Monomial bases related to the n! conjecture

Jean-Christophe Aval

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Jean-Christophe Aval
Laboratoire A2X
Université Bordeaux 1
351 cours de la Libération
33405 Talence cedex
e-mail : aval@math.u-bordeaux.fr

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Abstract: The purpose of this paper is to find a new way to prove the $n!$ conjecture for particular partitions. The idea is to construct a monomial and explicit basis for the space M_μ. We succeed completely for hook-shaped partitions, i.e., $\mu = (K+1, 1^L)$. We are able to exhibit a basis and to verify that its cardinality is indeed $n!$, that it is linearly independent and that it spans M_μ. We derive from this study an explicit and simple basis for I_μ, the annihilator ideal of Δ_μ. This method is also successful for giving directly a basis for the homogeneous subspace of M_μ consisting of elements of 0 x-degree.

1 Introduction

Let $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0)$ be a partition of n. We shall identify μ with its Ferrers diagram (using the French notation). To each cell s of the Ferrers diagram, we associate its coordinates (i, j), where i is the height of s and j the position of s in its row. The pairs $(i-1, j-1)$ occurring while s describes μ will be briefly referred to as the set of the biexponents of μ. Now let $(p_1, q_1), \ldots, (p_n, q_n)$ denote the set of biexponents arranged in lexicographic order and set

$$\Delta_\mu(x, y) = \Delta_\mu(x_1, \ldots, x_n; y_1, \ldots, y_n) = \det(x_i^{p_j} y_i^{q_j})_{i,j=1\ldots n}.$$

Let M_μ be the collection of polynomials in the variables $x_1, \ldots, x_n; y_1, \ldots, y_n$ obtained by taking the linear span of all the partial derivatives of Δ_μ. For-

mally we may write

$$M_\mu = \mathcal{L}\{\partial_x^a \partial_y^b \Delta_\mu(x,y);\ a,b \in \mathbb{N}^n\}$$

where $\partial_x^a = \partial_{x_1}^{a_1} \dots \partial_{x_n}^{a_n}$ and $\partial_y^b = \partial_{y_1}^{b_1} \dots \partial_{y_n}^{b_n}$. Then the $n!$ conjecture can be stated as follows.

Conjecture 1 ($n!$ conjecture): *Let μ be a partition of n, then* $\dim M_\mu = n!$.

This conjecture, stated by A. Garsia and M. Haiman is central for their study of Macdonald polynomials (cf. [5], [6]). To be more precise, Macdonald introduced in [12] a new symmetric function basis and associated Macdonald-Kostka coefficients $K_{\lambda\mu}(q,t)$, which are a priori rational functions in q,t. Macdonald conjectured that:

Conjecture 2 (MPK conjecture): *The functions $K_{\lambda\mu}(q,t)$ are polynomials with non-negative integer coefficients.*

Looking for a representation theoretical setting for the Macdonald basis, A. Garsia and M. Haiman made the following conjecture:

Conjecture 3 ($C = \tilde{H}$ conjecture): *For the diagonal action of S_n, M_μ is a bigraded version of the left regular representation. Moreover, if $C_{\lambda\mu}(q,t)$ denotes the bigraded multiplicity of the character χ_λ in the bigraded character of the module M_μ then: $C_{\lambda\mu}(q,t) = K_{\lambda\mu}(q,1/t)t^{n(\mu)}$, where $n(\mu) = \sum_{i=1}^{k}(i-1)\mu_i$.*

Conjecture 3 clearly implies Conjecture 1 and 2. M. Haiman [9] using Hilbert schemes theory recently proved that the $n!$ conjecture actually implies the $C = \tilde{H}$ conjecture. A part of the MPK conjecture is that the $K_{\lambda\mu}(q,t)$ are polynomials, which is not obvious from their definition. This part has been recently proved in several independent papers (cf. [7], [8], [10], [11], [14]).

When $\mu = (1^n)$ or $\mu = (n)$, Δ_μ reduces to the Vandermonde determinant in x and y respectively. In these cases, it is a classical result (see [2]) that $\dim M_\mu = n!$. But although this conjecture has been verified by computer for small partitions up to $n = 8$ and proved for some special cases (cf. [1], [4], [6], [13]), it has not been established in full generality. Several methods have been developed to prove the $n!$ conjecture but none of them has been able to give a proof in more than some special cases.

In this paper our goal is to propose a new method to prove the $n!$ conjecture for some particular partitions. We want to construct explicit bases for the space M_μ. These bases are made of monomial derivatives of Δ_μ. We present here how we are able to do it for hook-shaped partitions, i.e.,

$\mu = (K+1, 1^L)$ with $K + L + 1 = n$. In section 2 we describe the way to construct the basis and prove that its cardinality is $n!$. In the third section we show that our family spans M_μ. Moreover, we derive from that proof an explicit and simple basis for I_μ, the annihilator ideal of Δ_μ. In the fourth section we prove by a completely new method that the elements of our basis are linearly independent. In section 5 we explain how this method is also successful for the homogeneous subspace of M_μ consisting of elements of 0 x-degree. We obtain in fact a direct way to construct a basis for this subspace.

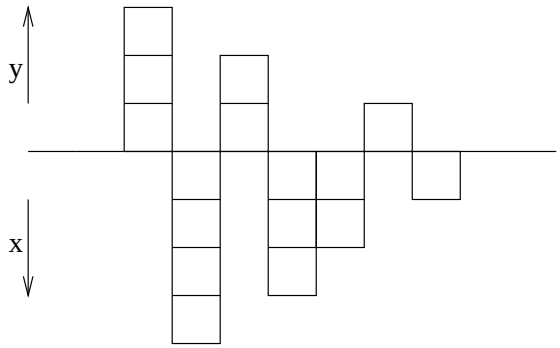
2 Construction and enumeration

Let μ be a partition of n whose Ferrers diagram is a hook, i.e., $\mu = (K+1, 1^L)$ with $K + L + 1 = n$.

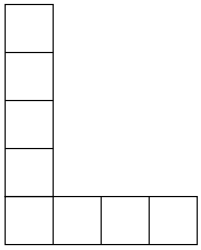
2.1 Construction

Let us take an horizontal axis. A "shape" associated to μ is constructed the following way: suppose the line has room for $K + L$ spaces. Choose K of these spaces to be y-columns and L to be x-columns. In the y-columns place stacks of boxes above the line of height $K, K-1, \ldots, 1$ arranged in decreasing order. In the x-columns place stacks of boxes of decreasing depth $L, L-1, \ldots, 1$ below the line.

