



Quasi-invariant and super-coinvariant polynomials for the generalized symmetric group

Jean-Christophe Aval

► To cite this version:

Jean-Christophe Aval. Quasi-invariant and super-coinvariant polynomials for the generalized symmetric group. Formal Power Series and Algebraic Combinatorics, 2003, Linköping, Sweden. hal-00185530

HAL Id: hal-00185530

<https://hal.science/hal-00185530>

Submitted on 6 Nov 2007

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

QUASI-INVARIANT AND SUPER-COINVARIANT POLYNOMIALS FOR THE GENERALIZED SYMMETRIC GROUP

J.-C. AVAL

ABSTRACT. The aim of this work is to extend the study of super-coinvariant polynomials, introduced in [2, 3], to the case of the generalized symmetric group $G_{n,m}$, defined as the wreath product $C_m \wr \mathcal{S}_n$ of the symmetric group by the cyclic group. We define a quasi-symmetrizing action of $G_{n,m}$ on $\mathbb{Q}[x_1, \dots, x_n]$, analogous to those defined in [12] in the case of \mathcal{S}_n . The polynomials invariant under this action are called quasi-invariant, and we define super-coinvariant polynomials as polynomials orthogonal, with respect to a given scalar product, to the quasi-invariant polynomials with no constant term. Our main result is the description of a Gröbner basis for the ideal generated by quasi-invariant polynomials, from which we deduce that the dimension of the space of super-coinvariant polynomials is equal to $m^n C_n$ where C_n is the n -th Catalan number.

RÉSUMÉ. Le but de ce travail est d'étendre l'étude des polynômes super-coinvariants (définis dans [2]), au cas du groupe symétrique généralisé $G_{n,m}$, défini comme le produit en couronne $C_m \wr \mathcal{S}_n$ du groupe symétrique par le groupe cyclique. Nous définissons ici une action quasi-symétrisante de $G_{n,m}$ sur $\mathbb{Q}[x_1, \dots, x_n]$, analogue à celle définie dans [12] dans le cas de \mathcal{S}_n . Les polynômes invariants sous cette action sont dits quasi-invariants, et les polynômes super-coinvariants sont les polynômes orthogonaux aux polynômes quasi-invariants sans terme constant (pour un certain produit scalaire). Notre résultat principal est l'obtention d'une base de Gröbner pour l'idéal engendré par les polynômes quasi-invariants. Nous en déduisons alors que la dimension de l'espace des polynômes super-coinvariants est $m^n C_n$ où C_n est le n -ième nombre de Catalan.

1. INTRODUCTION

Let X denote the alphabet in n variables (x_1, \dots, x_n) and $\mathbb{C}[X]$ denote the space of polynomials with complex coefficients in the alphabet X . Let $G_{n,m} = C_m \wr \mathcal{S}_n$ denote the wreath product of the symmetric group \mathcal{S}_n by the cyclic group C_m . This group is sometimes known as the *generalized symmetric group* (cf. [17]). It may be seen as the group of $n \times n$ matrices in which each row and each column has exactly one non-zero entry (pseudo-permutation matrices), and such that the non-zero entries are m -th roots of unity. The order of $G_{n,m}$ is $m^n n!$. When $m = 1$, $G_{n,m}$ reduces to the symmetric group \mathcal{S}_n , and when $m = 2$, $G_{n,m}$ is the hyperoctahedral group B_n , *i.e.* the group of signed permutations, which is the Weyl group of type B (see [14]

Date: November 6, 2007.

Research financed by EC's IHRP Programme, within the Research Training Network "Algebraic Combinatorics in Europe," grant HPRN-CT-2001-00272.

