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Central limit theorem for locally interacting Fermi gas

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Abstract

We consider a locally interacting Fermi gas in its natural non-equilibrium steady state and prove the Quantum Central Limit Theorem (QCLT) for a large class of observables. A special case of our results concerns finitely many free Fermi gas reservoirs coupled by local interactions. The QCLT for flux observables, together with the Green-Kubo formulas and the Onsager reciprocity relations previously established [JOP4], complete the proof of the Fluctuation-Dissipation Theorem and the development of linear response theory for this class of models.

1 Introduction

This paper and its companion [AJPP3] are first in a series of papers dealing with fluctuation theory of non-equilibrium steady states in quantum statistical mechanics. They are part of a wider program initiated in [Ru2, Ru3, JP1, JP2, JP4] which deals with the development of a mathematical theory of non-equilibrium statistical mechanics in the framework of algebraic quantum statistical mechanics [BR1, BR2, Pi]. For additional information about this program we refer the reader to the reviews [Ru4, JP3, AJPP1].

In this paper we study the same model as in [JOP4]: A free Fermi gas in a quasi-free state perturbed by a sufficiently regular local interaction. It is well-known that under the influence of such a perturbation this system approaches, as time $t \to +\infty$, a steady state commonly called the natural non-equilibrium steady state (NESS) [BM1, AM, BM2, FMU, JOP4]. Our main result is that under very general conditions the Quantum Central Limit Theorem (QCLT) holds for this NESS. Combined with the results of [JOP4], the QCLT completes the proof of the near-equilibrium Fluctuation-Dissipation Theorem and the development of linear response theory for this class of models.

The rest of this introduction is organized as follows. In Subsection 1.1 for notational purposes we review a few basic concepts of algebraic quantum statistical mechanics. In this subsection the reader can find the definition of QCLT for quantum dynamical systems and a brief review of related literature. Our main result is stated in Subsection 1.2. In Subsection 1.3 we discuss our results in the context of linear response theory.

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1.1 Central limit theorem for quantum dynamical systems

Let $\mathcal{O}$ be a C^*-algebra with identity $\mathbb{1}$ and let τ^t, $t \in \mathbb{R}$, be a strongly continuous group of $*$-automorphisms of $\mathcal{O}$. The pair $(\mathcal{O}, \tau)$ is called a C^*-dynamical system. A positive normalized element of the dual $\mathcal{O}^*$ is called a state on $\mathcal{O}$. In what follows ω is a given τ-invariant state on $\mathcal{O}$. The triple $(\mathcal{O}, \tau, \omega)$ is called a quantum dynamical system.

The system $(\mathcal{O}, \tau, \omega)$ is called ergodic if

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \omega\left(B^* \tau^s(A) B\right) \, \mathrm{d}s = \omega(B^*B)\omega(A),$$

and mixing if

$$\lim_{|t|\to\infty} \omega\left(B^* \tau^t(A) B\right) = \omega(B^*B)\omega(A),$$

for all $A, B \in \mathcal{O}$.

We denote by $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ the GNS-representation of the C^*-algebra $\mathcal{O}$ associated to the state ω. The state ω is called modular if Ω_ω is a separating vector for the enveloping von Neumann algebra $\pi_\omega(\mathcal{O})''$. The states of thermal equilibrium are described by the (τ, β)-KMS condition where $\beta > 0$ is the inverse temperature. Any (τ, β)-KMS state on $\mathcal{O}$ is τ-invariant and modular.

Let f be a bounded Borel function on $\mathbb{R}$ and $A \in \mathcal{O}_{\mathrm{self}}$. With a slight abuse of notation in the sequel we will often denote $f(\pi_\omega(A))$ by $f(A)$ and write $\omega(f(A)) = (\Omega_\omega, f(\pi_\omega(A))\Omega_\omega)$. With this convention, $1_{[a,b]}(A)$ denotes the spectral projection on the interval $[a, b]$ of $\pi_\omega(A)$. We shall use the same convention for the products $f_1(\pi_\omega(A_1)) \cdots f_n(\pi_\omega(A_n))$, etc.

An involutive skew $*$-automorphism Θ of $\mathcal{O}$ is called time-reversal if $\Theta \circ \tau^t = \tau^{-t} \circ \Theta$. A state η on $\mathcal{O}$ is called time-reversal invariant if $\eta \circ \Theta(A) = \eta(A^*)$ holds for all $A \in \mathcal{O}$.

For any subset $\mathcal{A} \subset \mathcal{O}$ we denote by $\mathcal{A}_{\mathrm{self}} = \{A \in \mathcal{A} \,|\, A = A^*\}$ the set of self-adjoint elements of $\mathcal{A}$. We say that a subset $\mathcal{A} \subset \mathcal{O}$ is L^1-asymptotically abelian for τ if for all $A, B \in \mathcal{A}$,

$$\int_{-\infty}^{\infty} \left\| [A, \tau^t(B)] \right\| \, \mathrm{d}t < \infty.$$

Throughout the paper we shall use the shorthand

$$\tilde{A}_t \equiv \frac{1}{\sqrt{t}} \int_0^t \left(\tau^s(A) - \omega(A)\right) \, \mathrm{d}s.$$

Let $\mathfrak{C}$ be a $*$-vector subspace of $\mathcal{O}$. We say that $\mathfrak{C}$ is *CLT-admissible* if for all $A, B \in \mathfrak{C}$,

$$\int_{-\infty}^{\infty} \left|\omega(\tau^t(A)B) - \omega(A)\omega(B)\right| \, \mathrm{d}t < \infty.$$

For $A, B \in \mathfrak{C}$ we set

$$L(A,B) \equiv \int_{-\infty}^{\infty} \omega\left((\tau^t(A) - \omega(A))(B - \omega(B))\right) \, \mathrm{d}t = \int_{-\infty}^{\infty} \left(\omega\left(\tau^t(A)B\right) - \omega(A)\omega(B)\right) \mathrm{d}t,$$

$$\varsigma(A,B) \equiv \frac{1}{2\mathrm{i}} \int_{-\infty}^{\infty} \omega\left([\tau^t(A), B]\right) \, \mathrm{d}t = \frac{1}{2\mathrm{i}} \left(L(A,B) - L(B,A)\right).$$

The functional $(A,B) \mapsto L(A,B)$ is obviously bilinear. Other properties of this functional are summarized in:

Proposition 1.1 *Suppose that $\mathfrak{C}$ is* CLT*-admissible and let $A, B \in \mathfrak{C}$. Then:*

(i) $L(A^*, A) \geq 0$.

(ii) $L(A,B) = \overline{L(B^*, A^*)}$. *In particular, if A and B are self-adjoint, then* $\varsigma(A,B) = \operatorname{Im} L(A,B)$.

(iii) $|L(A^*, B)|^2 \leq L(A^*, A) L(B^*, B)$.

(iv) $(A,B) \mapsto \varsigma(A,B)$ *is a symplectic form on the real vector space* $\mathfrak{C}_{\mathrm{self}}$.

(v) If ω is a mixing (τ, β)-KMS state, then $\varsigma = 0$.

(vi) Suppose that $\varsigma = 0$, that $\mathfrak{C}$ is dense in $\mathcal{O}$ and L^1-asymptotically abelian for τ, and that ω is either a factor state or 3-fold mixing: For all $A_1, A_2, A_3 \in \mathcal{O}$,

$$\lim_{\min_{i \neq j} |t_i - t_j| \to \infty} \omega\left(\tau^{t_1}(A_1)\tau^{t_2}(A_2)\tau^{t_3}(A_3)\right) = \omega(A_1)\omega(A_2)\omega(A_3).$$

Then ω is a (τ, β)-KMS state for some $\beta \in \mathbb{R} \cup \{\pm\infty\}$.

Proof. Note that

$$0 \leq \omega\left(\tilde{A}_t^* \tilde{A}_t\right) = \int_{-t}^{t} \left(1 - \frac{|s|}{t}\right) \omega\left((\tau^t(A^*) - \omega(A^*))(A - \omega(A))\right) \, \mathrm{d}s.$$

This identity and the dominated convergence theorem yield

$$L(A^*, A) = \lim_{t \to \infty} \omega\left(\tilde{A}_t^* \tilde{A}_t\right) \geq 0,$$

and (i) follows. Parts (ii) and (iv) are obvious. (i) and (ii) imply the Cauchy-Schwartz inequality (iii). Part (v) follows from Proposition 5.4.12 in [BR2]. Part (vi) is the celebrated stability result of Bratteli, Kishimoto and Robinson [BKR, BR2]. □

Definition 1.2 *Let $\mathfrak{C}$ be CLT-admissible. We shall say that the Simple Quantum Central Limit Theorem (SQCLT) holds for $\mathfrak{C}$ w.r.t. $(\mathcal{O}, \tau, \omega)$ if for all $A \in \mathfrak{C}_{\mathrm{self}}$,*

$$\lim_{t \to \infty} \omega\left(\mathrm{e}^{\mathrm{i}\tilde{A}_t}\right) = \exp\left(-\frac{1}{2} L(A,A)\right).$$

We shall say that the Quantum Central Limit Theorem (QCLT) holds for $\mathfrak{C}$ if for all n and all $A_1, \cdots, A_n$ in $\mathfrak{C}_{\mathrm{self}}$,

$$\lim_{t \to \infty} \omega\left(\mathrm{e}^{\mathrm{i}\tilde{A}_{1t}} \cdots \mathrm{e}^{\mathrm{i}\tilde{A}_{nt}}\right) = \exp\left(-\frac{1}{2} \sum_{1 \leq j,k \leq n} L\left(A_k, A_j\right) - \mathrm{i} \sum_{1 \leq j < k \leq n} \varsigma(A_j, A_k)\right). \tag{1.1}$$

The SQCLT is obviously a special case of the QCLT. Under sufficient ergodic assumptions, however, the QCLT can be deduced from the SQCLT.

Theorem 1.3 *Suppose that $\mathfrak{C}$ is CLT-admissible and L^1-asymptotically abelian for τ. Suppose also that the system $(\mathcal{O}, \tau, \omega)$ is ergodic and that the state ω is modular. If the SQCLT holds for $\mathfrak{C}$ w.r.t. $(\mathcal{O}, \tau, \omega)$ then the QCLT also holds for $\mathfrak{C}$.*

We shall prove Theorem 1.3 in Section 2 following the ideas of [GV].

The SQCLT has the same probabilistic interpretation as the classical central limit theorem. The probability of measuring a value of A in $[a, b]$ when the system is in the state ω is given by

$$\mathrm{Prob}_\omega\{A \in [a, b]\} = \omega(1_{[a,b]}(A)).$$

If SQCLT holds for A, then

$$\lim_{t\to\infty} \mathrm{Prob}_\omega \left\{ \frac{1}{t} \int_0^t \tau^s(A)\, \mathrm{d}s \in \omega(A) + \left[\frac{a}{\sqrt{t}}, \frac{b}{\sqrt{t}} \right] \right\} = \frac{1}{\sqrt{2\pi L(A,A)}} \int_a^b \mathrm{e}^{-x^2/2L(A,A)^2}\, \mathrm{d}x. \tag{1.2}$$

Except in trivial cases, the QCLT does not have a classical probabilistic interpretation. In this case the relevant concept is the CCR algebra over the symplectic space $(\mathfrak{C}_{\mathrm{self}}, \varsigma)$, often called the fluctuation algebra [GVV1]. The mathematical structure of the fluctuation algebra is discussed in many places in the literature, see e.g. [GVV1]–[GVV6] and [MSTV, BR2, Pe, OP, De2] for general results about CCR algebras. For notational and reference purposes we recall a few basic facts. Let $\mathcal{W}$ be the C^*-algebra generated by the elements $\{W(A) \mid A \in \mathfrak{C}_{\mathrm{self}}\}$ such that for all A, B in $\mathfrak{C}_{\mathrm{self}}$

$$W(-A) = W(A)^*, \qquad W(A)W(B) = \mathrm{e}^{-\mathrm{i}\varsigma(A,B)/2} W(A + B),$$

equipped with the minimal regular norm. The map

$$\omega_L(W(A)) \equiv \mathrm{e}^{-L(A,A)/2},$$

uniquely extends to a quasi-free state on $\mathcal{W}$ and (1.1) can be written as

$$\lim_{t\to\infty} \omega \left(\mathrm{e}^{\mathrm{i}\tilde{A}_{1t}} \cdots \mathrm{e}^{\mathrm{i}\tilde{A}_{nt}} \right) = \omega_L(W(A_1) \cdots W(A_n)). \tag{1.3}$$

