,x,y,z) of the character variety, exactly 3 essentially distinct representations, two of them in $\mathsf{SU}(2)$, and the third one in $\mathsf{SL}(2,\mathbf{R})$. It may happens (see [39]) that one of the two $\mathsf{SU}(2)$-representations is dihedral, while the other one is dense!

Proof. We shall prove that the $\mathsf{SL}(2,\mathbf{R})$-locus, i.e. the real semi-algebraic set X of $[-2,2]^4$ defined by

$$2(a^2+b^2+c^2+d^2)-abcd-16>\sqrt{(4-a^2)(4-b^2)(4-c^2)(4-d^2)},$$

consist in connected neighborhoods of those 8 vertices corresponding to the Cayley surface

$$(a,b,c,d)=(\varepsilon_1\cdot 2,\varepsilon_2\cdot 2,\varepsilon_3\cdot 2,\varepsilon_4\cdot 2),\quad \varepsilon_i=\pm 1,\ \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4=1.$$

Benedetto-Goldman symmetries act transitively on those components. On the other hand, the Cayley surface also arise for $(a,b,c,d)=(0,0,0,0)$, which is in the $\mathsf{SU}(2)$-locus: the Okamoto correspondence therefore sends any of the 8 components above into the $\mathsf{SU}(2)$-locus, thus proving the theorem.

By abuse of notation, still denote by Z the discriminant locus defined by $\{\Delta=0\}\subset(-2,2)^4$. The restriction $Z_{a,b}$ of Z to the slice

$$\Pi_{a,b}=\{(a,b,c,d)\ ;\ c,d\in(-2,2)\},\quad (a,b)\in(-2,2)^2,$$

is the union of two ellipses, namely those defined by

$$c^2+d^2-\delta cd+\delta^2-4=0,\quad \text{where } \delta=\frac{1}{2}\left(ab\pm\sqrt{(4-a^2)(4-b^2)}\right).$$

Those two ellipses are circumscribed into the square $\Pi_{a,b}$ (see figure 9.3) and, for generic parameters a and b, cut the square into 13 connected components. One easily verify that $\mathsf{SL}(2,\mathbf{R})$-components (namely those connected components of $X_{a,b}=X\cap\Pi_{a,b}$ defined by the inequality of the previous theorem) are those 4 neighborhoods of the vertices of the square.

This picture degenerates precisely when $a=\pm 2$, $b=\pm 2$ or $a=\pm b$. We do not need to consider the first two cases, since they are on the boudary of $(-2,2)^4$. Anyway, in these cases, the two ellipses coincide; they moreover degenerate to a double line when $a=\pm b$.

In the last case $a=\pm b$, the picture bifurcates. When $a=b$, one of the ellipses degenerates to the double line $c=d$, and the two components of $X_{a,b}$ near the vertices $(2,2)$ and $(-2,-2)$ collapse. When $a=-b$, the components of $X_{a,b}$ near the two other vertices collapse as well. This means that each component of $X_{a,b}$ stands for exactly two components of X: We finally obtain 8 connected components for the $\mathsf{SL}(2,\mathbf{R})$-locus $X\subset(-2,2)^4$. One easily verify that there are sixteen $\mathsf{SU}(2)$-components in $(-2,2)^4\setminus Z$. □

