Iterative schemes for computing fixed points of nonexpansive mappings in Banach spaces

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Abstract

Let X be a real Banach space with a normalized duality mapping uniformly norm-to-weak* continuous on bounded sets or a reflexive Banach space which admits a weakly continuous duality mapping J_Φ with gauge ϕ. Let f be an α-*contraction* and $\{T_n\}$ a sequence of nonexpansive mapping, we study the strong convergence of explicit iterative schemes

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T_n x_n \tag{1}$$

with a general theorem and then recover and improve some specific cases studied in the literature [17, 8, 13, 14, 3, 9].

1 Introduction and preliminaries

Let X be a real Banach space, C a nonempty closed convex subset of X. Recall that a mapping $T : C \mapsto C$ is *nonexpansive* if $\|T(x) - T(y)\| \leq \|x - y\|$ for all x, $y \in C$ and a mapping $f : C \mapsto C$ is an α-*contraction* if there exists $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for all x, $y \in C$.

We denote by $Fix(T)$ the set of fixed points of T, that is

$$Fix(T) \stackrel{\text{def}}{=} \{x \in C \,:\, Tx = x\} \tag{2}$$

and Π_C will denote the collection of contractions on C.

Let X be a real Banach space. The (normalized) duality map $J : X \mapsto X^\star$, where $X^\star$ is the dual space of X, is defined by :

$$J(x) \stackrel{\text{def}}{=} \left\{x^\star \in X^\star \,:\, \langle x, x^\star\rangle = \|x\|^2 = \|x^\star\|^2\right\}$$

and there holds the inequality

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle \text{ where } x, y \in X \text{ and } j(x+y) \in J(x+y)\,.$$

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Recall that if C and F are nonempty subsets of a Banach space X such that C is nonempty closed convex and $F \subset C$, then a map $R : C \mapsto F$ is called a *retraction* from C onto F if $R(x) = x$ for all $x \in F$. A retraction $R : C \mapsto F$ is *sunny* provided $R(x + t(x - R(x))) = R(x)$ for all $x \in C$ and $t \geq 0$ whenever $x + t(x - R(x)) \in C$. A *sunny nonexpansive retraction* is a sunny retraction, which is also nonexpansive.

Suppose that F is the non empty fixed point set of a nonexpansive mapping $T : C \mapsto C$, that is $F = Fix\,T \neq \emptyset$ and assume that F is closed. For a given $u \in C$ and every $t \in (0, 1)$ there exists a fixed point, denoted x_t, of the $(1 - t)$-contraction $tu + (1 - t)T$. Then we define $Q : C \mapsto F = Fix(T)$ by $Q(u) \stackrel{\text{def}}{=} \sigma\text{-}\lim_{t\to 0} x_t$ when this limit exists (σ-lim denotes the strong limit). Q will also be denoted by $Q_{Fix(T)}$ when necessary and note that it is easy to check that, when it exists, Q is a nonexpansive retraction.

Consider now f an *α-contraction*, then $Q_{Fix(T)} \circ f$ is also an *α-contraction* and admits therefore a unique fixed point $\tilde{x} = Q_T \circ f(\tilde{x})$. We define by $\mathbf{Q}(f)$ or $\mathbf{Q}_{Fix(T)}(f)$ the mapping $\mathbf{Q}(f) : \Pi_C \to Fix\,(T)$ such that :

$$\mathbf{Q}(f) \stackrel{\text{def}}{=} \tilde{x} \quad \text{where} \quad \tilde{x} = (Q_{Fix(T)} \circ f)(\tilde{x}). \tag{3}$$

For $t \in (0, 1)$ we can also find a fixed point, denoted x_t^f of the $(1-(1-t)\alpha)$-contraction $tf+(1-t)T$ and if $\lim_{t\to 0} x_t^f$ is well defined we can define a mapping $\widetilde{\mathbf{Q}} : \Pi_C \mapsto Fix(T)$ by :

$$\widetilde{\mathbf{Q}}(f) \stackrel{\text{def}}{=} \lim_{t\to 0} x_t^f \quad \text{where} \quad x_t^f = tf(x_t^f) + (1 - t)Tx_t^f \tag{4}$$

We then gather know theorems under which Q, $\mathbf{Q}$ and $\widetilde{\mathbf{Q}}$ are defined and give relations between them.

When X is a uniformly smooth Banach space, denoted by $\mathcal{B}_{\text{us}}$, It is known [17, Theorem 4.1] that $\widetilde{\mathbf{Q}}(f)$ is well defined and equal to $\mathbf{Q}(f)$ and $\tilde{x} = \mathbf{Q}(f)$ is characterized by :

$$\langle \tilde{x} - f(\tilde{x}), J(\tilde{x} - p)\rangle \leq 0 \text{ for all } p \in F = Fix\,(T). \tag{5}$$

A special case is when f is a constant function $\mathbf{u}(x) = u$. Then [17, Theorem 4.1] shows that Q is well defined and that $Q(u) = \mathbf{Q}(\mathbf{u}) = P_{Fix\,T}u$ (where P_S is the metric projection on S). If X is a smooth Banach space, $R : C \mapsto F$ is a sunny nonexpansive retraction [6] if and only if the following inequality holds :

$$\langle x - Rx, J(y - Rx)\rangle \leq 0 \text{ for all } x \in C \text{ and } y \in F. \tag{6}$$

Q is thus the unique sunny non expansive retraction from C to $Fix\,T$. [17, Theorem 4.1] was already known in the case f constant and in the context of Hilbert spaces [17, Theorem 3.1] and [11, Theorem 2.1].

The same existence and characterization results can be found firstly when X is a reflexive Banach space which admits a weakly continuous duality mapping J_Φ with gauge ϕ, denoted by $\mathcal{B}_{\text{rwsc}}$, in [18, Theorem 3.1] (with f constant) and

[14, Theorem 2.2] (where J is the (normalized) duality mapping). Note that the limitation of f constant in [18] can be relaxed with [15]. Secondly when X is a reflexive and a strictly convex Banach space with a uniformly Gâteaux differentiable norm, denoted by $\mathcal{B}_{\rm rug}$, [13, Theorem 3.1]. Note that in this three Banach spaces cases listed here the normalized duality mapping is shown to be single valued.

The aim of this paper is to study the strong convergence of iterative schemes :

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T_n x_n \tag{7}$$

when X can be a $\mathcal{B}_{\rm us}$, or a $\mathcal{B}_{\rm rwsc}$, or a $\mathcal{B}_{\rm rug}$ real Banach space and $\{T_n\}$ is a sequence of nonexpansive mappings which share at least a common fixed point. We give a general framework to show that $\{x_n\}$ will converge strongly to $\tilde{x}$ where $\tilde{x}$ is the unique solution of (5) for a fixed nonexpansive mapping T related to the sequence $\{T_n\}$. The key ingredient is the fact that Lemma 26 given in section 3 is valid in the three previous context. Then we show that by specifying the sequence T_n we can recover and extend some known convergence theorems [17, 8, 13, 14, 3, 9]. Note also that in equation (7), f is an α-contraction, but following [15] it is easy to show that f can be replaced by a Meir-Keeler contraction (Lemma 31 in section 3 is devoted to this extension). The paper is organized as follows : a main theorem is proved in section 3 using a set of lemmas which are postponed to the last section of the paper and which are verbatim or slight extensions of know results. Then in a collection of subsections, known convergence theorems are revisited with shorter proofs.

2 Main theorem

In the sequel a $\mathcal{B}$ *real Banach space*, will denote when not specifically stated a real Banach space with a normalized duality mapping uniformly norm-to-weak* continuous on bounded sets (which is the case for $\mathcal{B}_{\rm us}$ or $\mathcal{B}_{\rm rug}$) or a reflexive Banach space which admits a weakly continuous duality mapping J_Φ with gauge ϕ ($\mathcal{B}_{\rm rwsc}$).

$\mathbf{H_{1,N}}$: For a fixed given $N \geq 1$ and a given sequence $\{\alpha_n\}$, a sequence of mappings $\{T_n\}$ will be said to verify $\mathbf{H_{1,N}}$, if for a given bounded sequence $\{z_n\}$, we have

$$\|(1-\alpha_{n+N})T_{n+N}z_n - (1-\alpha_n)T_n z_n\| \leq \delta_n M \tag{8}$$

with either (i) $\sum_0^\infty |\delta_n| < \infty$ or (i') $\limsup_{n\to\infty} \delta_n/\alpha_n \leq 0$ and M a constant.