The pair $(\mathcal{W}, \omega_L)$ describes the fluctuations of $\mathfrak{C}$ w.r.t. the quantum dynamical system $(\mathcal{O}, \tau, \omega)$. Let $(\mathcal{H}_L, \pi_L, \Omega_L)$ be the GNS representation of $\mathcal{W}$ associated to ω_L. We shall also denote by ω_L the induced state on the enveloping von Neumann algebra $\pi_L(\mathcal{W})''$. Since for all $A \in \mathfrak{C}_{\mathrm{self}}$ the map

$$\mathbb{R} \ni x \mapsto \omega_L(W(xA)),$$

extends to an entire analytic function on $\mathbb{C}$, there exist self-adjoint operators $\varphi_L(A)$ on $\mathcal{H}_L$ such that

$$\pi_L(W(A)) = \mathrm{e}^{\mathrm{i}\varphi_L(A)}.$$

Moreover, the operators $\varphi_L(A)$, $A \in \mathfrak{C}_{\mathrm{self}}$ have a common dense set of analytic vectors $\mathcal{A} \subset \mathcal{H}_L$ and on this set

$$[\varphi_L(A), \varphi_L(B)] = \mathrm{i}\varsigma(A, B)\mathbb{1}.$$

The operators $\varphi_L(A)$ are the Bose fields associated by QCLT to $(\mathcal{O}, \tau, \omega)$. For any n and $A_1, \cdots, A_n \in \mathfrak{C}_{\mathrm{self}}$, Ω_L is in the domain of $\varphi_L(A_1) \cdots \varphi_L(A_n)$ and , as usual, we denote

$$\omega_L(\varphi_L(A_1) \cdots \varphi_L(A_n)) \equiv (\Omega_L, \varphi_L(A_1) \cdots \varphi_L(A_n)\Omega_L). \tag{1.4}$$

In particular $\omega_L(\varphi_L(A_1)\varphi_L(A_2)) = L(A_1, A_2)$. For any integer n we denote $\mathcal{P}_n$ the set of all permutations π of $\{1, \ldots, 2n\}$ such that

$$\pi(2j - 1) < \pi(2j), \quad \text{and} \quad \pi(2j - 1) < \pi(2j + 1), \tag{1.5}$$

for every $j \in \{1, \ldots, n\}$. The cardinality of $\mathcal{P}_n$ is $(2n)!/(2^n n!)$. Then

$$\omega_L(\varphi_L(A_1) \cdots \varphi_L(A_n)) = \begin{cases} \displaystyle\sum_{\pi \in \mathcal{P}_{n/2}} \prod_{j=1}^{n/2} \omega_L(\varphi_L(A_{\pi(2j-1)})\varphi_L(A_{\pi(2j)})), & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \tag{1.6}$$

With these preliminaries, we can formulate:

Theorem 1.4 *Suppose that QCLT holds for $\mathfrak{C}$ w.r.t. $(\mathcal{O},\tau,\omega)$, let $A_1,\cdots,A_n\in\mathfrak{C}_{\rm self}$, and let $f_1,\cdots,f_n$ be bounded Borel functions on $\mathbb{R}$. If $L(A_j,A_j)=0$, we assume in addition that f_j is continuous at zero. Then*

$$\lim_{t\to\infty}\omega(f_1(\tilde{A}_{1t})\cdots f_n(\tilde{A}_{nt}))=\omega_L(f_1(\varphi_L(A_1))\cdots f_n(\varphi_L(A_n))). \tag{1.7}$$

If $n=1$, then Theorem 1.4 is an immediate consequence of the classical Lévy-Cramér Continuity Theorem. The proof in the case $n>1$ is given in Subsection 2.2. For a probabilistic interpretation of Theorem 1.4 in the context of repeated quantum-mechanical measurements we refer the reader to Section 2 in [Da1].

The QCLT does not imply that

$$\lim_{t\to\infty}\omega(\tilde{A}_{1t}\cdots\tilde{A}_{nt})=\omega_L(\varphi_L(A_1)\cdots\varphi_L(A_n)), \tag{1.8}$$

and in principle the convergence of moments has to be established separately. In our model, the proof of (1.8) is an intermediate step in the proof of the QCLT.

To define Bose annihilation and creation operators associated with fields $\varphi_L(A)$, we need to assume that the symplectic form ς is non-degenerate (this implies that $\mathfrak{C}_{\rm self}$ is either even- or infinite-dimensional). In this case there exists a complex structure J on $\mathfrak{C}_{\rm self}$ satisfying $\varsigma(JA,JB)=\varsigma(A,B)$, and one can define the operators $a_L(A)/a_L^*(A)$ on $\mathcal{A}$ by

$$a_L(A)\equiv\frac{1}{\sqrt{2}}\left(\varphi_L(A)+{\rm i}\varphi_L(JA)\right),\qquad a_L^*(A)\equiv\frac{1}{\sqrt{2}}\left(\varphi_L(A)-{\rm i}\varphi_L(JA)\right). \tag{1.9}$$

These operators are closable and satisfy

$$[a_L(A),a_L^*(B)]={\rm i}\left(\varsigma(A,B)-{\rm i}\varsigma(A,JB)\right),$$

on $\mathcal{A}$.

We expect that in typical physical examples the symplectic form ς will be degenerate in which case the Bose annihilation and creation operators (1.9) cannot be defined globally. Let us assume, for simplicity, that L is non-degenerate (in the general case one has to further factor out the kernel of L). Consider first the extreme case $\varsigma=0$ (this will hold, for example, if ω is a mixing (τ,β)-KMS state). Let $\hat{\mathfrak{C}}_{\rm self}$ be the group of all characters of the discrete Abelian group $\mathfrak{C}_{\rm self}$. The dual group $\hat{\mathfrak{C}}_{\rm self}$ endowed with the topology of pointwise convergence is a compact topological group and the algebra $\mathcal{W}$ is isomorphic to the C^*-algebra of all continuous functions on $\hat{\mathfrak{C}}_{\rm self}$. The state ω_L is identified with the Gaussian measure on $\hat{\mathfrak{C}}_{\rm self}$ uniquely determined by

$$\int\chi(A)\,{\rm d}\mu_L(\chi)={\rm e}^{-L(A,A)/2}.$$

More generally, let

$$\mathfrak{C}_{\rm self}^{(1)}=\{A\,|\,\varsigma(A,B)=0\text{ for all }B\in\mathfrak{C}_{\rm self}\},$$

and suppose that there exist $\mathfrak{C}_{\rm self}^{(2)}$ such that $\mathfrak{C}_{\rm self}=\mathfrak{C}_{\rm self}^{(1)}\oplus\mathfrak{C}_{\rm self}^{(2)}$ and $L=L^{(1)}\oplus L^{(2)}$ where $L^{(j)}$ denotes the restriction of L to $\mathfrak{C}_{\rm self}^{(j)}$. This is certainly the case if $\mathfrak{C}_{\rm self}$ is finite dimensional, i.e., if we consider QCLT with respect to finitely many observables. The restriction of ς to $\mathfrak{C}_{\rm self}^{(2)}$ is non-degenerate, and if $W^{(j)}$, $\omega_L^{(j)}$, $j=1,2$ denote the respective CCR algebras and quasi-free states, then

$$\mathcal{W}=\mathcal{W}^{(1)}\otimes\mathcal{W}^{(2)},\qquad\omega_L=\omega_L^{(1)}\otimes\omega_L^{(2)}.$$

In particular, annihilation and creation operators can be associated to the elements of $\mathcal{W}^{(2)}$.

Besides QCLT one may consider the related and more general existence problem for the quantum hydrodynamic limit (QHL). For $\epsilon>0$ and $t>0$, let

$$\hat{A}_\epsilon(t)\equiv\epsilon\int_0^{t/\epsilon^2}\left(\tau^s(A)-\omega(A)\right)\,{\rm d}s.$$

We say that $\mathfrak{C}$ has QHL w.r.t. $(\mathcal{O},\tau,\omega)$ if for all $A_1,\cdots A_n\in\mathfrak{C}_{\rm self}$, and all $t_1>0,\cdots,t_n>0$,

$$\lim_{\epsilon\downarrow0}\omega\left({\rm e}^{{\rm i}\hat{A}_{1\epsilon}(t_1)}\cdots{\rm e}^{{\rm i}\hat{A}_{n\epsilon}(t_n)}\right)=\omega_L(W(\chi_{[0,t_1]}\otimes A_1)\cdots W(\chi_{[0,t_n]}\otimes A_n)). \tag{1.10}$$

where χ_I is the characteristic function of the interval I and, in the definition of the Weyl algebra, the bilinear form L must be replaced by

$$L_{\mathrm{QHL}}(\chi_{[0,s]} \otimes A, \chi_{[0,t]} \otimes B) = \min(s,t)\, L(A,B).$$

The special case where all t_j's are equal corresponds to QCLT. The QHL is interpreted as the weak convergence of the quantum stochastic process $\hat{A}_\epsilon(t)$ to a quantum Brownian motion. With the obvious reformulation, Theorem 1.4 holds for QHL. Convergence of moments

$$\lim_{\epsilon \downarrow 0} \omega(\hat{A}_{1\epsilon}(t_1) \cdots \hat{A}_{n\epsilon}(t_n)) = \omega_L(\varphi_L(\chi_{[0,t_1]} \otimes A_1) \cdots \varphi_L(\chi_{[0,t_n]} \otimes A_n), \tag{1.11}$$

is of independent interest. Even more generally, one may associate to a class $\mathfrak{F}$ of real valued integrable functions on $\mathbb{R}$ the observables

$$\hat{A}_\epsilon(f) \equiv \epsilon^{-1} \int_0^\infty f(\epsilon^2 t) \left(\tau^t(A) - \omega(A)\right) \mathrm{d}t,$$

with $f \in \mathfrak{F}$, $A \in \mathfrak{C}$ and study the limit $\epsilon \downarrow 0$ of

$$\omega\left(\mathrm{e}^{\mathrm{i}\hat{A}_{1\epsilon}(f_1)} \cdots \mathrm{e}^{\mathrm{i}\hat{A}_{n\epsilon}(f_n)}\right). \tag{1.12}$$

Note that QHL corresponds to the choice $\mathfrak{F} = \{\chi_{[0,t]} \,|\, t > 0\}$. For reasons of space and notational simplicity we will focus in the paper on the QCLT for locally interacting fermionic systems. With only notational changes our proofs can be extended to establish QHL and (1.11). It is likely that the proofs can be extended to a much larger class of functions $\mathfrak{F}$, but we shall not pursue this question here (see [De1] for a related discussion).

We finish this section with a few remarks about earlier quantum central limit type results.

First notice that since the law of a single observable is well-defined, the description of the limiting law of a family $\{\tilde{A}_t \,|\, t > 0\}$ of observables as the parameter $t \to \infty$, is covered by the classical Lévy- Cramèr theorem. Several results of interest exist, which are only of quantum nature insofar as the computation of $\lim_{t\to\infty} \omega(e^{\mathrm{i}\alpha\tilde{A}_t})$ is made more complicated by the quantum setting.

Truly quantum central limit theorems attempt to describe the joint limiting behavior, as $t \to \infty$, of a family of observables $\{(\tilde{A}_{1t}, \ldots, \tilde{A}_{nt}) \,|\, t > 0\}$. The earliest results of this type were quantum probabilistic versions of classical theorems concerning sums of independent, identically distributed random variables. Physically interesting applications of these results include the study of spatial fluctuations of local observables of quantum spin systems in a translation-invariant product state. The generality of the framework and the formulation of the limit vary. We mention in particular [GvW] which apply to general *-algebras but where only the convergence of moments is proved; [Kup] which works for general C*-algebras and where convergence in distribution (to a classical Gaussian family) is proved, but only with respect to a tracial state . We also mention [CH] which, although not a central limit theorem, is a first attempt to characterize the convergence in distribution of a family of non-commuting operators in terms of a (pseudo)-characteristic function.

The series of papers [GVV1]–[GVV6],[GV] is more directly oriented towards quantum statistical mechanics. Following ideas of [HL2, HL3],[W], a fluctuation algebra is associated to the spatial fluctuations of local observables for a quantum spin system in a translation-invariant state. That state does not have to be a product state but it must have very strong ergodic properties. It is therefore very difficult to apply these results beyond the product state case. Nevertheless, these works laid a solid conceptual ground and our construction owes much to them. The papers [Ma1]-[Ma2] are similar in spirit but require less stringent ergodic conditions. Their results apply, for example, to spatial fluctuations of local observables in XY-chains at thermal equilibrium.