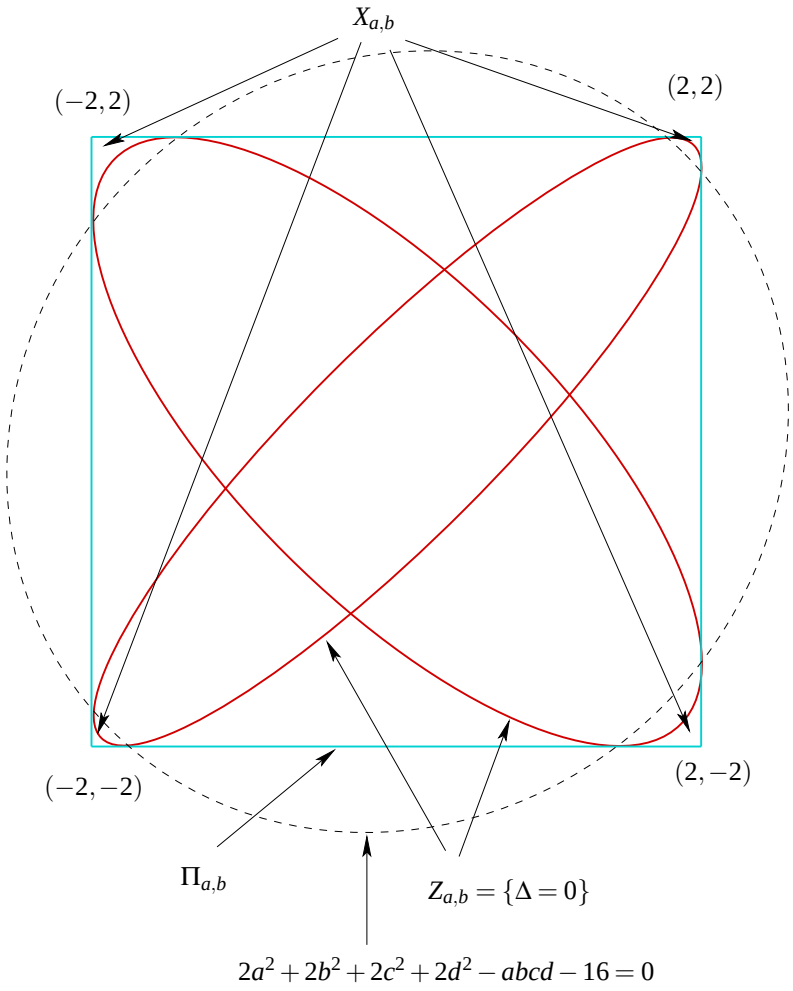

FIGURE 5. Z restricted to the slice $\Pi_{a,b}$.

9.4. **Ramified covers.** Here, we would like to describe other kinds of correspondences between surfaces $S_{(A,B,C,D)}$, that arise by lifting representations along a ramified cover of $\mathbb{S}^2_4$. Let $\rho \in \mathrm{Rep}(\mathbb{S}^2_4)$ be a representation with $a = d = 0$, so that $\rho(\alpha)^2 = \rho(\beta)^2 = -I$, and consider the two-fold cover $\pi : \mathbb{S}^2 \to \mathbb{S}^2$ ramifying over p_α and p_δ.

The four punctures lift-up as six punctures labelled in the obvious way

$$\pi \ : \ \begin{cases} \tilde{p}_\alpha & \mapsto & p_\alpha \\ \tilde{p}_\beta, \tilde{p}'_\beta & \mapsto & p_\beta \\ \tilde{p}_\gamma, \tilde{p}'_\gamma & \mapsto & p_\gamma \\ \tilde{p}_\delta & \mapsto & p_\delta \end{cases}$$

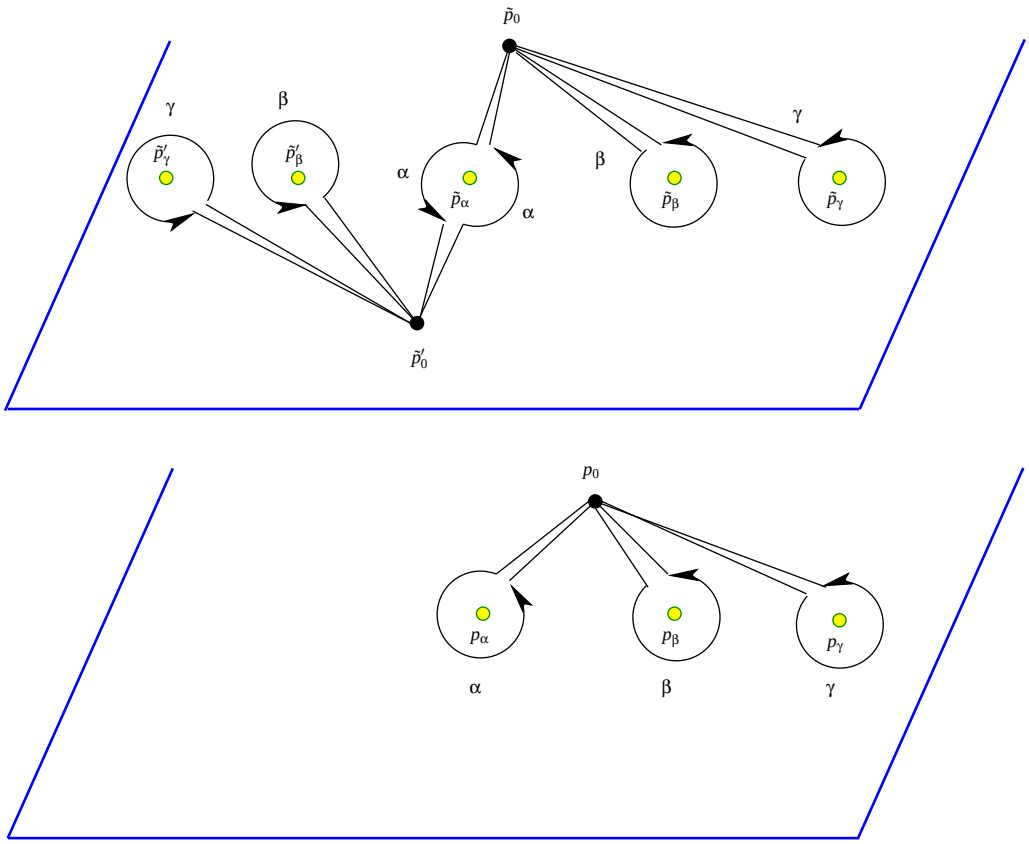

FIGURE 6. The two-fold cover. (the point p_δ is at infinity)