Remark 1 *Note that using Lemma 30 $\{\delta_n\}$ can be replaced by $\{\mu_n + \rho_n\}$ where $\{\mu_n\}$ satisfies (i) and $\{\rho_n\}$ satisfies (i').*

Remark 2 *Note that when $\alpha_n \in (0,1)$ we have :*

$$\|(1-\alpha_{n+N})T_{n+N}z_n - (1-\alpha_n)T_n z_n\| \leq |\alpha_{n+N}-\alpha_n|\|T_{n+N}z_n\| + \|T_{n+N}z_n - T_n z_n\|. \tag{9}$$

Thus, when $\{\alpha_n\}$ *satisfies* $\mathbf{H_{3,N}}$ *(given below), if for each bounded sequence* $\{z_n\}$, $\{T_n z_n\}$ *is bounded and either* (vi) $\sum_{n=0}^{\infty} \|T_{n+N} z_n - T_n z_n\| < \infty$ *or* (vi') $\|T_{n+N} z_n - T_n z_n\|/\alpha_n \to 0$ *then* $\mathbf{H_{1,N}}$ *is satisfied (again using previous remark about mixing between conditions with or without prime). In the previous case,* $\mathbf{H_{1,N}}$ *is thus implied by* $\mathbf{H_{1,N}^{'}}$ *which is stated now :*

$\mathbf{H_{1,N}^{'}}$: For a fixed given $N \geq 1$ and a given sequence $\{\alpha_n\}$ which satisfies $\mathbf{H_{3,N}}$ a sequence of mappings $\{T_n\}$ will be said to verify $\mathbf{H_{1,N}^{'}}$, if given bounded sequence $\{z_n\}$, we have $\|T_{n+N} z_n - T_n z_n\| \leq \rho_n$ with either (vi) $\sum_{n=0}^{\infty} \rho_n < \infty$ or (vi') $\rho_n/\alpha_n \to 0$.

$\mathbf{H_{2,p}}$: For a given $p \in X$, a sequence $\{x_n\}$ will be said to verify $\mathbf{H_{2,p}}$ if we have

$$\limsup_{n\to\infty} \langle f(p) - p, J(x_n - p)\rangle \leq 0\,. \tag{10}$$

$\mathbf{H_{3,N}}$: For a fixed given $N \geq 1$, a sequence of real numbers $\{\alpha_n\}$ will be said to verify $\mathbf{H_{3,N}}$ if the sequence $\{\alpha_n\}$ is such that (i) $\alpha_n \in (0,1)$, (ii) $\lim_{n\to\infty} \alpha_n = 0$, (iii) $\sum_{n=0}^{\infty} \alpha_n = \infty$ and either (iv) $\sum_{n=0}^{\infty} |\alpha_{n+N} - \alpha_n| < \infty$ or (iv') $\lim_{n\to\infty}(\alpha_{n+N}/\alpha_n) = 1$.

We can now formulate the main theorem of the paper :

Theorem 3 *Let* X *be a* $\mathcal{B}$ *real Banach space,* C *a closed convex subset of* X, $T_n : C \mapsto C$ *a sequence of nonexpansive mapping,* T *a nonexpansive mapping and* $f \in \Pi_C$. *We assume that* $Fix(T) \neq \emptyset$ *and that for all* $n \in \mathbb{N}$ $Fix(T) \subset Fix(T_n)$. *Let* $\{\alpha_n\}$ *be a sequence of real numbers for which there exists a fixed* $N \geq 1$ *such that* $\mathbf{H_{3,N}}$ *is satisfied and suppose that there exists* $p \in Fix(T)$ *such that* $\mathbf{H_{2,p}}$ *is satisfied, then the sequence* $\{x_n\}$ *defined by (34) converges strongly to* p.

Proof : The proof uses a set of Lemmas which are given in section 3. Since p is in $Fix(T_n)$ for all n we can use Lemma 23 to obtain the boundedness of the sequence $\{x_n\}$. Thus we can conclude using Lemma 28. □

Corollary 4 *Assume that the hypothesis of Theorem 3 except* $\mathbf{H_{2,p}}$ *are satisfied. Suppose that* $\mathbf{H_{1,N}}$ *or* $\mathbf{H_{1,N}^{'}}$ *is satisfied and that for each bounded sequence* $\{y_n\}$, *the sequence* $\|T_n y_n - T y_n\| \to 0$. *Then the conclusion of Theorem 3 remains for* $p = \mathbf{Q}(f)$.

Proof : We just need to prove that $\mathbf{H_{2,p}}$ is satisfied for $p = \mathbf{Q}(f)$. We first show that if $\mathbf{H_{1,N}^{'}}$ is satisfied then $\mathbf{H_{1,N}}$ is also satisfied. As in previous theorem $\{x_n\}$ is a bounded sequence. Then, let $p \in Fix(T)$, we have :

$$\begin{aligned} \|T_n x_n - T x_n\| &\leq \|T_n x_n - T_n p\| + \|T_n p - T p\| + \|T p - T x_n\| \\ &\leq 2\|x_n - p\| + \|T_n p - T p\|. \end{aligned}$$

and since $\|T_n p - T p\| \to 0$ by hypothesis we have that $\{T_n(x_n)\}$ is bounded. As shown in remark 2 we are within the case where $\mathbf{H_{1,N}}$ is implied by $\mathbf{H_{1,N}^{'}}$.

Applying Lemma 24 and Corollary 25 we obtain the convergence of $\|Tx_n - x_n\|$. We can then apply Lemma 26 to obtain $\mathbf{H_{2,p}}$ for $p = \mathbf{Q}(f)$. □

Corollary 4 can be extended as follows when a constant T cannot be found.

Corollary 5 *Assume that the hypothesis of Theorem 3 except* $\mathbf{H_{2,p}}$ *are satisfied. Suppose that* $\mathbf{H'_{1,N}}$ *is satisfied and that* $\{T_n x_n\}$ *is bounded and that from each subsequence* $\sigma(n)$ *we can extract a subsequence* $\mu(n)$ *and find a fixed mapping* T_μ *such that*

$$\|T_{\mu(n)} x_{\mu(n)} - T_\mu x_{\mu(n)}\| \to 0.$$

If $F = Fix\,(T_\mu)$ *does not depend on* μ*, then the conclusion of Theorem 3 remains for* $p = \mathbf{Q}_F(f)$*.*

Proof : We just need to prove that $\mathbf{H_{2,p}}$ is satisfied for $p = \mathbf{Q}(f)$. Using remark 2 we are in the case where $\mathbf{H_{1,N}}$ is implied by $\mathbf{H'_{1,N}}$. Using $\mathbf{H_{1,N}}$ we first easily obtain that $\|x_n - T_n x_n\| \to 0$ by an argument similar to Corollary 25. Then $\mathbf{H_{2,p}}$ for $p = \mathbf{Q}(f)$ follows from Corollary 27. □

We can now consider the case of composition. Assume that $\{T_n^1\}$ and $\{T_n^2\}$ satisfy $\mathbf{H'_{1,N}}$ with sequences denoted by ρ_n^i. Assume also that for a bounded sequence $\{z_n\}$ then the sequences $\{T_{n+N}^2 z_n\}$ and $\{T_{n+N}^1 T_{n+N}^2 z_n\}$ and also bounded. Then it is straightforward, since the mappings T_n^1 are nonexpansive, that :

$$\|T_{n+N}^1 T_{n+N}^2 z_n - T_n^1 \cdots T_n^2 z_n\| \quad \leq \quad \rho_n^1 + \|T_{n+N}^2 z_n - T_n^2 z_n\|\,.$$

Thus the composition $T_n^1 \circ T_n^2$ satisfy $\mathbf{H'_{1,N}}$ with $\rho_n \stackrel{\text{def}}{=} \rho_n^1 + \rho_n^2$. This lead us to propose the following Corollary for dealing with composition :

Corollary 6 *Assume that the hypothesis of corollary 5 are satisfied for the sequence* $\{T_n^1\}$ *with* $\mathbf{H'_{1,N}}$ *and for* $\{T_n^2\}$ *also with* $\mathbf{H'_{1,N}}$*. Then the conclusion of Theorem 3 remains for the sequence* $\{T_n^1 \circ T_n^2\}$ *with* $p = \mathbf{Q}_F(f)$ *and* $F = Fix(T_\mu^1 \circ T_\rho^2)$*.*