for example for further details). The group $G_{n,m}$ acts classically on $\mathbb{C}[X]$ by the rule

$$(1.1) \quad \forall g \in G_{n,m}, \forall P \in \mathbb{C}[X], g.P(X) = P(X.^t g),$$

where g is the transpose of the matrix g and X is considered as a row vector. Let

$$Inv_{n,m} = \{P \in \mathbb{C}[X] / \forall g \in G_{n,m}, g.P = P\}$$

denote the set of $G_{n,m}$ -invariant polynomials. Let us denote by $Inv_{n,m}^+$ the set of such polynomials with no constant term. We consider the following scalar product on $\mathbb{C}[X]$:

$$(1.2) \quad \langle P, Q \rangle = P(\partial X)Q(X) |_{X=0}$$

where ∂X stands for $(\partial x_1, \dots, \partial x_n)$ and $X = 0$ stands for $x_1 = \dots = x_n = 0$. The space of $G_{n,m}$ -coinvariant polynomials is then defined by

$$\begin{aligned} Cov_{n,m} &= \{P \in \mathbb{C}[X] / \forall Q \in Inv_{n,m}, Q(\partial X)P = 0\} \\ &= \langle Inv_{n,m}^+ \rangle^\perp \simeq \mathbb{C}[X] / \langle Inv_{n,m}^+ \rangle \end{aligned}$$

where $\langle S \rangle$ denotes the ideal generated by a subset S of $\mathbb{C}[X]$.

A classical result of Chevalley [6] states the following equality:

$$(1.3) \quad \dim Cov_{n,m} = |G_{n,m}| = m^n n!$$

which reduces when $m = 1$ to the theorem of Artin [1] that the dimension of the harmonic space $\mathbf{H}_n = Cov_{n,1}$ (cf. [9]) is $n!$.

Our aim is to give an analogous result in the case of quasi-symmetrizing action. The ring $Qsym$ of quasi-symmetric functions was introduced by Gessel [11] as a source of generating functions for P -partitions [18] and appears in more and more combinatorial contexts [5, 18, 19]. Malvenuto and Reutenauer [16] proved a graded Hopf duality between $QSym$ and the Solomon descent algebras and Gelfand *et. al.* [10] defined the graded Hopf algebra NC of non-commutative symmetric functions and identified it with the Solomon descent algebra.

In [2, 3], Aval *et. al.* investigated the space \mathbf{SH}_n of super-coinvariant polynomials for the symmetric group, defined as the orthogonal (with respect to (1.2)) of the ideal generated by quasi-symmetric polynomials with no constant term, and proved that its dimension as a vector space equals the n -th Catalan number:

$$(1.4) \quad \dim \mathbf{SH}_n = C_n = \frac{1}{n+1} \binom{2n}{n}.$$

Our main result is a generalization of the previous equation in the case of super-coinvariant polynomials for the group $G_{n,m}$.

In Section 2, we define and study a “quasi-symmetrizing” action of $G_{n,m}$ on $\mathbb{C}[X]$. We also introduce invariant polynomials under this action, which are called quasi-invariant, and polynomials orthogonal to quasi-invariant polynomials, which are called super-coinvariant. The Section 3 is devoted to the proof of our main result (Theorem 2.4), which gives the dimension of the space $SCov_{n,m}$ of super-coinvariant polynomials for $G_{n,m}$: we construct an explicit basis for $SCov_{n,m}$ from which we deduce its Hilbert series.

2. A QUASI-SYMMETRIZING ACTION OF $G_{n,m}$

We use vector notation for monomials. More precisely, for $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{N}^n$, we denote X^ν the monomial

$$(2.1) \quad x_1^{\nu_1} x_2^{\nu_2} \cdots x_n^{\nu_n}.$$

For a polynomial $P \in \mathbb{Q}[X]$, we further denote $[X^\nu] P(X)$ as the coefficient of the monomial X^ν in $P(X)$.

Our first task is to define a quasi-symmetrizing action of the group $G_{n,m}$ on $\mathbb{C}[X]$, which reduces to the quasi-symmetrizing action of Hivert (*cf.* [12]) in the case $n = 1$. This is done as follows. Let $A \subset X$ be a subalphabet of X with l variables and $K = (k_1, \dots, k_l)$ be a vector of positive (> 0) integers. If B is a vector whose entries are distinct variables x_i multiplied by roots of unity, the vector $(B)_<$ is obtained by ordering the elements in B with respect to the variable order. Now the quasi-symmetrizing action of $g \in G_{n,m}$ is given by

$$(2.2) \quad g \bullet A^K = w(g)^{c(K)} (A \cdot {}^t|g|)_<^K$$

where $w(g)$ is the weight of g , *i.e.* the product of its non-zero entries, $|g|$ is the matrix obtained by taking the modules of the entries of g , and the coefficient $c(K)$ is defined as follows:

$$c(K) = \begin{cases} 0 & \text{if } \forall i, k_i \equiv 0 [m] \\ 1 & \text{if not.} \end{cases}$$