Here is an example of shape:



associated to the partition:



We shall now put crosses in the cells of the shape to obtain "drawings". As we shall not distinguish two drawings with the same number of crosses in each column, we put the crosses near the axis. The rules for putting crosses in a drawing are the following:

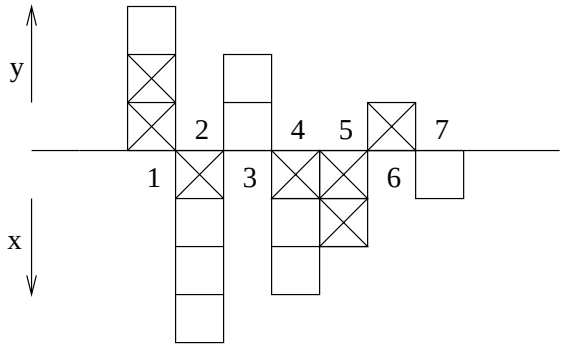
1. the number of crosses in the x-columns is any number (not greater than the depth of the column);

2. the number of crosses in the y-columns depends on the x-crosses. For a column which has no x-column to its right, the number of crosses is not greater than the height of the column. In the other case, we look at the first "plain" x-column on the right; i.e., the first column which has only crosses (full x-column) or only white cells (empty x-column). There is always one, at least the x-column of depth one. Then:

   - if it is all white, then we impose at least one cross in the y-column.
   - if it is all crossed, then we impose at least one white cell in the y-column.

Remark 1: The family of drawings that we defined is invariant under the operator that inverts the white cells and the crosses. We call this operator flip (it is different from the flip introduced by A. Garsia and M. Haiman in [6], that we denote from now on by Flip).

Here we give an example of drawing with crosses:



Once we have defined the drawings (with crosses), we define associated derivative operators. We give an index to the places of the drawing from left to right and from 1 to $n-1$. Then to each x-cross in place i, we derive once with respect to x_i. We do the same thing for the y-crosses. For example, for the last drawing, the associated derivative operator is: $\partial_D = \partial y_1^2 \partial x_2 \partial x_4 \partial x_5^2 \partial y_6$.

2.2 Enumeration

We shall denote by $\mathcal{D}$ the set of drawings that we defined in the previous subsection. We now verify that its cardinality is $n!$.

As the number of choices for the y-columns depends only on the shape of the drawing (and not on the x-crosses), we can write that the cardinality equals the following expression, where k_1 denotes the number of y-columns on the right of the last x-column:

$$\sum_{k_1+k_2=K} 2 \cdot 3 \cdots (k_1+1) \cdot (k_1+1) \cdots (k_1+k_2) \cdot (L+1)! \binom{k_2+L-1}{k_2}$$

$$= L(L+1)K! \sum_{k_2=0}^{K} \frac{(k_2+L-1)!}{k_2!}(K+1-k_2)$$

$$= (L+1)!K! \sum_{k_2=0}^{K} \binom{L-1+k_2}{L-1}\binom{K+1-k_2}{1}$$

$$= (L+1)!K!\binom{K+L+1}{L+1} = (K+L+1)!$$

by the Chu-Vandermonde formula ([3], p. 163).

3 Proof that the family spans M_μ

We show here that $\{\partial_D \Delta_\mu\}_{D\in\mathcal{D}}$ spans M_μ. We begin by studying I_μ, the annihilator ideal of Δ_μ.

3.1 Study of I_μ

For P, Q two polynomials, we write $P \equiv Q$ if $P(\partial)\Delta_\mu = Q(\partial)\Delta_\mu$, i.e., $P - Q \in I_\mu$ ($P(\partial)$ corresponds to the substitution: $x_i \to \partial x_i,\ y_i \to \partial y_i$). We denote as usual by h_k the k-th complete homogeneous symmetric function. Let also X denote a subset of $(x_1, x_2, \ldots, x_n)$, Y a subset of $(y_1, y_2, \ldots, y_n)$, $|X|$ and $|Y|$ their cardinality. We also set $\bar{X} = \prod_{x\in X} x$ and $\bar{Y} = \prod_{y\in Y} y$.

We first notice that:

1. for all $1 \le i \le n$, $x_i y_i \equiv 0$;
2. $\bar{X} \equiv 0$ as soon as $|X| > L$;
3. $\bar{Y} \equiv 0$ as soon as $|Y| > K$;
4. for any symmetric homogeneous polynomial P of positive degree, $P \equiv 0$.

The fourth relation is well known (cf. [2]). The others are clear by observing the elements in the determinantal form of Δ_μ when $\mu = (K+1, 1^L)$.

Proposition 1:

$$h_k(Y) \equiv 0$$

as soon as $k > 0$ *and* $k + |Y| > n$.

Proof. It is easily proved by an induction based on $h_k(y_1, \ldots, y_n) \equiv 0$ for all $k > 0$. We have indeed $h_1(Y_n) \equiv 0$, where $Y_n = (y_1, \ldots, y_n)$ and for any $y \notin Y$:

$$h_k(Y, y) = h_k(Y) + y h_{k-1}(Y, y).$$

□

Proposition 2:

$$\bar{Y} h_k(Y') \equiv 0$$

as soon as $k > 0$, $k + |Y| > K$ *and* $Y \subset Y'$.

Proof. Proposition 2 is proved by decreasing induction on $|Y'|$.

We observe that the result is true for $|Y'|$ equal to $K+1$ and K. Let Y and Y' satisfy the hypotheses and assume the result is true down to $|Y'|+1$. We write for all $y_i \notin Y'$:

$$h_k(Y', y_i) \equiv h_k(Y') + y_i h_{k-1}(Y', y_i),$$

thus, by induction if $k > 1$ we obtain the following relation; this relation is obvious if $k = 1$ because this implies that $|Y| \geq K$:

$$\bar{Y} h_k(Y', y_i) \equiv \bar{Y} h_k(Y').$$

Once we have this relation the conclusion easily follows by an increasing induction on $|Y'|$ (for example up to n).

□

Proposition 3:

$$h_k(Y) h_l(X) \equiv 0$$

as soon as $k > 0$, $l > 0$, $k + l + |Y| + |X| \geq 2n$ *and* $X \subset Y$ *or* $Y \subset X$.

Proof. We only show the result when $k + |Y| = n$ and $l + |X| = n$ (the other cases are consequences of Proposition 1).

It is in fact proved as Proposition 1 by a simple induction based on:

$$h_1(x_1, \ldots, x_{n-1}) h_1(y_1, \ldots, y_{n-1}) \equiv 0$$

which is a consequence of Proposition 1 and $x_n y_n \equiv 0$.

□

Proposition 4:

$$h_k(Y) h_l(X) \equiv 0$$

as soon as $k > 0$, $l > 0$ *and*

- *either* $Y \subset X$ *and* $k + l + |Y| > n$,
- *or* $X \subset Y$ *and* $k + l + |X| > n$.

Proof. This is proved by induction on $\alpha = 2n - (k + |Y| + l + |X|)$.

The case $\alpha \leq 0$ reduces to Proposition 3.