A distinct feature of our work is that we study QCLT with respect to the group τ^t describing the microscopic dynamics of the system. There is a number of technical and conceptual aspects of QCLT which are specific to the dynamical group. For example, the ergodic properties of the system (laws of large numbers), which have to be established prior to the study of fluctuations, are typically much harder to prove for the dynamical group than for the lattice translation group. As for the conceptual differences, we mention that if ω is a (τ, β)-KMS state, then by Proposition 1.1 (v), $\varsigma = 0$ and the CCR algebra of fluctuations $\mathcal{W}$ is commutative (Part (vi) provides a partial converse to this statement). This is in sharp contrast with QCLT w.r.t. the translation group, where even in the simple example of product states of spin systems the fluctuation algebra is non-commutative.

The CLT for classical dynamical systems is discussed in [Li]. For a review of results on dynamical CLT for interacting particle systems in classical statistical mechanics we refer the reader to [Sp] and [KL]. The CLT for classical spin systems is discussed in Section V.7 of [E].

After this paper was completed, we have learned of the work [De1] which is technically and conceptually related to ours. We shall comment on Dereziński's result at the end of Subsection 3.3.

1.2 QCLT for locally interacting fermions

A free Fermi gas is described by the C^*-dynamical system $(\mathcal{O}, \tau_0)$ where:

(i) $\mathcal{O} = \mathrm{CAR}(\mathfrak{h})$ is the CAR algebra over the single particle Hilbert space $\mathfrak{h}$;

(ii) τ_0^t is the group of Bogoliubov $*$-automorphisms generated by the single particle Hamiltonian h_0,

$$\tau_0^t(a^\#(f)) = a^\#(\mathrm{e}^{\mathrm{i}th_0} f),$$

where $a^*(f)/a(f)$ are the Fermi creation/annihilation operators associated to $f \in \mathfrak{h}$ and $a^\#$ stands for either a or a^*. We denote by δ_0 the generator of τ_0.

Let $\mathfrak{O}$ be the τ_0-invariant C^*-subalgebra of $\mathcal{O}$ generated by $\{a^*(f)a(g) \,|\, f, g \in \mathfrak{h}\}$ and $\mathbb{1}$. Physical observables are gauge invariant and hence belong to $\mathfrak{O}$.

Let $\mathfrak{v}$ be a vector subspace of $\mathfrak{h}$ and let $\mathfrak{O}(\mathfrak{v})$ be the collection of the elements of the form

$$A = \sum_{k=1}^{K} \prod_{j=1}^{n_k} a^*(f_{kj})a(g_{kj}), \tag{1.13}$$

where K and n_k's are finite and $f_{kj}, g_{kj} \in \mathfrak{v}$. We denote $\overline{n}_A \equiv \max_k n_k$ and $\mathcal{F}(A) \equiv \{f_{kj}, g_{kj}\}$ (to indicate the dependence of K on A we will also denote it by K_A). $\mathfrak{O}(\mathfrak{v})$ is a $*$-subalgebra of $\mathfrak{O}$, and if $\mathfrak{v}$ is dense in $\mathfrak{h}$, then $\mathfrak{O}(\mathfrak{v})$ is norm dense in $\mathfrak{O}$.

Our main assumption is :

(A) There exists a dense vector subspace $\mathfrak{d} \subset \mathfrak{h}$ such that the functions

$$\mathbb{R} \ni t \mapsto (f, \mathrm{e}^{\mathrm{i}th_0} g),$$

are in $L^1(\mathbb{R}, \mathrm{d}t)$ for all $f, g \in \mathfrak{d}$.

This assumption implies that h_0 has purely absolutely continuous spectrum. Specific physical models which satisfy this assumption are discussed at the end of this subsection.

Let $V \in \mathfrak{O}(\mathfrak{d})_{\mathrm{self}}$ be a self-adjoint perturbation. We shall always assume that $\overline{n}_V \geq 2$. The special case $\overline{n}_V = 1$ leads to quasi-free perturbed dynamics and is discussed in detail in the companion paper [AJPP3], see also [AJPP1, AJPP2, JKP] and Remark after Theorem 1.6 below.

Let $\lambda \in \mathbb{R}$ be a coupling constant and let τ_λ be the C^*-dynamics generated by $\delta_\lambda = \delta_0 + \mathrm{i}\lambda[V, \cdot\,]$. By rescaling λ, without loss of generality we may assume that

$$\max_{f \in \mathcal{F}(V)} \|f\| = 1. \tag{1.14}$$

We shall consider the locally interacting fermionic system described by $(\mathcal{O}, \tau_\lambda)$. Note that τ_λ preserves $\mathfrak{O}$ and that the pair $(\mathfrak{O}, \tau_\lambda)$ is also a C^*-dynamical system. Let

$$\lambda_V \equiv \frac{1}{2\overline{n}_V K_V \ell_V} \frac{(2\overline{n}_V - 2)^{2\overline{n}_V - 2}}{(2\overline{n}_V - 1)^{2\overline{n}_V - 1}}, \tag{1.15}$$

where

$$\ell_V \equiv \int_{-\infty}^{\infty} \sup_{f, g \in \mathcal{F}(V)} |(f, \mathrm{e}^{\mathrm{i}th_0} g)| \mathrm{d}t. \tag{1.16}$$

The following result was proven in [JOP4] (see also [BM1, AM, BM2, FMU]).

Theorem 1.5 *Suppose that (A) holds. Then:*

1. *For all $A \in \mathfrak{O}(\mathfrak{d})$ and any monomial $B = a^{\#}(f_1)\cdots a^{\#}(f_m)$ with $\{f_1,\ldots,f_m\} \subset \mathfrak{d}$, one has*

$$\sup_{|\lambda|\leq\lambda_V} \int_{\mathbb{R}} \left\|[\tau_\lambda^t(A), B]\right\| \,\mathrm{d}t < \infty.$$

2. *For $|\lambda| \leq \lambda_V$ the Møller morphisms*

$$\gamma_\lambda^+ \equiv \mathrm{s} - \lim_{t\to\infty} \tau_0^{-t} \circ \tau_\lambda^t,$$

exist and are $*$-automorphisms of $\mathcal{O}$.

In what follows we shall assume that (A) holds. Let T be a self-adjoint operator on $\mathfrak{h}$ satisfying $0 \leq T \leq I$ and $[T, \mathrm{e}^{\mathrm{i}th_0}] = 0$ for all t, and let ω_0 be the gauge invariant quasi-free state on $\mathcal{O}$ associated to T. We will sometimes call T the *density operator*. The state ω_0 is τ_0-invariant and is the initial (reference) state of our fermionic system. The quantum dynamical system $(\mathcal{O}, \tau_0, \omega_0)$ is mixing. We denote by $\mathcal{N}_0$ the set of all ω-normal states on $\mathcal{O}$. Theorem 1.5 yields that any state $\eta \in \mathcal{N}_0$ evolves to the limiting state $\omega_\lambda^+ = \omega_0 \circ \gamma_\lambda^+$, i.e., for $A \in \mathcal{O}$ and $|\lambda| \leq \lambda_V$,

$$\lim_{t\to\infty} \eta(\tau_\lambda^t(A)) = \omega_\lambda^+(A),$$

see, e.g., [Ro, AJPP1]. The state ω_λ^+ is the NESS (non-equilibrium steady state) of $(\mathcal{O}, \tau_\lambda)$ associated to the initial state ω_0. Clearly, ω_λ^+ is τ_λ-invariant and γ_λ^+ is an isomorphism of the quantum dynamical systems $(\mathcal{O}, \tau_0, \omega_0)$ and $(\mathcal{O}, \tau_\lambda, \omega_\lambda^+)$. In particular, the system $(\mathcal{O}, \tau_\lambda, \omega_\lambda^+)$ is mixing.

In what follows we shall always assume that $\mathrm{Ker}\, T = \mathrm{Ker}\,(I - T) = \{0\}$. This assumption ensures that the states ω_0 and ω_λ^+ are modular.

Let $\mathfrak{c} \subset \mathfrak{d}$ be a vector subspace such that the functions

$$\mathbb{R} \ni t \mapsto (f, \mathrm{e}^{\mathrm{i}th_0} T g),$$

are in $L^1(\mathbb{R}, dt)$ for all $f, g \in \mathfrak{c}$. In general, it may happen that $\mathfrak{c} = \{0\}$, and so the existence of a non-trivial $\mathfrak{c}$ is a dynamical regularity property of the pair (T, h_0). If $T = F(h_0)$, where $F \in L^1(\mathbb{R}, \mathrm{d}x)$ is such that its Fourier transform

$$\hat{F}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx} F(x) \mathrm{d}x,$$

is also in $L^1(\mathbb{R}, \mathrm{d}t)$, then one can take $\mathfrak{c} = \mathfrak{d}$.

Let

$$\tilde{\lambda}_V \equiv 2^{-8(\overline{n}_V - 1)} \lambda_V, \tag{1.17}$$

and

$$\mathfrak{C} \equiv \mathfrak{O}(\mathfrak{c}).$$

The main result of this paper is:

Theorem 1.6 *Suppose that (A) holds, that $V \in \mathfrak{C}_{\mathrm{self}}$, and that $|\lambda| \leq \tilde{\lambda}_V$. Then $\mathfrak{C}$ is CLT-admissible and the QCLT holds for $\mathfrak{C}$ w.r.t. $(\mathcal{O}, \tau_\lambda, \omega_\lambda^+)$.*

Remark. If $n_V = 1$, then Theorem 1.5 holds for any $0 < \lambda_V < (2K_V\ell_V)^{-1}$, see [JOP4]. With this change, Theorem 1.6 holds with $\tilde{\lambda}_V = \lambda_V$. The case $n_V = 1$ is however very special. If $V = \sum_k a^*(f_k)a(g_k)$, then τ_λ is quasi-free dynamics generated by $h_\lambda = h_0 + \lambda \sum_k (g_k, \cdot) f_k$ and Theorem 1.5 can be derived from the scattering theory of the pair (h_λ, h_0), see [Ro, AJPP1]. This alternative approach is technically simpler, yields better constants, and can be also used to prove a Large Deviation Principle and to discuss additional topics like Landauer-Büttiker formula which cannot be handled by the method of [JOP4] and this paper. For this reason, we shall discuss this special case separately in the companion paper [AJPP3].

As we have already remarked, our proof of Theorem 1.6 also yields the convergence of moments (see Theorem 3.2), and is easily extended to the proof of existence of QHL for locally interacting fermionic systems (recall (1.10), (1.11)).

We finish this subsection with some concrete models to which Theorem 1.6 applies. The models on graphs are the same as in [JOP4]. Let $\mathcal{G}$ be the set of vertices of a connected graph of bounded degree, $\Delta_{\mathcal{G}}$ the discrete Laplacian acting on $l^2(\mathcal{G})$, and δ_x the Kronecker delta function at $x \in \mathcal{G}$. We shall call a graph $\mathcal{G}$ *admissible* if there exists $\gamma > 1$ such that for all $x, y \in \mathcal{G}$,

$$|(\delta_x, \mathrm{e}^{-\mathrm{i}t\Delta_{\mathcal{G}}}\delta_y)| = O(|t|^{-\gamma}), \tag{1.18}$$

as $t \to \infty$. Examples of admissible graphs are $\mathcal{G} = \mathbb{Z}^d$ for $d \geq 3$, $\mathcal{G} = \mathbb{Z}_+ \times \mathbb{Z}^{d-1}$ where $\mathbb{Z}_+ = \{0, 1, \cdots\}$ and $d \geq 1$, tubular graphs of the type $\mathbb{Z}_+ \times \Gamma$, where $\Gamma \subset \mathbb{Z}^{d-1}$ is finite, a rooted Bethe lattice, etc. Assumption (A) holds and Theorem 1.6 holds with $\mathfrak{c} = \mathfrak{d}$ if:

(i) $\mathcal{G}$ is an admissible graph;
(ii) $\mathfrak{h} = \ell^2(\mathcal{G})$ (or more generally $\ell^2(\mathcal{G}) \otimes \mathbb{C}^L$) and $h_0 = -\Delta_{\mathcal{G}}$;
(iii) $\mathfrak{d}$ is the subspace of finitely supported elements of $\mathfrak{h}$;
(iv) $T = F(h_0)$ where $\hat{F} \in L^1(\mathbb{R}, \mathrm{d}t)$ and $0 < F(x) < 1$ for $x \in \mathrm{sp}(h_0)$.

