Central and non-central limit theorems for weighted power variations of fractional Brownian motion

Ivan Nourdin, David Nualart, Ciprian A. Tudor

Central and non-central limit theorems for weighted power variations of fractional Brownian motion

Ivan Nourdin*, David Nualart†‡ and Ciprian A. Tudor§

Abstract: In this paper, we prove some central and non-central limit theorems for renormalized weighted power variations of order $q \geq 2$ of the fractional Brownian motion with Hurst parameter $H \in (0,1)$, where q is an integer. The central limit holds for $\frac{1}{2q} < H \leq 1 - \frac{1}{2q}$, the limit being a conditionally Gaussian distribution. If $H < \frac{1}{2q}$ we show the convergence in L^2 to a limit which only depends on the fractional Brownian motion, and if $H > 1 - \frac{1}{2q}$ we show the limit in L^2 to a stochastic integral with respect to the Hermite process of order q.

Key words: fractional Brownian motion, central limit theorem, non-central limit theorem, Hermite process.

2000 Mathematics Subject Classification: 60F05, 60H05, 60G15, G0H07.

1 Introduction

The study of single path behavior of stochastic processes is often based on the study of their power variations and there exists a very extensive literature on the subject. Recall that, a real $q > 0$ being given, the q-power variation of a stochastic process X, with respect to a subdivision $\pi_n = \{0 = t_{n,0} < t_{n,1} < \ldots < t_{n,n} = 1\}$ of $[0,1]$, is defined to be the sum

$$\sum_{k=0}^{n-1} |X_{t_{n,k+1}} - X_{t_{n,k}}|^q.$$

For simplicity, consider from now on the case where $t_{n,k} = k2^{-n}$, for $n \in \mathbb{N}$ and $k \in \{0, \ldots, 2^n\}$. In the present paper we wish to point out some interesting phenomena when $X = B$ is a fractional Brownian motion of Hurst index $H \in (0,1)$ and when $q \geq 1$ is an integer. In fact,

*Laboratoire de Probabilités et Modèles Aléatoires, Université Pierre et Marie Curie, Boîte courrier 188, 4 Place Jussieu, 75252 Paris Cedex 5, France, nourdin@ccr.jussieu.fr

†Department of Mathematics, University of Kansas, 405 Snow Hall, Lawrence, Kansas 66045-2142, USA, nualart@math.ku.edu

‡The work of D. Nualart is supported by the NSF Grant DMS-0604207

§SAMOS/MATISSE, Centre d'Économie de La Sorbonne, Université de Panthéon-Sorbonne Paris 1, 90 rue de Tolbiac, 75634 Paris Cedex 13, France, tudor@univ-paris1.fr

we will also drop the absolute value (when q is odd) and we will introduce some weights. More precisely, we will consider

$$\sum_{k=1}^{2^n} f(B_{(k-1)2^{-n}})(\Delta B_{k2^{-n}})^q, \quad q \in \{1,2,3,4,\ldots\}, \tag{1.1}$$

where the function $f : \mathbb{R} \to \mathbb{R}$ is assumed to be smooth enough and where $\Delta B_{k2^{-n}}$ denotes the increment $B_{k2^{-n}} - B_{(k-1)2^{-n}}$.

The analysis of the asymptotic behavior of quantities of type (1.1) is motivated, for instance, by the study of the exact rates of convergence of some approximation schemes of scalar stochastic differential equations driven by B (see [6], [12] and [14]), besides, of course, the traditional applications of quadratic variations to parameter estimation problems.

Now, let us recall some known results concerning q-power variations (for $q = 1,2,3,4,\ldots$), which are today more or less classical. First, assume that the Hurst index is $H = \frac{1}{2}$, that is B is a standard Brownian motion. Let μ_q denote the q-moment of a standard Gaussian random variable $G \sim \mathcal{N}(0,1)$. By the scaling property of the Brownian motion and using the central limit theorem, it is immediate that, as $n \to \infty$:

$$2^{-n/2} \sum_{k=1}^{2^n} \left[(2^{n/2} \Delta B_{k2^{-n}})^q - \mu_q \right] \overset{\text{Law}}{\longrightarrow} \mathcal{N}(0, \mu_{2q} - \mu_q^2). \tag{1.2}$$

When weights are introduced, an interesting phenomenon appears: instead of Gaussian random variables, we rather obtain mixing random variables as limit in (1.2). Indeed, when q is even, it is a very particular case of a more general result by Jacod [9] that we have, as $n \to \infty$:

$$2^{-n/2} \sum_{k=1}^{2^n} f(B_{(k-1)2^{-n}}) \left[(2^{n/2} \Delta B_{k2^{-n}})^q - \mu_q \right] \overset{\text{Law}}{\longrightarrow} \sqrt{\mu_{2q} - \mu_q^2} \int_0^1 f(B_s) dW_s. \tag{1.3}$$

Here, W denotes another standard Brownian motion, independent of B. When q is odd, we have, this time, as $n \to \infty$:

$$2^{-n/2} \sum_{k=1}^{2^n} f(B_{(k-1)2^{-n}})(2^{n/2} \Delta B_{k2^{-n}})^q \overset{\text{Law}}{\longrightarrow} \int_0^1 f(B_s) \big(\sqrt{\mu_{2q} - \mu_{q+1}^2}\, dW_s + \mu_{q+1}\, dB_s \big), \tag{1.4}$$

see for instance [15].

Secondly, assume that $H \neq \frac{1}{2}$, that is the case where the fractional Brownian motion B has not independent increments anymore. Then (1.2) has been extended by Breuer and Major [1], Dobrushin and Major [4], Giraitis and Surgailis [5] or Taqqu [20]. Precisely, four cases are considered according to the evenness of q and the value of H:

- if q is even and if $H \in (0, \frac{3}{4})$, as $n \to \infty$,

$$2^{-n/2} \sum_{k=1}^{2^n} \left[(2^{nH} \Delta B_{k2^{-n}})^q - \mu_q \right] \overset{\text{Law}}{\longrightarrow} \mathcal{N}(0, \sigma_{H,q}^2); \tag{1.5}$$

- if q is even and if $H \in (\frac{3}{4}, 1)$, as $n \to \infty$,

$$2^{n-2nH} \sum_{k=1}^{2^n} \left[(2^{nH} \Delta B_{k2^{-n}})^q - \mu_q \right] \stackrel{\text{Law}}{\longrightarrow} \text{“Rosenblatt r.v.”} \tag{1.6}$$

- if q is odd and if $H \in (0, \frac{1}{2}]$, as $n \to \infty$,

$$2^{-n/2} \sum_{k=1}^{2^n} (2^{nH} \Delta B_{k2^{-n}})^q \stackrel{\text{Law}}{\longrightarrow} \mathcal{N}(0, \sigma_{H,q}^2). \tag{1.7}$$

- if q is odd and if $H \in (\frac{1}{2}, 1)$, as $n \to \infty$,

$$2^{-nH} \sum_{k=1}^{2^n} (2^{nH} \Delta B_{k2^{-n}})^q \stackrel{\text{Law}}{\longrightarrow} \mathcal{N}(0, \sigma_{H,q}^2). \tag{1.8}$$

Here and from now on, $\sigma_{H,q} > 0$ will denote a constant depending only on H and q, which may be different form one formula to another one, and which can be computed explicitly. The term “Rosenblatt r.v.” denotes a random variable whose distribution is the same as that of $Z^{(2)}$ at time one, for $Z^{(2)}$ defined in Definition 6 below.

Now, let us proceed with the results concerning the *weighted* power variations in the case where $H \neq \frac{1}{2}$. In what follows, f denotes a regular enough function such that f together with its derivatives have subexponential growth. If q is even and $H \in (\frac{1}{2}, \frac{3}{4})$, then by Theorem 2 in León and Ludeña [11] (see also Corcuera *et al* [3] for related results on the asymptotic behavior of the p-variation of stochastic integrals with respect to B) we have, as $n \to \infty$:

$$2^{-n/2} \sum_{k=1}^{2^n} f(B_{(k-1)2^{-n}}) \left[(2^{nH} \Delta B_{k2^{-n}})^q - \mu_q \right] \stackrel{\text{Law}}{\longrightarrow} \sigma_{H,q} \int_0^1 f(B_s) dW_s, \tag{1.9}$$

where, once again, W denotes a standard Brownian motion independent of B. Thus, (1.9) shows for (1.1) a similar behavior to that observed in the standard Brownian case, compare with (1.3). In contradistinction, the asymptotic behavior of (1.1) can be completely different of (1.3) or (1.9) for other values of H. The first result in this direction has been observed by Gradinaru *et al* [8] and continued in [6]. Namely, if q is odd and $H \in (0, \frac{1}{2})$, we have, as $n \to \infty$:

$$2^{nH-n} \sum_{k=1}^{2^n} f(B_{(k-1)2^{-n}}) (2^{nH} \Delta B_{k2^{-n}})^q \stackrel{\text{L}^2}{\longrightarrow} -\frac{\mu_{q+1}}{2} \int_0^1 f'(B_s) ds. \tag{1.10}$$

Also, when $q = 2$ and $H \in (0, \frac{1}{4})$, Nourdin [13] proved that we have, as $n \to \infty$:

$$2^{2Hn-n} \sum_{k=1}^{2^n} f(B_{(k-1)2^{-n}}) \left[(2^{nH} \Delta B_{k2^{-n}})^2 - 1 \right] \stackrel{\text{L}^2}{\longrightarrow} \frac{1}{4} \int_0^1 f''(B_s) ds. \tag{1.11}$$

In view of (1.3), (1.4), (1.9), (1.10) and (1.11), we observe that the asymptotic behaviors of the power variations of fractional Brownian motion (1.1) can be really different, depending on the values of q and H. The aim of the present paper is to investigate what happens in the whole generality with respect to q and H. Our main tool is the Malliavin calculus that appeared, in several recent papers, to be very useful in the study of the power variations for stochastic processes. As we will see, the Hermite polynomials play a crucial role in this analysis. In the sequel, for an integer $q \geq 1$, we write H_q for the Hermite polynomial with degree q defined by

$$H_q(x) = \frac{(-1)^q}{q!} \mathrm{e}^{-\frac{x^2}{2}} \frac{d^q}{dx^q} \left(\mathrm{e}^{-\frac{x^2}{2}} \right),$$

and we consider, when $f : \mathbb{R} \to \mathbb{R}$ is a deterministic function, the sequence of *weighted Hermite variation of order q* defined by

$$V_n^{(q)}(f) := \sum_{k=1}^{2^n} f\left(B_{(k-1)2^{-n}}\right) H_q\big(2^{nH} \Delta B_{k2^{-n}}\big). \tag{1.12}$$

The following is the main result of this paper.

Theorem 1 *Fix an integer $q \geq 1$. Let $f : \mathbb{R} \to \mathbb{R}$ be a deterministic function that satisfies the following condition:*

$(\mathbf{H_q})$ *f belongs to $\mathscr{C}^{2q}$ and, for any $p \in (0, \infty)$ and $0 \leq i \leq 2q$: $\sup_{t\in[0,1]} E\left\{|f^{(i)}(B_t)|^p\right\} < \infty$.*

1. *Assume that $0 < H < \frac{1}{2q}$. Then, as $n \to \infty$, it holds*

$$2^{nqH-n} V_n^{(q)}(f) \xrightarrow{L^2} \frac{(-1)^q}{2^q} \int_0^1 f^{(q)}(B_s) ds. \tag{1.13}$$

2. *Assume that $\frac{1}{2q} < H < 1 - \frac{1}{2q}$. Then, as $n \to \infty$, it holds*

$$2^{-n/2} V_n^{(q)}(f) \xrightarrow{\text{Law}} c_{H,q} \int_0^1 f(B_s) dW_s, \tag{1.14}$$

where W is a standard Brownian motion independent by B and $c_{H,q}$ is an explicit constant depending only on q and H.

3. *Assume that $H = 1 - \frac{1}{2q}$. Then, as $n \to \infty$, it holds*

$$\frac{1}{\sqrt{n}} 2^{-n/2} V_n^{(q)}(f) \xrightarrow{\text{Law}} c_q \int_0^1 f(B_s) dW_s, \tag{1.15}$$

where W is a standard Brownian motion independent by B and c_q is an explicit constant depending only on q.