After twisting the lifted representation $\rho \circ \pi$ by $-I$ at $\tilde{p}_\alpha$ and $\tilde{p}_\delta$, we get a new representation $\tilde{\rho} \in \mathrm{Rep}(\mathbb{S}^2_4)$; the new punctures are respectively $\tilde{p}'_\gamma$, $\tilde{p}_\gamma$, $\tilde{p}_\beta$ and $\tilde{p}'_\beta$ and the new generators for the fundamental group are given by $\alpha\beta\gamma\beta^{-1}\alpha^{-1}$, $\alpha\beta\alpha^{-1}$, β, and γ. After computation, we get a map

$$\begin{cases} 0 & \mapsto & c \\ b & \mapsto & b \\ c & \mapsto & b \\ 0 & \mapsto & c \end{cases} \quad \text{and} \quad \begin{cases} x & \mapsto & y \\ y & \mapsto & 2-x^2 \\ z & \mapsto & x^2y+xz-y+bc \end{cases}$$

defining a two-fold cover

$$\mathrm{Quad} \ : \ S_{(0,B,0,D)} \to S_{(2B,4-D,2B,2D-B^2-4)}$$

where $B = bc$ and $D = 4-b^2-c^2$. This map corresponds to the so-called "quadratic transformation" of Painlevé VI equation.

When moreover $c = 0$, we can iterate twice this transformation and we deduce a 4-fold cover

$$\mathrm{Quad} \circ \mathrm{Quad} \ : \ S_{(0,0,0,D)} \to S_{(8-2D,8-2D,8-2D,-28+12D-D^2)}$$

$$\begin{cases} 0 & \mapsto & b \\ b & \mapsto & b \\ 0 & \mapsto & b \\ 0 & \mapsto & b \end{cases} \quad \text{and} \quad \begin{cases} x & \mapsto & 2-x^2 \\ y & \mapsto & 2-y^2 \\ z & \mapsto & 2-z^2 \end{cases}$$

For instance, when $D = 0$, we get a covering $S_{(0,0,0,0)} \to S_{(8,8,8,-28)}$. Another particular case arise when $D = 4$ where Quad defines an endomorphism of the Cayley cubic surface $S_{(0,0,0,4)} \to S_{(0,0,0,4)}$, namely that one induced by the regular cover

$$\mathbf{C}^* \times \mathbf{C}^* \to \mathbf{C}^* \times \mathbf{C}^* \ : \ (u,v) \mapsto (v,u^2).$$

Example 9.12. By the way, we note that, up to the action of Q, the following traces data are related :

$$\begin{array}{ccccc} (0,0,0,d) & \leftrightarrow & (d'',d'',d'',-d'') & \to & S_{(0,0,0,4-d^2)} \\ \downarrow & & & & \downarrow \\ (0,0,d,d) & \leftrightarrow & (2,2,d',-d') & \to & S_{(d^2,0,0,4-2d^2)} \\ \downarrow & & & & \downarrow \\ (d,d,d,d) & \leftrightarrow & (2,2,2,d^2-2) & \to & S_{(2d^2,2d^2,2d^2,4-4d^2-d^4)} \end{array}$$

where $d' = \sqrt{4-d^2}$ and $d'' = \sqrt{2+d'}$. In the previous diagram, horizontal correspondences arise from Okamoto symmetries, while vertical arrows, from quadratic transformation Q.

More generally, we have related

$$\begin{array}{ccccc} (0,0,c,d) & \leftrightarrow & (c'',c'',d'',-d'') & \to & S_{(cd,0,0,4-c^2-d^2)} \\ \downarrow & & & & \downarrow \\ (c,c,d,d) & \leftrightarrow & (2,2,c',d') & \to & S_{(c^2+d^2,2cd,2cd,4-2c^2-2d^2-c^2d^2)} \end{array}$$

where $c' = \frac{cd+\sqrt{(c^2-4)(d^2-4)}}{2}$, $d' = \frac{cd-\sqrt{(c^2-4)(d^2-4)}}{2}$, $c'' = \sqrt{2+c'}$ and $d'' = \sqrt{2-c'}$.

Remark 9.13. One can check by direct computations that the quadratic tranformation Quad is equivariant, up to finite index, with respect to the Γ_2^*-actions. Precisely, we have

$$\begin{cases} \mathrm{Quad} \circ B_1^2 & = & B_2^{-1} \circ \mathrm{Quad}, \\ \mathrm{Quad} \circ B_2 & = & B_1^{-2} \circ \mathrm{Quad}, \\ \mathrm{Quad} \circ s_z & = & s_z \circ \mathrm{Quad}. \end{cases}$$

The group generated by B_1^2, B_2 and s_z, acting on both sides, contains Γ_2^* as an index 2 subgroup (recall that $B_1^2 = g_x = s_z \circ s_y$ and $B_2^2 = g_y = s_x \circ s_z$). Therefore, if $q = \mathrm{Quad}(p)$ (for some parameters $(0, B, 0, D)$), then p is Γ_2^*-periodic (resp. bounded) if, and only if, q is.

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