Suppose the result is true up to $\alpha - 1$ and $2n - (k + |Y| + l + |X|) = \alpha > 0$. By symmetry, we shall assume that $Y \subset X$ and $k + l + |Y| > n$. If $l > 1$, then for any $x_i \notin X$, we write:

$$h_k(Y)h_l(X) \equiv h_k(Y)h_l(X,x_i) - x_i h_k(Y)h_{l-1}(X,x_i)$$
$$\equiv h_k(Y)h_l(X,x_i) - x_i h_k(Y,y_i)h_{l-1}(X,x_i) \equiv 0$$

by induction.

If $l = 1$, then $|Y| + k \geq n$ and we write for any $x_i \notin X$:

$$h_k(Y)h_1(X) \equiv h_k(Y,y_i)h_1(X) - y_i h_{k-1}(Y)h_1(X,x_i).$$

The first term is zero by Proposition 1. The second term is proved to be also zero by increasing induction on $|X|$ (up to n), since $n - k \leq |Y| \leq |X| \Rightarrow n - |X| \leq k$.

□

3.2 Application

We shall show here that any monomial derivative of Δ_μ is a linear combination of the derivatives: $\{\partial_D \Delta_\mu\}_{D\in\mathcal{D}}$ (derivatives corresponding to drawings, i.e., the family defined in section 2).

Theorem 1: $\{\partial_D \Delta_\mu\}_{D\in\mathcal{D}}$ *spans* M_μ.a

Proof. It is clear that any monomial can be associated to a diagram of crosses (by the same process as in paragraph 2.1), and let D be such a diagram which is not a drawing.

We look at the rightmost "anomaly", that is the rightmost place where the diagram D associated to the monomial can not be a drawing (we call this place "guilty").

- Case 1: the diagram D could not be put in a set of ordered columns (i.e., in the shape of a drawing). This case gives four subcases. Assume the guilty column is a y-column. We can not put another y-column on the right. Either because each y-column on the left has a cross (case 1a), or because there is no first plain and white x-column on the right (case 1b). If the guilty column is an x-column, we are led to cases 1c (each x-column on the left has a cross) and 1d (there is no first plain and white y-column on the right). Since the rules are not involved here the problems are symmetric for x or y.

- Case 2: the diagram D could be put in a set of ordered columns but the rules are broken. Either for the white cells (case 2a), or for the crosses (case 2b).

We shall prove, using the propositions of the last section, that the monomial associated to the diagram D can be written modulo I_μ as a linear combination of monomials strictly smaller with respect to the lexicographic order ($x_1 < x_2 < \cdots < x_n < y_1 < \cdots < y_n$). We look at each case that we have mentioned above.

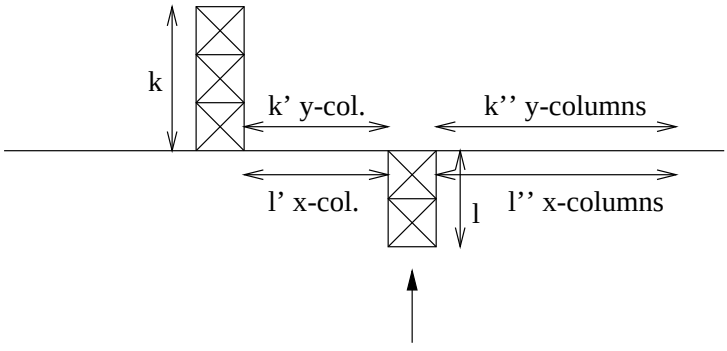
- Case 1b with no x-column on the right is solved by Proposition 1, as well as case 1d with no y-column on the right.

- Cases 1a and 1c are symmetric and treated by Proposition 2: we note that the height of the h-th y-column is $K-h+1$. If it has $k+1$ crosses, there is a problem if $k+h > K$. It then can be treated by Proposition 2: we take $Y' = Y = \{i_1 < \ldots < i_h\}$, to be the places of the first h y-columns, each of which has at least one cross. The monomial is a multiple of
$$\bar{Y} y_{i_h}^k \equiv \bar{Y}(y_{i_h}^k - h_k(Y))$$
and all monomials in the expansion of the right side are lexicographically smaller than the monomial on the left side.

- Case 2a is immediately settled by inverting the involved columns.

Therefore the only remaining cases are case 1b (resp. 1d) with a first full x- (resp. y-) column on the right and case 2b.

- Let us first study the case 2b.



We observe that there is a problem if one has simultaneously:

- $k = k' + k'' + 1$,
- $l = l'' + 1$,
- there is a cross in each of the l' x-columns between the two columns appearing on the figure.

Let:

- Y denote the places on the left of the y-column plus the place of the y-column plus the l' places of the x-columns between the y- and the x-column on D plus the place of the x-column,
- X denote the places on the left of the x-column plus the place of the x-column itself,
- X' denote the places of the l' x-columns between the y- and the x-column of D.

We shall be able to express the monomial corresponding to this D as a linear combination of monomials stricly smaller with respect to the lexicographic order if we establish that

$$h_k(Y)h_l(X) \equiv 0.$$

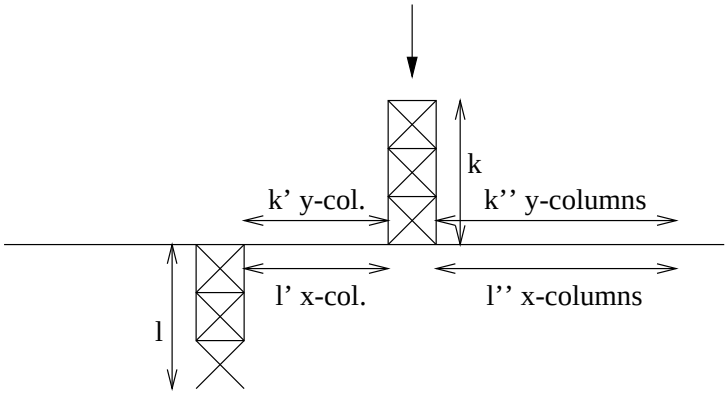
Indeed the leading monomial of $\bar{X}'h_k(Y)h_l(X)$ (for the lexicographic order), in which we delete the multiples of x_iy_i for any i, is a divisor of the monomial associated to D.

We want to apply Proposition 4 with $|Y| = n - (k' + k'' + l'' + 1)$ and $|X| = n - (k'' + l'' + 1)$. We have $Y \subset X$ and we calculate:

$$k + l + |Y| - n = 1 > 0.$$

Hence we are done in this case.

- Let us now consider the case 1d with a first full y-column.



Here a problem occurs if:

- $k = k'' + 1$,
- $l \geq l' + l'' + 2$,
- there is a cross in each of the k' y-columns between the two columns appearing on the figure.

We proceed as in the previous case. We want to use Proposition 4 to show that

$$h_k(Y)h_l(X) \equiv 0$$

with Y corresponding to all the places strictly left of the y-column on the diagram D and X corresponding to all the places up to the x-column, plus the places of the k' y-columns between the x-column and the y-column.