4. *Assume that* $H > 1 - \frac{1}{2q}$. *Then*

$$2^{nq(1-H)-n}\, V_n^{(q)}(f) \xrightarrow{L^2} \int_0^1 f(B_s) dZ_s^{(q)}, \tag{1.16}$$

where $Z^{(1)} = B$ *and, for* $q \geq 2$, $Z^{(q)}$ *denotes the Hermite process of order* q *(see Definition 6 below).*

Remark 1 When $q = 1$, $V_n^{(1)}(f)$ is the forward Riemann sum $\sum_{k=1}^{2^n} f\left(B_{(k-1)2^{-n}}\right) \Delta B_{k2^{-n}}$. In this case, for $H = \frac{1}{2}$, the limit in L^2 is the Itô stochastic integral $\int_0^1 f(B_s) dB_s$ and for $H > \frac{1}{2}$, we also have convergence in L^2 and almost surely to the Riemann-Stieltjes integral $\int_0^1 f(B_s) dB_s$.

Remark 2 We do not know what happens in the critical case $H = \frac{1}{2q}$ when $q \geq 2$. The case $q = 2$ et $H = \frac{1}{4}$ is related to a conjecture by Swanson [19] on the asymptotic behavior of the Riemann sums with alternating signs, for the solution of the stochastic heat equation driven by a space-time white noise. The reader is also referred to [15] for the study of the weighted variations associated with iterated Brownian motion, which is a non-Gaussian self-similar process of order $\frac{1}{4}$. Note that the weighted variations of iterated Brownian motion are customarily defined by means of a random partition composed of Brownian hitting times (see also [10]).

Our Theorem 1 allows to complete the missing cases in the understanding of the asymptotic behavior of power variations of fractional Brownian motion:

Corollary 2 *Let* $q \geq 1$ *be an integer, and* $f : \mathbb{R} \to \mathbb{R}$ *be a function such that* $(\mathbf{H_q})$ *holds. Then, as* $n \to \infty$*:*

1. *When* $H < \frac{1}{2}$ *and* q *is odd,*

$$2^{-nH} \sum_{k=1}^{2^n} f(B_{(k-1)2^{-n}}) (2^{nH} \Delta B_{k2^{-n}})^q \xrightarrow{L^2} q\mu_{q-1} \int_0^1 f(B_s) dB_s = q\mu_{q-1} \int_0^{B_1} f(x) dx. \tag{1.17}$$

2. *When* $H < \frac{1}{4}$ *and* q *is even,*

$$2^{2nH-n} \sum_{k=1}^{2^n} f(B_{(k-1)2^{-n}}) \left[(2^{nH} \Delta B_{k2^{-n}})^q - \mu_q \right] \xrightarrow{L^2} \frac{1}{4} \binom{q}{2} \mu_{q-2} \int_0^1 f''(B_s) ds. \tag{1.18}$$

(We recover (1.11) by choosing $q = 2$*).*

3. *When* $\frac{1}{4} < H < \frac{3}{4}$ *and* q *is even,*

$$2^{-n/2}\sum_{k=1}^{2^n} f(B_{(k-1)2^{-n}})\big[(2^{nH}\Delta B_{k2^{-n}})^q - \mu_q\big] \xrightarrow{\text{Law}} \sigma_{H,q}\int_0^1 f(B_s)dW_s, \tag{1.19}$$

for $\sigma_{H,q}$ *an explicit constant and* W *a standard Brownian motion independent of* B*.*

4. *When* $H = \frac{3}{4}$ *and* q *is even,*

$$\frac{1}{\sqrt{n}}2^{-n/2}\sum_{k=1}^{2^n} f(B_{(k-1)2^{-n}})\big[(2^{nH}\Delta B_{k2^{-n}})^q - \mu_q\big] \xrightarrow{\text{Law}} \sigma_{\frac{3}{4},q}\int_0^1 f(B_s)dW_s, \tag{1.20}$$

for $\sigma_{\frac{3}{4},q}$ *an explicit constant and* W *a standard Brownian motion independent of* B*.*

5. *When* $H > \frac{3}{4}$ *and* q *is even,*

$$2^{n-2Hn}\sum_{k=1}^{2^n} f(B_{(k-1)2^{-n}})\big[(2^{nH}\Delta B_{k2^{-n}})^q - \mu_q\big] \xrightarrow{L^2} \mu_{q-2}\binom{q}{2}\int_0^1 f(B_s)dZ_s^{(2)}, \tag{1.21}$$

for $Z^{(2)}$ *the Rosenblatt process introduced in Definition 6.*

Finally, we can also give a new proof of the following result, stated and showed independently by Gradinaru *et al.* [7] and Cheridito and Nualart [2] in a *continuous* setting:

Theorem 3 *Assume that* $H > \frac{1}{6}$*, and that* $f : \mathbb{R} \to \mathbb{R}$ *verifies* ($\mathbf{H_6}$)*. Then the limit in probability, as* $n \to \infty$*, of the symmetric Riemann sums*

$$\frac{1}{2}\sum_{k=1}^{2^n}\left(f'(B_{k2^{-n}}) + f'(B_{(k-1)2^{-n}})\right)\Delta B_{k2^{-n}} \tag{1.22}$$

exists and is given by $f(B_1) - f(0)$*.*

Remark 3 When $H \leq \frac{1}{6}$, quantity (1.22) does not converge in general. As a counterexample, one can consider the case where $f(x) = x^3$, see Gradinaru *et al.* [7] or Cheridito and Nualart [2].

2 Preliminaries and notations

We briefly recall some basic facts about stochastic calculus with respect to a fractional Brownian motion. One refers to [17] for further details. Let $B = (B_t)_{t\in[0,1]}$ be a fractional Brownian motion with Hurst parameter $H \in (0,1)$. That is B is a zero mean Gaussian process, defined on a complete probability space $(\Omega, \mathcal{A}, P)$, with the covariance function

$$R_H(t,s) = E(B_tB_s) = \frac{1}{2}\big(s^{2H} + t^{2H} - |t-s|^{2H}\big), \quad s,t \in [0,1].$$

We denote by $\mathscr{E}$ the set of step functions on $[0,T]$. Let $\mathfrak{H}$ be the Hilbert space defined as the closure of $\mathscr{E}$ with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathfrak{H}} = R_H(t,s).$$

The mapping $\mathbf{1}_{[0,t]} \mapsto B_t$ can be extended to an isometry between $\mathfrak{H}$ and the Gaussian space $\mathcal{H}_1$ associated with B. We will denote this isometry by $\varphi \mapsto B(\varphi)$.

The covariance kernel can be written as

$$R_H(t,s) = \int_0^{s\wedge t} K_H(t,r) K_H(s,r) dr,$$

where K_H is a square integrable kernel given by

$$K_H(t,s) = \Gamma(H+\frac{1}{2})^{-1}(t-s)^{H-\frac{1}{2}} F(H-\frac{1}{2}, \frac{1}{2}-H, H+\frac{1}{2}, 1-\frac{t}{s}),$$

$F(a,b,c,z)$ being the Gauss hypergeometric function. Consider the linear operator K_H^* from $\mathscr{E}$ to $L^2([0,1])$ defined by

$$(K_H^*\varphi)(s) = K_H(1,s)\varphi(s) + \int_s^1 (\varphi(r)-\varphi(s)) \frac{\partial K_H}{\partial r}(r,s) dr.$$

For any pair of step functions φ and ψ in $\mathscr{E}$, we have

$$\langle K_H^*\varphi, K_H^*\psi \rangle_{L^2([0,1])} = \langle \varphi, \psi \rangle_{\mathfrak{H}}.$$

As a consequence, the operator K_H^* provides an isometry between the Hilbert spaces and $L^2([0,1])$. Hence, the process $W = (W_t)_{t\in[0,1]}$ defined by

$$W_t = B\left((K_H^*)^{-1}(\mathbf{1}_{[0,t]})\right) \tag{2.23}$$

is a Wiener process, and the process B has an integral representation of the form

$$B_t = \int_0^t K_H(t,s) dW_s,$$

because $(K_H^* \mathbf{1}_{[0,t]})(s) = K_H(t,s)$.

Let $\mathscr{S}$ be the set of all smooth cylindrical random variables, *i.e.* of the form

$$F = \phi(B_{t_1}, \ldots, B_{t_m})$$

where $m \geq 1$, $\phi : \mathbb{R}^m \to \mathbb{R} \in \mathscr{C}_b^\infty$ and $0 \leq t_1 < \ldots < t_m \leq 1$. The derivative of F with respect to B is the element of $L^2(\Omega, \mathfrak{H})$ defined by

$$D_s F \;=\; \sum_{i=1}^m \frac{\partial \phi}{\partial x_i}(B_{t_1}, \ldots, B_{t_m}) \mathbf{1}_{[0,t_i]}(s), \quad s \in [0,1].$$

In particular $D_s B_t = \mathbf{1}_{[0,t]}(s)$. For any integer $k \geq 1$, we denote by $\mathbb{D}^{k,2}$ the closure of the set of smooth random variables with respect to the norm

$$\|F\|_{k,2}^2 \;=\; E\left[F^2\right] + \sum_{j=1}^{k} E\left[|D^j F|_{\mathfrak{H}^{\otimes j}}^2\right].$$

The Malliavin derivative D verifies the chain rule: if $\varphi : \mathbb{R}^n \to \mathbb{R}$ is $\mathscr{C}_b^1$ and if $(F_i)_{i=1,\ldots,n}$ is a sequence of elements of $\mathbb{D}^{1,2}$ then $\varphi(F_1,\ldots,F_n) \in \mathbb{D}^{1,2}$ and we have,

$$D\,\varphi(F_1,\ldots,F_n) = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(F_1,\ldots,F_n) DF_i.$$

The divergence operator $I = I_1$ is the adjoint of the derivative operator D. If a random variable $u \in L^2(\Omega,\mathfrak{H})$ belongs to the domain of the divergence operator, that is if it verifies

$$|E\langle DF, u\rangle_{\mathfrak{H}}| \leq c_u \|F\|_{L^2} \quad \text{for any } F \in \mathscr{S},$$

then $I(u)$ is defined by the duality relationship

$$E\big(FI(u)\big) = E\big(\langle DF, u\rangle_{\mathfrak{H}}\big),$$

for every $F \in \mathbb{D}^{1,2}$.

For every $n \geq 1$, let $\mathcal{H}_n$ be the nth Wiener chaos of B, that is, the closed linear subspace of $L^2(\Omega,\mathcal{A},P)$ generated by the random variables $\{H_n(B(h)), h \in H, |h|_{\mathfrak{H}} = 1\}$, where H_n is the nth Hermite polynomial. The mapping $I_n(h^{\otimes n}) = n! H_n(B(h))$ provides a linear isometry between the symmetric tensor product $\mathfrak{H}^{\odot n}$ and $\mathcal{H}_n$. For $H = \frac{1}{2}$, I_n coincides with the multiple stochastic integral. The following duality formula holds

$$E\left(F I_n(h)\right) = E\left(\langle D^n F, h\rangle_{\mathfrak{H}^{\otimes n}}\right), \tag{2.24}$$

for any element $h \in \mathfrak{H}^{\odot n}$ and any random variable $F \in \mathbb{D}^{n,2}$.