Example 2.1. If $m = 3$ and $n = 3$, and we denote by j the complex number $j = e^{\frac{2i\pi}{3}}$, then for example

$$\begin{aligned} & \begin{pmatrix} 0 & 0 & j \\ 1 & 0 & 0 \\ 0 & j & 0 \end{pmatrix} \bullet (x_1^2 x_2) \\ &= (j^2)^1 \left[\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot (x_1, x_2) \right]_{<}^{(2,1)} \\ &= j^2 (x_3, x_1)_{<}^{(2,1)} \\ &= j^2 (x_1, x_3)^{(2,1)} \\ &= j^2 x_1^2 x_3. \end{aligned}$$

It is clear that this defines an action of the generalized symmetric group $G_{n,m}$ on $\mathbb{C}[X]$, which reduces to Hivert's quasi-symmetrizing action (*cf.* [12], Proposition 3.4) in the case $m = 1$.

Let us now study its invariant and coinvariant polynomials. We need to recall some definitions.

A *composition* $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ of a positive integer d is an ordered list of positive integers (> 0) whose sum is d . For a vector $\nu \in \mathbb{N}^n$, let $c(\nu)$ represent the composition obtained by erasing zeros (if any) in ν . A polynomial $P \in \mathbb{Q}[X]$ is said

to be *quasi-symmetric* if and only if, for any ν and μ in \mathbb{N}^n , we have

$$[X^\nu]P(X) = [X^\mu]P(X)$$

whenever $c(\nu) = c(\mu)$. The space of quasi-symmetric polynomials in n variables is denoted by $QSym_n$.

The polynomials invariant under the action (2.2) of $G_{n,m}$ are said to be *quasi-invariant* and the space of quasi-invariant polynomials is denoted by $QInv_{n,m}$, i.e.

$$P \in QInv_{n,m} \Leftrightarrow \forall g \in G_{n,m}, g \bullet P = P.$$

Let us recall (cf. [12], Proposition 3.15) that $QInv_{n,1} = QSym_n$. The following proposition gives a characterization of $QInv_{n,m}$.

Proposition 2.2. *One has*

$$P \in QInv_{n,m} \Leftrightarrow \exists Q \in QSym_n / P(X) = Q(X^m)$$

where $Q(X^m) = Q(x_1^m, \dots, x_n^m)$.

Proof. Let P be an element of $QInv_{n,m}$. Let us denote by ζ the m -th root of unity $\zeta = e^{\frac{2i\pi}{m}}$ and by g_j the element of $G_{n,m}$ whose matrix is

$$\begin{pmatrix} \zeta & & 0 \\ & 1 & \\ 0 & & \ddots \\ & & & 1 \end{pmatrix}$$

with the ζ in place j . Then we observe that the identities

$$\forall j = 1, \dots, n, \frac{1}{m}(P + g_j \bullet P + g_j^2 \bullet P + \dots + g_j^{m-1} \bullet P) = P$$

imply that every exponents appearing in P are multiples of m . Thus there exists a polynomial $Q \in \mathbb{C}[X]$ such that $P(X) = Q(X^m)$. To conclude, we note that $\mathcal{S}_n \subset G_{n,m}$ implies that P is quasi-symmetric, whence Q is also quasi-symmetric.

The reverse implication is obvious. \square

Let us now define *super-coinvariant* polynomials:

$$\begin{aligned} SCov_{n,m} &= \{P \in \mathbb{C}[X] / \forall Q \in QInv_{n,m}, Q(\partial X)P = 0\} \\ &= \langle QInv_{n,m}^+ \rangle^\perp \simeq \mathbb{C}[X] / \langle QInv_{n,m}^+ \rangle \end{aligned}$$

with the scalar product defined in (1.2). This is the natural analogous to Cov_n in the case of quasi-symmetrizing actions and $SCov_{n,m}$ reduces to the space of super-harmonic polynomials \mathbf{SH}_n (cf. [3]) when $m = 1$.