We want to apply Proposition 4 with $|X| = n - (l' + k'' + l'' + 2)$ and $|Y| = n - (k'' + l'' + 1)$. We have $X \subset Y$ and we compute:

$$k + l + |X| - n \geq 1.$$

Thus this case is also settled.

- It remains to observe that the case 1b with a first full x-column is treated by case 2b.

The proof of Theorem 1 is now complete.

□

3.3 Conclusion

We can deduce from what precedes a basis for the ideal I_μ when μ is a hook, since the first relations exposed at the beginning of the study of I_μ were sufficient to prove that our family is a basis of M_μ.

Theorem 2: *If we denote by $\langle G \rangle$ the ideal generated by a set G, then for μ a hook partition of n, we have:*

$$I_\mu = \langle h_i(X_n),\ 1 \leq i \leq n;\ h_i(Y_n),\ 1 \leq i \leq n;$$
$$x_i y_i,\ 1 \leq i \leq n;\ \bar{X},\ |X| = L+1;\ \bar{Y},\ |Y| = K+1 \rangle.$$

Proof. To prove this we assume that the previous ideal (we denote it by I) is not equal to I_μ, so that there is a polynomial P in $I_\mu \backslash I$. According to the proof of Theorem 1, we can decompose it as $P = A + Q$, where A is a linear combination of monomials of our family and Q is an element of I. Taking the derivatives and applying it to Δ_μ, we obtain $A(\partial)\Delta_\mu = 0$. As we shall see in section 4, this implies $A = 0$, and $P = Q \in I$.

□

4 Proof of the independence

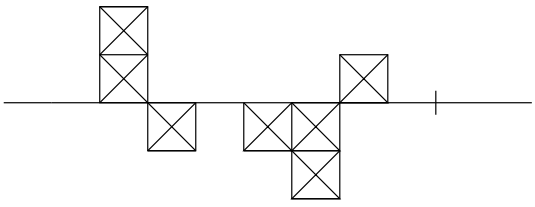
4.1 Exposition and reduction of the problem

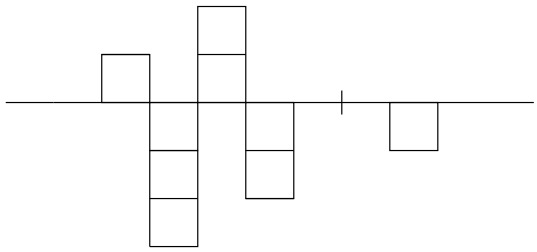
We shall now prove that our family is an independent set.

Since the derivative operator associated to a drawing D depends only on the crosses and not on the shape of the drawing, we define S as the diagram consisting only of the crosses of D. We also define T as the diagram consisting of the white cells (a "complement" of S).

Let $\mathcal{S}$ denote the set of S when D varies in $\mathcal{D}$, the set of drawings defined in section 2.

For example, for the drawing in section 2, we have:

S = 

T = 

Let now ∂_S and ∂_T denote the derivative operators associated to S and T (after putting crosses in all the cells of T).

Theorem 3:
The family $\{\partial_S.\Delta_\mu\}_{S\in\mathcal{S}}$ is an independent set.

Lemma 1:
S or T determines the drawing from which it comes.

Proof. Indeed, we can recontruct the shape of the drawing from S by proceeding from left to right. The method is the following: if there are crosses at the place we are looking at, we complete the column with respect to the size of the successive columns. If there is no cross, we look at the x-crosses on the right: if they can fit in with one x-column missing, then we put an x-column at the empty place, else we put a y-column.

The method is the same for T since the family is invariant under flip.

□

Let us now show that the family is linearly independent. Let us begin with some definitions. Let $D = (S, T)$ and $D_1 = (S_1, T_1)$ be two different drawings; we shall say that D_1 is a son of D if $\partial_T \circ \partial_{S_1}.\Delta_\mu \in \mathbb{Z}\backslash\{0\}$. We shall denote by $T + S_1$ the figure corresponding to the superposition (place by place) of the cells of T and S_1 (all these cells being crossed). If we repeat this process, we obtain the notion of descendant.

Lemma 2:
To show the independence, it is sufficient to prove that a drawing can not be its own descendant (i.e., there is no "loop").

Proof. We assume we have a relation of dependence: $\sum_S c_S \partial_S.\Delta_\mu = 0$, that the coefficients are not all zero, and that there is no loop. Then we take a S_0 for which $c_{S_0} \neq 0$. If S_0 has no son or if they have all c_S equal to zero, we obtain a contradiction by applying ∂_{T_0} to the relation and by looking at

the constant term of the result. If S_0 has a son S_1 for which $c_{S_1} \neq 0$, we repeat with S_1. As the set is finite and there is no loop, we certainly obtain a S' which gives a contradiction.

□

So we have to prove that there is no loop. It is sufficient to show that a drawing $D = (S, T)$ is different from all its descendants that have the same shape (i.e., the x-columns at the same places). Let $D' = (S', T')$ be a descendant of D that has the same shape. We want to show that $D \neq D'$.

4.2 Definition of completeness

To explain this notion, let D_1 denotes a drawing and D_2 one of its sons. We define on the places of D_2 a notion of "completeness" (relative to D_1 too) as follows: We say that the first k places of D_2 are complete if the heights of the y-columns of $T_1 + S_2$ in these k places and read from left to right are $K, K-1, K-2, \ldots$ and if we have the same for x-columns.

We want now to obtain a (more quantitative) characterization of the completeness. To do this we need to introduce some more definitions.

We look at the left parts (made of the first $k-1$ places) of D_1 and D_2. We define d as the difference between the number of times where a y-column of D_1 has been replaced in D_2 by a white x-column and the number of times where an x-column of D_1 has been replaced in D_2 by a white y-column. We also define d' as the difference between the number of times where a crossed y-column of D_1 has been replaced in D_2 by an x-column and the number of times where a crossed x-column of D_1 has been replaced in D_2 by a y-column. We should note that d and d' are relative to $k-1$.

Since the problem is symmetric with respect to x and y (as long as we do not use the rules of construction), we shall only examine the case where we derive with respect to y_k, i.e., where there is a y-column at the k-th place of $T_1 + S_2$. The symmetric case has a similar characterization (with opposite signs for d and d'). We now introduce the following notations: b_1 (resp. b_2) denotes the number of white cells at place k in D_1 (resp. D_2) and c_1 (resp. c_2) the number of crosses. The characterization can now be stated as follows:

Characterization :

If the first $k-1$ places are complete, the k-th is complete if one of the following conditions is verified:

1. *at place k in D_1 and D_2 there is a y-column and $b_2 = b_1 + d$ and $c_2 = c_1 + d'$ (each of these equalities easily implies the other);*

2. *at place k, there is a crossed x-column in D_1 (i.e., $b_1 = 0$) and a y-column in D_2, and $b_2 = d$;*

3. *at place k, there is a y-column in D_1 and a white x-column in D_2 ($c_2 = 0$), and $c_1 = -d'$.*

Proof. To prove this result, we begin by observing that we can not have x- and y- cells at the same place in $T_1 + S_2$: when μ is a hook, we have $\partial x_i \partial y_i \Delta_\mu = 0$. There are in fact three possibilities for the columns at place k:

1. D_1 and D_2 have a y-column;

2. D_1 has a crossed x-column and D_2 a y-column;

3. D_1 has a y-column and D_2 a white x-column.

We deal with these three cases.