We recall the following simple formula for any $s < t$ and $u < v$:

$$E\left((B_t - B_s)(B_v - B_u)\right) = \frac{1}{2}\left(|t-r|^{2H} + |s-u|^{2H} - |t-u|^{2H} - |s-r|^{2H}\right). \tag{2.25}$$

We will need the following lemma:

Lemma 4 *We have:*

1. *Let $s < t$ and u belong to $[0,1]$. Then, if $H < 1/2$*

$$\left|E\big(B_u(B_t - B_s)\big)\right| \leq |t-s|^{2H}. \tag{2.26}$$

2. For any $1 \leq k, \ell \leq 2^n$, and for all $H \in (0,1)$

$$\sum_{k,\ell=1}^{2^n} \left| E\left(B_{(k-1)2^{-n}} \Delta B_{\ell 2^{-n}} \right) \right| = O(2^n). \tag{2.27}$$

3. For any $r \geq 1$, we have, if $H < 1 - \frac{1}{2r}$

$$\sum_{k,\ell=1}^{2^n} \left| E\left(\Delta B_{k2^{-n}} \Delta B_{\ell 2^{-n}} \right) \right|^r = O(2^{n-2rHn}). \tag{2.28}$$

4. For any $r \geq 1$, we have if $H = 1 - \frac{1}{2r}$

$$\sum_{k,\ell=1}^{2^n} \left| E\left(\Delta B_{k2^{-n}} \Delta B_{\ell 2^{-n}} \right) \right|^r = O(n 2^{2n-2rn}). \tag{2.29}$$

Proof. To prove inequality (2.26), we just write

$$E(B_u(B_t - B_s)) = \frac{1}{2}(t^{2H} - s^{2H}) + \frac{1}{2}\left(|s-u|^{2H} - |t-u|^{2H} \right),$$

and observe that we have $|b^{2H} - a^{2H}| \leq |b-a|^{2H}$ for any $a, b \in [0,1]$, because $H < \frac{1}{2}$. To show (2.27) using (2.25) we write

$$\begin{aligned} \sum_{k,\ell=1}^{2^n} \left| E\left(B_{(k-1)2^{-n}} \Delta B_{\ell 2^{-n}} \right) \right| &= 2^{-2Hn-1} \sum_{k,\ell=1}^{2^n} \left| |k-1|^{2H} - k^{2H} - |\ell - k + 1|^{2H} + |\ell - k|^{2H} \right| \\ &\leq C 2^n. \end{aligned}$$

Finally, to show (2.28) and (2.29), we write

$$\begin{aligned} 2^{-2nrH} \sum_{k,\ell=1}^{2^n} \left| |k-\ell+1|^{2H} + |k-\ell-1|^{2H} - 2|k-\ell|^{2H} \right|^r \\ \leq 2^{n-2nrH} \sum_{p=-\infty}^{\infty} \left| |p+1|^{2H} + |p-1|^{2H} - 2|p|^{2H} \right|^r, \end{aligned}$$

and observe that, since the function $\left| |p+1|^{2H} + |p-1|^{2H} - 2|p|^{2H} \right|$ behaves as $C_H p^{2H-2}$ for large p, the series in the right-hand side is convergent because $H < 1 - \frac{1}{2r}$. In the critical case $H = 1 - \frac{1}{2r}$, this series is divergent, and

$$\sum_{p=-2^n}^{2^n} \left| |p+1|^{2H} + |p-1|^{2H} - 2|p|^{2H} \right|^r$$

behaves as a constant time n. ■

Lemma 5 *Assume that $H > \frac{1}{2}$.*

1. *For every $u, s, t \in [0,1]$ we have*

$$\left|E\big(B_u(B_t - B_s)\big)\right| \leq 2H|t-s|. \tag{2.30}$$

2. *Assume that $H > 1 - \frac{1}{2l}$ for some $l \geq 1$. Let $u < v$ and $s < t$ belong to $[0,1]$. Then*

$$|E(B_u - B_v)(B_t - B_s)| \leq H(2H-1)\left(\frac{2}{2Hl+1-2l}\right)^{\frac{1}{l}} |u-v|^{\frac{l-1}{l}}|t-s|. \tag{2.31}$$

3. *Assume that $H > 1 - \frac{1}{2l}$ for some $l \geq 0$ (of course, this condition is immaterial when $l = 0$). Then*

$$\sum_{i,j=1}^{2^n} \left|E\big(\Delta B_{i2^{-n}}\, \Delta B_{j2^{-n}}\big)\right|^l = O(2^{2n-2ln}). \tag{2.32}$$

Proof: By symmetry, we can assume that $s \leq t$. We have

$$E\big(B_u(B_t - B_s)\big) = \frac{1}{2}\big(t^{2H} - s^{2H}\big) + \frac{1}{2}\big(|s-u|^{2H} - |t-u|^{2H}\big).$$

But, when $0 \leq a < b \leq 1$:

$$b^{2H} - a^{2H} = 2H\int_0^{b-a} (u+a)^{2H-1} du \leq 2H\, b^{2H-1}\,(b-a) \leq 2H(b-a).$$

Thus, $|b^{2H} - a^{2H}| \leq 2H|b-a|$ and the first point follows.

Concerning the second point, using Hölder inequality, we can write

$$\begin{aligned}
|E(B_u - B_v)(B_t - B_s)| &= H(2H-1)\int_u^v \int_s^t |y-x|^{2H-2} dy dx \\
&\leq H(2H-1)|u-v|^{\frac{l-1}{l}} \left(\int_0^1 \left(\int_s^t |y-x|^{2H-2} dy\right)^l dx\right)^{\frac{1}{l}} \\
&\leq H(2H-1)|u-v|^{\frac{l-1}{l}}|t-s|^{\frac{l-1}{l}} \left(\int_0^1 \int_s^t |y-x|^{(2H-2)l} dy dx\right)^{\frac{1}{l}}.
\end{aligned}$$

Denote by $H' = 1 + (H-1)l$ and observe that $H' > \frac{1}{2}$ (because $H > 1 - \frac{1}{2l}$). We can write

$$H'(2H'-1)\int_0^1 \int_s^t |y-x|^{(2H-2)l} dy dx = E\left|B_1^{H'}(B_t^{H'} - B_s^{H'})\right| \leq 2H'|t-s|$$

by the first point of this lemma. This gives the desired bound.

We prove now the third point. We have

$$\begin{aligned}\sum_{i,j=1}^{2^n}\left|E\left(\Delta B_{i2^{-n}}\,\Delta B_{j2^{-n}}\right)\right|^l &= 2^{-2nHl-l}\sum_{i,j=1}^{2^n}\left||i-j+1|^{2H}+|i-j-1|^{2H}-2|i-j|^{2H}\right|^l\\ &\leq 2^{n-2Hln+1-l}\sum_{k=-2^n+1}^{2^n-1}\left||k+1|^{2H}+|k-1|^{2H}-2|k|^{2H}\right|^l\end{aligned}$$

and the function $|k+1|^{2H}+|k-1|^{2H}-2|k|^{2H}$ behaves as $|k|^{2H-2}$ for large k. As a consequence the series

$$\sum_{k=-\infty}^{\infty}\left||k+1|^{2H}+|k-1|^{2H}-2|k|^{2H}\right|^l$$

is divergent for $H>1-\frac{1}{2q}$ and the third point follows. ■

Let us introduce the Hermite process of order q.

Definition 6 *Fix $q\geq 2$ and $H>1-\frac{1}{2q}$. The Hermite process of order q is defined by $Z_t^{(q)}=I_q^W(L_t)$, where I_q^W is the multiple stochastic integral with respect to the Wiener process W given by the transfer equation (2.23), and for every $t\in[0,1]$, L_t is the symmetric square integrable kernel given by*

$$L_t(y_1,\ldots,y_q)=\frac{1}{q!}\mathbf{1}_{[0,t]^q}(y_1,\ldots,y_q)\int_{y_1\vee\ldots\vee y_q}^t\partial_1K_H(u,y_1)\cdots\partial_1K_H(u,y_q)du, \tag{2.33}$$

where ∂_1K_H denotes the partial derivative of K_H with respect to the first variable.

We have that

$$\begin{aligned}E(Z_t^{(q)}Z_s^{(q)}) &= \frac{1}{q!}\int_{[0,t\wedge s]^q}\left(\int_{y_1\vee\ldots\vee y_q}^s\int_{y_1\vee\ldots\vee y_q}^t\partial_1K_H(u,y_1)\partial_1K_H(v,y_1)\right.\\ &\quad\left.\times\cdots\times\partial_1K_H(u,y_q)\partial_1K_H(v,y_q)dudv\right)dy_1\cdots dy_q\\ &= \frac{1}{q!}\int_0^t\int_0^s\left(\int_0^{u\wedge v}\partial_1K_H(u,y_1)\partial_1K_H(v,y_1)dy_1\right)^q dudv\\ &= \frac{(H(2H-1))^q}{q!}\int_0^t\int_0^s|u-v|^{q(2H-2)}dudv\\ &= \frac{c_q(H)}{q!}\frac{1}{2}\left(t^{q(2H-2)+2}+s^{q(2H-2)+2}-|t-s|^{q(2H-2)+2}\right),\end{aligned}$$

for some constant $c_q(H)$. It can be easily seen that this process is $q(H-1)+1$ self-similar, has stationary increments and admits moments of all orders. On the other hand, it has Hölder continuous paths of order less than $q(H-1)+1$. We will see later (see Remark 4) that this process coincides with the classical Hermite process that appears in [4] or [20]. For other properties of related processes, we refer to Tudor [21].

3 Proof of the main results

In this section we will provide the proofs of the main results. For simplicity, we write $\varepsilon_{(k-1)2^{-n}}$ (resp. $\delta_{k2^{-n}}$) instead of $\mathbf{1}_{[0,(k-1)2^{-n}]}$ (resp. $\mathbf{1}_{[(k-1)2^{-n},k2^{-n}]}$). The following proposition provides information on the asymptotic behavior of $E\left(V_n^{(q)}(f)^2\right)$, as n tends to infinity, for $H \leq 1-\frac{1}{2q}$.

Proposition 7 *Fix an integer $q \geq 1$. Suppose that f satisfies ($\mathbf{H}_q$). Then, if $H \leq \frac{1}{2q}$, then*

$$E\left(V_n^{(q)}(f)^2\right) = O(2^{n(-2Hq+2)}). \tag{3.34}$$

If $\frac{1}{2q} \leq H < 1-\frac{1}{2q}$, then

$$E\left(V_n^{(q)}(f)^2\right) = O(2^n). \tag{3.35}$$

Finally, if $H = 1-\frac{1}{2q}$, then

$$E\left(V_n^{(q)}(f)^2\right) = O(n2^n). \tag{3.36}$$

Proof. Using the relation between Hermite polynomials and multiple stochastic integrals, we have $H_q\left(2^{nH}\Delta B_{k2^{-n}}\right) = \frac{1}{q!}2^{qnH}I_q\left(\delta_{k2^{-n}}^{\otimes q}\right)$. In this way we obtain

$$\begin{aligned}
& E\left(V_n^{(q)}(f)^2\right) \\
= & \sum_{k,\ell=1}^{2^n} E\left\{f(B_{(k-1)2^{-n}})\, f(B_{(\ell-1)2^{-n}})\, H_q\left(2^{nH}\Delta B_{k2^{-n}}\right)\, H_q\left(2^{nH}\Delta B_{\ell 2^{-n}}\right)\right\} \\
= & \frac{1}{q!^2}\, 2^{2Hqn} \sum_{k,\ell=1}^{2^n} E\left\{f(B_{(k-1)2^{-n}})\, f(B_{(\ell-1)2^{-n}})\, I_q\left(\delta_{k2^{-n}}^{\otimes q}\right) I_q\left(\delta_{\ell 2^{-n}}^{\otimes q}\right)\right\}.
\end{aligned}$$