The continuous examples are similar. Let $\mathcal{D} \subset \mathbb{R}^d$ be a domain and let $\Delta_{\mathcal{D}}$ be the Dirichlet Laplacian on $L^2(\mathcal{D}, \mathrm{d}x)$. We shall say that a domain $\mathcal{D}$ is admissible if there exists $\gamma > 1$ such that

$$|(f, \mathrm{e}^{-\mathrm{i}t\Delta_{\mathcal{D}}}g)| = O(|t|^{-\gamma}),$$

for all bounded f and g with compact support. Examples of admissible domains are $\mathcal{D} = \mathbb{R}^d$ for $d \geq 3$, $\mathcal{D} = \mathbb{R}_+ \times \mathbb{R}^{d-1}$ for $d \geq 1$, tubular domains of the type $\mathbb{R}_+ \times \Gamma$, where $\Gamma \subset \mathbb{R}^{d-1}$ is a bounded domain, etc. Assumption (A) holds and Theorem 1.6 holds with $\mathfrak{c} = \mathfrak{d}$ if:

(i) $\mathcal{D}$ is an admissible domain;
(ii) $\mathfrak{h} = L^2(\mathcal{D}, \mathrm{d}x)$ (or more generally $L^2(\mathcal{D}, \mathrm{d}x) \otimes \mathbb{C}^L$) and $h_0 = -\Delta_{\mathcal{D}}$;
(iii) $\mathfrak{d}$ is the subspace of bounded compactly supported elements of $\mathfrak{h}$;
(iv) $T = F(h_0)$ where $\hat{F} \in L^1(\mathbb{R}, \mathrm{d}t)$ and $0 < F(x) < 1$ for $x \in \mathrm{sp}(h_0)$.

1.3 QCLT and linear response

In addition to the assumptions of the previous subsection, we assume that $\mathfrak{h}, h_0, T$ have the composite structure

$$\mathfrak{h} = \bigoplus_{j=1}^{M} \mathfrak{h}_j, \qquad h_0 = \bigoplus_{j=1}^{M} h_j, \qquad T = \bigoplus_{j=1}^{M} \frac{1}{1 + \mathrm{e}^{\beta_j(h_j - \mu_j)}}, \tag{1.19}$$

where h_j's are bounded from below self-adjoint operators on the Hilbert subspaces $\mathfrak{h}_j$, $\beta_j > 0$, and $\mu_j \in \mathbb{R}$. We denote by p_j the orthogonal projections onto $\mathfrak{h}_j$. The subalgebras $\mathcal{O}_j = \mathrm{CAR}(\mathfrak{h}_j)$ describe Fermi gas reservoirs $\mathcal{R}_j$ which are initially in equilibrium at inverse temperatures β_j and chemical potentials μ_j. The perturbation λV describes the interaction between the reservoirs and allows for the flow of heat and charges within the system.

The non-equilibrium statistical mechanics of this class of models has been studied recently in [JOP4] (see also [FMU] for related models and results). We briefly recall the results we need.

Suppose that $p_j\mathcal{F}(V) \subset \mathrm{Dom}\,(h_j)$ for all j. The entropy production observable of $(\mathcal{O}, \tau_\lambda)$ associated to the reference state ω_0 is

$$\sigma_\lambda \equiv -\sum_{j=1}^{M} \beta_j(\Phi_j - \mu_j\mathcal{J}_j),$$

where $\Phi_j \equiv \mathrm{i}\lambda[\mathrm{d}\Gamma(h_jp_j), V]$ and $\mathcal{J}_j \equiv \mathrm{i}\lambda[\mathrm{d}\Gamma(p_j), V]$. Explicitly,

$$\Phi_j = \lambda \sum_{k=1}^{K_V}\sum_{l=1}^{n_k} \left(\prod_{i=1}^{l-1} a^*(f_{ki})a(g_{ki})\right) \{a^*(\mathrm{i}h_jp_jf_{kl})a(g_{kl}) + a^*(f_{kl})a(\mathrm{i}h_jp_jg_{kl})\} \left(\prod_{i=l+1}^{n_k} a^*(f_{ki})a(g_{ki})\right),$$

$$\mathcal{J}_j = \lambda \sum_{k=1}^{K_V}\sum_{l=1}^{n_k} \left(\prod_{i=1}^{l-1} a^*(f_{ki})a(g_{ki})\right) \{a^*(\mathrm{i}p_jf_{kl})a(g_{kl}) + a^*(f_{kl})a(\mathrm{i}p_jg_{kl})\} \left(\prod_{i=l+1}^{n_k} a^*(f_{ki})a(g_{ki})\right).$$

The observable $\Phi_j/\mathcal{J}_j$ describes the heat/charge flux out of the reservoir $\mathcal{R}_j$ (note that $\Phi_j, \mathcal{J}_j \in \mathfrak{O}$). The conservation laws

$$\sum_{j=1}^{M} \omega_\lambda^+(\Phi_j) = 0, \qquad \sum_{j=1}^{M} \omega_\lambda^+(\mathcal{J}_j) = 0,$$

hold. By the general result of [JP1, Ru2, JP4], the entropy production of the NESS ω_λ^+ is non-negative,

$$\mathrm{Ep}(\omega_\lambda^+) \equiv \omega_\lambda^+(\sigma_\lambda) = -\sum_{j=1}^{M} \beta_j(\omega_\lambda^+(\Phi_j) - \mu_j \omega_\lambda^+(\mathcal{J}_j)) \geq 0.$$

If all β_j's and μ_j's are equal, i.e. $\beta_1 = \cdots = \beta_M = \beta$ and $\mu_1 = \cdots = \mu_M = \mu$, then $\omega_0 \restriction \mathfrak{O}$ is a (τ_0, β)-KMS state and so the reference state is a thermal equilibrium state of the unperturbed system. Then $\omega_\lambda^+ \restriction \mathfrak{O}$ is a (τ_λ, β)-KMS state, $\omega_\lambda^+(\Phi_j) = \omega_\lambda^+(\mathcal{J}_j) = 0$ for all j, and in particular $\mathrm{Ep}(\omega_\lambda^+) = 0$, see [JOP2]. On physical grounds, vanishing of the fluxes and the entropy production in thermal equilibrium is certainly an expected result. It is also expected that if either β_j's or μ_j's are not all equal, then $\mathrm{Ep}(\omega_\lambda^+) > 0$. For specific interactions V one can compute $\omega_\lambda^+(\sigma_\lambda)$ to the first non-trivial order in λ and hence establish the strict positivity of entropy production by a perturbative calculation (see [FMU, JP6] and [JP3] for a related results). The strict positivity of the entropy production for a generic perturbation λV has been established in [JP5].

To establish QCLT for the flux observables in addition to the Assumption (A) we need:

(B) For all j, $h_j p_j \mathfrak{d} \subset \mathfrak{d}$.

This assumption and the specific form of density operator ensure that one may take $\mathfrak{c} = \mathfrak{d}$ and that if $V \in \mathfrak{C}_{\mathrm{self}}$, then $\{\Phi_j, \mathcal{J}_j\} \subset \mathfrak{C}_{\mathrm{self}}$. Hence, for $|\lambda| \leq \tilde{\lambda}_V$ the QCLT holds for the flux observables.

We finish with a discussion of linear response theory (for references and additional information about linear response theory in algebraic formalism of quantum statistical mechanics we refer the reader to [AJPP1] and [JOP1]-[JOP4]). We will need the following two assumptions:

(C) The operators h_j are bounded.

(D) There exists a complex conjugation c on $\mathfrak{h}$ which commutes with all h_j and satisfies $cf = f$ for all $f \in \mathcal{F}(V)$.

Assumption (C) is of technical nature and can be relaxed. Assumption (D) ensures that the system $(\mathcal{O}, \tau_\lambda, \omega_0)$ is time-reversal invariant. Time-reversal invariance is of central importance in linear response theory.

Let $\beta_{\mathrm{eq}} > 0$ and $\mu_{\mathrm{eq}} \in \mathbb{R}$ be given equilibrium values of the inverse temperature and chemical potential. We denote $\vec{\beta} = (\beta_1, \cdots, \beta_M)$, $\vec{\mu} = (\mu_1, \cdots, \mu_M)$, $\vec{\beta}_{\mathrm{eq}} = (\beta_{\mathrm{eq}}, \cdots, \beta_{\mathrm{eq}})$, $\vec{\mu}_{\mathrm{eq}} = (\mu_{\mathrm{eq}}, \cdots, \mu_{\mathrm{eq}})$, and we shall indicate explicitly the dependence of ω_λ^+ on $\vec{\beta}$ and $\vec{\mu}$ by $\omega_{\lambda,\vec{\beta},\vec{\mu}}^+$. Similarly, we shall indicate explicitly the dependence of $L(A,B)$ on $\lambda, \vec{\beta}, \vec{\mu}$ by $L_{\lambda,\vec{\beta},\vec{\mu}}$. Since $\omega_{\lambda,\vec{\beta}_{\mathrm{eq}},\vec{\mu}_{\mathrm{eq}}}^+(\Phi_j) = \omega_{\lambda,\vec{\beta}_{\mathrm{eq}},\vec{\mu}_{\mathrm{eq}}}^+(\mathcal{J}_j) = 0$,

$$L_{\lambda,\vec{\beta}_{\mathrm{eq}},\vec{\mu}_{\mathrm{eq}}}(A,B) = \int_{-\infty}^{\infty} \omega_{\lambda,\vec{\beta}_{\mathrm{eq}},\vec{\mu}_{\mathrm{eq}}}^+ \left(A\tau_\lambda^t(B)\right) \mathrm{d}t,$$

for $A, B \in \{\Phi_j, \mathcal{J}_j \,|\, 1 \leq j \leq M\}$.

Assuming the existence of derivatives, the kinetic transport coefficients are defined by

$$\begin{aligned} \mathcal{L}_{\lambda\mathrm{hh}}^{kj} &\equiv -\partial_{\beta_j} \omega_{\lambda,\vec{\beta},\vec{\mu}}^+(\Phi_k)\big|_{\vec{\beta}=\vec{\beta}_{\mathrm{eq}},\vec{\mu}=\vec{\mu}_{\mathrm{eq}}}, \quad \mathcal{L}_{\lambda\mathrm{hc}}^{kj} \equiv \beta_{\mathrm{eq}} \partial_{\mu_j} \omega_{\lambda,\vec{\beta},\vec{\mu}}^+(\Phi_k)\big|_{\vec{\beta}=\vec{\beta}_{\mathrm{eq}},\vec{\mu}=\vec{\mu}_{\mathrm{eq}}}, \\ \mathcal{L}_{\lambda\mathrm{ch}}^{kj} &\equiv -\partial_{\beta_j} \omega_{\lambda,\vec{\beta},\vec{\mu}}^+(\mathcal{J}_k)\big|_{\vec{\beta}=\vec{\beta}_{\mathrm{eq}},\vec{\mu}=\vec{\mu}_{\mathrm{eq}}}, \quad \mathcal{L}_{\lambda\mathrm{cc}}^{kj} \equiv \beta_{\mathrm{eq}} \partial_{\mu_j} \omega_{\lambda,\vec{\beta},\vec{\mu}}^+(\mathcal{J}_k)\big|_{\vec{\beta}=\vec{\beta}_{\mathrm{eq}},\vec{\mu}=\vec{\mu}_{\mathrm{eq}}}, \end{aligned} \tag{1.20}$$

where the indices h/c stand for heat/charge. We then have

Theorem 1.7 *Suppose that Assumptions (A)-(D) hold. Then, for any $|\lambda| < \lambda_V$, the functions*

$$(\vec{\beta}, \vec{\mu}) \mapsto \omega^+_{\lambda,\vec{\beta},\vec{\mu}}(\Phi_j), \qquad (\vec{\beta}, \vec{\mu}) \mapsto \omega^+_{\lambda,\vec{\beta},\vec{\mu}}(\mathcal{J}_j),$$

are analytic in a neighborhood of $(\vec{\beta}_{\rm eq}, \vec{\mu}_{\rm eq})$. Moreover,
(1) *The Green-Kubo formulas hold:*

$$\begin{aligned} \mathcal{L}^{kj}_{\lambda\rm hh} &= \frac{1}{2} L_{\lambda,\vec{\beta}_{\rm eq},\vec{\mu}_{\rm eq}}(\Phi_k, \Phi_j), \quad \mathcal{L}^{kj}_{\lambda\rm hc} = \frac{1}{2} L_{\lambda,\vec{\beta}_{\rm eq},\vec{\mu}_{\rm eq}}(\Phi_k, \mathcal{J}_j), \\ \mathcal{L}^{kj}_{\lambda\rm ch} &= \frac{1}{2} L_{\lambda,\vec{\beta}_{\rm eq},\vec{\mu}_{\rm eq}}(\mathcal{J}_k, \Phi_j), \quad \mathcal{L}^{kj}_{\lambda\rm cc} = \frac{1}{2} L_{\lambda,\vec{\beta}_{\rm eq},\vec{\mu}_{\rm eq}}(\mathcal{J}_k, \mathcal{J}_j). \end{aligned} \tag{1.21}$$