1. Case 1: if in $T_1 + S_2$ the heights of the y-columns in the first $k-1$ places are K, $K-1$, ..., $K-l+1$ and if our y-column is the h-th of D_2, we observe that $l = h - 1 + d$. The height of the y-column of $T_1 + S_2$ at place k is at most $K - l$. But if we observe that the height of the h-th y-column of D_2 is $K - h + 1$, we obtain:

$$b_1 + c_2 \leq K - l = K - h + 1 - d = b_2 + c_2 - d.$$

   Hence $b_2 \geq b_1 + d$ and equality holds when it is complete. As $b_2 + c_2 = b_1 + c_1 + d + d'$, the equality $c_2 = c_1 + d'$ holds too.

2. Case 2: this case is treated like Case 1.

3. Case 3: the reasoning is similar to Case 1. If our y-column is the h-th of D_1 and if in the first $k-1$ places of $T_1 + S_2$ the successive y-columns have height K, $K-1$, ..., $K-l+1$, then $l = h - 1 - d'$. As the height of the y-column at place k of $T_1 + S_2$ is at most $K - l$, we deduce that $c_1 \geq -d'$, with equality corresponding to completion.

□

Remark 2: If the first $k-1$ places are complete but not the k-th, we observe easily that it corresponds to an increasing of the number of white cells in D_2. We have indeed seen in the proof that $b_2 \geq b_1 + d$ and $c_2 \leq c_1 + d'$ in Case 1 and similar inequalities in Cases 2 and 3.

Remark 3: We observe that the Cases 2 and 3 can not happen simultaneously since we can not have at the same place a crossed column in D_1 and a white column in D_2 (there is at least one cell at each place).

Once we have obtained this characterization of completeness, we shall use it to progress in the proof of Theorem 3.

4.3 Application

Lemma 3:

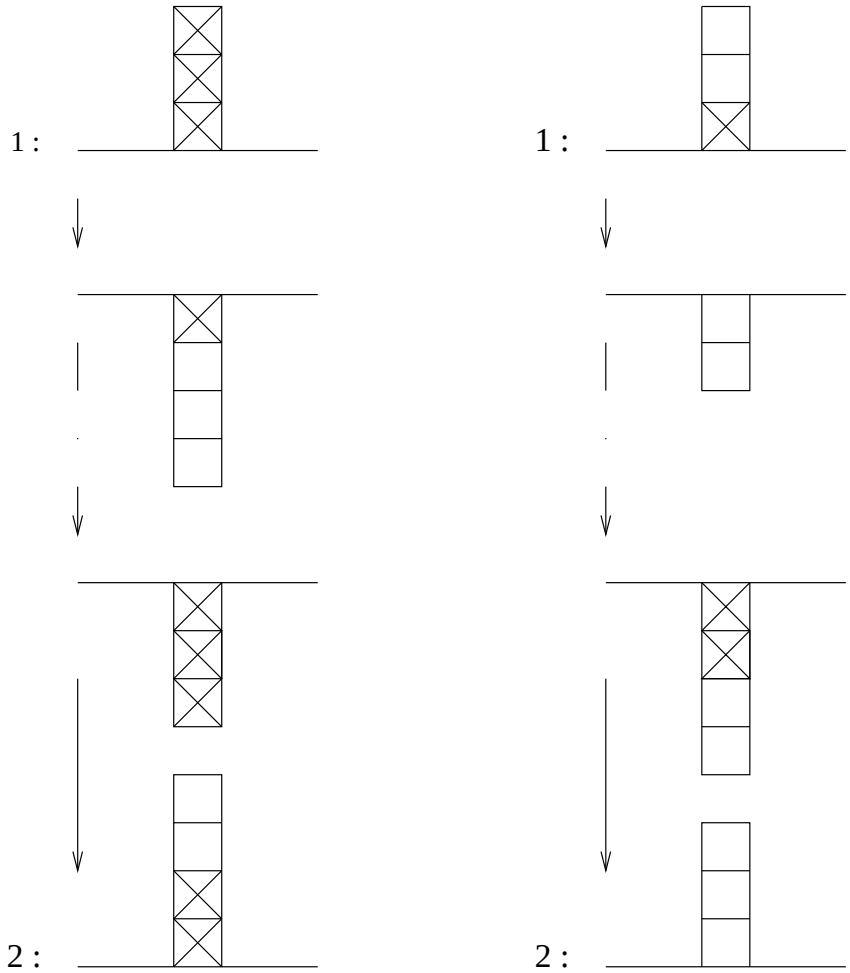
If we have completeness on the first k places along the chain between two drawings D and D′ with the same shape, then the sum of the d along the chain is equal to zero, as well as the sum of the d′ (d and d′ relative to the first k places).

We will first apply this result in the following lemma and prove it after Lemma 4.

Lemma 4:

If we have completeness on the first k places between D and D′, then these two drawings are identical on the first k places.

Proof. To prove this result we shall use Lemma 3. Indeed we notice that if we keep either an x- or a y-column at place k along the chain between D and D', the result is obvious since (by Lemma 3) the sum of the d is equal to zero. With natural notations, we have: $b' = b + \sum d = b$. Now, if the "shape" of the column at place k changes, let us observe the two following cases (by symmetry we look at the changes for a y-column):



(simple arrows mean single generation, broken arrows mean possibly several generations, but at fixed shape at place k).

In view of the characterization of completeness, we observe that we have in both cases: $b_2 = b_1 + d,\ c_2 = c_1 + d'$, as if we had not changed the shape (it is easily seen by looking at the d on the left and at the d' on the right).