Now we apply the product formula for multiple stochastic integrals (see [16], Proposition 1.1.3) and the duality relationship between the multiple stochastic integral I_N and the iterated derivative operator D^N, obtaining

$$\begin{aligned}
& E\left(V_{1,n}^{(q)}(f)^2\right) \\
= & 2^{2Hqn} \sum_{k,\ell=1}^{2^n} \sum_{r=0}^{q} \frac{r!}{q!^2}\binom{q}{r}^2 \\
& \times E\left\{f(B_{(k-1)2^{-n}})\, f(B_{(\ell-1)2^{-n}})\, I_{2q-2r}\left(\delta_{k2^{-n}}^{\otimes q-r} \widetilde{\otimes} \delta_{\ell 2^{-n}}^{\otimes q-r}\right)\right\} \langle \delta_{k2^{-n}}, \delta_{\ell 2^{-n}}\rangle_{\mathfrak{H}}^r \\
= & 2^{2Hqn} \sum_{k,\ell=1}^{2^n} \sum_{r=0}^{q} \frac{1}{r!(q-r)!^2} \\
& \times E\left\{\left\langle D^{2q-2r}\left(f(B_{(k-1)2^{-n}})\, f(B_{(\ell-1)2^{-n}})\right), \delta_{k2^{-n}}^{\otimes q-r} \widetilde{\otimes} \delta_{\ell 2^{-n}}^{\otimes q-r}\right\rangle_{\mathfrak{H}^{\otimes(2q-2r)}}\right\} \langle \delta_{k2^{-n}}, \delta_{\ell 2^{-n}}\rangle_{\mathfrak{H}}^r,
\end{aligned}$$

where $\widetilde{\otimes}$ denotes the symmetrization of the tensor product. The derivative of the product $D^{2q-2r}\left(f(B_{(k-1)2^{-n}})\, f(B_{(\ell-1)2^{-n}})\right)$ will be equal to a sum of derivatives:

$$\begin{aligned} D^{2q-2r}\left(f(B_{(k-1)2^{-n}})\, f(B_{(\ell-1)2^{-n}})\right) &= \sum_{a+b=2q-2r} f^{(a)}(B_{(k-1)2^{-n}})\, f^{(b)}(B_{(\ell-1)2^{-n}}) \\ &\quad \times(2q-2r)!\left(\varepsilon_{(k-1)2^{-n}}^{\otimes a}\widetilde{\otimes}\varepsilon_{(\ell-1)2^{-n}}^{\otimes b}\right). \end{aligned}$$

We make the decomposition

$$E\left(V_n^{(q)}(f)^2\right) = A_n + B_n + C_n + D_n, \tag{3.37}$$

where

$$A_n = 2^{2Hqn}\sum_{k,\ell=1}^{2^n} E\left\{f^{(q)}(B_{(k-1)2^{-n}})\, f^{(q)}(B_{(\ell-1)2^{-n}})\right\}\langle\varepsilon_{(k-1)2^{-n}}, \delta_{k2^{-n}}\rangle^q\ \langle\varepsilon_{(\ell-1)2^{-n}}, \delta_{\ell 2^{-n}}\rangle^q,$$

$$\begin{aligned} B_n &= 2^{2Hqn}\sum_{\substack{c+d+e+f=2q\\ d+e\geq 1}}\sum_{k,\ell=1}^{2^n} E\left\{f^{(q)}(B_{(k-1)2^{-n}})\, f^{(q)}(B_{(\ell-1)2^{-n}})\right\}\alpha(c,d,e,f) \\ &\quad \times\langle\varepsilon_{(k-1)2^{-n}}, \delta_{k2^{-n}}\rangle_{\mathfrak{H}}^{c}\langle\varepsilon_{(k-1)2^{-n}}, \delta_{\ell 2^{-n}}\rangle_{\mathfrak{H}}^{d}\langle\varepsilon_{(\ell-1)2^{-n}}, \delta_{k2^{-n}}\rangle_{\mathfrak{H}}^{e}\langle\varepsilon_{(\ell-1)2^{-n}}, \delta_{\ell 2^{-n}}\rangle_{\mathfrak{H}}^{f}, \\ C_n &= 2^{2Hqn}\sum_{\substack{a+b=2q\\ (a,b)\neq(q,q)}}\sum_{k,\ell=1}^{2^n} E\left\{f^{(a)}(B_{(k-1)2^{-n}})\, f^{(b)}(B_{(\ell-1)2^{-n}})\right\}\frac{(2q)!}{q!^2} \\ &\quad \times\langle\varepsilon_{(k-1)2^{-n}}^{\otimes a}\widetilde{\otimes}\varepsilon_{(\ell-1)2^{-n}}^{\otimes b}, \delta_{k2^{-n}}^{\otimes q}\widetilde{\otimes}\delta_{\ell 2^{-n}}^{\otimes q}\rangle_{\mathfrak{H}^{\otimes(2q)}}, \end{aligned}$$

and

$$\begin{aligned} D_n &= 2^{2Hqn}\sum_{r=1}^{q}\sum_{a+b=2q-2r}\sum_{k,\ell=1}^{2^n} E\left\{f^{(a)}(B_{(k-1)2^{-n}})\, f^{(b)}(B_{(\ell-1)2^{-n}})\right\}\frac{(2q-2r)!}{r!(q-r)!^2} \\ &\quad \times\langle\varepsilon_{(k-1)2^{-n}}^{\otimes a}\widetilde{\otimes}\varepsilon_{(\ell-1)2^{-n}}^{\otimes b}, \delta_{k2^{-n}}^{\otimes q-r}\widetilde{\otimes}\delta_{\ell 2^{-n}}^{\otimes q-r}\rangle_{\mathfrak{H}^{\otimes(2q-2r)}}\langle\delta_{k2^{-n}}, \delta_{\ell 2^{-n}}\rangle_{\mathfrak{H}}^{r}, \end{aligned}$$

for some combinatorial constants $\alpha(c,d,e,f)$. That is, A_n and B_n contain all the terms with $r=0$ and $(a,b)=(q,q)$; C_n contains the terms with $r=0$ and $(a,b)\neq(q,q)$; and D_n contains the terms with $1\leq r\leq q$.

For any integer $r\geq 1$, we set

$$\alpha_{r,n} = \sup_{k,\ell}\left|\langle\varepsilon_{(k-1)2^{-n}}, \delta_{\ell 2^{-n}}\rangle_{\mathfrak{H}}\right|^r, \tag{3.38}$$

$$\beta_{r,n} = \sum_{k,\ell=1}^{2^n}\left|\langle\delta_{k2^{-n}}, \delta_{\ell 2^{-n}}\rangle_{\mathfrak{H}}\right|^r, \tag{3.39}$$

$$\gamma_n = \sum_{k,\ell=1}^{2^n}\left|\langle\varepsilon_{(k-1)2^{-n}}, \delta_{\ell 2^{-n}}\rangle_{\mathfrak{H}}\right|. \tag{3.40}$$

Then we have the following estimates:

$$\begin{aligned} |A_n| &\leq C2^{2Hqn+2n}\alpha_{2q,n}, \\ |B_n|+|C_n| &\leq C2^{2Hqn}\alpha_{2q-1,n}\gamma_n, \\ |D_n| &\leq C2^{2Hqn}\sum_{r=1}^{q}\alpha_{2q-2r,n}\beta_{r,n}, \end{aligned}$$

where C is a constant depending on the function f. Notice that the second inequality follows from the fact that when $(a,b)\neq(q,q)$, or $(a,b)=(q,q)$ and $c+d+e+f=2q$ with $d\geq 1$ or $e\geq 1$, there will be at least a factor of the form $\langle \varepsilon_{(k-1)2^{-n}}, \delta_{\ell 2^{-n}}\rangle_{\mathfrak{H}}$ in the expression of B_n or C_n.

In the case $H<\frac{1}{2}$, we have, by Lemma 4, (2.26), $\alpha_{r,n}\leq 2^{-2nrH}$, by Lemma 4, (2.28), $\beta_{r,n}\leq C2^{n-2rHn}$ and by Lemma 4, (2.27), $\gamma_n\leq C2^n$. As a consequence, we obtain

$$|A_n| \leq C2^{n(-2Hq+2)}, \tag{3.41}$$

$$|B_n|+|C_n| \leq C2^{n(-2Hq+2H+1)}, \tag{3.42}$$

$$|D_n| \leq C\sum_{r=1}^{q}2^{n(-2(q-r)H+1)}, \tag{3.43}$$

which implies the estimates (3.34) and (3.35).

In the case $\frac{1}{2}\leq H<1-\frac{1}{2q}$, by Lemma 5, (2.30), $\alpha_{r,n}\leq C2^{-nr}$, by Lemma 4, (2.28) $\beta_{r,n}\leq C2^{n-2rHn}$ and by Lemma 4, (2.27), $\gamma_n\leq C2^n$. As a consequence, we obtain

$$\begin{aligned} |A_n|+|B_n|+|C_n| &\leq C2^{n(2q(H-1)+2)}, \\ |D_n| &\leq C\sum_{r=1}^{q}2^{n((2q-2r)(H-1)+1)}, \end{aligned}$$

which also implies (3.35).

Finally, if $H=1-\frac{1}{2q}$, by Lemma 5, (2.30), $\alpha_{r,n}\leq C2^{-nr}$, by Lemma 4, (2.29) $\beta_{r,n}\leq Cn2^{2n-2rn}$ and by Lemma 4, (2.27), $\gamma_n\leq C2^n$. As a consequence, we obtain

$$\begin{aligned} |A_n|+|B_n|+|C_n| &\leq C2^n, \\ |D_n| &\leq C\sum_{r=1}^{q}n2^{n\frac{r}{q}}, \end{aligned}$$

which implies (3.36). ■

3.1 Proof of Theorem 1 in the case $0<H<\frac{1}{2q}$

In this subsection we are going to prove the first point of Theorem 1. The proof will be done in three steps. Set

$$V_{1,n}^{(q)}(f)=2^{n(qH-1)}V_n^{(q)}(f).$$

We first study the asymptotic behavior of $E\left(V_{1,n}^{(q)}(f)^2\right)$, using Proposition 7.

Step 1. The decomposition (3.37) leads to

$$E\left(V_{1,n}^{(q)}(f)^2\right) = 2^{2n(qH-1)}\left(A_n + B_n + C_n + D_n\right).$$

From the estimate (3.42) we obtain

$$2^{2n(qH-1)}\left(|B_n| + |C_n|\right) \leq C2^{n(2H-1)},$$

which converges to zero as n goes to infinity since $H < \frac{1}{2q} \leq \frac{1}{2}$. On the other hand (3.43) yields

$$2^{2n(qH-1)}\,|D_n| \leq C\sum_{r=1}^{q} 2^{n(2rH-1)},$$

which tends to zero as n goes to infinity since $2rH - 1 \leq 2qH - 1 < 0$.

In order to handle the term A_n we make use of the following estimate, which follows from Lemma 4, (2.26) and (2.25)

$$\begin{aligned}
&\left|\langle \varepsilon_{(k-1)2^{-n}}, \delta_{k2^{-n}}\rangle_{\mathfrak{H}}^{q} - \left(-\frac{2^{-2Hn}}{2}\right)^{q}\right| \\
= &\left|\langle \varepsilon_{(k-1)2^{-n}}, \delta_{k2^{-n}}\rangle_{\mathfrak{H}} + \frac{2^{-2Hn}}{2}\right| \left|\sum_{s=0}^{q-1}\langle \varepsilon_{(k-1)2^{-n}}, \delta_{k2^{-n}}\rangle_{\mathfrak{H}}^{s}\left(-\frac{2^{-2Hn}}{2}\right)^{q-1-s}\right| \\
\leq &\; C\left(k^{2H} - (k-1)^{2H}\right)2^{-2Hqn}. \qquad (3.44)
\end{aligned}$$

Thus,

$$\begin{aligned}
\Bigg|2^{4Hqn-2n}\sum_{k,\ell=1}^{2^n} E\left\{f^{(q)}(B_{(k-1)2^{-n}})\, f^{(q)}(B_{(\ell-1)2^{-n}})\right\}\langle \varepsilon_{(k-1)2^{-n}}, \delta_{k2^{-n}}\rangle_{\mathfrak{H}}^{q}\,\langle \varepsilon_{(\ell-1)2^{-n}}, \delta_{\ell 2^{-n}}\rangle_{\mathfrak{H}}^{q} \\
- 2^{-2n-2q}\sum_{k,\ell=1}^{2^n} E\left\{f^{(q)}(B_{(k-1)2^{-n}})\, f^{(q)}(B_{(\ell-1)2^{-n}})\right\}\Bigg| \leq C2^{2Hn-n},
\end{aligned}$$

which implies, as $n \to \infty$:

$$E\big(V_{1,n}^{(q)}(f)^2\big) = 2^{-2n-2q}\sum_{k,\ell=1}^{2^n} E\left\{f^{(q)}(B_{(k-1)2^{-n}})\, f^{(q)}(B_{(\ell-1)2^{-n}})\right\} + o(1). \qquad (3.45)$$