(2) *The Onsager reciprocity relations hold:*

$$\mathcal{L}^{kj}_{\lambda\rm hh} = \mathcal{L}^{jk}_{\lambda\rm hh}, \qquad \mathcal{L}^{kj}_{\lambda\rm cc} = \mathcal{L}^{jk}_{\lambda\rm cc}, \qquad \mathcal{L}^{kj}_{\lambda\rm hc} = \mathcal{L}^{jk}_{\lambda\rm ch}. \tag{1.22}$$

(3) *For $|\lambda| \leq \tilde{\lambda}_V$ the QCLT holds for the linear span of $\{\Phi_j, \mathcal{J}_j \,|\, 1 \leq j \leq M\}$ w.r.t. $(\mathcal{O}, \tau_\lambda, \omega_{\lambda,\vec{\beta}_{\rm eq},\vec{\mu}_{\rm eq}})$. The associated fluctuation algebra $\mathcal{W}$ is commutative.*

Remark 1. Parts (1) and (2) of Theorem 1.7 are proven in [JOP4]. Part (3) is a special case of Theorem 1.6. Parts (1) and (3) relate linear response to thermodynamical forces to fluctuations in thermal equilibrium and constitute the Fluctuation-Dissipation Theorem for our model. The physical aspects of linear response theory and Fluctuation-Dissipation Theorem are discussed in classical references [DGM, KTH].
Remark 2. The arguments in [JOP4] do not establish that the functions

$$t \mapsto \omega^+_{\lambda,\vec{\beta}_{\rm eq},\vec{\mu}_{\rm eq}} \left(A\tau^t_\lambda(B) \right), \tag{1.23}$$

are absolutely integrable for $A, B \in \{\Phi_j, \mathcal{J}_j \,|\, 1 \leq j \leq M\}$ and in Part (2) $L_{\lambda,\vec{\beta}_{\rm eq},\vec{\mu}_{\rm eq}}(A, B)$ is defined by

$$L_{\lambda,\vec{\beta}_{\rm eq},\vec{\mu}_{\rm eq}}(A, B) = \lim_{t\to\infty} \int_{-t}^{t} \omega^+_{\lambda,\vec{\beta}_{\rm eq},\vec{\mu}_{\rm eq}} \left(A\tau^s_\lambda(B) \right) \, {\rm d}s.$$

The absolute integrability of the correlation functions (1.23) is a delicate dynamical problem resolved in Part (3) for $|\lambda| \leq \tilde{\lambda}_V$.
Remark 3. Remarks 4 and 6 after Theorem 1.5 in [JOP4] apply without changes to Theorem 1.7. Remark 7 is also applicable and allows to extend the Fluctuation-Dissipation Theorem to a large class of so called *centered* observables.
Remark 4. Although the time-reversal Assumption (D) plays no role in Part (3) of Theorem 1.7, it is a crucial ingredient in proofs of Parts (1) and (2) (see [JOP4, AJPP3] for a discussion). The Fluctuation-Dissipation Theorem fails for locally interacting open fermionic systems which are not time-reversal invariant.

A class of concrete models for which (A)-(B)-(D) hold is easily constructed following the examples discussed at the end of Subsection 1.2. Let $\mathcal{G}_1, \ldots, \mathcal{G}_M$ be admissible graphs. Then (A)-(D) hold if $\mathfrak{h}_j = \ell^2(\mathcal{G}_j)$ (or $\ell^2(\mathcal{G}_j) \otimes \mathbb{C}^L$), $h_j = -\Delta_{\mathcal{G}_j}$, and $\mathfrak{d}$ is the subspace of finitely supported elements of $\mathfrak{h}$. A physically important class of allowed interactions is $V = V^{\rm hop} + V^{\rm int}$ where

$$V^{\rm hop} = \sum_{x,y} t(x,y) \left(a^*(\delta_x)a(\delta_y) + a^*(\delta_y)a(\delta_x) \right),$$

and $t : \mathcal{G} \times \mathcal{G} \to \mathbb{R}$ is a finitely supported function ($\mathcal{G} = \cup_j \mathcal{G}_j$), and

$$V^{\rm int} = \sum_{x,y} v(x,y) a^*(\delta_x) a^*(\delta_y) a(\delta_y) a(\delta_x),$$

where $v : \mathcal{G} \times \mathcal{G} \to \mathbb{R}$ is finitely supported. $V^{\rm hop}$ describes tunneling junctions between the reservoir and $V^{\rm int}$ is a local pair interaction.

2 General aspects of CLT

2.1 Proof of Theorem 1.3

Our argument follows the ideas of [GV]. We set

$$D(A,B) \equiv \mathrm{e}^{\mathrm{i}A}\mathrm{e}^{\mathrm{i}B} - \mathrm{e}^{\mathrm{i}(A+B)}\mathrm{e}^{-\frac{1}{2}[A,B]}.$$

The first ingredient of the proof is:

Proposition 2.1 *If $\{A,B\} \subset \mathcal{O}_{\mathrm{self}}$ is L^1-asymptotically abelian for τ then the asymptotic 2nd-order Baker-Campbell-Hausdorff formula*

$$\lim_{t\to\infty} \|D(\tilde{A}_t, \tilde{B}_t)\| = 0,$$

holds.

Note that Proposition 2.1 is not a simple consequence of the BCH formula because its hypothesis does not ensure that the double commutator $[\tilde{A}_t, [\tilde{A}_t, \tilde{B}_t]]$ vanishes as $t \to \infty$. To prove Proposition 2.1 we need the following estimate.

Lemma 2.2 *If A, B, a, b are bounded self-adjoint operators then*

$$\|D(A+a, B+b)\| \leq \|D(A,B)\| + 4\left(\|a\|^3 + \|b\|^3\right) + \|[[A,B],[a,b]]\| + (2 + \|a\| + \|b\|) \sum_{\substack{X\in\{A,B\}\\ y\in\{a,b\}}} \|[X,y]\|.$$

Proof. We decompose $D(A+a, B+b) = \sum_{j=1}^{9} D_j$ according to the following table and get an upper bound of the norm of each term using the elementary estimates

$$\left\|\mathrm{e}^{\mathrm{i}(x+y)} - \mathrm{e}^{\mathrm{i}x}\right\| \leq \|y\|, \qquad \left\|\mathrm{e}^{\mathrm{i}(x+y)} - \mathrm{e}^{\mathrm{i}x}\mathrm{e}^{\mathrm{i}y}\right\| \leq \frac{1}{2}\|[x,y]\|, \qquad \left\|\mathrm{e}^{\mathrm{i}x}\mathrm{e}^{\mathrm{i}y} - \mathrm{e}^{\mathrm{i}y}\mathrm{e}^{\mathrm{i}x}\right\| \leq \|[x,y]\|.$$

j	D_j	upper bound on $\|D_j\|$
1	$\left(\mathrm{e}^{\mathrm{i}(A+a)} - \mathrm{e}^{\mathrm{i}a}\mathrm{e}^{\mathrm{i}A}\right)\mathrm{e}^{\mathrm{i}(B+b)}$	$\frac{1}{2}\|[A,a]\|$
2	$\mathrm{e}^{\mathrm{i}a}\mathrm{e}^{\mathrm{i}A}\left(\mathrm{e}^{\mathrm{i}(B+b)} - \mathrm{e}^{\mathrm{i}b}\mathrm{e}^{\mathrm{i}B}\right)$	$\frac{1}{2}\|[B,b]\|$
3	$\mathrm{e}^{\mathrm{i}a}\left(\mathrm{e}^{\mathrm{i}A}\mathrm{e}^{\mathrm{i}b} - \mathrm{e}^{\mathrm{i}b}\mathrm{e}^{\mathrm{i}A}\right)\mathrm{e}^{\mathrm{i}B}$	$\|[A,b]\|$
4	$\mathrm{e}^{\mathrm{i}a}\mathrm{e}^{\mathrm{i}b}\left(\mathrm{e}^{\mathrm{i}A}\mathrm{e}^{\mathrm{i}B} - \mathrm{e}^{\mathrm{i}(A+B)}\mathrm{e}^{-\frac{1}{2}[A,B]}\right)$	$\|D(A,B)\|$
5	$\left(\mathrm{e}^{\mathrm{i}a}\mathrm{e}^{\mathrm{i}b} - \mathrm{e}^{\mathrm{i}(a+b)}\mathrm{e}^{-\frac{1}{2}[a,b]}\right)\mathrm{e}^{\mathrm{i}(A+B)}\mathrm{e}^{-\frac{1}{2}[A,B]}$	$\|D(a,b)\|$
6	$\mathrm{e}^{\mathrm{i}(a+b)}\left(\mathrm{e}^{-\frac{1}{2}[a,b]}\mathrm{e}^{\mathrm{i}(A+B)} - \mathrm{e}^{\mathrm{i}(A+B)}\mathrm{e}^{-\frac{1}{2}[a,b]}\right)\mathrm{e}^{-\frac{1}{2}[A,B]}$	$\frac{1}{2}\|[A+B,[a,b]]\|$
7	$\mathrm{e}^{\mathrm{i}(a+b)}\mathrm{e}^{\mathrm{i}(A+B)}\left(\mathrm{e}^{-\frac{1}{2}[a,b]}\mathrm{e}^{-\frac{1}{2}[A,B]} - \mathrm{e}^{-\frac{1}{2}[A,B]-\frac{1}{2}[a,b]}\right)$	$\frac{1}{8}\|[[A,B],[a,b]]\|$
8	$\mathrm{e}^{\mathrm{i}(a+b)}\mathrm{e}^{\mathrm{i}(A+B)}\left(\mathrm{e}^{-\frac{1}{2}[A,B]-\frac{1}{2}[a,b]} - \mathrm{e}^{-\frac{1}{2}[A+a,B+b]}\right)$	$\frac{1}{2}(\|[A,b]\| + \|[B,a]\|)$
9	$\left(\mathrm{e}^{\mathrm{i}(a+b)}\mathrm{e}^{\mathrm{i}(A+B)} - \mathrm{e}^{\mathrm{i}(A+B+a+b)}\right)\mathrm{e}^{-\frac{1}{2}[A+a,B+b]}$	$\frac{1}{2}(\|[A,a]\| + \|[A,b]\| + \|[B,a]\| + \|[B,b]\|)$

From the BCH estimate we further get

$$\|D_5\| \leq \|D(a,b)\| \leq \|[a,[a,b]]\| + \|[b,[a,b]]\| \leq 4(\|a\|^3 + \|b\|^3),$$

and the Jacobi identity yields

$$\|D_6\| \leq \|a\|(\|[A,b]\| + \|[B,b]\|) + \|b\|(\|[A,a]\| + \|[B,a]\|).$$

The result follows. □

Proof of Proposition 2.1. For $t > 0$ and $j \in \mathbb{N}$ set $p(t) \equiv \log(1+t)$ and $I_j(t) \equiv [jp(t), (j+1)p(t)[$. For $X \in \mathcal{O}_{\rm self}$ define

$$X_t^{(j)} \equiv t^{-1/2} \int_{I_j(t)\cap[0,t]} \tau^s(X)\,\mathrm{d}s, \qquad X_t^{(<k)} \equiv \sum_{0\leq j<k} X_t^{(j)}.$$

If $N(t)$ denotes the integer such that $N(t)p(t) \leq t < (N(t)+1)p(t)$ then repeated use of Lemma 2.2 yields

$$\begin{aligned}\|D(\tilde{A}_t, \tilde{B}_t)\| \leq & 4\sum_{j=0}^{N(t)} \left(\|A_t^{(j)}\|^3 + \|B_t^{(j)}\|^3\right) + \sum_{j=0}^{N(t)} \left(2 + \|A_t^{(j)}\| + \|B_t^{(j)}\|\right) \sum_{X,Y\in\{A,B\}} \|[X_t^{(<j)}, Y_t^{(j)}]\| \\ & + \sum_{j=0}^{N(t)} \|[[A_t^{(<j)}, B_t^{(<j)}], [A_t^{(j)}, B_t^{(j)}]]\|.\end{aligned} \tag{2.24}$$

We now estimate the right hand side of this inequality. We first note that

$$\|X_t^{(j)}\| \leq \|X\|\, t^{-1/2} p(t) \leq \|X\|, \tag{2.25}$$

and hence

$$\sum_{j=0}^{N(t)} \left(\|A_t^{(j)}\|^3 + \|B_t^{(j)}\|^3\right) \leq \left(\|A\|^3 + \|B\|^3\right)(N(t)+1)t^{-3/2}p(t)^3 \leq 2\left(\|A\|^3 + \|B\|^3\right) t^{-1/2} p(t)^2 \to 0,$$

as $t \to \infty$. Next consider

$$\sum_{j=0}^{N(t)} \sum_{k=0}^{j-1} \|[X_t^{(k)}, Y_t^{(j)}]\| \leq \frac{1}{t} \sum_{j=0}^{N(t)} \int_0^{jp(t)} \int_{jp(t)}^{(j+1)p(t)} \|[X, \tau^{v-u}(Y)]\|\,\mathrm{d}v\mathrm{d}u.$$