Remark 2.3. It is clear that any polynomial invariant under (2.2) is also invariant under (1.1), i.e. $Inv_{n,m} \subset QInv_{n,m}$. By taking the orthogonal, this implies that $SCov_{n,m} \subset Cov_{n,m}$. These observations somewhat justify the terminology.

Our main result is the following theorem which is a generalization of equality (1.4).

Theorem 2.4. *The dimension of the space $Scov_{n,m}$ is given by*

$$(2.3) \quad \dim SCov_{n,m} = m^n C_n = m^n \frac{1}{n+1} \binom{2n}{n}.$$

Remark 2.5. In the case of the hyperoctahedral group $B_n = G_{n,2}$, C.-O. Chow [7] defined a class $BQSym(x_0, X)$ of quasi-symmetric functions of type B in the alphabet (x_0, X) . His approach is quite different from ours. In particular, one has the equality:

$$BQSYm(x_0, X) = QSym(X) + QSym(x_0, X).$$

In the study of the coinvariant polynomials, it is not difficult to prove that the quotient $\mathbb{C}[x_0, X]/\langle BQSym^+ \rangle$ is isomorphic to the quotient $\mathbb{C}[X]/\langle QSym^+ \rangle$ studied in [3]. To see this, we observe that if \mathcal{G} is the Gröbner basis of $\langle QSym^+ \rangle$ constructed in [3] (see also the next section), then the set $\{x_0, \mathcal{G}\}$ is a Gröbner basis (any syzygy is reducible thanks to Buchberger's first criterion, *cf.* [8]).

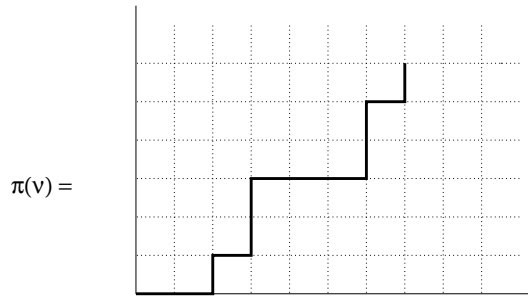
The next section is devoted to give a proof of Theorem 2.4 by constructing an explicit basis for the quotient $\mathbb{C}[X]/\langle QInv_{n,m}^+ \rangle$.

3. PROOF OF THE MAIN THEOREM

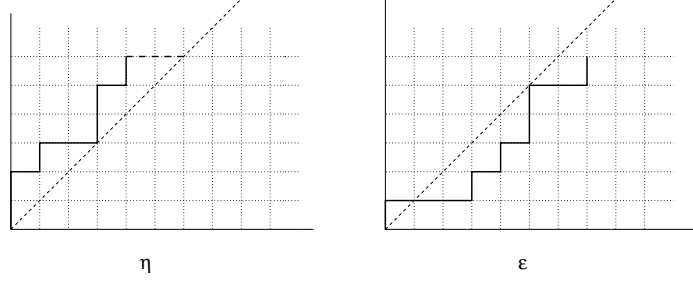
Our task is here to construct an explicit monomial basis for the quotient space $\mathbb{C}[X]/\langle QInv_{n,m}^+ \rangle$. Let us first recall (*cf.* [3]) the following bijection which associates to any vector $\nu \in \mathbb{N}^n$ a path $\pi(\nu)$ in the $\mathbb{N} \times \mathbb{N}$ plane with steps going north or east as follows. If $\nu = (\nu_1, \dots, \nu_n)$, the path $\pi(\nu)$ is

$$(0, 0) \rightarrow (\nu_1, 0) \rightarrow (\nu_1, 1) \rightarrow (\nu_1 + \nu_2, 1) \rightarrow (\nu_1 + \nu_2, 2) \rightarrow \dots \\ \rightarrow (\nu_1 + \dots + \nu_n, n-1) \rightarrow (\nu_1 + \dots + \nu_n, n).$$

For example the path associated to $\nu = (2, 1, 0, 3, 0, 1)$ is



We distinguish two kinds of paths, thus two kinds of vectors, with respect to their “behavior” regarding the diagonal $y = x$. If the path remains above the diagonal, we call it a *Dyck path*, and say that the corresponding vector is *Dyck*. If not, we say that the path (or equivalently the associated vector) is *transdiagonal*. For example $\eta = (0, 0, 1, 2, 0, 1)$ is Dyck and $\varepsilon = (0, 3, 1, 1, 0, 2)$ is transdiagonal.