Step 2: We need the asymptotic behavior of the double product

$$J_n := E\left(V_{1,n}^{(q)}(f)2^{-n}\sum_{\ell=1}^{2^n} f^{(q)}(B_{(\ell-1)2^{-n}})\right).$$

Using the same arguments as in Step 1 we obtain

$$\begin{aligned}
J_n &= 2^{Hqn-2n}\sum_{k,\ell=1}^{2^n} E\left\{f(B_{(k-1)2^{-n}})\, f^{(q)}(B_{(\ell-1)2^{-n}})\, H_q\big(2^{nH}\Delta B_{k2^{-n}}\big)\right\} \\
&= \frac{1}{q!}2^{2Hqn-2n}\sum_{k,\ell=1}^{2^n} E\left\{f(B_{(k-1)2^{-n}})\, f^{(q)}(B_{(\ell-1)2^{-n}})\, I_q\big(\delta_{k2^{-n}}^{\otimes q}\big)\right\} \\
&= \frac{1}{q!}2^{2Hqn-2n}\sum_{k,\ell=1}^{2^n} E\left\{\left\langle D^q\big(f(B_{(k-1)2^{-n}})\, f^{(q)}(B_{(\ell-1)2^{-n}})\big), \delta_{k2^{-n}}^{\otimes q}\right\rangle\right\} \\
&= 2^{2Hqn-2n}\sum_{k,\ell=1}^{2^n}\sum_{a=0}^{q}\frac{1}{q!}\binom{q}{a}^2 E\left\{f^{(a)}(B_{(k-1)2^{-n}})\, f^{(2q-a)}(B_{(\ell-1)2^{-n}})\right\} \\
&\quad \times\langle\varepsilon_{(k-1)2^{-n}}, \delta_{k2^{-n}}\rangle_{\mathfrak{H}}^{a}\,\langle\varepsilon_{(\ell-1)2^{-n}}, \delta_{k2^{-n}}\rangle_{\mathfrak{H}}^{q-a}.
\end{aligned}$$

It turns out that only the term with $a = q$ will contribute to the limit as n tends to infinity. For this reason we make the decomposition

$$J_n = 2^{2Hqn-2n}\sum_{k,\ell=1}^{2^n}\frac{1}{q!}E\left\{f^{(a)}(B_{(k-1)2^{-n}})\, f^{(q)}(B_{(\ell-1)2^{-n}})\right\}\,\langle\varepsilon_{(k-1)2^{-n}}, \delta_{k2^{-n}}\rangle_{\mathfrak{H}}^{q} + S_n,$$

where

$$\begin{aligned}
S_n &= 2^{2Hqn-2n}\sum_{k,\ell=1}^{2^n}\langle\varepsilon_{(\ell-1)2^{-n}}, \delta_{k2^{-n}}\rangle_{\mathfrak{H}}\sum_{a=0}^{q-1}\frac{1}{q!}\binom{q}{a}^2 E\left\{f^{(a)}(B_{(k-1)2^{-n}})\, f^{(2q-a)}(B_{(\ell-1)2^{-n}})\right\} \\
&\times\langle\varepsilon_{(k-1)2^{-n}}, \delta_{k2^{-n}}\rangle_{\mathfrak{H}}^{a}\,\langle\varepsilon_{(\ell-1)2^{-n}}, \delta_{k2^{-n}}\rangle_{\mathfrak{H}}^{q-a-1}.
\end{aligned}$$

By (2.26) and (2.27), we have

$$|S_n| \leq C 2^{2Hn-n},$$

which tends to zero as n goes to infinity. Moreover, by (3.44), we have

$$\begin{aligned}
&\Bigg|2^{2Hqn-2n}\sum_{k,\ell=1}^{2^n} E\left\{f^{(q)}(B_{(k-1)2^{-n}})\, f^{(q)}(B_{(\ell-1)2^{-n}})\right\}\,\langle\varepsilon_{(k-1)2^{-n}}, \delta_{k2^{-n}}\rangle_{\mathfrak{H}}^{q} \\
&\quad -(-1)^q\, 2^{-2n-q}\sum_{k,\ell=1}^{2^n} E\left\{f^{(q)}(B_{(k-1)2^{-n}})\, f^{(q)}(B_{(\ell-1)2^{-n}})\right\}\Bigg| \leq C\, 2^{2Hn-n},
\end{aligned}$$

which also tends to zero as n goes to infinity. Thus, finally, as $n \to \infty$:

$$A_n = (-1)^q\, 2^{-2n-q}\sum_{k,\ell=1}^{2^n} E\left\{f^{(q)}(B_{(k-1)2^{-n}})\, f^{(q)}(B_{(\ell-1)2^{-n}})\right\} + o(1). \tag{3.46}$$

Step 3: By combining (3.45) and (3.46), we obtain that

$$E\left|V_{1,n}^{(q)}(f)-\frac{(-1)^q}{2^q}\,2^{-n}\sum_{k=1}^{2^n}f^{(q)}(B_{(k-1)2^{-n}})\right|^2=o(1),$$

as $n\to\infty$. Thus, the proof of the first point of Theorem 1 is done. ■

3.2 Proof of Theorem 1 in the case $H>1-\frac{1}{2q}$: the weighted Non Central Limit Theorem

We can assume that $q\geq 2$, the case where $q=1$ being straightforward. We prove here that the sequence $V_{3,n}(f)$ given by

$$V_{3,n}^{(q)}(f)=2^{n(1-H)q-n}\,V_n^{(q)}(f)=2^{qn-n}\frac{1}{q!}\sum_{k=1}^{2^n}f\left(B_{(k-1)2^{-n}}\right)I_q\left(\delta_{k2^{-n}}^{\otimes q}\right)$$

converges in $L^2(\Omega)$ as $n\to\infty$ to the pathwise integral $\int_0^1 f(B_s)dZ_s^{(q)}$ with respect to the Hermite process of order q.

First we will show the result when the function f is one. It is a generalization of a result in [22] where the case $q=2$ has been proven.

Lemma 8 *For each $t\in[0,1]$, the sequence*

$$T_n(t)=2^{qn-n}\frac{1}{q!}\sum_{k=1}^{[2^nt]}I_q\left(\delta_{k2^{-n}}^{\otimes q}\right)\tag{3.47}$$

converges in $L^2(\Omega)$ to the Hermite random variable $Z_t^{(q)}$.

Proof: By the transfer principle the multiple stochastic integral $I_q\left(\delta_{k2^{-n}}^{\otimes q}\right)$ can be written as a multiple stochastic integral with respect to the Wiener process W, $I_q\left(\delta_{k2^{-n}}^{\otimes q}\right)=I_q^W\left(A_{k2^{-n}}^{\otimes q}\right)$, where

$$\begin{aligned}A_{k2^{-n}}(s)&=K_H\left(k2^{-n},s\right)\mathbf{1}_{[0,k2^{-n}]}(s)-K_H\left((k-1)2^{-n},s\right)\mathbf{1}_{[0,(k-1)2^{-n}]}(s)\\&=\left(K_H\left(k2^{-n},s\right)-K_H\left((k-1)2^{-n},s\right)\right)\varepsilon_{(k-1)2^{-n}}(s)+K_H\left(k2^{-n},s\right)\delta_{k2^{-n}}(s)\\&:=D_{k2^{-n}}(s)+C_{k2^{-n}}(s).\end{aligned}$$

We will consider the case $t=1$, the proof in the general case being similar. It suffices to prove that

$$2^{qn-n}\frac{1}{q!}\sum_{k=1}^{2^n}\left(D_{k2^{-n}}+C_{k2^{-n}}\right)^{\otimes q}\to L_1$$

in $L^2([0,1]^q)$ as $n \to \infty$, where L_t denotes the kernel introduced in (2.33). We will show first that $2^{qn-n}\frac{1}{q!}\sum_{k=1}^{2^n} D_{k2^{-n}}^{\otimes q}$ converges in $L^2([0,1]^q)$ to L_1 as n tends to infinity. We can write, for every $s \in [0,1]$ and $k = 1, \ldots, n$

$$D_{k2^{-n}}(s) = 2^{-n}\partial_1 K_H\left(\xi_k(s), s\right) \varepsilon_{(k-1)2^{-n}}(s),$$

where $\xi_k(s)$ is a point in the interval $[(k-1)2^{-n}, k2^{-n}]$. Hence

$$\begin{aligned} 2^{qn-n}\frac{1}{q!}\sum_{k=1}^{2^n} D_{k2^{-n}}(s_1)\cdots D_{k2^{-n}}(s_q) &= 2^{-n}\frac{1}{q!}\sum_{k=1}^{2^n} \partial_1 K_H\left(\xi_k(s_1), s_1\right)\ldots\partial_1 K_H\left(\xi_k(s_q), s_q\right) \\ &\quad \times \mathbf{1}_{[0,(k-1)2^{-n}]^q}(s_1, \ldots s_q), \end{aligned}$$

is a Riemann sum that converges a.e. to L_1. To obtain the convergence in $L^2([0,1]^q)$ we will apply the dominated convergence theorem. In fact,

$$\begin{aligned} &\int_{[0,1]^q} \left| 2^{qn-n}\sum_{k=1}^{2^n} D_{k2^{-n}}(s_1)\cdots D_{k2^{-n}}(s_q)\right|^2 ds_1\cdots ds_q \\ = \;& 2^{2qn-2n}\sum_{k,\ell=1}^{2^n}\left(\int_0^1 D_{k2^{-n}}(s) D_{\ell 2^{-n}}(s) ds\right)^q \\ \leq \;& 2^{2qn-2n}\sum_{k,\ell=1}^{2^n}\left(E\left(\Delta B_{k2^{-n}}\,\Delta B_{\ell 2^{-n}}\right)\right)^q, \end{aligned}$$

which is bounded by Lemma 5, (2.32). Then, it only remains to show that for every $j \leq q-1$, the kernel

$$R_n(s_1, \ldots, s_q) = 2^{qn-n}\sum_{k=1}^{2^n} D_{k2^{-n}}(s_1)\ldots D_{k2^{-n}}(s_{q-j}) C_{k2^{-n}}(s_{q-j+1})\ldots C_{k2^{-n}}(s_q)$$

converges to zero in $L^2([0,1]^q)$ as $n \to \infty$. Let us compute the L^2 norm of this kernel (note that the presence of at least one term C_i implies that the mixed terms in the above expression vanish)

$$\begin{aligned} \|R_n\|^2 &= 2^{2qn-2n}\int_{[0,1]^q} ds_1\cdots ds_q \sum_{k=1}^{2^n} D_{k2^{-n}}^2(s_1)\ldots D_{k2^{-n}}^2(s_{q-j}) \\ &\quad \times C_{k2^{-n}}^2(s_{q-j+1})\ldots C_{k2^{-n}}^2(s_q), \end{aligned}$$

and since

$$\int_0^1 D_{k2^{-n}}^2(s) ds \leq E\left|\Delta B_{k2^{-n}}\right|^2 = 2^{-2Hn},$$

and

$$\int_0^1 C_{k2^{-n}}^2(s) ds \leq E\left|\Delta B_{i2^{-n}}\right|^2 = 2^{-2Hn},$$

we get $\|R_n\|^2 = O(2^{2qn-n-2Hqn})$ and this goes to zero because $H > 1 - \frac{1}{2q}$. ■

Remark 4 It follows from Taqqu [20] and Dobrushin and Major [4] that the sequence T_n in Lemma 8 converges in law to the "standard" historical Hermite process defined through its moving average representation as a multiple integral with respect to a Wiener process with time horizon $\mathbb{R}$. From Lemma 8, it is clear that the process introduced in Definition 6 has the same finite dimensional distributions as the historical Hermite process.