The change of variables $\xi = v - jp(t)$, $\eta = v - u$, leads to

$$\sum_{j=0}^{N(t)} \sum_{k=0}^{j-1} \|[X_t^{(k)}, Y_t^{(j)}]\| \leq \frac{1}{t} \sum_{j=0}^{N(t)} \int_0^{p(t)} \int_\xi^{\xi+jp(t)} \|[X, \tau^\eta(Y)]\|\,\mathrm{d}\eta\mathrm{d}\xi \leq \frac{N(t)+1}{t} \int_0^{p(t)} \int_\xi^\infty \|[X, \tau^\eta(Y)]\|\,\mathrm{d}\eta\mathrm{d}\xi.$$

Since $(N(t)+1)/t \leq 2/p(t)$ we obtain, for $X, Y \in \{A, B\}$,

$$\lim_{t\to\infty} \sum_{j=0}^{N(t)} \sum_{k=0}^{j-1} \|[X_t^{(k)}, Y_t^{(j)}]\| \leq \lim_{p\to\infty} \frac{2}{p} \int_0^p \left[\int_\xi^\infty \|[X, \tau^\eta(Y)]\|\,\mathrm{d}\eta\right] \mathrm{d}\xi = 0. \tag{2.26}$$

Combining this with (2.25) we get

$$\sum_{j=0}^{N(t)} \left(2 + \|A_t^{(j)}\| + \|B_t^{(j)}\|\right) \sum_{X,Y\in\{A,B\}} \|[X_t^{(<j)}, Y_t^{(j)}]\| \leq (2 + \|A\| + \|B\|) \sum_{X,Y\in\{A,B\}} \sum_{j=0}^{N(t)} \sum_{k=0}^{j-1} \|[X_t^{(k)}, Y_t^{(j)}]\| \to 0,$$

as $t \to \infty$. To estimate the last term on the right hand side of (2.24) we write

$$\sum_{j=0}^{N(t)} \|[[A_t^{(<j)}, B_t^{(<j)}], [A_t^{(j)}, B_t^{(j)}]]\| \leq \sum_{j=0}^{N(t)} \sum_{k=0}^{j-1} \sum_{l=0}^{j-1} \|[[A_t^{(k)}, B_t^{(l)}], [A_t^{(j)}, B_t^{(j)}]]\| = Z_1 + Z_2 + Z_3,$$

where

$$Z_1 \equiv \sum_{j=0}^{N(t)} \sum_{k=0}^{j-1} \|[[A_t^{(k)}, B_t^{(k)}], [A_t^{(j)}, B_t^{(j)}]]\|,$$

and

$$Z_2 \equiv \sum_{j=0}^{N(t)} \sum_{k=0}^{j-1} \sum_{l=0}^{k-1} \|[[A_t^{(k)}, B_t^{(l)}], [A_t^{(j)}, B_t^{(j)}]]\|, \quad Z_3 \equiv \sum_{j=0}^{N(t)} \sum_{l=0}^{j-1} \sum_{k=0}^{l-1} \|[[A_t^{(k)}, B_t^{(l)}], [A_t^{(j)}, B_t^{(j)}]]\|.$$

Combined with (2.25) and (2.26), the identity

$$[[A_k, B_k], [A_j, B_j]] = [[[A_k, A_j], B_k], B_j] + [[[A_j, B_k], A_k], B_j] + [[[B_k, B_j], A_k], A_j] + [[[B_j, A_k], B_k], A_j],$$

yields

$$Z_1 \leq 4(\|A\|^2 + \|B\|^2) \sum_{X,Y \in \{A,B\}} \sum_{j=0}^{N(t)} \sum_{k=0}^{j-1} \|[X_t^{(k)}, Y_t^{(j)}]\| \to 0,$$

as $t \to \infty$. The estimate

$$\begin{aligned}
\sum_{j=0}^{N(t)} \|[A_t^{(j)}, B_t^{(j)}]\| &\leq \frac{1}{t} \sum_{j=0}^{N(t)} \int_{I_j(t)} \int_{I_j(t)} \|[A, \tau^{u-v}(B)]\| \, \mathrm{d}u \mathrm{d}v \\
&\leq \frac{1}{t} \sum_{j=0}^{N(t)} \int_0^{p(t)} \int_{-v}^{p(t)-v} \|[A, \tau^u(B)]\| \, \mathrm{d}u \mathrm{d}v \\
&\leq \frac{N(t)+1}{t} p(t) \int_{-p(t)}^{p(t)} \|[A, \tau^u(B)]\| \mathrm{d}u \leq 2 \int_{\mathbb{R}} \|[A, \tau^u(B)]\| \, \mathrm{d}u,
\end{aligned}$$

together with (2.26) yield

$$Z_2 \leq 2 \sum_{j=0}^{N(t)} \|[A_t^{(j)}, B_t^{(j)}]\| \sum_{k=0}^{N(t)} \sum_{l=0}^{k-1} \|[A_t^{(k)}, B_t^{(l)}]\| \to 0,$$

as $t \to \infty$. The same argument applies to Z_3 and completes the proof.□

Let $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ be the GNS-representation of the algebra $\mathcal{O}$ associated to the state ω. The second ingredient of the proof of Theorem 1.3 is:

Proposition 2.3 *Suppose that $(\mathcal{O}, \tau, \omega)$ is an ergodic quantum dynamical system and that ω is a modular state. If $\{A, B\}$ is an L^1-asymptotically abelian pair for τ, then*

$$\mathrm{s}-\lim_{t\to\infty} \pi_\omega \left([\tilde{A}_t, \tilde{B}_t]\right) = \int_{-\infty}^{\infty} \omega([\tau^s(A), B]) \, \mathrm{d}s.$$

Proof. We shall first prove that

$$\lim_{t\to\infty} \pi_\omega \left([\tilde{A}_t, \tilde{B}_t]\right) \Omega_\omega = \left(\int_{-\infty}^{\infty} \omega([\tau^s(A), B]) \, \mathrm{d}s\right) \Omega_\omega. \tag{2.27}$$

Writing

$$\pi_\omega \left([\tilde{A}_t, \tilde{B}_t]\right) = \frac{1}{t} \int_0^t \int_0^t \pi_\omega \left(\tau^{s_1}([\tau^{s_2-s_1}(A), B])\right) \, \mathrm{d}s_1 \mathrm{d}s_2,$$

the change of variable $u = s_1$, $v = s_2 - s_1$ yields that

$$\pi_\omega \left([\tilde{A}_t, \tilde{B}_t]\right) = \int_{-t}^{t} f_t(v) \, \mathrm{d}v, \tag{2.28}$$

where

$$f_t(v) = \frac{1}{t}\int_{\max(-v,0)}^{\min(t,t-v)} \pi_\omega\left(\tau^u([\tau^v(A),B])\right)\,\mathrm{d}u.$$

Clearly,

$$\|f_t(v)\| \leq \|[\tau^v(A),B])\| \in L^1(\mathbb{R},\mathrm{d}v), \tag{2.29}$$

and so, by the dominated convergence theorem, it suffices to show that

$$\lim_{t\to\infty} f_t(v)\Omega_\omega = \omega\left([\tau^v(A),B]\right)\Omega_\omega, \tag{2.30}$$

for all $v \in \mathbb{R}$ to prove (2.27). Let $\mathcal{L}_\omega$ be the standard Liouvillean associated to ω. We recall that $\mathcal{L}_\omega$ is the unique self-adjoint operator on $\mathcal{H}_\omega$ such that

$$\pi_\omega(\tau^t(A)) = \mathrm{e}^{\mathrm{i}t\mathcal{L}_\omega}\pi_\omega(A)\mathrm{e}^{-\mathrm{i}t\mathcal{L}_\omega}, \qquad \mathcal{L}_\omega\Omega_\omega = 0.$$

Then

$$\pi_\omega(\tau^u([\tau^v(A),B]))\Omega_\omega = \mathrm{e}^{\mathrm{i}u\mathcal{L}_\omega}\pi_\omega([\tau^v(A),B])\Omega_\omega,$$

implies

$$f_t(v)\Omega_\omega = \frac{1}{t}\int_{\max(-v,0)}^{\min(t,t-v)} \mathrm{e}^{\mathrm{i}u\mathcal{L}_\omega}\pi_\omega\left([\tau^v(A),B]\right)\Omega_\omega\,\mathrm{d}u.$$

Since $(\mathcal{O},\tau,\omega)$ is ergodic, zero is a simple eigenvalue of $\mathcal{L}_\omega$, and von Neumann's mean ergodic theorem yields

$$\mathrm{s}-\lim_{t\to\infty}\frac{1}{t}\int_{\max(-v,0)}^{\min(t,t-v)} \mathrm{e}^{\mathrm{i}u\mathcal{L}_\omega}\,\mathrm{d}u = \mathrm{s}-\lim_{t\to\infty}\frac{1}{t}\int_0^t \mathrm{e}^{\mathrm{i}u\mathcal{L}_\omega}\,\mathrm{d}u = \Omega_\omega(\Omega_\omega|\,\cdot\,),$$

for all $v \in \mathbb{R}$. This implies (2.30) and (2.27) follows.

To finish the proof note that for any $X \in \pi_\omega(\mathcal{O})'$ one has

$$\pi_\omega\left([\tilde{A}_t,\tilde{B}_t]\right)X\Omega_\omega = X\pi_\omega\left([\tilde{A}_t,\tilde{B}_t]\right)\Omega_\omega,$$

and so for all $\Psi \in \pi_\omega(\mathcal{O})'\Omega_\omega$

$$\lim_{t\to\infty}\pi_\omega\left([\tilde{A}_t,\tilde{B}_t]\right)\Psi = \left(\int_{-\infty}^{\infty}\omega([\tau^s(A),B])\,\mathrm{d}s\right)\Psi. \tag{2.31}$$

Since ω is modular $\pi_\omega(\mathcal{O})'\Omega_\omega$ is dense in $\mathcal{H}_\omega$ and it follows from the estimate

$$\sup_{t>0}\|[\tilde{A}_t,\tilde{B}_t]\| \leq \int_{\mathbb{R}}\|[\tau^s(A),B]\|\,\mathrm{d}s < \infty,$$

that (2.31) extends to all $\Psi \in \mathcal{H}_\omega$. □

We are now ready to complete:

Proof of Theorem 1.3. Let $\{A_1,\cdots,A_n\} \in \mathfrak{C}_{\mathrm{self}}$. For $j = 1,\ldots,n-1$, we set

$$U_{jt} = \exp\left(-\frac{1}{2}[\tilde{A}_{jt},\tilde{A}_{(j+1)t}+\cdots+\tilde{A}_{nt}]\right),$$

and $U_t = U_{1t}\cdots U_{(n-1)t}$. Clearly, the U_{jt}'s are unitary and repeated use of Proposition 2.1 yields that

$$\lim_{t\to\infty}\left\|\mathrm{e}^{\mathrm{i}\tilde{A}_{1t}}\cdots\mathrm{e}^{\mathrm{i}\tilde{A}_{nt}} - \mathrm{e}^{\mathrm{i}(\tilde{A}_{1t}+\cdots+\tilde{A}_{nt})}U_t\right\| = 0,$$

and hence,

$$\lim_{t\to\infty}\left|\omega\left(\mathrm{e}^{\mathrm{i}\tilde{A}_{1t}}\cdots\mathrm{e}^{\mathrm{i}\tilde{A}_{nt}}\right) - \omega\left(\mathrm{e}^{\mathrm{i}(\tilde{A}_{1t}+\cdots+\tilde{A}_{nt})}U_t\right)\right| = 0. \tag{2.32}$$

Proposition 2.3 implies that

$$\mathrm{s}-\lim_{t\to\infty}\pi_\omega(U_{jt}) = \exp\left(-\mathrm{i}\sum_{k=j+1}^{n}\varsigma(A_j,A_k)\right),$$

and so

$$\mathrm{s}-\lim_{t\to\infty}\pi_\omega(U_t)=\exp\left(-\mathrm{i}\sum_{1\le j<k\le n}\varsigma(A_j,A_k)\right). \tag{2.33}$$