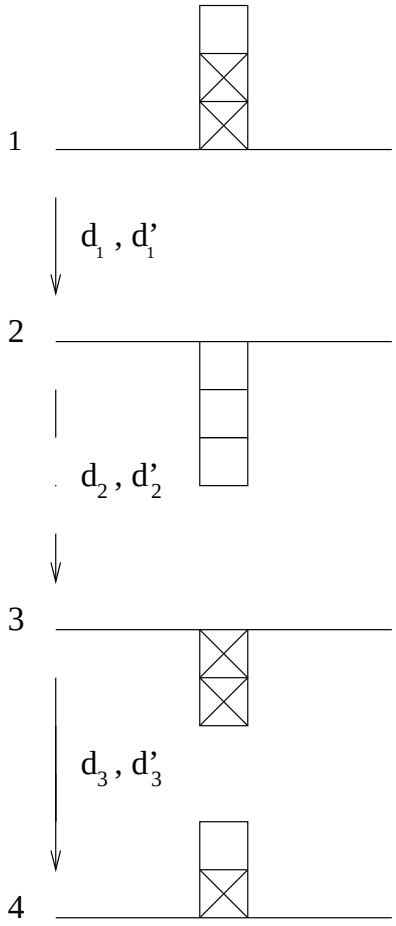
By Lemma 3, we are now able to remove the condition that the shape does not change at the broken arrows. Indeed, we begin by reasoning about chains as above, then we can ignore the change of shape. By this method we obtain the general result (analogy with a Dick path for which we repeat the removing of sequences $\vee$ and $\wedge$).

□

Proof of Lemma 3. This will be done by induction on k.

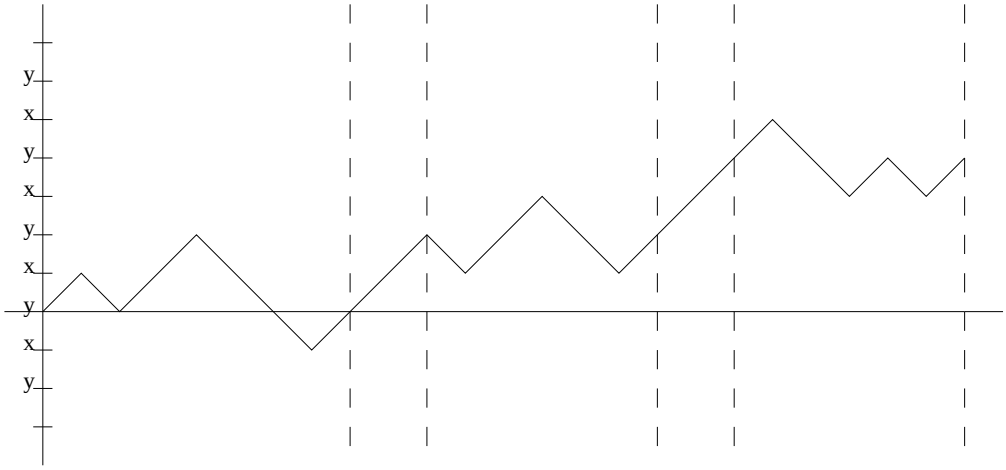
- If $k = 1$, the result is obvious.
- To prove the result for k, we have to show that along the chain between D and D', the shape of the k-th column has changed as many times by appearance of a white x-column as by appearance of a white y-column (i.e., sum of d equal to zero) and as many times by disappearance of a crossed x-column as by disappearance of a crossed y-column (i.e., sum of d' equal to zero).

  We suppose that our column (assume it is a y-column in D and D') changes more times by appearance of a white x-column than by appearance of a white y-column. Let us observe the subchain on the figure below:



Let h_1 denote the height of the y-column of drawing 1 and h'_1 the depth of the first x-column on its right. We observe that $b_4 = d_3 \geq 0$ (Case 2 of the Characterization) and that $d_2 = h'_1 - d_1 - d'_1$ since $b_3 = 0 = b_2 - d_2$ (Case 1). Thus: $d_1 + d_2 = h'_1 - d'_1$.

We now visualize the changes of shape at place k between D and D' on the following representation.



The even coordinates correspond to a y-column at place k, the odd ones to an x-column. A north-east line is either the appearance of a white x-column or disappearance of a crossed x-column (according to odd or even coordinate) and a south-east line is either the appearance of a white y-column or disappearance of a crossed y-column. The vertical dotted lines are defined as follows. The first is placed at the last point for which the coordinate is equal to zero. Then we have clearly two north-east lines and we put another dotted line. Then we restart with taking coordinate 2 as a new zero for the coordinates.

Let us suppose that between D and D' there is a single ascent (i.e., a subchain like 1-2-3-4). If we verify that $d_0 + d_1 + d_2 > 0$, where d_0 is the sum of the d before the ascent, then since $\sum d = 0$ between D and D', we have necesseraly some $d < 0$ after this sequence, which is impossible without a disappearance of the crossed y-column. That is what we wanted to show.

Let us prove that $d_0 + d_1 + d_2 > 0$.

Let b denote the number of white cells at place k of D then $b_1 = b + d_0$.

Hence:

$$d_0 + d_1 + d_2 = d_0 + h'_1 - d'_1 = b_1 - b + h'_1 - d'_1 = h_1 + h'_1 - b.$$

It is easy to check that $h_1 + h'_1 - b > 0$.

It remains to observe that when there are several ascents, the previous reasoning is still true, by looking at the last one. Indeed, it suffices to replace the equality $b_1 = b + d_0$ by $b_1 \leq b + d_0$ (thanks to what precedes), which keeps the result unchanged.

The proof of Lemma 3 is almost complete. It remains to observe that the symmetries between x and y and between crossed and white cells allow us to deal with the other cases.

□

Lemma 5 :
If there is no total completeness along the chain between D and D', then $D \neq D'$ which implies Theorem 3.

Proof. This is an easy consequence of Lemmas 2 and 4 and Remark 2. It suffices to look at the leftmost place for which the completeness fails: D' has more white cells (and less crosses) than D at this place.

□

4.4 End of the proof

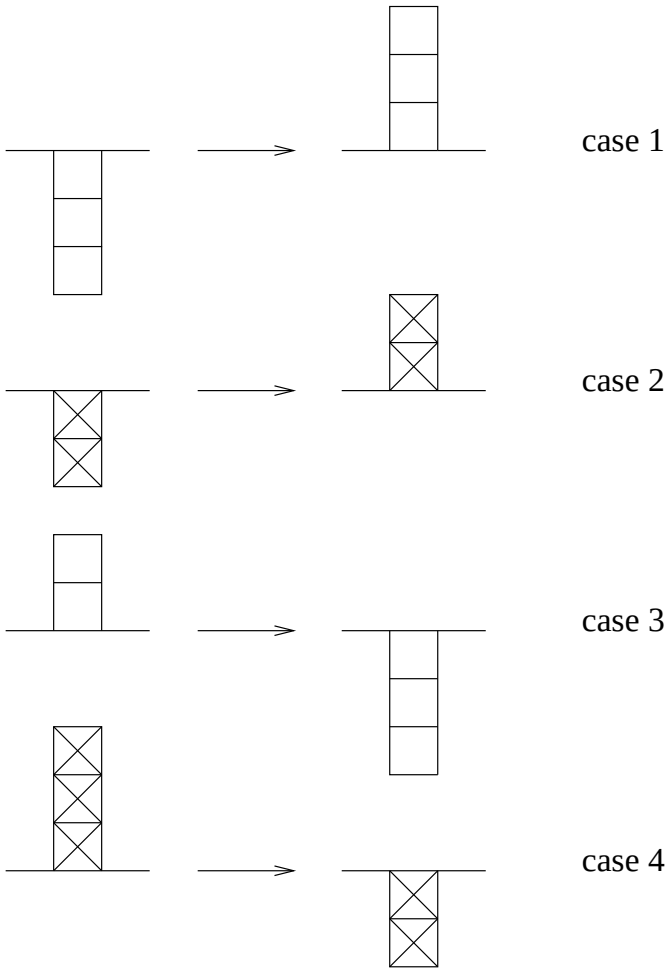
It is now sufficient to show that there is at least one generation between D and D' that is not complete. We shall in fact show that each generation is not complete.