Remark 5 We can also work directly with the fractional Brownian motion B without using its associated Brownian motion W. The sequence of functions

$$\varphi_n = 2^n \sum_{j=1}^{[2^n t]} \delta_{j2^{-n}}^{\otimes q}$$

is a Cauchy sequence in the space $\mathfrak{H}^{\otimes q}$. Indeed, for any $m \geq n$ we have

$$\langle \varphi_n, \varphi_m \rangle_{\mathfrak{H}^{\otimes q}} = (H(2H-1))^2 2^{n+m} \sum_{j=1}^{[2^m t]} \sum_{k=1}^{[2^n t]} \left(\int_{(j-1)2^{-m}}^{j2^{-m}} \int_{(k-1)2^{-n}}^{k2^{-n}} |s-s'|^{2H-2} ds ds' \right)^q,$$

and,

$$\begin{aligned} \lim_{n\to\infty} \lim_{m\to\infty} \langle \varphi_n, \varphi_m \rangle_{\mathfrak{H}^{\otimes 2}} &= (H(2H-1))^2 \\ \times \int_0^t \int_0^t |s-s'|^{(2H-2)q} ds ds' &= c_{q,H} t^{(2H-2)q+2}, \end{aligned}$$

where $c_{q,H} = \frac{(H(2H-1))^2}{(2H-2)q(2(2H-2)q+1)}$. In the same way we can prove that $\lim_{n\to\infty} \langle \varphi_n, \varphi_n \rangle_{\mathfrak{H}^{\otimes 2}} = c_{q,H} t^{(2H-2)q+2}$. For every $t \in [0,1]$, we denote by μ_t the uniform measure on the diagonal of $[0,t]^q$, normalized to have a total mass equals t. That is, for any continuous function f on $[0,1]^q$ we have

$$\int_{[0,1]^q} f d\mu_t = \int_0^t f(s, \ldots, s) ds.$$

Then, it is clear that the limit in $\mathfrak{H}^{\otimes q}$ of the sequence of functions φ_n can be identified with μ_t. We define the stochastic process X_t by $X_t = I_q(\mu_t)$. Then, as a consequence of the previous computations we have

$$\lim_{n\to\infty} V_n^{(q)}(\mathbf{1}_{[0,t]}) = X_t,$$

in L^2. It is clear that the process X coincides with the Hermite process $Z^{(q)}$.

Proof of the third point in Theorem 1. Consider now the case of a general function f. We fix two integers $m \geq n$, and decompose the sequence $V_{3,m}^{(q)}(f)$ as follows:

$$V_{3,m}^{(q)}(f) = A^{(m,n)} + B^{(m,n)},$$

where

$$A^{(m,n)} = \frac{1}{q!} 2^{m(q-1)} \sum_{j=1}^{2^n} f\left(B_{(j-1)2^{-n}}\right) \sum_{i=(j-1)2^{m-n}+1}^{j2^{m-n}} I_q\left(\delta_{i2^{-m}}^{\otimes q}\right),$$

and

$$B^{(m,n)} = \frac{1}{q!} 2^{m(q-1)} \sum_{j=1}^{2^n} \sum_{i=(j-1)2^{m-n}+1}^{j2^{m-n}} \Delta_{i,j}^{m,n} f(B) I_q\left(\delta_{i2^{-m}}^{\otimes q}\right),$$

with the notation $\Delta_{i,j}^{m,n} f(B) = f(B_{(i-1)2^{-m}}) - f(B_{(j-1)2^{-n}})$. We shall study $A^{(m,n)}$ and $B^{(m,n)}$ separately.

Study of $A^{(m,n)}$. When n is fixed, Lemma 8 yields that the random vector

$$\left(\frac{1}{q!} 2^{m(q-1)} \sum_{i=(j-1)2^{m-n}+1}^{j2^{m-n}} I_q\left(\delta_{i2^{-m}}^{\otimes q}\right); j = 1, \ldots, 2^n\right)$$

converges in L^2, as $m \to \infty$, to the vector

$$\left(Z_{j2^{-n}}^{(q)} - Z_{(j-1)2^{-n}}^{(q)}; j = 1, \ldots, 2^n\right).$$

Then, as $m \to \infty$, $A^{(m,n)} \overset{\mathrm{L}^2}{\to} A^{(\infty,n)}$, where

$$A^{(\infty,n)} := \sum_{j=1}^{2^n} f(B_{(j-1)2^{-n}}) \left(Z_{j2^{-n}}^{(q)} - Z_{(j-1)2^{-n}}^{(q)}\right).$$

Finally, we claim that when n tends to infinity, $A^{(\infty,n)}$ converges in L^2 to $\int_0^1 f(B_s)\, dZ_s^{(q)}$. Observe that the stochastic integral $\int_0^1 f(B_s)\, dZ_s^{(q)}$ is a pathwise Riemann-Stieltjes integral. So, to get the convergence in L^2 it suffices to show that the sequence $A^{(\infty,n)}$ is bounded in L^p for some $p \geq 2$. The integral $\int_0^1 f(B_s)\, dZ_s^{(q)}$ has moments of all orders, because for all $p \geq 2$

$$E\left[\sup_{0 \leq s < t \leq 1} \left(\frac{\left|Z_t^{(q)} - Z_s^{(q)}\right|}{|t-s|^\gamma}\right)^p\right] < \infty$$

and

$$E\left[\sup_{0 \leq s < t \leq 1} \left(\frac{|B_t - B_s|}{|t-s|^\beta}\right)^p\right] < \infty,$$

if $\gamma < q(H-1)+1$ and $\beta < H$. On the other hand, Young's inequality implies

$$\left|A^{(\infty,n)} - \int_0^1 f(B_s)\, dZ_s^{(q)}\right| \leq c_{\rho,\nu} \mathrm{Var}_\rho\big(f(B)\big) \mathrm{Var}_\nu\big(Z^{(q)}\big),$$

where Var_ρ denotes the variation of order ρ, and with $\rho, \nu > 1$ such that $\frac{1}{\rho} + \frac{1}{\nu} > 1$. Choosing $\rho > \frac{1}{H}$ and $\nu > \frac{1}{q(H-1)+1}$, the result follows.

This proves that, by letting m and then n go to infinity, $A^{(m,n)}$ converges in L^2 to $\int_0^1 f(B_s)\, dZ_s^{(q)}$.

Study of the term $B^{(m,n)}$: We prove that

$$\lim_{n\to\infty} \sup_m E\left|B^{(m,n)}\right|^2 = 0. \tag{3.48}$$

We have, using the product formula for multiple stochastic integrals

$$\begin{aligned} E\left|B^{(m,n)}\right|^2 &= 2^{2m(q-1)} \sum_{j=1}^{2^n} \sum_{i=(j-1)2^{m-n}+1}^{j2^{m-n}} \sum_{j'=1}^{2^n} \sum_{i'=(j'-1)2^{m-n}+1}^{j'2^{m-n}} \sum_{l=0}^{q} \frac{l!}{q!^2} \binom{q}{l}^2 \\ &\quad \times b_l^{(m,n)} \langle \delta_{i2^{-m}}, \delta_{i'2^{-m}} \rangle_{\mathfrak{H}}^l, \end{aligned} \tag{3.49}$$

where

$$b_l^{(m,n)} = E\left(\Delta_{i,j}^{m,n} f(B) \Delta_{i',j'}^{m,n} f(B) I_{2(q-l)}\left(\delta_{i2^{-m}}^{\otimes(q-l)} \widetilde{\otimes} \delta_{i'2^{-m}}^{\otimes(q-l)}\right)\right). \tag{3.50}$$

By the duality relationship between the multiple stochastic integral and the iterated derivative operator, we obtain

$$\begin{aligned} b_l^{(m,n)} &= E\left\langle D^{2(q-l)}\left(\Delta_{i,j}^{m,n} f(B) \Delta_{i',j'}^{m,n} f(B)\right), \delta_{i2^{-m}}^{\otimes(q-l)} \widetilde{\otimes} \delta_{i'2^{-m}}^{\otimes(q-l)} \right\rangle_{\mathfrak{H}^{\otimes 2(q-l)}} \\ &= \sum_{a+b=2(q-l)} \left\langle E\left[D^{(a)}\left(\Delta_{i,j}^{m,n} f(B)\right) \widetilde{\otimes} D^{(b)}\left(\Delta_{i',j'}^{m,n} f(B)\right)\right], \delta_{i2^{-m}}^{\otimes(q-l)} \widetilde{\otimes} \delta_{i'2^{-m}}^{\otimes(q-l)} \right\rangle_{\mathfrak{H}^{\otimes 2(q-l)}} \\ &= \sum_{a+b=2(q-l)} (2q-2l)! \left\langle E\left(\left(f^{(a)}(B_{(i-1)2^{-m}}) \varepsilon_{(i-1)2^{-m}}^{\otimes a} - f^{(a)}(B_{(j-1)2^{-n}}) \varepsilon_{(j-1)2^{-n}}^{\otimes a}\right) \widetilde{\otimes} \right.\right. \\ &\quad \left.\left.\left(f^{(b)}(B_{(i'-1)2^{-m}}) \varepsilon_{(i'-1)2^{-m}}^{\otimes b} - f^{(b)}(B_{(j'-1)2^{-n}}) \varepsilon_{(j'-1)2^{-m}}^{\otimes b}\right)\right), \delta_{i2^{-m}}^{\otimes(q-l)} \widetilde{\otimes} \delta_{i'2^{-m}}^{\otimes(q-l)} \right\rangle_{\mathfrak{H}^{\otimes 2(q-l)}}, \end{aligned}$$

where $D^{(a)}$ denotes the iterated Malliavin derivative. The term in (3.49) corresponding to $l = q$ can be estimated by

$$\frac{1}{q!} 2^{2m(q-1)} \sup_{|x-y|\le 2^{-n}} E\left|f(B_x) - f(B_y)\right|^2 \beta_{q,m},$$

where $\beta_{q,m}$ has been introduced in (3.39), which converges to zero as n tends to infinity, uniformly in m, because by Lemma 5, (2.32), and using that $H > 1 - \frac{1}{2q}$, we have

$$\sup_m 2^{2m(q-1)} \beta_{q,m} < \infty.$$

In order to handle the terms with $0 \leq l \leq q-1$, we make the decomposition

$$\left| b_l^{(m,n)} \right| \leq \sum_{a+b=2(q-l)} (2q-2l)! \sum_{h=1}^{4} B_h, \tag{3.51}$$

where

$$\begin{aligned}
B_1 &= E\left|\Delta_{i,j}^{m,n} f(B) \Delta_{i',j'}^{m,n} f(B)\right| \langle \varepsilon_{(i-1)2^{-m}}^{\otimes a} \widetilde{\otimes} \varepsilon_{(i'-1)2^{-m}}^{\otimes b}, \delta_{i2^{-m}}^{\otimes(q-l)} \widetilde{\otimes} \delta_{i'2^{-m}}^{\otimes(q-l)} \rangle_{\mathfrak{H}^{\otimes 2(q-l)}}, \\
B_2 &= E\left|f^{(a)}(B_{(j-1)2^{-n}}) \Delta_{i',j'}^{m,n} f(B)\right| \\
&\quad \times \langle \left(\varepsilon_{(i-1)2^{-m}}^{\otimes a} - \varepsilon_{(j-1)2^{-n}}^{\otimes a}\right) \widetilde{\otimes} \varepsilon_{(i'-1)2^{-m}}^{\otimes b}, \delta_{i2^{-m}}^{\otimes(q-l)} \widetilde{\otimes} \delta_{i'2^{-m}}^{\otimes(q-l)} \rangle_{\mathfrak{H}^{\otimes 2(q-l)}}, \\
B_3 &= E\left|\Delta_{i,j}^{m,n} f(B) f^{(b)}(B_{(j'-1)2^{-n}})\right| \\
&\quad \times \langle \varepsilon_{(i-1)2^{-m}}^{\otimes a} \widetilde{\otimes} \left(\varepsilon_{(i'-1)2^{-m}}^{\otimes b} - \varepsilon_{(j'-1)2^{-n}}^{\otimes b}\right), \delta_{i2^{-m}}^{\otimes(q-l)} \widetilde{\otimes} \delta_{i'2^{-m}}^{\otimes(q-l)} \rangle_{\mathfrak{H}^{\otimes 2(q-l)}}, \\
B_4 &= E\left|f^{(a)}(B_{(j-1)2^{-n}}) f^{(b)}(B_{(j'-1)2^{-n}})\right| \\
&\quad \times \langle \left(\varepsilon_{(i-1)2^{-m}}^{\otimes a} - \varepsilon_{(j-1)2^{-n}}^{\otimes a}\right) \widetilde{\otimes} \left(\varepsilon_{(i'-1)2^{-m}}^{\otimes b} - \varepsilon_{(j'-1)2^{-n}}^{\otimes b}\right), \delta_{i2^{-m}}^{\otimes(q-l)} \widetilde{\otimes} \delta_{i'2^{-m}}^{\otimes(q-l)} \rangle_{\mathfrak{H}^{\otimes 2(q-l)}}.
\end{aligned} \tag{3.52}$$