*Proof :*As pointed out before the statement of the corollary the composition $T_n^1 \circ T_n^2$ satisfy $\mathbf{H'_{1,N}}$. Consider a subsequence $\sigma(n)$ we can find first a subsequence $\mu_2(n)$ and μ_2 such that :

$$\|T_{\mu(n)}^2 x_{\mu(n)} - T_\mu^2 x_{\mu(n)}\| \to 0.$$

Then, using properties of the T_n^1 sequence, we can re-extract a new subsequence $\rho(n)$ and ρ such that :

$$\|T_{\rho(n)}^1 T_{\rho(n)}^2 x_{\rho(n)} - T_\rho^1 T_{\rho(n)}^2 x_{\rho(n)}\| \to 0.$$

Since we have :

$$\begin{aligned}\|T^1_{\rho(n)}T^2_{\rho(n)}x_{\rho(n)} - T^1_\rho T^2_\mu x_{\rho(n)}\| &\leq \|T^1_{\rho(n)}T^2_{\rho(n)}x_{\rho(n)} - T^1_\rho T^2_{\rho(n)}x_{\rho(n)}\| \\ &+ \|T^2_{\rho(n)}x_{\rho(n)} - T^2_\rho x_{\rho(n)}\|\end{aligned}$$

When obtain the conclusion for the composition. □

Recall that a mapping T is *attracting non expansive* if it is nonexpansive and satisfies :

$$\|Tx - p\| < \|x - p\| \text{ for all } x \notin Fix\, T \text{ and } p \in Fix\, T. \tag{11}$$

In particular a *firmly nonexpansive* mapping, *i.e* $\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle$ is attracting nonexpansive [6].

Remark 7 *In the previous corollary, we obtain a fixed point of a composition and in practice the aim is to obtain a common fixed point of two mappings. If the mappings T^1_μ and T^2_ρ are attracting, have a common fixed point and T^1_μ or T^2_ρ is attracting then we will have $Fix\, T^1_\mu \cap Fix\, T^2_\rho = Fix\, T^1_\mu \circ T^2_\rho$. The proof is contained in [1, Proposition 2.10 (i)] and given in Lemma 32 for completeness.*

Remark 8 *Note that if X is a strictly convex Banach space, then for $\lambda \in (0,1)$ the mapping $T_\lambda \stackrel{def}{=} (1-\lambda)I + \lambda T$ is attracting nonexpansive when T is nonexpansive. Extension to a set of N operators is immediate by induction. This gives a way to build attracting nonexpansive mappings and mixed with previous remark it gives [16, Proposition 3.1].*

Remark 9 *Note also that, when X is strictly convex, an other way to obtain $F = \cap_i Fix\,(T_i)$ for a sequence of nonexpansive mappings $\{T_i\}$ is to use $T = \sum_i \lambda_i T_i$ with a sequence $\{\lambda_i\}$ of real positive numbers such that $\sum_i \lambda_i = 1$ [2, Lemma 3].*

2.1 Example 1

Theorem 10 *[17, Theorem 4.2] Let X be a B real Banach space, C a closed convex subset of X, $T : C \mapsto C$ a nonexpansive mapping with $Fix(T) \neq \emptyset$, and f an α-contraction. Then when the sequence $\{\alpha_n\}$ satisfies $\mathbf{H_{3,1}}$ the sequence $\{x_n\}$ defined by (34) with $T_n \stackrel{def}{=} T$ converges strongly to $\mathbf{Q}(f)$.*

Proof : Here the sequence T_n does not depend on n. We just apply Corollary 4 to get the result. Of course, if the sequence $\{x_n\}$ is bounded then $\{T_n(x_n) = Tx_n\}$ is bounded and equation (8) of $\mathbf{H_{1,1}}$ is then satisfied with $\delta_n = |\alpha_n - \alpha_{n+1}|$. Since $\{\alpha_n\}$ satisfies $\mathbf{H_{3,1}}$, $\{\delta_n\}$ satisfies $\mathbf{H_{1,1}}$. We also have $\|T_n x_n - Tx_n\| = 0 \to 0$ and the conclusion follows. □

Remark 11 *Suppose now that* $T \stackrel{def}{=} \sum_i \lambda_i T_i$ *where* $\{\lambda_i\}$ *is a sequence of positive real numbers such that* $\sum_i \lambda_i = 1$ *and the* T_i *mappings are all supposed nonexpansive. Then, we can apply Theorem 10 to obtain the strong convergence of the sequence* $\{x_n\}$ *to* $\mathbf{Q}_{Fix\,T}(f)$*. Moreover, If we assume that* X *is strictly convex then using remark 9 we obtain a strong convergence to* $\mathbf{Q}_F(f)$ *with* $F \stackrel{def}{=} \cap_{i\in I} Fix(T_i)$*.*

This can be extended to the case when the λ_i also depends on n and recover [9, Theorem 4] as follows :

Corollary 12 *Let* X *be a strictly convex B real Banach space,* C *a closed convex subset of* X*,* $T_i : C \mapsto C$ *for* $i \in I$ *a finite family of nonexpansive mapping with* $\cap_{i\in I} Fix(T_i) \neq \emptyset$*, and* f *an* α*-contraction. For a sequence* $\{\alpha_n\}$ *satisfying* $\mathbf{H_{3,1}}$ *we consider the sequence* $\{x_n\}$ *defined by* (34) *with* $T_n \stackrel{def}{=} \sum_{i\in I} \lambda_{i,n} T_i$*. Assume that for all* i *and* n $\lambda_{i,n} \in [a, b]$ *with* $a > 0$ *and* $b < \infty$ *either* $\sum_n \lambda_{i,n} < \infty$ *or* $\lambda_{i,n}/\alpha_n \to 0$ *then* $\{x_n\}$ *converges strongly to* $\mathbf{Q}_F(f)$ *with* $F = \cap_{i\in I} Fix(T_i)$

*Proof :*The proof is given by an application of corollary 5. Indeed since the $\lambda_{i,n}$ are bounded $T_n x_n$ remains bounded for a bounded sequence x_n. Then T_n satisfies $\mathbf{H'_{1,1}}$ with $\rho_n = \sum_{i\in I} \lambda_{i,n}$. By extracting from each given subsequence $\sigma(n)$ a subsequence $\mu(n)$ such that $\lim_{n\to\infty} \lambda_{i,\mu(n)} = \overline{\lambda}_i$ for all $i \in I$ we can use corollary 5. Finally, noting that, for a strictly convex space X, the fixed points of $T_{\overline{\lambda}} \stackrel{\text{def}}{=} \sum_{i\in I} \overline{\lambda}_i T_i$ does not depend on $\overline{\lambda}$ and is equal to $\cap_{i\in I} Fix(T_i)$ we conclude the proof. □

2.2 Example 1′

In [14] The following algorithm is considered :

$$y_{n+1} = P(\alpha_n f(y_n) + (1 - \alpha_n) T y_n) \tag{12}$$

Where $P : X \mapsto C$ is a sunny nonexpansive retraction, $f : C \mapsto X$ an α-contraction and $T : C \mapsto X$ a nonexpansive mapping such that $Fix(T) \neq \emptyset$.

If we consider the sequence $x_{n+1} = \alpha_n f(y_n) + (1 - \alpha_n) T y_n$ then we have $y_{n+1} = P x_{n+1}$ and thus

$$x_{n+1} = \alpha_n f(P(x_n)) + (1 - \alpha_n) T(P(x_n)) \tag{13}$$

Since $f \circ P$ is an α-contraction from X onto X and $T \circ P$ a non expansive mapping from X onto X we can use the previous theorem to obtain the strong convergence of the sequence $\{x_n\}$ to x a fixed point of $T \circ P$ such that $x = P_{Fix(T\circ P)} f(T(x))$ (P_S is the metric projection on S). We thus obtain now the strong convergence of the initial sequence $\{y_n\}$ to $y = P(x)$ and since x is a fixed point of $T \circ P$, y is a fixed point of $P \circ T$.

If we suppose in addition that X is such that J (or J_ϕ) is norm-to-weak* continuous (i.e X is smooth) and that T satisfy the weakly inward condition then

we can use the result of [14, Lemma 1.2] which state that $Fix(T) = Fix(P \circ T)$ to conclude that y is in fact a fixed point of T and recover the result of [14, Theorem 2.4].