Let again $D_1 = (S_1, T_1)$ and $D_2 = (S_2, T_2)$ denote two different drawings, father and son.

If D_1 and D_2 have the same shape, the result is obvious.

It then remains to study the case where D_1 and D_2 have different shape. We suppose that completeness holds and reduce it to the absurd.

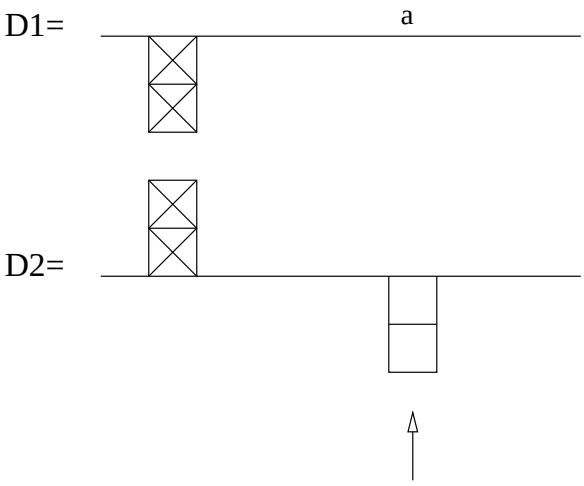
By looking at the place at most on the left where the shape changes, we can consider only the case where the shape changes at place 1. The only changes for which the non-completeness is not obvious are the following (remark that here $d = d' = 0$):



The following remark allows us to divide by two the number of cases:

Remark 4: D_2 is a son of D_1 if and only if flip(D_1) is a son of flip(D_2). This allows us to only consider cases 2 and 4.

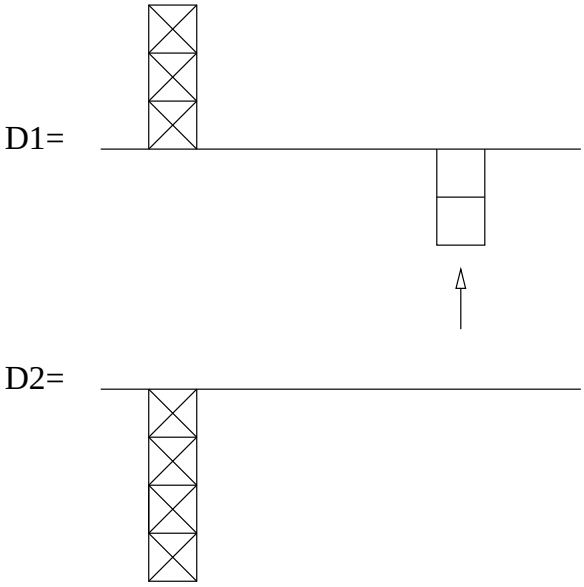
1. Case 2 :



If at place "a"(corresponding to the first plain crossed x-column in D_2), there is

- an x-column: we first verify that at each place on the left of "a" we have $d = 0$; then we show that the x-column in D_1 is smaller than the one in D_2, which contradicts $b_2 = b_1 - d = b_1$.
- a y-column: we first show that in each x-column of $T_1 + S_2$ there is at least one cell coming from D_1 and one coming from D_2. This is absurd since there are not enough x-columns.

2. Case 4:



In this case, if the first plain x-column of D_1 is the l-th x-column of D_1, we begin by observing that the x-column on its left have at least one white cell, hence have a contribution to $T_1 + S_2$. Thus on the left of this place there is already an x-column of depth $L - l + 1$ (there are at least l x-columns in $T_1 + S_2$ on the left). This is absurd.

5 Elements of 0 x-degree

5.1 Description

Let $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0)$ be any partition of n. The goal of this section is to give an explicit basis for M_μ^0, which denotes the homogeneous subspace of M_μ of elements of 0 x-degree. We construct this basis with the same visual objects as in the case of hooks. We also obtain a basis for the subspace of $n(\mu)$ x-degree which we shall denote by $M_\mu^{n(\mu)}$.

The space M_μ^0 has already been studied in [2] and [6]. In particular it is proved that its dimension is $n!/\mu'!$, where $\mu! = \mu_1! \dots \mu_k!$ and μ' is the conjugate of μ. In fact our basis is related to a family introduced in [2]. But we obtain here a direct (and not recursive) method of construction. Moreover we apply the monomial derivatives to Δ_μ itself and therefore obtain a simple and explicit basis for M_μ^0.

We use again the drawings introduced for hook-shaped partitions, here in the case of any partition of n. A shape is then made of $n-1$ bars. Each of these bars has n_x x-cells and n_y y-cells. The set of pairs (n_x, n_y) is the set of biexponents of the partition (the biexponent is omitted). We again put crosses in the shapes and the set of rules for these drawings is the following:

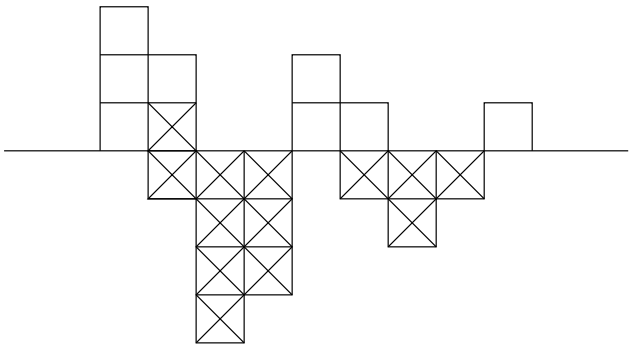
1. the bars with the same number of x-cells are arranged in decreasing height;

2. there are crosses in every x-cell;

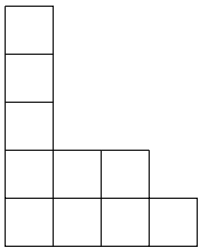
3. if a bar B is on the left of a bar with more x-cells than B and q y-cells, then the bar B must have at least $q+1$ y-white cells.

Remark 5: By applying flip we obtain a family of drawings with no x-crosses.