By using Lemma 5, (2.30), and the conditions imposed on the function f, one can bound the terms B_1, B_2 and B_3 as follows:

$$|B_1| \leq c(q,f,H) \sup_{|x-y| \leq \frac{1}{2^n}, 0 \leq a \leq 2q} E\left|f^{(a)}(B_x) - f^{(a)}(B_y)\right|^2 2^{-2m(q-l)},$$

$$|B_2| + |B_3| \leq c(q,f,H) \sup_{|x-y| \leq \frac{1}{2^n}, 0 \leq a \leq 2q} E\left|f^{(b)}(B_x) - f^{(b)}(B_y)\right| 2^{-2m(q-l)},$$

and by using Lemma 5, (2.31), we obtain that

$$|B_4| \leq c(q,f,H) 2^{-n\frac{q-1}{q} - 2m(q-l)}.$$

By setting

$$R_n = \frac{1}{q!} \sup_{|x-y| \leq 2^{-n}} E\left|f(B_x) - f(B_y)\right|^2 \sup_m 2^{2m(q-1)} \beta_{q,m},$$

we can finally write, by the estimate (2.32) in Lemma 5

$$\begin{aligned}
E\left|B^{(m,n)}\right|^2 &\leq R_n + c(H,f,q)2^{2m(q-1)}\left(\sup_{|x-y|\leq\frac{1}{2^n},0\leq a\leq 2q}\left|f^{(b)}(B_x)-f^{(b)}(B_y)\right|+(2^{-n})^{\frac{q-1}{q}}\right)\\
&\quad\times\sum_{j=1}^{2^n}\sum_{i=(j-1)2^{m-n}+1}^{j2^{m-n}}\sum_{j'=1}^{2^n}\sum_{i'=(j'-1)2^{m-n}+1}^{j'2^{m-n}}\sum_{l=0}^{q-1}2^{-2m(q-l)}\langle\delta_{i2^{-m}},\delta_{i'2^{-m}}\rangle_{\mathfrak{H}}^{l}\\
&\leq R_n + c(H,f,q)2^{2m(q-1)}\left(\sup_{|x-y|\leq\frac{1}{2^n},0\leq a\leq 2q}\left|f^{(b)}(B_x)-f^{(b)}(B_y)\right|+(2^{-n})^{\frac{q-1}{q}}\right)\\
&\quad\times\sum_{l=0}^{q-1}2^{-2m(q-l)}\sum_{i,j=0}^{2^m}\langle\delta_{i2^{-m}},\delta_{i'2^{-m}}\rangle_{\mathfrak{H}}^{l}\\
&\leq R_n + c(H,f,q)\left(\sup_{|x-y|\leq\frac{1}{2^n},0\leq a\leq 2q}\left|f^{(b)}(B_x)-f^{(b)}(B_y)\right|+(2^{-n})^{\frac{q-1}{q}}\right)
\end{aligned}$$

and this converges to zero due to the continuity and B and since $q>1$. ■

3.3 Proof of Theorem 1 in the case $\frac{1}{2q} < H \leq 1-\frac{1}{2q}$: the weighted central limit theorem

Suppose first that $\frac{1}{2q} < H < 1-\frac{1}{2q}$. We study the convergence in law of the sequence $V_{2,n}^{(q)}(f) = 2^{-\frac{n}{2}}V_n^{(q)}(f)$. We fix two integers $m \geq n$, and decompose this sequence as follows:

$$V_{2,m}^{(q)}(f) = A^{(m,n)} + B^{(m,n)},$$

where

$$A^{(m,n)} = 2^{-\frac{m}{2}}\sum_{j=1}^{2^n} f\left(B_{(j-1)2^{-n}}\right)\sum_{i=(j-1)2^{m-n}+1}^{j2^{m-n}} H_q\left(2^{nH}\Delta B\right),$$

and

$$B^{(m,n)} = \frac{1}{q!}2^{m(Hq-\frac{1}{2})}\sum_{j=1}^{2^n}\sum_{i=(j-1)2^{m-n}+1}^{j2^{m-n}}\Delta_{i,j}^{m,n}f(B)I_q\left(\delta_{i2^{-n}}^{\otimes q}\right),$$

and where as before we make use of the notation $\Delta_{i,j}^{m,n}f(B) = f(B_{(i-1)2^{-m}}) - f(B_{(j-1)2^{-n}})$.

Let us first consider the term $A^{(m,n)}$. From Theorem 1 in Breuer and Major (1983), and taking into account that $H < 1-\frac{1}{2q}$, it follows that the random vector

$$\left(2^{-\frac{m}{2}}\sum_{i=(j-1)2^{m-n}+1}^{j2^{m-n}} H_q\left(2^{nH}\Delta B_{i2^{-n}}\right);\quad j=1,\ldots,2^n\right)$$

converges in law, as $m \to \infty$, to

$$\left(c_{H,q}\Delta W_{j2^{-n}}; \quad j = 1, \ldots, 2^n\right)$$

where W is a standard Brownian motion independent of B (the independence is a consequence of the central limit theorem for multiple stochastic integrals proved, for instance, in Peccati and Tudor [18]). Since

$$\sum_{j=1}^{2^n} f\left(B_{(j-1)2^{-n}}\right) \Delta W_{j2^{-n}}$$

converges in L^2 as $n \to \infty$ to the Itô integral $\int_0^1 f(B_s)dW_s$ we conclude that, by letting $m \to \infty$ and then $n \to \infty$, we have

$$A^{(m,n)} \xrightarrow{\text{law}} c_{H,q} \int_0^1 f(B_s)dW_s.$$

Then it suffices to show that

$$\lim_{n\to\infty} \sup_{m\to\infty} E\left|B^{(m,n)}\right|^2 = 0. \tag{3.53}$$

We have, as in (3.49)

$$\begin{aligned} E\left|B^{(m,n)}\right|^2 &= 2^{m(2Hq-1)} \sum_{j=1}^{2^n} \sum_{i=(j-1)2^{m-n}+1}^{j2^{m-n}} \sum_{j'=1}^{2^n} \sum_{i'=(j'-1)2^{m-n}+1}^{j'2^{m-n}} \sum_{l=0}^{q} \frac{l!}{q!^2}\binom{q}{l}^2 \\ &\quad \times b_l^{(m,n)} \langle \delta_{i2^{-m}}, \delta_{i'2^{-m}} \rangle_{\mathfrak{H}}^l, \end{aligned} \tag{3.54}$$

where $b_l^{(m,n)}$ has been defined in (3.50). The term in (3.54) corresponding to $l = q$ can be estimated by

$$\frac{1}{q!} 2^{m(2Hq-1)} \sup_{|x-y|\le 2^{-n}} E\left|f(B_x) - f(B_y)\right|^2 \beta_{q,m},$$

which converges to zero as n tends to infinity, uniformly in m, because by Lemma 4, (2.28), and using that $H < 1 - \frac{1}{2q}$

$$\sup_m 2^{m(2Hq-1)} \beta_{q,m} < \infty.$$

In order to handle the terms with $0 \le l \le q-1$, we will distinguish two different cases, depending on the value of H.

Case $H < 1/2$. Suppose $0 \le l \le q-1$. By Lemma 4, (2.27), we can majorize $b_l^{(m,n)}$ as follows

$$|b_l^{(m,n)}| \le C 2^{-4Hm(q-l)} ,$$

and, as a consequence, applying again Lemma 4, (2.28) the corresponding term in (3.54) will be bounded by

$$C 2^{m(2Hq-1)} 2^{-4Hm(q-l)} \beta_{l,m} \le C 2^{2mH(l-q)},$$

which converges to zero as m tends to infinity because $l < q$.

Case $H > 1/2$. Suppose $0 \leq l \leq q-1$. By Lemma 5, (2.30), we get the estimate

$$|b_l^{(m,n)}| \leq C 2^{-2m(q-l)} ,$$

and, as a consequence, applying again Lemma 4, (2.28) the corresponding term in (3.54) will be bounded by

$$C 2^{m(2Hq-1)} 2^{-2m(q-l)} \beta_{l,m}.$$

If $H < 1 - \frac{1}{2l}$, applying Lemma 4, (2.28) this is bounded by $C 2^{m(2H(q-l)-2(q-l))}$, which converges to zero as m tends to infinity because $H < 1$ and $l < q$. In the case $H = 1 - \frac{1}{2l}$, applying Lemma 4, (2.29) we get the estimate $Cm 2^{m(2H(q-l)-2(q-l))}$, which converges to zero again as m tends to infinity because $H < 1$ and $l < q$. In the case $H > 1 - \frac{1}{2l}$ we apply Lemma 5, (2.30) and we get the estimate $C 2^{m(2H2+1-2q)}$, which converges to zero as m tends to infinity because $H < 1 - \frac{1}{2q}$.

The proof in the case $H = 1 - \frac{1}{2q}$ is similar. The convergence of the term $A^{(m,n)}$ is obtained by applying Theorem 1' in Breuer and Major (1983), and the convergence to zero in L^2 of the term $B^{(m,n)}$ follows the same lines as before.

3.4 Proof of Corollary 2

For any integer q, we have

$$\begin{aligned} \left(2^{nH} \Delta B_{k2^{-n}}\right)^q - \mu_q &= \sum_{p=1}^{q} \frac{1}{p!} \binom{q}{p} \mu_{q-p} 2^{Hnp} I_p(\delta_{k2^{-n}}^{\otimes p}) \\ &= \sum_{p=1}^{q} \binom{q}{p} \mu_{q-p} H_p\left(2^{nH} \Delta B_{k2^{-n}}\right) \end{aligned}$$

Indeed, the pth kernel in the chaos representation of $\left(2^{nH} \Delta B_{k2^{-n}}\right)^q$ will be

$$\frac{1}{p!} E(D^p \left(2^{nH} \Delta B_{k2^{-n}}\right)^q) = \frac{1}{p!} \binom{q}{p} 2^{nHp} \mu_{q-p} \delta_{k2^{-n}}^{\otimes p}.$$

Suppose first that q is odd and $H > \frac{1}{2}$. In this case we have

$$2^{-nH} \sum_{k=1}^{2^n} f(B_{(k-1)2^{-n}}) (2^{nH} \Delta B_{k2^{-n}})^q = \sum_{p=1}^{q} \binom{q}{p} \mu_{q-p} 2^{-nH} V_n^{(p)}(f).$$

The term with $p = 1$ converges in L^2 to $q\mu_{q-1} \int_0^1 f(B_s) dB_s$. For $p \geq 2$, the limit in L^2 is zero. In fact, if $H \leq 1 - \frac{1}{2p}$, $E\left(V_n^{(p)}(f)^2\right)$ is bounded by a constant times $n 2^n$, by Proposition

7, and, by Theorem 1, (1.16), if $H > 1 - \frac{1}{2p}$, $E\left(V_n^{(p)}(f)^2\right)$ is bounded by a constant times $2^{n2(1-H)p-2n}$, and $2(1-H)p - 2 - 2H < 0$.

Suppose now that q is even, then

$$2^{2nH-n} \sum_{k=1}^{2^n} f(B_{(k-1)2^{-n}}) \left[(2^{nH} \Delta B_{k2^{-n}})^q - \mu_q \right] = 2^{2nH-n} \sum_{p=2}^{q} \binom{q}{p} \mu_{q-p} V_n^{(p)}(f) \ .$$

If $H < \frac{1}{4}$, by Theorem 1, (1.13), $2^{2nH-n}\binom{q}{2}\mu_{q-2} V_n^{(2)}(f)$ converges in L^2 as n tends to infinity to $\frac{1}{4}\binom{q}{2}\mu_{q-2} \int_0^1 f''(B_s)ds$. On the other hand, for $p \geq 4$, $2^{2nH-n} V_n^{(p)}(f)$ converges to zero in L^2. In fact, if $H < \frac{1}{2p}$, then by (3.34), $E\left(V_n^{(p)}(f)^2\right) = O(2^{n(-2Hp+2)})$, and $-2Hp + 2 + 4H - 2 < 0$, and if $H \geq \frac{1}{2p}$, then by (3.35), $E\left(V_n^{(p)}(f)^2\right) = O(2^n)$, and $4H - 1 < 0$. Therefore (1.18) holds.