2.3 Example 2

We consider now the example given in [8] where the sequence $\{x_n\}$ is given by :

$$y_n = \beta_n x_n + (1-\beta_n)Tx_n$$
$$x_{n+1} = \alpha_n u + (1-\alpha_n)y_n$$

With a sequence of mappings $T_n x \stackrel{\text{def}}{=} \beta_n x + (1-\beta_n)Tx$. This problem is rewritten as follows :

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T_n x_n \tag{14}$$

Theorem 13 *Let X be a $\mathcal{B}$ real Banach space, C a closed convex subset of X, $T : C \mapsto C$ a nonexpansive mapping with $Fix(T) \neq \emptyset$, and f an α-contraction. When the sequence $\{\alpha_n\}$ satisfies $\mathbf{H_{3,1}}$ and the sequence $\{\beta_n\}$ converges to zero and satisfy either $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ or $|\beta_{n+1} - \beta_n|/\alpha_n \to 0$. Then, the sequence $\{x_n\}$ defined by (14) converges strongly to $\mathbf{Q}(f)$.*

This theorem is very similar to [8, Theorem 1] where f was supposed to be constant. It could be covered by corollary 12 but here strict convexity is not needed.

Proof : We easily check that the fixed points p of T are fixed points of T_n for all $n \in \mathbb{N}$ and T_n is nonexpansive for all n. Thus by Lemma 23 the sequence $\{x_n\}$ is bounded . If the sequence $\{x_n\}$ is bounded then $\|T_n(x_n)\| \leq max(\|x_n\|, \|Tx_n\|)\}$ is bounded too. Since :

$$\|T_n y_n - Ty_n\| \leq \beta_n(\|y_n\| + \|Ty_n\|) \tag{15}$$

we have $\|T_n y_n - Ty_n\| \to 0$ for each bounded sequence $\{y_n\}$. It is easily checked that $\mathbf{H_{1,1}}$ is satisfied with $\delta_n = |\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n|$. The conclusion follows from Corollary 4. □

2.4 Example 3

We consider here the accretive operators example given in [8] or [18] :

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T_n x_n \tag{16}$$

Where $T_n x = J_{r_n} x$ and J_λ is the resolvent of an m-accretive operator A, $J_\lambda x = (I + \lambda A)^{-1}$. The following theorem is similar to [18, Theorem 4.2, Theorem 4.4] or [8, Theorem 2].

Theorem 14 *Let X be a $\mathcal{B}$ real Banach space, A an m-accretive operator in X such that $A^{-1}(0) \neq \emptyset$. We assume here that $C \stackrel{def}{=} \overline{D(A)}$ where $D(A)$ is the domain of A and suppose that C is convex. Suppose that $\mathbf{H_{3,1}}$ is satisfied by the sequence $\{\alpha_n\}$ and that the sequence r_n is such that $r_n \geq \epsilon > 0$ and either $\sum_0^\infty |1 - r_n/r_{n+1}| < \infty$ or $|1 - r_n/r_{n+1}|/\alpha_n \to 0$, then the sequence $\{x_n\}$ defined by (16) converges strongly to a zero of A.*

Proof : We first note that [18, p 632], for $\lambda > 0$, $Fix\,(J_\lambda) = F$ where F is the set of zero of A and for an m-accretive operator A, J_λ is non expansive from $X \mapsto \overline{D(A)}$. Using the resolvent identity $J_\lambda x = J_\mu((\mu/\lambda)x + (1 - \mu/\lambda)J_\lambda x)$ we obtain :

$$\|T_{n+1}z_n - T_n z_n\| \leq \left|1 - \frac{r_n}{r_{n+1}}\right| (\|z_n\| + \|T_n z_n\|) \tag{17}$$

and since the sequence $T_n y_n$ is bounded for a bounded sequence y_n (for $p \in A^{-1}(0)$ we have $\|T_n y_n - p\| \leq \|y_n - p\|$) we can apply remark 2 in order to obtain $\mathbf{H_{1,1,.}}$ We thus have $\|x_{n+1} - x_n\| \to 0$ by Lemma 24 and $\|x_n - T_n x_n\| \to 0$ by :

$$\begin{aligned} \|x_n - T_n x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_n x_n\| \\ &\leq \|x_n - x_{n+1}\| + \alpha_n(\|f(x_n)\| + \|T_n(x_n)\|) \end{aligned}$$

Take now r such that $0 < r < \epsilon$ and define $T \stackrel{\text{def}}{=} J_r$ then we have :

$$\|T_n x_n - T x_n\| \leq \left|1 - \frac{r}{r_n}\right| \|x_n - T_n x_n\| \tag{18}$$

We thus obtain that $x_n - T x_n \to 0$ from :

$$\|x_n - T x_n\| \leq \|x_n - T_n x_n\| + \|T_n x_n - T x_n\| \tag{19}$$

The conclusion is obtained through Corollary 4. □

2.5 Example 4

We consider here the example given in [13]

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T_n y_n \tag{20}$$

where $T_n = Q_{n \bmod N}$, where $N \geq 1$ is a fixed integer and the $(Q_l)_{l=0,\ldots,N-1}$ is a family of nonexpansive mappings.

Theorem 15 *Let X be a $\mathcal{B}$ real Banach space, C a closed convex subset of X, $Q_l : C \mapsto C$ for $l \in \{1, \ldots, N\}$ a family of nonexpansive mappings such that $F \stackrel{def}{=} \cap_{l=0}^{N-1} Fix(Q_l)$ is not empty and*

$$\cap_{l=0}^{N-1} Fix(Q_l) = Fix(T_{n+N} T_{n+N-1} \cdots T_{n+1}) \text{ for all } n \in \mathbb{N} \tag{21}$$

and f an α-contraction. When the sequence $\{\alpha_n\}$ satisfies $\mathbf{H_{3,N}}$ then the sequence $\{x_n\}$ defined by (20) converges strongly to $\mathbf{Q}_F(f)$.

Proof : By Lemma 23, since the T_n have a common fixed point, the sequence $\{x_n\}$ is bounded. Since the sequence of mappings T_n is periodic, the sequence $\{T_n x_n\}$ is bounded and equation (8) of $\mathbf{H_{1,N}}$ is obtained for $\delta_n = |\alpha_n - \alpha_{n+N}|$ using (9). Since $\{\alpha_n\}$ satisfies $\mathbf{H_{3,N}}$, $\{\delta_n\}$ satisfies $\mathbf{H_{1,N}}$. Thus, using Lemma 24 we obtain that $\|x_{n+N} - x_n\| \to 0$. Since $\|x_{n+1} - T_n x_n\| \leq \alpha_n(\|f(x_n)\| + \|T_n x_n\|)$, we have $\|x_{n+1} - T_n x_n\| \to 0$. We introduce the sequence of mappings $A_n^{(N,\alpha)} \stackrel{\text{def}}{=} T_{n+N-1} \cdots T_{n+\alpha}$ for $\alpha \neq N$ and $A_n^{(N,N)} = Id.$ Using Lemma 16, given just after this proof, we conclude that : $\|x_{n+N} - A_n^{(N,0)} x_n\| \to 0$. This combined with $\|x_{n+N} - x_n\| \to 0$ gives $\|x_{n+N} - A_n^{(N,0)} x_n\| \to 0$. Note now that the mappings $A_n^{(N,0)}$ are in finite number are all nonexpansive and share common fixed points by hypothesis. Thus we can prove that $\mathbf{H_{2,p}}$ is satisfied for $p = \mathbf{Q}_F(f)$. Let $p = \mathbf{Q}_F(f)$ we suppose that $\mathbf{H_{2,p}}$ is not satisfied, then it possible to extract a subsequence of $\{x_{\sigma(n)}\}$ such that :

$$\lim_{n\to\infty} \left\langle f(p) - p, J(x_{\sigma(n)} - p) \right\rangle \leq 0 \tag{22}$$

But it is then possible to find $q \in \{0, \ldots, N-1\}$ and an extracted new subsequence $\mu(n)$ from $\sigma(n)$ such that $\mu_{(n)} \mod N = q$. We thus have $\|x_{\mu(n)} - T x_{\mu(n)}\| \to 0$, with $T \stackrel{\text{def}}{=} A_q^{(N,0)}$ which is now a fixed mapping and $Fix\,(T) = F$. Then $\mathbf{H_2}$p should be true by Lemma 26 and this leads to a contradiction. The conclusion follows by 28. □

Lemma 16 *Let $N \in \mathbb{N}$, $\alpha \in \{0, \ldots, N\}$ and $A_n^{(N,\alpha)} \stackrel{def}{=} T_{n+N-1} \cdots T_{n+\alpha}$ for $\alpha \neq N$ and $A_n^{(N,N)} = Id.$ Assume that $\|x_{n+1} - T_n x_n\| \to 0$ then $\|x_{n+N} - A_n^{(N,0)} x_n\| \to 0$.*