We then have the following result which generalizes Theorem 4.1 of [3] and which clearly implies the Theorem 2.4.

Theorem 3.1. *The set of monomials*

$$\mathcal{B}_{n,m} = \{(X_n)^{m\eta+\alpha} / \pi(\eta) \text{ is a Dyck path, } 0 \leq \alpha_i < m\}$$

is a basis for the quotient $\mathbb{C}[X_n]/\langle QInv_{n,m}^+ \rangle$.

To prove this result, the goal is here to construct a Gröbner basis for the ideal $\mathcal{J}_{n,m} = \langle QInv_{n,m}^+ \rangle$. We shall use results of [2, 3].

Recall that the *lexicographic order* on monomials is

$$(3.1) \quad X^\nu >_{\text{lex}} X^\mu \quad \text{iff} \quad \nu >_{\text{lex}} \mu,$$

if and only if the first non-zero part of the vector $\nu - \mu$ is positive.

For any subset \mathcal{S} of $\mathbb{Q}[X]$ and for any positive integer m , let us introduce $\mathcal{S}^m = \{P(X^m), P \in \mathcal{S}\}$. If we denote by $G(I)$ the unique reduced monic Gröbner basis (cf. [8]) of an ideal I , then the simple but crucial fact in our context is the following.

Proposition 3.2. *With the previous notations,*

$$(3.2) \quad G(\langle \mathcal{S}^m \rangle) = G(\langle \mathcal{S} \rangle)^m.$$

Proof. This is a direct consequence of Buchberger's criterion. Indeed, if for every pair g, g' in $G(\langle \mathcal{S} \rangle)$, the syzygy

$$S(g, g')$$

reduces to zero, then the syzygy

$$S(g(X^m), g'(X^m))$$

also reduces to zero in $G(\langle \mathcal{S}^m \rangle)$ by exactly the same computation. \square

Let us recall that in [2] is constructed a family \mathcal{G} of polynomials G_ε indexed by transdiagonal vectors ε . This family is constructed by using recursive relations of the fundamental quasi-symmetric functions and one of its property (cf. [2]) says that the leading monomial of G_ε is: $LM(G_\varepsilon) = X^\varepsilon$. Since \mathcal{G} is a Gröbner basis of $\mathcal{J}_{n,1}$, the following result is a consequence of Propositions 2.2 and 3.2.

Proposition 3.3. *The set \mathcal{G}^m is a Gröbner basis of the ideal $\mathcal{J}_{n,m}$.*

To conclude the proof of Theorem 3.1, it is sufficient to observe that the monomials not divisible by a leading monomial of an element of \mathcal{G}^m , i.e. by a $X^{m\varepsilon}$ for ε transdiagonal, are precisely the monomials appearing in the set $\mathcal{B}_{n,m}$.

As a corollary of Theorem 3.1, one gets an explicit formula for the Hilbert series of $SCov_{n,m}$. For $k \in \mathbb{N}$, let $SCov_{n,m}^{(k)}$ denote the projection

$$(3.3) \quad SCov_{n,m}^{(k)} = SCov_{n,m} \cap \mathbb{Q}^{(k)}[X]$$

where $\mathbb{Q}^{(k)}[X]$ is the vector space of homogeneous polynomials of degree k together with zero.

Let us denote by $F_{n,m}(t)$ the Hilbert series of $SCov_{n,m}$, *i.e.*

$$(3.4) \quad F_{n,m}(t) = \sum_{k \geq 0} \dim SCov_{n,m}^{(k)} t^k.$$

Let us recall that in [3] is given an explicit formula for $F_{n,1}$:

$$(3.5) \quad F_{n,1}(t) = F_n(t) = \sum_{k=0}^{n-1} \frac{n-k}{n+k} \binom{n+k}{k} t^k$$

using the number of Dyck paths with a given number of factors (*cf.* [13]).