In the case $\frac{1}{4} < H < \frac{3}{4}$, for any $p = 2, 4, \ldots, q$, by Theorem 1, (1.14), $2^{-n/2} V_n^{(p)}(f)$ converges in law to $\sigma_{H,p} \int_0^1 f(B_s) dW_s^{(p)}$. The Brownian motions $W^{(p)}$ are independent. This is a consequence of the central limit theorems for multiple stochastic integrals proved, for instance, in Peccati and Tudor [18]. This implies the convergence (1.19). The proof of (1.20) is analogous.

Finally, consider the case $H > \frac{3}{4}$. For $p = 2$, $2^{n-2Hn} V_n^{(2)}(f)$ converges in L^2 to $\mu_{q-2}\binom{q}{2} \int_0^1 f(B_s) dZ_s^{(2)}$ by Theorem 1, (1.16). If $p \geq 4$, then $2^{n-2Hn} V_n^{(p)}(f)$ converges in L^2 to zero because again by Theorem 1, (1.16), $E\left(V_n^{(p)}(f)^2\right) = O(2^{n(2-2(1-H)p)})$.

3.5 Proof of Theorem 3

We can assume $H < \frac{1}{2}$, the case where $H \geq \frac{1}{2}$ being straightforward. By a Taylor's formula, we have

$$\begin{aligned} f(B_1) &= f(0) + \frac{1}{2} \sum_{k=1}^{2^n} \left(f'(B_{k2^{-n}}) + f'(B_{(k-1)2^{-n}}) \right) \Delta B_{k2^{-n}} \\ &\quad - \frac{1}{12} \sum_{k=1}^{2^n} f^{(3)}(B_{(k-1)2^{-n}}) \left(\Delta B_{k2^{-n}} \right)^3 - \frac{1}{24} \sum_{k=1}^{2^n} f^{(4)}(B_{(k-1)2^{-n}}) \left(\Delta B_{k2^{-n}} \right)^4 \\ &\quad - \frac{1}{80} \sum_{k=1}^{2^n} f^{(5)}(B_{(k-1)2^{-n}}) \left(\Delta B_{k2^{-n}} \right)^5 + R_n, \end{aligned} \tag{3.55}$$

with R_n converging towards 0 in probability as $n \to \infty$, because $H > 1/6$. We can expand the monomials x^m, $m = 2, 3, 4, 5$, in terms of the Hermite polynomials:

$$\begin{aligned} x^2 &= H_2(x) + 1 \\ x^3 &= H_3(x) + 3\, H_1(x) \\ x^4 &= H_4(x) + 6\, H_2(x) + 3 \\ x^5 &= H_5(x) + 10\, H_3(x) + 15\, H_1(x). \end{aligned}$$

In this way we obtain

$$\sum_{k=1}^{2^n} f^{(3)}(B_{(k-1)2^{-n}})\,(\Delta B_{k2^{-n}})^3 = 2^{-3Hn} V_n^{(3)}(f^{(3)}) + 3 \times 2^{-2Hn} V_n^{(1)}(f^{(3)}), \quad (3.56)$$

$$\begin{aligned} \sum_{k=1}^{2^n} f^{(4)}(B_{(k-1)2^{-n}})\,(\Delta B_{k2^{-n}})^4 &= 2^{-4Hn} V_n^{(4)}(f^{(4)}) \\ &\quad + 6 \times 2^{-4Hn} V_n^{(2)}(f^{(4)}) + 3 \times 2^{-4Hn} \sum_{k=1}^{2^n} f^{(4)}(B_{(k-1)2^{-n}}), \end{aligned} \quad (3.57)$$

$$\begin{aligned} \sum_{k=1}^{2^n} f^{(5)}(B_{(k-1)2^{-n}})\,(\Delta B_{k2^{-n}})^5 &= 2^{-5Hn} V_n^{(5)}(f^{(5)}) \\ &\quad + 10 \times 2^{-5Hn} V_n^{(3)}(f^{(5)}) + 15 \times 2^{-4Hn} V_n^{(1)}(f^{(5)}). \end{aligned} \quad (3.58)$$

By (3.35) and using that $H > \frac{1}{6}$, we have $E\left(V_n^{(3)}(f^{(3)})^2\right) \leq C2^n$, $E\left(V_n^{(3)}(f^{(5)})^2\right) \leq C2^n$, and, as a consequence, the first summand in (3.56) and the second one in (3.58) converge to zero in L^2 as n tends to infinity. Also, by (3.35), $E\left(V_n^{(4)}(f^{(4)})^2\right) \leq C2^n$, and $E\left(V_n^{(5)}(f^{(5)})^2\right) \leq C2^n$, and the first summand in (3.57) and the first summand in (3.58) converge to zero in L^2 as n tends to infinity. If $\frac{1}{6} < H < \frac{1}{4}$, by (3.34), $E\left(V_n^{(2)}(f^{(4)})^2\right) \leq C2^{n(-4H+2)}$, and $2^{-4Hn} V_n^{(2)}(f^{(4)})$ converges to zero in L^2 as n tends to infinity. If $\frac{1}{4} \leq H < \frac{1}{2}$, by (3.35), $E\left(V_n^{(2)}(f)^2\right) \leq C2^n$, and $2^{-4Hn} V_n^{(2)}(f^{(4)})$ converges to zero in L^2 as n tends to infinity.

Moreover, using the identity, valid for regular functions $h : \mathbb{R} \to \mathbb{R}$:

$$\sum_{k=1}^{2^n} h'(B_{(k-1)2^{-n}})\,\Delta B_{k2^{-n}} = h(B_1) - h(0) - \frac{1}{2} \sum_{k=1}^{2^n} h''(B_{\theta_{k2^{-n}}})\,(\Delta B_{k2^{-n}})^2$$

for some $\theta_{k2^{-n}}$ lying between $(k-1)2^{-n}$ and $k2^{-n}$, we deduce that $2^{-4Hn} V_n^{(1)}(f^{(5)})$ tends to

zero, because $H > \frac{1}{6}$. In the same way, we have

$$\begin{aligned} 2^{-2Hn} V_n^{(1)}(f^{(3)}) &= -\frac{1}{2} 2^{-2Hn} \sum_{k=1}^{2^n} f^{(4)}(B_{(k-1)2^{-n}}) (\Delta B_{k2^{-n}})^2 \\ &\quad -\frac{1}{6} 2^{-2Hn} \sum_{k=1}^{2^n} f^{(5)}(B_{(k-1)2^{-n}}) (\Delta B_{k2^{-n}})^3 + o(1). \end{aligned}$$

We have obtained

$$\begin{aligned} f(B_1) &= f(0) + \frac{1}{2} \sum_{k=1}^{2^n} \left(f'(B_{k2^{-n}}) + f'(B_{(k-1)2^{-n}}) \right) \Delta B_{k2^{-n}} \\ &\quad + \frac{1}{8} 2^{-4Hn} \sum_{k=1}^{2^n} f^{(4)}(B_{(k-1)2^{-n}}) H_2\left(2^{nH} \Delta B_{k2^{-n}}\right) \\ &\quad - \frac{1}{6} 2^{-2Hn} \sum_{k=1}^{2^n} f^{(5)}(B_{(k-1)2^{-n}}) (\Delta B_{k2^{-n}})^3 + o(1). \end{aligned}$$

As before $2^{-4Hn} V_n^{(2)}(f^{(4)})$ converges to zero in L^2. Finally,

$$2^{-2Hn} \sum_{k=1}^{2^n} f^{(5)}(B_{(k-1)2^{-n}}) (\Delta B_{k2^{-n}})^3 = 2^{-5Hn} V_n^{(5)}(f^{(8)}) + 3 \times 2^{-4Hn} V_n^{(3)}(f^{(5)}),$$

which also converges to zero by the same arguments as above. This completes the proof. ■

References

[1] P. Breuer and P. Major (1983): *Central limit theorems for nonlinear functionals of Gaussian fields.* J. Multivariate Anal. **13** (3), 425-441.

[2] P. Cheridito and D. Nualart (2005): *Stochastic integral of divergence type with respect to fractional Brownian motion with Hurst parameter H in $(0, 1/2)$.* Ann. Inst. H. Poincar Probab. Statist. **41**, 1049-1081.

[3] J.M. Corcuera, D. Nualart and J.H.C. Woerner (2006): *Power variation of some integral fractional processes.* Bernoulli **12**, 713-735.

[4] R.L. Dobrushin and P. Major (1979): *Non-central limit theorems for nonlinear functionals of Gaussian fields.* Z. Wahrsch. verw. Gebiete **50**, 27-52.

[5] L. Giraitis and D. Surgailis (1985): *CLT and other limit theorems for functionals of Gaussian processes.* Z. Wahrsch. verw. Gebiete **70**, 191-212.

[6] M. Gradinaru and I. Nourdin (2007): *Convergence of weighted power variations of fractional Brownian motion.* Preprint 2007/05, University of Nancy.

[7] M. Gradinaru, I. Nourdin, F. Russo and P. Vallois (2005): *m-order integrals and Itô's formula for non-semimartingale processes; the case of a fractional Brownian motion with any Hurst index.* Ann. Inst. H. Poincaré Probab. Statist. **41**, 781-806.

[8] M. Gradinaru, F. Russo and P. Vallois (2001): *Generalized covariations, local time and Stratonovich Itô's formula for fractional Brownian motion with Hurst index* $H \geq \frac{1}{4}$. Ann. Probab. **31**, 1772-1820.

[9] J. Jacod (1994): *Limit of random measures associated with the increments of a Brownian semimartingale.* Preprint. University of Paris VI (revised version, unpublished work).

[10] D. Khoshnevisan and T.M. Lewis (1999). *Stochastic calculus for Brownian motion on a Brownian fracture.* Ann. Appl. Probab. **9**, no. 3, 629-667.

[11] J. León and C. Ludeña (2006): *Limits for weighted p-variations and likewise functionals of fractional diffusions with drift.* Stoch. Proc. Appl. **117** (3), 271-296.

[12] A. Neuenkirch and I. Nourdin (2007): *Exact rate of convergence of some approximation schemes associated to SDEs driven by a fractional Brownian motion.* J. Theor. Probab., to appear.

[13] I. Nourdin (2007): *Asymptotic behavior of some weighted quadratic and cubic variations of the fractional Brownian motion.* Preprint. University of Paris VI.

[14] I. Nourdin (2007): *A simple theory for the study of SDEs driven by a fractional Brownian motion, in dimension one.* Séminaire de Probabilités **XLI**, to appear.

[15] I. Nourdin and G. Peccati (2007): *Weighted power variations of iterated Brownian motion.* Preprint. University of Paris VI.

[16] D. Nualart (2005): *Malliavin calculus and related topics. Second Edition.* Springer-Verlag, New-York.

[17] D. Nualart (2003): *Stochastic calculus with respect to the fractional Brownian motion and applications.* Contemp. Math. **336**, 3-39.

[18] G. Peccati and C. A. Tudor (2005): *Gaussian limits for vector-valued multiple stochastic integrals.* Séminaire de Probabilités **XXXVIII**, 247–262, Lecture Notes in Math., 1857, Springer, Berlin.

[19] J. Swanson (2007): *Variations of the solution to a stochastic heat equation.* Ann. Probab. **35**, no. 6, 2122-2159.

[20] M. Taqqu (1979): *Convergence of integrated processes of arbitrary Hermite rank.* Z. Wahrsch. verw. Gebiete **50**, 53-83.

[21] C.A. Tudor (2007): *Analysis of the Rosenblatt process*. ESAIM Probability and Statistics. To appear.

[22] C.A Tudor and F. Viens (2007): *Variations and estimators for the selsimilarity order through Malliavin calculus.* Preprint.