Proof : We have for $\alpha \in \{0, \ldots, N-1\}$ by definition of $A_n^{(N,\alpha)}$ and using the fact that $A_n^{(N,\alpha)}$ is nonexpansive :

$$\begin{aligned} \|A_n^{(N,\alpha+1)} x_{n+\alpha+1} - A_n^{(N,\alpha)} x_{n+\alpha}\| &= \|A_n^{(N,\alpha+1)} x_{n+\alpha+1} - A_n^{(N,\alpha+1)} T_{n+\alpha} x_{n+\alpha}\| \\ &\leq \|x_{n+\alpha+1} - T_{n+\alpha} x_{n+\alpha}\| \end{aligned}$$

Thus :

$$\|x_{n+N} - A_n^{(N,0)} x_n\| \leq \sum_{\alpha=0}^{N-1} \|x_{n+\alpha+1} - T_{n+\alpha} x_{n+\alpha}\|$$

and the result follows. □

2.6 Example 5

Let $\Gamma_n^{(j)}$ for $j \in \{1, \ldots, m\}$ be a sequence of mappings defined recursively as follows :

$$\Gamma_n^{(j)} x \stackrel{\text{def}}{=} \beta_n^{(j)} x + (1 - \beta_n^{(j)}) T_j \Gamma_n^{(j+1)} x \text{ and } \Gamma_n^{(m+1)} x = x \tag{23}$$

where the sequences $\{\beta_n^{(j)}\} \in (0,1)$, and $\{T_j\}$ for $j \in \{1,\ldots,m\}$ are nonexpansive mappings. We want to prove here the convergence of the sequence generated by the iterations :

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\Gamma_n^{(1)} x_n \tag{24}$$

Theorem 17 *Let X be a $\mathcal{B}$ real Banach space, C a closed convex subset of X, $T_j : C \mapsto C$ for $j \in \{1,\ldots,m\}$ a family of nonexpansive mappings such that $\cap_{l=1}^m Fix(T_j)$ is not empty and f an α-contraction. When the sequence $\{\alpha_n\}$ satisfies $\mathbf{H_{3,N}}$ and for $j \in \{1,\ldots,m\}$ the sequences $\{\beta_n^{(j)}\}$ satisfy $\lim_{n\to\infty} \beta_n^{(j)} = 0$ and either $\sum_{n=0}^{\infty} |\beta_{n+1}^{(j)} - \beta_n^{(j)}| < \infty$ or $|\beta_{n+1}^{(j)} - \beta_n^{(j)}|/\alpha_n \to 0$ then the sequence defined by (24) converges strongly to $\mathbf{Q}_F(f)$ associated to $F = Fix\,(T_1 \cdots T_m)$.*

Proof : Note first that by an elementary induction $\Gamma_n^{(1)}$ is a nonexpansive mapping. If we assume that p is a common fixed point to the mappings T_i then p is a fixed point of the mappings $\Gamma_n^{(j)}$. By Lemma 23 the sequence $\{x_n\}$ is bounded. Then using Lemma 19 , given just after this proof, combined with the boundedness of $\{x_n\}$, $\mathbf{H_{1,1}}$ is valid with

$$\delta_n = \sum_{p=1}^{m} |\beta_{n+1}^{(p)} - \beta_n^{(p)}| + |\alpha_{n+1} - \alpha_n| \tag{25}$$

Now if we can prove that

$$\|\Gamma_n^{(1)} x_n - T_1 T_2 \cdots T_m x_n\| \to 0 \tag{26}$$

the conclusion will be given by Corollary 4. The last assetion can easily be obtained by induction on $\|\Gamma_n^{(j)} x_n - T_j \cdots T_m x_n\|$, since we have :

$$\begin{aligned} \|\Gamma_n^{(j)} x_n - T_j \cdots T_m x_n\| &\leq \beta_n^{(j)}(\|x_n\| + \|T_j \cdots T_m x_n\|) \\ &\quad +(1-\beta_n)\|T_j \Gamma_n^{(j+1)} x_n - T_j \cdots T_m x_n\| \\ &\leq \beta_n^{(j)}(\|x_n\| + \|T_j \cdots T_m x_n\|) + \|\Gamma_n^{(j+1)} x_n - T_{j+1} \cdots T_m x_n\|. \end{aligned}$$

□

Remark 18 *For $m = 1$ we obtain the same result as Theorem 13.*

Lemma 19 *Let $\Gamma_n^{(j)}$ be the sequence of mappings defined by (23) Then we have for $j \in \{1,\ldots,m\}$:*

$$\|\Gamma_{n+1}^{(j)} x - \Gamma_n^{(j)} x\| \leq \left\{ \sum_{p=j}^{m} |\beta_{n+1}^{(p)} - \beta_n^{(p)}| \right\} K \tag{27}$$

where K is a constant which depends on the mappings $(T_p)_{p\geq j}$ and x.

Proof : Note first that :

$$\|\Gamma_n^{(j)}x\| \leq \|x\| + \|T_j(\Gamma_n^{(j+1)}x)\| \tag{28}$$

which applied recursively shows that $\|\Gamma_n^{(j)}x\|$ is bounded by a constant which depends on the mappings $(T_p)_{p\geq j}$ and x and not on n. Then, using the definition of $\Gamma_n^{(j)}$ we have :

$$\begin{aligned}\|\Gamma_{n+1}^{(j)}x - \Gamma_n^{(j)}\| &\leq |\beta_{n+1}^{(j)} - \beta_n^{(j)}|(\|x\| + \|T_j\Gamma^{(j+1)}x\|) \\ &\quad + \|T_j\Gamma_{n+1}^{(j+1)}(x) - T_j\Gamma_n^{(j+1)}(x)\| \end{aligned} \tag{29}$$

since T_j is nonexpansive mappings :

$$\|\Gamma_{n+1}^{(j)}x - \Gamma_n^{(j)}\| \leq |\beta_{n+1}^{(j)} - \beta_n^{(j)}|(\|x\| + \|T_j\Gamma^{(j+1)}x\|) + \|\Gamma_{n+1}^{(j+1)}(x) - \Gamma_n^{(j+1)}(x)\|$$

by recursion and since the last term $\Gamma_{n+1}^{(m+1)}(x) - \Gamma_n^{(m+1)}(x) = 0$ we obtain the result. □

Note that Lemma 19 remains valid for the sequence

$$\Gamma_n^{(j)}x \stackrel{\text{def}}{=} \beta_n^{(j)}g(x) + (1-\beta_n^{(j)})T_j\Gamma_n^{(j+1)}x \text{ and } \Gamma_n^{(m+1)}x = x \tag{30}$$

if g is a nonexpansive mapping.

2.7 Example 6

We consider here the example given in [3]

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T_n x_n$$

where $T_n x \stackrel{\text{def}}{=} P_C(x - \lambda_n Ax)$ and P_C is the metric projection from X to C. The aim is to find a solution of the variational inequality problem which is to find $x \in C$ such that $\langle Ax, y - x\rangle \geq 0$ for all $y \in C$. The set of solution of the variational inequality problem is denoted by $\mathrm{VI}(C, A)$. The operator A is said to be *μ-inverse-strongly monotone* if

$$\langle x - y, Ax - Ay\rangle \geq \mu\|Ax - Ay\|^2 \text{ for all } x, y \in C$$

The next theorem is similar to [3, Proposition 3.1].