The Theorem 3.1 then implies the

Corollary 3.4. *With the notations of (3.5), the Hilbert series of $SCov_{n,m}$ is given by*

$$F_{n,m}(t) = \frac{1-t^m}{1-t} F_n(t^m)$$

from which one deduces the close formula

$$\sum_n F_{n,m}(t) x^n = \frac{(1-t) - \sqrt{(1-t)(1-t-4t^m x(1-t^m))} - 2x(1-t^m)}{(1-t)(2t^m - 1) - x(1-t^m)}.$$

REFERENCES

- [1] E. ARTIN, *Galois Theory*, Notre Dame Mathematical Lecture **2** (1944), Notre Dame, IN.
- [2] J.-C. AVAL AND N. BERGERON, *Catalan Paths and Quasi-Symmetric Functions*, Proc. Amer. Math. Soc., to appear.
- [3] J.-C. AVAL, F. BERGERON AND N. BERGERON, *Ideals of Quasi-Symmetric Functions and Super-Coinvariant Polynomials for \mathcal{S}_n* , Adv. in Math., to appear.
- [4] F. BERGERON, A. GARSIA AND G. TESLER, *Multiple Left Regular Representations Generated by Alternants*, J. of Comb. Th., Series A, **91**, **1-2** (2000), 49–83.
- [5] N. BERGERON, S. MYKYTIUK, F. SOTTILE, AND S. VAN WILLIGENBURG, *Pieri Operations on Posets*, J. of Comb. Theory, Series A, **91** (2000), 84–110.
- [6] C. CHEVALLEY, *Invariants of finite groups generated by reflections*, Amer. J. Math., **77** (1955), 778–782.
- [7] C.-O. CHOW, *Noncommutative Symmetric Functions of Type B*, Thesis, Massachusetts Institute of Technology, 2001.
- [8] D. COX, J. LITTLE AND D. O’SHEA, *Ideals, Varieties, and Algorithms*, Springer-Verlag, New-York, 1992.
- [9] A. GARSIA, M. HAIMAN, *Orbit Harmonics and Graded Representations*, Éditions du Lacim, to appear.
- [10] I. GELFAND, D. KROB, A. LASCoux, B. LECLERC, V. RETAKH, AND J.-Y. THIBON, *Non-commutative symmetric functions*, Adv. in Math., **112** (1995), 218–348.

- [11] I. GESSEL, *Multipartite P -partitions and products of skew Schur functions*, in Combinatorics and Algebra (Boulder, Colo., 1983), C. Greene, ed., vol. 34 of Contemp. Math., AMS, 1984, pp. 289–317.
- [12] F. HIVERT, *Hecke algebras, difference operators, and quasi-symmetric functions*, Adv. in Math., **155** (2000), 181–238.
- [13] G. KREWERAS, *Sur les éventails de segments*, Cahiers du BURO, **15** (1970), 3–41.
- [14] G. LUSZTIG, *Irreducible representations of finite reflection groups*, Invent. Math., **43** (1977), 125–175.
- [15] I. MACDONALD, *Symmetric Functions and Hall Polynomials*, Oxford Univ. Press, 1995, second edition.
- [16] C. MALVENUTO AND C. REUTENAUER, *Duality between quasi-symmetric functions and the Solomon descent algebra*, Journal of Algebra, **177** (1995), 967–982.
- [17] M. OSIMA, *On the representations of the generalized symmetric group*, Math. J. Okayama Univ., **4** (1954), 39–56.
- [18] R. STANLEY, *Enumerative Combinatorics, Vol. 1*, Wadsworth and Brooks/Cole, 1986.
- [19] R. STANLEY, *Enumerative Combinatorics Vol. 2*, no. 62 in Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1999. Appendix 1 by Sergey Fomin.

(Jean-Christophe Aval) LABRI, UNIVERSITÉ BORDEAUX 1, 351 COURS DE LA LIBÉRATION,
33405 TALENCE CEDEX, FRANCE

E-mail address: aval@labri.fr