Since SQCLT holds, Relations (2.32) and (2.33) yield

$$\begin{aligned}\lim_{t\to\infty}\omega\left(\mathrm{e}^{\mathrm{i}\tilde{A}_{1t}}\cdots\mathrm{e}^{\mathrm{i}\tilde{A}_{nt}}\right)&=\lim_{t\to\infty}\omega\left(\mathrm{e}^{\mathrm{i}(\tilde{A}_{1t}+\cdots+\tilde{A}_{nt})}U_t\right)\\&=\lim_{t\to\infty}\omega\left(\mathrm{e}^{\mathrm{i}(\tilde{A}_{1t}+\cdots+\tilde{A}_{nt})}\right)\exp\left(-\mathrm{i}\sum_{1\le j<k\le n}\varsigma(A_j,A_k)\right)\\&=\exp\left(-\frac{1}{2}\,L\left(\sum_{k=1}^n A_k,\sum_{j=1}^n A_j\right)\right)\exp\left(-\mathrm{i}\sum_{1\le j<k\le n}\varsigma(A_j,A_k)\right),\end{aligned}$$

and the theorem follows. □

2.2 Proof of Theorem 1.4

Let $f_1,\dots,f_n$ be as in the theorem and set $M=\sup_{1\le j\le n,x\in\mathbb{R}}|f_j(x)|$. Then, for each $j=1,\dots,n$, there exists a sequence $(g_{jk})_{k\in\mathbb{N}}$ of compactly supported continuous functions such that

$$\sup_{k\in\mathbb{N},x\in\mathbb{R}}|g_{jk}(x)|\le M\qquad\text{and}\qquad\lim_{k\to\infty}g_{jk}(x)=f_j(x), \tag{2.34}$$

for all $x\in\mathbb{R}$. For $\eta>0$ set

$$g_{jk\eta}(x)=\int_{\mathbb{R}}\mathrm{e}^{-(x-y)^2/\eta}g_{jk}(y)\,\frac{\mathrm{d}y}{\sqrt{\pi\eta}}.$$

$\hat{g}_{jk\eta}$ denotes the Fourier transform of $g_{jk\eta}$. Note that $\hat{g}_{jk\eta}\in L^1(\mathbb{R})$ and

$$\sup_{k\in\mathbb{N},\eta>0,x\in\mathbb{R}}|g_{jk\eta}(x)|\le M\qquad\text{and}\qquad\lim_{\eta\to0}g_{jk\eta}(x)=g_{jk}(x), \tag{2.35}$$

for all $x\in\mathbb{R}$.

Lemma 2.4

$$\lim_{k_1\to\infty}\lim_{\eta_1\to0}\cdots\lim_{k_n\to\infty}\lim_{\eta_n\to0}\lim_{t\to\infty}\omega(g_{1k_1\eta_1}(\tilde{A}_{1t})\cdots g_{nk_n\eta_n}(\tilde{A}_{nt}))=\omega_L(f_1(\varphi_L(A_1))\cdots f_n(\varphi_L(A_n))).$$

Proof. Write

$$\omega(g_{1k_1\eta_1}(\tilde{A}_{1t})\cdots g_{nk_n\eta_n}(\tilde{A}_{nt}))=\int_{\mathbb{R}^n}\hat{g}_{1k_1\eta_1}(\xi_1)\cdots\hat{g}_{nk_n\eta_n}(\xi_n)\omega\left(\mathrm{e}^{-\mathrm{i}\xi_1\tilde{A}_{1t}}\cdots\mathrm{e}^{-\mathrm{i}\xi_n\tilde{A}_{nt}}\right)\frac{\mathrm{d}\xi_1\cdots\mathrm{d}\xi_n}{(2\pi)^n}.$$

The QCLT implies that

$$\lim_{t\to\infty}\omega\left(\mathrm{e}^{-\mathrm{i}\xi_1\tilde{A}_{1t}}\cdots\mathrm{e}^{-\mathrm{i}\xi_n\tilde{A}_{nt}}\right)=\omega_L(W(-\xi_1A_1)\cdots W(-\xi_nA_n)),$$

and the dominated convergence theorem yields

$$\begin{aligned}\lim_{t\to\infty}\omega(g_{1k_1\eta_1}(\tilde{A}_{1t})\cdots g_{nk_n\eta_n}(\tilde{A}_{nt}))&=\int_{\mathbb{R}^n}\hat{g}_{1k_1\eta_1}(\xi_1)\cdots\hat{g}_{nk_n\eta_n}(\xi_n)\omega_L(W(-\xi_1A_1)\cdots W(-\xi_nA_n))\frac{\mathrm{d}\xi_1\cdots\mathrm{d}\xi_n}{(2\pi)^n}\\&=\omega_L(g_{1k_1\eta_1}(\varphi_L(A_1))\cdots g_{nk_n\eta_n}(\varphi_L(A_n))).\end{aligned}$$

By the functional calculus for self-adjoint operators (2.34) and (2.35) imply

$$\lim_{k_1\to\infty}\lim_{\eta_1\to 0}\cdots\lim_{k_n\to\infty}\lim_{\eta_n\to 0}\omega_L(g_{1k_1\eta_1}(\varphi_L(A_1))\cdots g_{nk_n\eta_n}(\varphi_L(A_n)))=\omega_L(f_1(\varphi_L(A_1))\cdots f_n(\varphi_L(A_n))),$$

and the statement follows. □

Lemma 2.5 *For any given* $\xi_1,\dots,\xi_{j-1}\in\mathbb{R}$,

$$\lim_{k\to\infty}\lim_{\eta\to 0}\lim_{t\to\infty}\omega\left(\mathrm{e}^{-\mathrm{i}\xi_1\tilde{A}_{1t}}\cdots\mathrm{e}^{-\mathrm{i}\xi_{j-1}\tilde{A}_{(j-1)t}}\left|f_j(\tilde{A}_{jt})-g_{jk\eta}(\tilde{A}_{jt})\right|^2\mathrm{e}^{\mathrm{i}\xi_{j-1}\tilde{A}_{(j-1)t}}\cdots\mathrm{e}^{\mathrm{i}\xi_1\tilde{A}_{1t}}\right)=0.$$

Proof. Let μ_t be the spectral measure for $\pi_\omega(\tilde{A}_{jt})$ and the unit vector $\pi_\omega(\mathrm{e}^{\mathrm{i}\xi_{j-1}\tilde{A}_{(j-1)t}}\cdots\mathrm{e}^{\mathrm{i}\xi_1\tilde{A}_{1t}})\Omega_\omega$. Then

$$\omega\left(\mathrm{e}^{-\mathrm{i}\xi_1\tilde{A}_{1t}}\cdots\mathrm{e}^{-\mathrm{i}\xi_{j-1}\tilde{A}_{(j-1)t}}\left|f_j(\tilde{A}_{jt})-g_{jk\eta}(\tilde{A}_{jt})\right|^2\mathrm{e}^{\mathrm{i}\xi_{j-1}\tilde{A}_{(j-1)t}}\cdots\mathrm{e}^{\mathrm{i}\xi_1\tilde{A}_{1t}}\right)=\int_{\mathbb{R}}|f_j(x)-g_{jk\eta}(x)|^2\mathrm{d}\mu_t(x).$$

The QCLT yields that for any $\alpha\in\mathbb{R}$,

$$\begin{aligned}\lim_{t\to\infty}\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}\alpha x}\mathrm{d}\mu_t(x)&=\lim_{t\to\infty}\omega\left(\mathrm{e}^{-\mathrm{i}\xi_1\tilde{A}_{1t}}\cdots\mathrm{e}^{-\mathrm{i}\xi_{j-1}\tilde{A}_{(j-1)t}}\mathrm{e}^{\mathrm{i}\alpha\tilde{A}_{jt}}\mathrm{e}^{\mathrm{i}\xi_{j-1}\tilde{A}_{(j-1)t}}\cdots\mathrm{e}^{\mathrm{i}\xi_1\tilde{A}_{1t}}\right)\\&=\exp\left(-\frac{\alpha^2}{2}L(A_j,A_j)-2\mathrm{i}\alpha\varsigma\left(A_j,\sum_{1\le k\le j-1}\xi_kA_k\right)\right)=\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}\alpha x}\mathrm{d}\mu(x),\end{aligned}$$

where μ is a Gaussian measure with variance $L(A_j,A_j)$ and expectation $-2\varsigma\left(A_j,\sum_{k=1}^{j-1}\xi_kA_k\right)$ (note that if $L(A_j,A_j)=0$ then μ is the δ-measure at 0). This fact and the Lévy-Cramér Continuity Theorem yield that

$$\lim_{t\to\infty}\int_{\mathbb{R}}|f_j(x)-g_{jk\eta}(x)|^2\mathrm{d}\mu_t(x)=\int_{\mathbb{R}}|f_j(x)-g_{jk\eta}(x)|^2\mathrm{d}\mu(x),$$

and the statement follows from (2.34), (2.35) and the dominated convergence theorem. □

We are now ready to complete:

Proof of Theorem 1.4. Write

$$\omega(f_1(\tilde{A}_{1t})\cdots f_n(\tilde{A}_{nt}))-\omega(g_{1k_1\eta_1}(\tilde{A}_{1t})\cdots g_{nk_n\eta_n}(\tilde{A}_{nt}))=\sum_{j=1}^n T_j,$$

where

$$\begin{aligned}T_j&=\omega\left(g_{1k_1\eta_1}(\tilde{A}_{1t})\cdots g_{(j-1)k_{j-1}\eta_{j-1}}(\tilde{A}_{(j-1)t})\left[f_j(\tilde{A}_{jt})-g_{jk_j\eta_j}(\tilde{A}_{jt})\right]f_{j+1}(\tilde{A}_{(j+1)t})\cdots f_n(\tilde{A}_{nt})\right)\\&=\int_{\mathbb{R}^{j-1}}\hat{g}_{1k_1\eta_1}(\xi_1)\cdots\hat{g}_{(j-1)k_{j-1}\eta_{j-1}}(\xi_{j-1})D_j(\xi_1,\dots,\xi_{j-1})\frac{\mathrm{d}\xi_1\cdots\mathrm{d}\xi_{j-1}}{(2\pi)^{j-1}},\end{aligned}$$

with

$$D_j(\xi_1,\dots,\xi_{j-1})=\omega\left(\mathrm{e}^{-\mathrm{i}\xi_1\tilde{A}_{1t}}\cdots\mathrm{e}^{-\mathrm{i}\xi_{j-1}\tilde{A}_{(j-1)t}}\left[f_j(\tilde{A}_{jt})-g_{jk_j\eta_j}(\tilde{A}_{jt})\right]f_{j+1}(\tilde{A}_{(j+1)t})\cdots f_n(\tilde{A}_{nt})\right).$$

The Cauchy-Schwartz inequality yields that

$$|D_j(\xi_1,\dots,\xi_{j-1})|\le M^{n-j}\left|\omega\left(\mathrm{e}^{-\mathrm{i}\xi_1\tilde{A}_{1t}}\cdots\mathrm{e}^{-\mathrm{i}\xi_{j-1}\tilde{A}_{(j-1)t}}\left|f_j(\tilde{A}_{jt})-g_{jk_j\eta_j}(\tilde{A}_{jt})\right|^2\mathrm{e}^{\mathrm{i}\xi_{j-1}\tilde{A}_{(j-1)t}}\cdots\mathrm{e}^{\mathrm{i}\xi_1\tilde{A}_{1t}}\right)\right|^{1/2},$$

and so Lemma 2.5 and the dominated convergence theorem gives

$$\limsup_{k_1\to\infty}\limsup_{\eta_1\to 0}\cdots\limsup_{k_j\to\infty}\limsup_{\eta_j\to 0}\limsup_{t\to\infty}|T_j| = 0,$$

for all j. Hence,

$$\limsup_{k_1\to\infty}\limsup_{\eta_1\to 0}\cdots\limsup_{k_n\to\infty}\limsup_{\eta_n\to 0}\limsup_{t\to\infty}|\omega(f_1(\tilde{A}_{1t})\cdots f_n(\tilde{A}_{nt})) - \omega(g_{1k_1\eta_1}(\tilde{A}_{1t})\cdots g_{nk_n\eta_n}(\tilde{A}_{nt}))| = 0,$$

and the statement follows from Lemma 2.4. □

2.3 Norm localization

For $\epsilon > 0$ we denote $D_\epsilon = \{z \in \mathbb{C} \,|\, |z| < \epsilon\}$.

Proposition 2.6 *Let $A \in \mathcal{O}_{\rm self}$ be such that*

$$\int_{-\infty}^{\infty} |\omega(A\tau^t(A)) - \omega(A)^2|\,\mathrm{d}t < \infty.$$

Suppose that there exists $\epsilon > 0$ such that

$$\lim_{t\to\infty}\omega(\mathrm{e}^{\mathrm{i}\alpha\tilde{A}_t}) = \mathrm{e}^{-L(A,A)\alpha^2/2}, \tag{2.36}$$

for $\alpha \in D_\epsilon$. Then (2.36) holds for all $\alpha \in \mathbb{R}$.