Theorem 20 *Let X be a real Hilbert space, C a nonempty closed convex, f an α-contraction, and let A be a μ-inverse-strongly monotone mapping of H into itself such that $VI(C, A) \neq \emptyset$. Assume that $\mathbf{H_{3,1}}$ is satisfied and that $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\mu$ and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$. then the sequence $\{x_n\}$ generated by (31) converges strongly to $\mathbf{Q}_F(f)$ associated to $F = Fix\,(T_\lambda)$ where $T_\lambda(x) \stackrel{def}{=} P_C(x - \lambda Ax)$. $F = Fix\,(T_\lambda)$ does not depend on λ for $\lambda > 0$ and equals $VI(C, A)$.*

Proof :For $\lambda > 0$, let $T_\lambda x \stackrel{\text{def}}{=} P_C(x - \lambda Ax)$. When X is an Hilbert space we have $Fix(T_\lambda) = \text{VI}(C, A)$. When A is μ-inverse-strongly monotone then for, $\lambda \leq 2\mu$, $I - \lambda A$ is nonexpansive. Thus the mappings T_n are non expansive and $Fix\,(T_n) = \text{VI}(C, A) \neq \emptyset$. By Lemma 23 the sequence $\{x_n\}$ is bounded. Since $\|T_n z\| \leq K(\|z\| + 2\mu\|Az\|)$, the sequence $\{T_n x_n\}$ is bounded too. We also have $\|T_{n+1}z_n - T_n z_n\| \leq |\lambda_{n+1} - \lambda_n|\|Az_n\|$ which gives $\mathbf{H_{1,N}}$ with $\delta_n = |\lambda_{n+1} - \lambda_n| + |\alpha_{n+1} - \alpha_n|$ by remark 2. The result follows now from Corollary 5. Indeed, since $\lambda_{\sigma(n)} \in [a, b]$ it is possible to extract a converging subsequence $\lambda_{\mu(n)} \to \overline{\lambda} \in [a, b]$ and we then have $\|T_{\mu(n)}z - T_{\overline{\lambda}}z\| \leq |\lambda_{\mu(n)} - \overline{\lambda}|\|Az\|$. Thus $\|T_{\mu(n)}x_{\mu(n)} - T_{\overline{\lambda}}x_{\mu(n)}\| \to 0$. □

Remark 21 *We can note that for $\lambda < 2\alpha$, $I - \lambda A$ is in fact attracting nonexpansive since :*

$$\|(I - \lambda A)x - (I - \lambda A)y\| \leq \|x - y\| + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2.$$

Thus it is also the case for $P_C \circ (I - \lambda A)$ [1]. For a nonexpansive mapping S we can consider the previous theorem with $T_\lambda x \stackrel{def}{=} S \circ P_C(x - \lambda Ax)$ and using Remark 7 (an Hilbert space is strcitly convex) to obtain a strong convergence to a point in $Fix\,(T_\lambda) = Fix\,S \cap VI(C, A)$ and thus fully recover [3, Proposition 3.1]

2.8 Example 7

We consider here the equilibrium problem for a bifunction $F : C \times C \mapsto \mathbb{R}$ where C is a closed convex subset of a real Hilbert space X. The problem is to find $x \in C$ such that $F(x, y) \geq 0$ for all $y \in C$. The set of solutions if denoted by $\text{EP}(F)$. It is proved in [5] (See also [4]) that for $r > 0$, the mapping $T_r : X \mapsto C$ defined as follows :

$$T_r(x) \stackrel{\text{def}}{=} \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\} \tag{31}$$

is such that T_r is singled valued, firmly nonexpansive ($i.e$ $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$ for any $x, y \in X$), $Fix(T_r) = \text{EP}(F)$ and $\text{EP}(F)$ is closed and convex if the bifunction F satisfies $(A_1) F(x, x) = 0$ for all $x \in C$, $(A_2) F(x, y) + F(y, x) \leq 0$ for all x, $y \in C$, (A_3) for each x, y, $z \in C$ $\lim_{t\to 0} F(tz+(1-t)x, y) \leq F(x, y)$ and (A_4) for each $x \in C$ $y \mapsto F(x, y)$ is convex and lower semicontinuous.

we can now consider the sequence $\{x_n\}$ given by :

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T_n x_n$$

where $T_n \stackrel{\text{def}}{=} T_{r_n}$ for a given sequence of real numbers $\{r_n\}$.

Theorem 22 *Let X be a real Hilbert space,C a nonempty closed convex, f an α-contraction, assume that EP(F) $\neq \emptyset$, $\mathbf{H_{3,1}}$ is satisfied and the sequence $\{r_n\}$ is such that $\liminf_{n\to\infty} r_n > 0$ and either $\sum_n |r_{n+1}-r_n| < \infty$ of $|r_{n+1}-r_n|/\alpha_n \to 0$. Then, the sequence $\{x_n\}$ generated by (32) converges strongly to $\mathbf{Q}_{EP(F)}(f)$.*

Proof : Since the r_n are strictly positive the mappings T_{r_n} are non expansive and share the same fixed points EP(F) which was supposed non empty. By Lemma 23 the sequence $\{x_n\}$ is bounded.

Using the definition of $T_r(x)$ and the monotonicity of F (A_2) easy computations leads to the following inequality [12, p 464] :

$$\|T_r(x) - T_s(y)\| \leq \|x-y\| + \left|1 - \frac{s}{r}\right| \|T_r(y) - y\| \tag{32}$$

Using $r > 0$ such that $r_n > r$ for all $n \in N$ and $y \in Fix\,(T_r)$ we obtain $\|T_{r_n}(x_n) - T_r(y)\| \leq \|x_n - y\|$ which gives the boundedness of the sequence $\{T_{r_n}(x_n)\}$. Moreover, for a bounded sequence $\{y_n\}$ we obtain :

$$\|T_{r_{n+1}}(y_n) - T_{r_n}(y_n)\| \leq \frac{|r_{n+1} - r_n|}{r} \|T_{r_n}(y_n) - y_n\| \tag{33}$$

We thus obtain $\mathbf{H_{1,1}}$ with $\delta_n = |r_{n+1} - r_n| + |\alpha_{n+1} - \alpha_n|$ using remark 2. The result follows now from Corollary 5. Indeed, since $r_{\sigma(n)} > r$ it is possible to extract a converging subsequence $r_{\mu(n)} \to \overline{r} > r$ and we then have $\|T_{r_{\mu(n)}} z - T_{\overline{r}} z\| \leq |r_{\mu(n)} - \overline{r}|K$. Thus

$$\|T_{r_{\mu(n)}} x_{\mu(n)} - T_{\overline{r}} x_{\mu(n)}\| \to 0\,.$$

□

3 A collection of Lemma

The first Lemma can be used to derive boundedness of the sequence $\{x_n\}$ generated by 34.

Lemma 23 *Let $\{x_n\}$, the sequence generated by the iterations*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T_n x_n \tag{34}$$

where f is contraction of parameter α, T_n is a family of nonexpansive mappings and α_n is a sequence in $(0,1)$. Suppose that there exists p a common fixed point of T_n for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ is bounded.

Proof : The proof exactly follows the proof of [17, theorem 3.2], the only difference is that here the mappings T_n are indexed by n but it does not change the

proof. Obviously we have :

$$\begin{aligned}\|x_{n+1}-p\| &\leq \alpha_n\|f(x_n)-p\|+(1-\alpha_n)\|T_nx_n-p\| \\ &\leq \alpha_n\left(\alpha\|x_n-p\|+\|f(p)-p\|\right)+(1-\alpha_n)\|x_n-p\| \\ &\leq (1-\alpha_n(1-\alpha))\|x_n-p\|+\alpha_n(1-\alpha)\frac{\|f(p)-p\|}{(1-\alpha)} \\ &\leq \max\left(\|x_n-p\|,\frac{\|f(p)-p\|}{(1-\alpha)}\right).\end{aligned}$$

And, by induction, $\{x_n\}$ is bounded. □

The next lemma aims at proving that the sequence $\{x_n\}$ is asymptotically regular *i.e* for a given $N \geq 1$, we have $\|x_{n+N}-x_n\| \to 0$.

Lemma 24 *With the same assumptions as in Lemma 23 and assuming that there exists $N \geq 1$ such that $\mathbf{H_{1,N}}$ and $\mathbf{H_{3,N}}$ are fulfilled then, for the sequence $\{x_n\}$ given by iterations (34), we have $\|x_{n+N}-x_n\| \to 0$.*

Proof : Using the definition of $\{x_n\}$ we have :

$$\begin{aligned}x_{n+N+1}-x_{n+1} &= \alpha_{n+N}(f(x_{n+N})-f(x_n))+(\alpha_{n+N}-\alpha_n)f(x_n) \\ &\quad +(1-\alpha_{n+N})(T_{n+N}x_{n+N}-T_{n+N}x_n) \\ &\quad +\left((1-\alpha_{n+N})T_{n+N}x_n-(1-\alpha_n)T_nx_n\right).\end{aligned}$$

By Lemma 23 the sequence $\{x_n\}$ is bounded, we can therefore use $\mathbf{H_{1,N}}$ with $\{x_n\}$. Since $\{f(x_n)\}$ is bounded too, we can find three constants such that :

$$\begin{aligned}\|x_{n+N+1}-x_{n+1}\| &\leq \alpha_{n+N}\alpha\|x_{n+N}-x_n\|+|\alpha_{n+N}-\alpha_n|K_1 \\ &\quad +(1-\alpha_{n+N})\|x_{n+N}-x_n\|+\delta_nM \\ &\leq (1-(1-\alpha)\alpha_{n+N})\|x_{n+N}-x_n\|+(|\alpha_{n+N}-\alpha_n|+\delta_n)K_2\end{aligned}$$

The proof then follows easily using the properties of α_n i.e $\mathbf{H_{3,N}}$ and Lemma 30. □

The next step is to prove that we can find a fixed mapping T such that $\|x_n-Tx_n\| \to 0$. The next corollary gives a simple example for which the property can be derived from Lemma 24. Indeed, we have seen specific proofs in previous sections on illustrated examples.