We now give an example of a drawing:



associated to the partition:

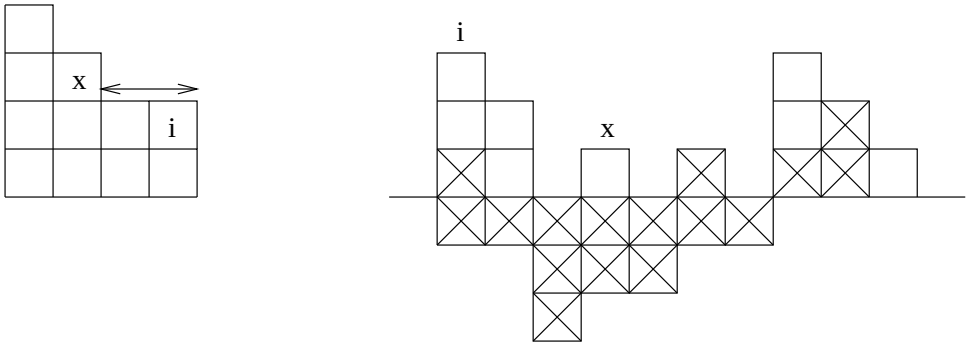


5.2 Enumeration

We verify that the number of drawings introduced in subsection 1 is $n!/\mu!$. We consider the drawing from the left to the right. The bar we are looking at corresponds to a corner of the Ferrers diagram of the partition from which we have removed the cells corresponding to the bars on the left.

Number the cells of the partition μ by writing i in the cell associated to the bar at place $n-i+1$ in the drawing. By the preceding paragraph, this gives a standard tableau.

We now look at the following figures:



We observe that the number of choices for cell i is the length of the arrow that we denote by $\mathrm{coarm}_i^*(T^{i+1})$, where T^{i+1} is the (standard) tableau T from which we have removed the cells numbered from $i+1$ to n.

We thus obtain that the cardinality is:

$$\sum_{T \text{standard}} \prod_{i=n}^{1} \mathrm{coarm}_i^*(T^{i+1}). \tag{1}$$

We show that this number equals $n!/\mu'!$ by induction on n. The result is obvious when $n=1$. We write $\mu' = (c_1^{\alpha_1}, \ldots, c_h^{\alpha_h})$, where the c_j's are the height of columns of μ and α_j their multiplicities. In particular, μ has h corners, $\mu'! = \prod_{i=j}^{h} (c_j!)^{\alpha_j}$, $n = \sum_{j=1}^{h} \alpha_j c_j$ and α_j is the contribution of corner j in the product of (1). We then rewrite this formula as:

$$\sum_{j=1}^{h} \alpha_j . \sum_{T'} \prod_{i=n-1}^{1} \mathrm{coarm}_i^*(T'^{i+1})$$

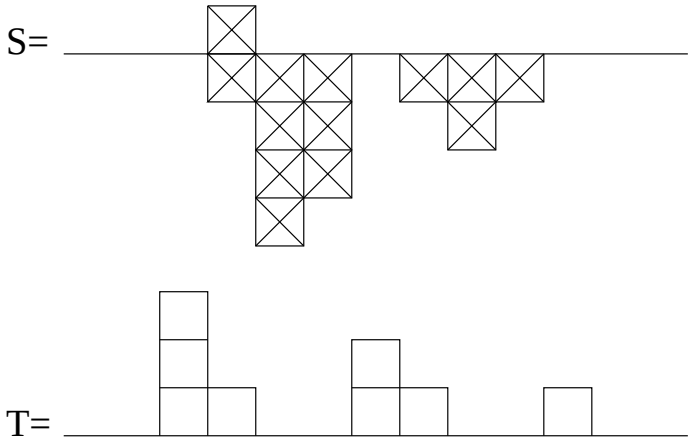
where T' varies amongst every standard tableaux of the Ferrers diagram from which we have removed its j-th corner (let μ^j denote the corresponding

partition). We are now able to conclude, since $\mu'^j! = \mu'!/c_j$:

$$\sum_{j=1}^{h} \alpha_j \frac{(n-1)!}{\mu'^j!} = \frac{(n-1)!}{\mu'!} \sum_{j=1}^{h} \alpha_j.c_j = \frac{n!}{\mu'!}.$$

5.3 Independence and conclusion

As in the case of the hook-shaped partitions, we denote by S (respectively T) the diagram consisting only of the crosses (respectively of the white cells) of a given drawing. For example in the case of the drawing of subsection 5.1, we have:



Let now ∂_S and ∂_T denote the derivative operators associated to S and T. Let also $\mathcal{S}$ and $\mathcal{T}$ denote the set of all S's and T's constructed by this way. We also associate to S and T a monomial by the same way. For example, in the case of the previous figures, we have: $M_S = x_2 y_2 x_3^4 x_4^3 x_6 x_7^2 x_8$ and $M_T = y_1^3 y_2 y_5^2 y_6 y_9$.

Theorem 4:
The set $\{\partial_S \Delta_\mu\}_{S \in \mathcal{S}}$ is linearly independent and hence is a basis of M_μ^0.
The set $\{\partial_T \Delta_\mu\}_{T \in \mathcal{T}}$ is linearly independent and hence is a basis of $M_\mu^{n(\mu)}$.

The theorem is a consequence of the two following lemmas.

Lemma 6:
We can reconstruct the drawing from S or T.

Lemma 7:
For the lexicographic order ($x_1 < x_2 < \cdots < x_n < y_1 < \cdots < y_n$) M_T is the minimal monomial for $\partial_S \Delta_\mu$ and M_S for $\partial_T \Delta_\mu$.

Proof. The proof of Lemma 6 is easy: we reconstruct the drawing from the left to the right, as in the case of hooks, thanks to the rules.

The proof of Lemma 7 requires attention only in the case of T, so we develop this point. Once the crossed cells have been fixed, we have to show that the white cells are at most on the left. It suffices in fact to show that the x-white cells can not be moved to the left. We show it by looking at the drawing from the left to the right. Let k and l denote the number of x-cells and of y-crossed cells at place p. We have to prove that a bar with $l' > l$ x-cells and $k' \geq k$ y-crossed cells is forbidden at place p. If the couple (k', l') is not a biexponent of the partition or if it is present on the left, we are done. To conclude we observe that this couple can not be a biexponent of the partition appearing on the right of the initial drawing. Indeed, because of the rules we should have: $k > k'$.

□

Remark 6: It is possible to show that our family of monomials $\{M_S\}_{S \in \mathcal{S}}$ is equal to the family B_μ of [2], section 4. But whereas B_μ was constructed recursively, our construction is direct. Moreover we apply it directly to Δ_μ and obtain simple and explicit bases for M_μ^0 and $M_\mu^{n(\mu)}$, whereas N. Bergeron and A. Garsia were dealing in [2] with linear translates of Garnir polynomials.

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