Corollary 25 *Using the same hypothesis as in Lemma 24 and assuming that $\{T_nx_n\}$ is bounded and that $\|T_nx_n-Tx_n\| \to 0$ we also have $\|x_n-Tx_n\| \to 0$.*

Proof :

$$\begin{aligned}\|x_n-Tx_n\| &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-Tx_n\| \\ &\leq \|x_n-x_{n+1}\|+\alpha_nK_1+(1-\alpha_n)\|T_nx_n-Tx_n\|\end{aligned}$$

and the result follows. □

The next Lemma gives assumptions to obtain $\mathbf{H_{2,p}}$ for a given p.

Lemma 26 *Suppose that X is a $\mathcal{B}$ real Banach space. Let T be a nonexpansive mapping with $Fix(T) \neq \emptyset$, f an α-contraction and $\{x_n\}$ a bounded sequence such that $\|Tx_n - x_n\| \to 0$. Then for $\tilde{x} = \mathbf{Q}(f)$ we have :*

$$\limsup_{n\to\infty} \langle f(\tilde{x}) - \tilde{x}, J(x_n - \tilde{x})\rangle \leq 0 \tag{35}$$

*Proof :*When X is a $\mathcal{B}_{\text{us}}$ or a $\mathcal{B}_{\text{rug}}$ the key point is the fact that J is uniformly norm-to-weak* continuous on bounded sets.

The proof of this Lemma can be found in the proof of Theorem [17, Theorem 4.2] or [13, Theorem 3.1]. We just summarize the line of the proof here. Let $\tilde{x} \stackrel{\text{def}}{=} \sigma\text{-}\lim_{t\to 0} x_t$ where x_t solves $x_t = tf(x_t) + (1-t)Tx_t$, we thus have :

$$\begin{aligned}
\|x_t - x_n\|^2 &\leq (1-t)^2\|Tx_t - x_n\|^2 + 2t\,\langle f(x_t) - x_n, J(x_t - x_n)\rangle \\
&\leq (1-t)^2(\|Tx_t - Tx_n\| + \|Tx_n - x_n\|)^2 \\
&\quad +2t\,\langle f(x_t) - x_t, J(x_t - x_n)\rangle + 2t\|x_t - x_n\|^2 \\
&\leq (1+t^2)\|x_t - x_n\|^2 + a_n(t) \\
&\quad +2t\,\langle f(x_t) - x_t, J(x_t - x_n)\rangle
\end{aligned} \tag{36}$$

where $a_n(t) = 2\|Tx_n - x_n\|\|x_t - x_n\| + \|Tx_n - x_n\|^2 \to 0$ when n tends to infinity. Thus :

$$\langle f(x_t) - x_t, J(x_n - x_t)\rangle \leq \frac{a_n(t)}{2t} + \frac{t}{2}\|x_t - x_n\|^2 \tag{37}$$

and we have :

$$\lim_{t\to 0}\limsup_{n\to\infty} \langle f(x_t) - x_t, J(x_n - x_t)\rangle \leq 0 \tag{38}$$

We consider now a sequence $t_p \to 0$ and $y_p \stackrel{\text{def}}{=} x_{t_p}$, then we have $y_p \to \tilde{x}$ and with $g(x) \stackrel{\text{def}}{=} (x) - x$ we have

$$\begin{aligned}
\langle g(\tilde{x}), J(x_n - \tilde{x})\rangle &\leq \langle g(y_p), J(x_n - y_p)\rangle \\
&+ |\,\langle g(\tilde{x}), J(x_n - \tilde{x}) - J(x_n - y_p)\rangle\,| + (1+\alpha)\|\tilde{x} - y_p\|\|x_n - y_p\|
\end{aligned}$$

Since J is uniformly norm-to-weak* continuous on bounded sets and $y_p \to \tilde{x}$, for $\epsilon > 0$, we can find $\tilde{p}$ such that for all $p \geq \tilde{p}$ and all $n \in \mathbb{N}$ we have :

$$\langle g(\tilde{x}), J(x_n - \tilde{x})\rangle \leq \langle g(y_p), J(x_n - y_p)\rangle + \epsilon(1+\alpha)\|\tilde{x} - y_p\|\|x_n - yp\| \tag{39}$$

Thus :

$$\begin{aligned}\limsup_{n\to\infty}\langle g(\tilde{x}), J(x_n-\tilde{x})\rangle &\leq \limsup_{n\to\infty}\langle g(y_p), J(x_n-y_p)\rangle + \epsilon + \|\tilde{x}-y_p\|K \\ &\leq \lim_{p\to\infty}(\limsup_{n\to\infty}\langle g(y_p), J(x_n-y_p)\rangle + \epsilon\|\tilde{x}-y_p\|K) \leq \epsilon\end{aligned}$$

Suppose now that X is a $\mathcal{B}_{\text{rwsc}}$. We follow the proof of [Theorem 2.2]song-chen-1 or [18, Theorem 3.1]. Let $\tilde{x} = \mathbf{Q}(f)$ and consider a subsequence $\{x_{\sigma(n)}\}$ such that $\limsup_{n\to\infty}\langle f(\tilde{x})-\tilde{x}, J(x_n-\tilde{x})\rangle = \lim_{n\to\infty}\langle f(\tilde{x})-\tilde{x}, J(x_{\sigma(n)}-\tilde{x})\rangle$. It is then possible to re-extract a subsequence $x_{\mu(n)}$ weakly converging to $x^\star$. Since we have $x_{\mu(n)} - Tx_{\mu(n)} \to 0$ then $x^\star \in Fix(T)$ using the key property that X satisfies Opial's condition [7, Theorem 1] and the fact that $I-T$ is demi-closed at zero [13, Lemma 2.2]. Thus by definition of $\tilde{x}$ we must have $\langle f(\tilde{x})-\tilde{x}, J(x^\star-\tilde{x})\rangle \leq 0$. □

Corollary 27 *Suppose that X is a $\mathcal{B}_{us}$, or a $\mathcal{B}_{rug}$, or a $\mathcal{B}_{rwsc}$. let f a contraction and $\{x_n\}$ a bounded sequence such that $x_n - T_n x_n \to 0$. From each subsequence $\sigma(n)$ we can extract a subsequence $\mu(n)$ and find a fixed mapping T_μ such that $\|T_{\mu(n)}x_{\mu(n)} - T_\mu x_{\mu(n)}\| \to 0$. Then, if $F = Fix\,T_\mu$ does not depend on μ, for $\tilde{x} = \mathbf{Q}(f)$ associated to F, we have :*

$$\limsup_{n\to\infty}\langle f(\tilde{x})-\tilde{x}, J(x_n-\tilde{x})\rangle \leq 0 \tag{40}$$

*Proof :*The proof is by contradiction using Lemma 26. Assume that the result is false, then we can find a subsequence $\sigma(n)$ such that

$$\limsup_{n\to\infty}\left\langle f(\tilde{x})-\tilde{x}, J(x_{\mu(n)}-\tilde{x})\right\rangle \geq \epsilon > 0 \tag{41}$$

by hypothesis we can extract from $\sigma(n)$ a sub-sequence $\mu(n)$ such that $\|T_{\mu(n)}x_{\mu(n)} - Tx_{\mu(n)}\| \to 0$. Thus, since

$$\|x_{\mu(n)} - Tx_{\mu(n)}\| \leq \|x_{\mu(n)} - T_{\mu(n)}x_{\mu(n)}\| + \|T_{\mu(n)}x_{\mu(n)} - Tx_{\mu(n)}\|,$$

we have $x_{\mu(n)} - Tx_{\mu(n)} \to 0$ we can then apply Lemma 26 to the sequence $\{x_{\mu(n)}\}$ and mapping T_μ to derive that :

$$\limsup_{n\to\infty}\left\langle f(\tilde{x})-\tilde{x}, J(x_{\mu(n)}-\tilde{x})\right\rangle \leq 0$$

for $\tilde{x} = \mathbf{Q}(f)$ corresponding to $F = Fix\,T_\mu$ and since F does not depend on μ, this gives a contradiction with (41). □

The next Lemma helps concluding the proof.

Lemma 28 *Assume that the sequence $\{x_n\}$ given by iterations (34) is bounded and assume that for p, a common fixed point of the mappings T_n,$\mathbf{H_{2,p}}$ is satisfied and that (i, ii, iii) items of $\mathbf{H_{3,N}}$ is also satisfied*[1]*. Then the sequence $\{x_n\}$ converges to p.*

Proof :

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq (1-\alpha_n)^2\|T_nx_n-p\|^2+2\alpha_n\langle f(x_n)-p, J(x_{n+1}-p)\rangle\\ &\leq (1-\alpha_n)^2\|x_n-p\|^2+2\alpha_n\langle f(x_n)-f(p), J(x_{n+1}-p)\rangle\\ &\quad +2\alpha_n\langle f(p)-p, J(x_{n+1}-p)\rangle\\ &\leq (1-\alpha_n)^2\|x_n-p\|^2+2\alpha_n\alpha\|x_n-p\|\|x_{n+1}-p)\|\\ &\quad +2\alpha_n\langle f(p)-p, J(x_{n+1}-p)\rangle\end{aligned}$$

Note that $\|x_{n+1}-p\| \leq \|x_n-p\|+\alpha_nK$. Thus :

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq (1-\alpha_n)^2\|x_n-p\|^2+2\alpha_n\alpha\|x_n-p\|^2\\ &\quad +2\alpha_n^2K+2\alpha_n\langle f(p)-p, J(x_{n+1}-p)\rangle\\ &\leq (1-\alpha_n(1-\alpha)+\alpha_n^2)\|x_n-p\|^2\\ &\quad +2\alpha_n^2K+2\alpha_n\langle f(p)-p, J(x_{n+1}-p)\rangle\end{aligned} \tag{42}$$

And we conclude with Lemma 29. □

Lemma 29 *.[8, Lemma 2.1] Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying the property*

$$s_{n+1} \leq (1-\alpha_n)s_n+\alpha_n\beta_n \text{ for } n \geq 0\,,$$

where $\alpha_n \in (0,1)$ and β_n are sequences of real numbers such that : (i) $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$ (ii) either $\limsup_{n\to\infty}\beta_n \leq 0$ or $\sum_{n=0}^{\infty}|\alpha_n\beta_n| < \infty$. Then $\{s_n\}$ converges to zero.

Corollary 30 *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying the property*

$$s_{n+1} \leq (1-\alpha_n)s_n+\alpha_n\beta_n+\alpha_n\gamma_n \text{ for } n \geq 0\,,$$

where $\alpha_n \in (0,1)$, β_n and γ_n are sequences of real numbers such that : (i) $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$ (ii) $\limsup_{n\to\infty}\beta_n \leq 0$ and (iv) $\sum_{n=0}^{\infty}|\alpha_n\delta_n| < \infty$. Then $\{s_n\}$ converges to zero.

*Proof :*The proof is similar to the proof of Lemma 29 [8, Lemma 2.1]. Fix $\epsilon > 0$ and N such that $\beta_n \leq \epsilon/2$ for $n \geq N$ and $\sum_{j=N}^{\infty}|\alpha_n\delta_n| \leq \epsilon/2$. Then

[1]Note that (i, ii, iii) of $\mathbf{H_{3,N}}$ do not use the value of N

following [8] we have for $n > N$:

$$\begin{aligned} s_{n+1} &\leq \prod_{j=N}^{n}(1-\alpha_j)s_N + \frac{\epsilon}{2}(1-\prod_{j=N}^{n}(1-\alpha_j)) + \sum_{j=N}^{n}|\alpha_n\delta_n| \\ &\leq \prod_{j=N}^{n}(1-\alpha_j)s_N + \frac{\epsilon}{2}(1-\prod_{j=N}^{n}(1-\alpha_j)) + \frac{\epsilon}{2} \end{aligned} \tag{43}$$

and then by taking the limit sup when $n \to \infty$ we obtain $\limsup_{n\to\infty} s_{n+1} \leq \epsilon$. □

A contraction is said to be a *Meir-Keeler contraction* (MKC) if for every $\epsilon > 0$ there exits $\delta > 0$ such that $\|x - y\| < \epsilon + \delta$ implies $\|\Phi(x) - \Phi(y)\| < \epsilon$.

Lemma 31 *[15] Suppose that the sequence $\{x_n\}$ defined by equation (34) strongly converges for an α-contraction f (or a constant function f) to the fixed point of $P_F \circ f$ then the results remains valid for a Meir-Keeler contraction Φ.*

Proof :Suppose that we have proved that (34) converges for an α-contraction f to the fixed point of $P_F \circ f$. Then indeed, the result is true when f is a constant mapping. Let Φ be a Meir-Keeler contraction, fix $y \in C$, when f is constant and equal to $\Phi(y)$ then $\{x_n\}$ defined by (34) converges to $P_F(\Phi(y))$. If Φ is a MKC then since P_F is nonexpansive $P_F \circ \Phi$ is also MKC (Proposition 3 of [15]) and has a unique fixed point [10]. We can consider $z = P_F(\Phi(z))$ and consider two sequences :

$$x_{n+1} = \alpha_n\Phi(x_n) + (1-\alpha_n)T_nx_n \tag{44}$$

$$y_{n+1} = \alpha_n\Phi(z) + (1-\alpha_n)T_ny_n \tag{45}$$

Of course $\{y_n\}$ converges strongly to z. We now prove that $\{x_n\}$ also converges strongly to z following [15]. Fix $\epsilon > 0$, by Proposition 2 of [15], we can find $r \in (0,1)$ such that $\|x - y\| \leq \epsilon$ implies $\|\Phi(x) - \Phi(y)\| \leq r\|x - y\|$. Choose now N such that $\|y_n - z\| \leq \epsilon(1-r)/r$. Assume now that for all $n \geq N$ we have $\|x_n - y_n\| > \epsilon$ then

$$\begin{aligned} \|x_{n+1} - y_{n+1}\| &\leq (1-\alpha_n)\|x_n - y_n\| + \alpha_n\|\Phi(x_n) - \Phi(y_n)\| + \alpha_n\|\Phi(y_n) - z\| \\ &\leq (1-\alpha_n(1-r))\|x_n - y_n\| + \alpha_n\epsilon \end{aligned}$$

We cannot use here directly Lemma 29 but following the proof of this Lemma we obtain that $\limsup \|x_n - y_n\| \leq \epsilon$. Assume now that for a given value of n we have $\|x_n - y_n\| \leq \epsilon$. Since Φ is a MKC we have $\|\Phi(x) - \Phi(y)\| \leq \max(r\|x - y\|, \epsilon)$ and since we have

$$r\|x_n - z\| \leq r\|x_n - y_n\| + r\|y_n - z\| \leq \epsilon \tag{46}$$

we obtain

$$\|x_{n+1} - y_{n+1}\| \leq (1-\alpha_n)\|T_n x_n - T_n y_n\| + \alpha_n \max(r\|x_n - z\|, \epsilon) \leq \epsilon\,. \quad (47)$$

Thus we have in both cases $\limsup_{n\to\infty} \|x_n - y_n\| \leq \epsilon$ and the conclusion follows. □

Lemma 32 *[1, Proposition 2.10 (i)] Suppose that X is strictly convex, T_1 an attracting non expansive mapping and T_2 a non expansive mapping which have a common fixed point. Then :*

$$Fix(T_1 \circ T_2) = Fix(T_2 \circ T_1) = Fix(T_2) \cap Fix(T_1)\,.$$

*Proof :*We have $Fix(T_2) \cap Fix(T_1) \subset Fix(T_2 \circ T_1)$ and $Fix(T_2) \cap Fix(T_1) \subset Fix(T_1 \circ T_2)$. Let x be a common fixed point of T_1 and T_2. If y, a fixed point of $T_1 \circ T_2$, is such that $y \notin Fix(T_2)$ then since T_1 is attracting non expansive we have :

$$\|y - x\| = \|T_1 \circ T_2(y) - x\| < \|T_2(y) - x\| \leq \|y - x\|$$

which gives a contradiction. Thus y is a fixed point of T_2 and then also of T_1. If now y a fixed point of $T_2 \circ T_1$ and assume that $y \notin Fix(T_1)$ then we have

$$\|y - x\| = \|T_2 \circ T_1(y) - x\| \leq \|T_1(y) - x\| < \|y - x\|$$

which gives also a contradiction and same conclusion. □

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