Proof. This follows from well-known results in classical probability. See section 30 in [Bil]. □

3 Locally interacting fermions

In this section we describe the strategy of the proof of our main result, Theorem 1.6, and establish a number of preliminary results needed for the proof. In particular, we shall reduce the proof of Theorem 1.6 to the proof of Theorem 3.5 (stated in Subsection 3.3 and proven in Section 4). Theorem 3.5, which is the main technical result of our paper, concerns only the unperturbed system $(\mathcal{O}, \tau_0, \omega_0)$.

3.1 Strategy

Suppose that the assumptions of Theorem 1.6 hold and let

$$A = \sum_{k=1}^{K_A}\prod_{j=1}^{n_k} a^*(f_{kj})a(g_{kj}),$$

be an element of $\mathfrak{C}$. Clearly,

$$\omega_\lambda^+\left((\tilde{A}_t)^n\right) = t^{-n/2}\int_{[0,t]^n}\omega_\lambda^+\left(\prod_{j=1}^n\left(\tau_\lambda^{t_j}(A) - \omega_\lambda^+(A)\right)\right)\,\mathrm{d}t_1\cdots\mathrm{d}t_n.$$

The first ingredient of the proof of Theorem 1.6 is:

Theorem 3.1 *There exists a finite constant $C_{V,A}$ such that for all n,*

$$\sup_{|\lambda|\le\tilde{\lambda}_V, t>0} t^{-n/2}\int_{[0,t]^n}\left|\omega_\lambda^+\left(\prod_{j=1}^n\left(\tau_\lambda^{t_j}(A) - \omega_\lambda^+(A)\right)\right)\right|\,\mathrm{d}t_1\cdots\mathrm{d}t_n \le C_{V,A}^n\, n!. \tag{3.37}$$

Remark 1. Our proof also gives an explicit estimate on the constant $C_{V,A}$, see Formula (3.51) below.
Remark 2. In the special case $n = 2$, Theorem 3.1 yields that for all $t > 0$ and $|\lambda| \leq \tilde{\lambda}_V$,

$$\int_{-t}^{t} \left(1 - \frac{|s|}{t}\right) \left|\omega_\lambda^+ \left((\tau_\lambda^s(A) - \omega_\lambda^+(A))(A - \omega_\lambda^+(A))\right)\right| \, \mathrm{d}s \leq 2C_{V,A}^2.$$

As $t \to \infty$ the monotone convergence theorem yields

$$\int_{-\infty}^{\infty} \left|\omega_\lambda^+ \left((\tau_\lambda^s(A) - \omega_\lambda^+(A))(A - \omega_\lambda^+(A))\right)\right| \, \mathrm{d}s \leq 2C_{V,A}^2.$$

In particular, we derive that $\mathfrak{C}$ is CLT-admissible.

The second ingredient of the proof of Theorem 1.6 is:

Theorem 3.2 *For $|\lambda| \leq \tilde{\lambda}_V$ and all $n \geq 1$,*

$$\lim_{t\to\infty} \omega_\lambda^+ \left((\tilde{A}_t)^n\right) = \begin{cases} \dfrac{n!}{2^{n/2}(n/2)!} L(A,A)^{n/2} & \textit{if } n \textit{ is even}, \\ 0 & \textit{if } n \textit{ is odd}. \end{cases}$$

Remark. With only notational changes the proof of Theorem 3.2 yields that for all $A_1, \cdots, A_n \in \mathfrak{C}$,

$$\lim_{t\to\infty} \omega_\lambda^+ \left(\tilde{A}_{1t} \cdots \tilde{A}_{nt}\right) = \omega_L \left(\varphi_L(A_1) \cdots \varphi_L(A_n)\right),$$

where the r.h.s. is defined by (1.6).

Given Theorems 3.1 and 3.2, we can complete:

Proof of Theorem 1.6. Let $A \in \mathfrak{C}_{\mathrm{self}}$. For $\alpha \in \mathbb{C}$ one has

$$\omega_\lambda^+ \left(\mathrm{e}^{\mathrm{i}\alpha\tilde{A}_t}\right) = \sum_{n\geq 0} \frac{(\mathrm{i}\alpha)^n}{n!} \omega_\lambda^+ \left((\tilde{A}_t)^n\right). \tag{3.38}$$

Let $\epsilon = 1/(2C_{V,A})$ and suppose that $|\lambda| \leq \tilde{\lambda}_V$. Theorems 3.1 and 3.2 yield that

$$\sup_{|\alpha|<\epsilon, t>0} \left|\omega_\lambda^+ \left(\mathrm{e}^{\mathrm{i}\alpha\tilde{A}_t}\right)\right| < \infty,$$

and that for $|\alpha| < \epsilon$,

$$\lim_{t\to\infty} \omega_\lambda^+ \left(\mathrm{e}^{\mathrm{i}\alpha\tilde{A}_t}\right) = \mathrm{e}^{-L(A,A)\,\alpha^2/2}. \tag{3.39}$$

Proposition 2.6 yields that (3.39) holds for all $\alpha \in \mathbb{R}$, and so SQCLT holds for $\mathfrak{C}$ w.r.t. $(\mathcal{O}, \tau_\lambda, \omega_\lambda^+)$. Our standing assumption $\mathrm{Ker}\,(T) = \mathrm{Ker}\,(I - T) = \{0\}$ ensures that the state ω_0 is modular, and since $\omega_\lambda^+ = \omega_0 \circ \gamma_\lambda^+$, the state ω_λ^+ is also modular. By Theorem 1.5, if $|\lambda| \leq \lambda_V$, then $\mathfrak{C}$ is L^1-asymptotically Abelian for τ_λ and it follows from Theorem 1.3 that the QCLT also holds.□

Notice that in the initial step of the proof we did not use the assumption that A is self-adjoint, and so the following weak form of QCLT holds for *any* $A \in \mathfrak{C}$:

Corollary 3.3 *For any $A \in \mathfrak{C}$ there exists $\epsilon > 0$ such that for $|\lambda| \leq \tilde{\lambda}_V$ and $|\alpha| < \epsilon$,*

$$\lim_{t\to\infty} \omega_\lambda^+ \left(\mathrm{e}^{\mathrm{i}\alpha\tilde{A}_t}\right) = \mathrm{e}^{-L(A,A)\alpha^2/2}.$$

In the rest of this section we shall describe the strategy of the proof of Theorems 3.1 and 3.2.

3.2 The commutator estimate

We shall need the following result

Theorem 3.4 *Suppose that Assumption (A) holds. Let $V \in \mathfrak{O}(\mathfrak{d})_{\rm self}$ be a perturbation such that $\overline{n}_V \geq 2$ and*

$$\max_{f \in \mathcal{F}(V)} \|f\| = 1.$$

Let $A = a^{\#}(f_1) \cdots a^{\#}(f_m)$ be a monomial such that $\mathcal{F}(A) = \{f_1, \cdots, f_m\} \subset \mathfrak{d}$, and let

$$\mathcal{C}_A^{(n)}(s_1, \ldots, s_n) = [\tau_0^{s_n}(V), [\cdots, [\tau_0^{s_1}(V), A] \cdots]].$$

Then for all $n \geq 0$ there exist a finite index set $\mathcal{Q}_n(A)$, monomials $F_{A,q}^{(n)} \in \mathcal{O}$, and scalar functions $G_{A,q}^{(n)}$ such that

$$\mathcal{C}_A^{(n)}(s_1, \ldots, s_n) = \sum_{q \in \mathcal{Q}_n(A)} G_{A,q}^{(n)}(s_1, \ldots, s_n) F_{A,q}^{(n)}(s_1, \ldots, s_n). \tag{3.40}$$

Moreover,

1. *The order of the monomial $F_{A,q}^{(n)}$ does not exceed $2n(\overline{n}_V - 1) + m$.*
2. *If m is even then the order of $F_{A,q}^{(n)}$ is also even.*
3. *The factors of $F_{A,q}^{(n)}$ are from*

$$\left\{ a^{\#}(\mathrm{e}^{\mathrm{i}sh_0} g) \,\middle|\, g \in \mathcal{F}(V), s \in \{s_1, \ldots, s_n\} \right\} \cup \left\{ a^{\#}(g) \,\middle|\, g \in \mathcal{F}(A) \right\},$$

   The number of factors from the first set does not exceed $n(2\overline{n}_V - 1)$ while the number of factors from the second set does not exceed $m - 1$. In particular, $\|F_{A,q}^{(n)}\| \leq \max(1, \max_{f \in \mathcal{F}(A)} \|f\|^{m-1})$.
4. *Let λ_V be given by (1.15). Then*

$$W_{V,A} \equiv \sum_{n=1}^{\infty} |\lambda_V|^n \sum_{q \in \mathcal{Q}_n(A)} \int_{-\infty < s_n \leq \cdots \leq s_1 \leq 0} \left| G_{A,q}^{(n)}(s_1, \ldots, s_n) \right| \mathrm{d}s_1 \cdots \mathrm{d}s_n < \infty. \tag{3.41}$$

The proof of Theorem 3.4 is identical to the proof of Theorem 1.1 in [JOP4]. Parts 1–3 are simple and are stated for reference purposes. The Part 4 is a relatively straightforward consequence of the fundamental Botvich-Guta-Maassen integral estimate [BGM] which also gives an explicit estimate on $W_{V,A}$. A pedagogical exposition of the Botvich-Guta-Maassen estimate can be found in [JP6].

If A is as in Theorem 3.4 then

$$\gamma_\lambda^+(A) = \lim_{t \to \infty} \tau_0^{-t} \circ \tau_\lambda^t(A),$$

can be expanded in a power series in λ which converges for $|\lambda| \leq \lambda_V$. Indeed, it follows from the Araki-Dyson expansion that

$$\tau_0^{-t} \circ \tau_\lambda^t(A) = A + \sum_{n=1}^{\infty} (\mathrm{i}\lambda)^n \int_{-t \leq s_n \leq \cdots \leq s_1 \leq 0} [\tau_0^{s_n}(V), [\cdots, [\tau_0^{s_1}(V), A] \cdots]] \, \mathrm{d}s_1 \cdots \mathrm{d}s_n.$$

Hence, for $|\lambda| \leq \lambda_V$,

$$\gamma_\lambda^+(A) = A + \sum_{n=1}^{\infty} (\mathrm{i}\lambda)^n \sum_{q \in \mathcal{Q}_n(A)} \int_{-\infty < s_n \leq \cdots \leq s_1 \leq 0} G_{A,q}^{(n)}(s_1, \ldots, s_n) F_{A,q}^{(n)}(s_1, \ldots, s_n) \, \mathrm{d}s_1 \cdots \mathrm{d}s_n, \tag{3.42}$$

where the series on the right-hand side is norm convergent by Parts 3 and 4 of Theorem 3.4. This expansion will be used in the proof of Theorems 3.1 and 3.2.

3.3 Quasi-free correlations

Let $\mathcal{O}$, τ_0 and ω_0 be as in Subsection 1.2. We denote by

$$\varphi(f) = \frac{1}{\sqrt{2}}\left(a(f) + a^*(f)\right),$$

the Fermi field operator associated to $f \in \mathfrak{h}$. The Fermi field operators satisfy the commutation relation

$$\varphi(f)\varphi(g) + \varphi(g)\varphi(f) = \mathrm{Re}(f,g)\mathbb{1},$$

and the CAR algebra $\mathcal{O}$ is generated by $\{\varphi(f) \,|\, f \in \mathfrak{h}\}$. Clearly,

$$a(f) = \frac{1}{\sqrt{2}}\left(\varphi(f) + \mathrm{i}\varphi(\mathrm{i}f)\right), \qquad a^*(f) = \frac{1}{\sqrt{2}}\left(\varphi(f) - \mathrm{i}\varphi(\mathrm{i}f)\right). \tag{3.43}$$

We recall that ω_0, the gauge invariant quasi-free state associated to the density operator T, is uniquely specified by

$$\omega_0(a^*(f_n)\cdots a^*(f_1)a(g_1)\cdots a(g_m)) = \delta_{n,m}\det\{(g_i, Tf_j)\}.$$

Alternatively, ω_0 can be described by its action on the Fermi field operators. Let $\mathcal{P}_n$ be the set of all permutations π of $\{1,\ldots,2n\}$ described in Subsection 1.1 (recall (1.5)). Denote by $\epsilon(\pi)$ the signature of $\pi \in \mathcal{P}_n$. ω_0 is the unique state on $\mathcal{O}$ such that

$$\omega_0(\varphi(f_1)\varphi(f_2)) = \frac{1}{2}(f_1, f_2) - \mathrm{i}\,\mathrm{Im}(f_1, Tf_2),$$

and

$$\omega_0(\varphi(f_1)\cdots\varphi(f_n)) = \begin{cases} \displaystyle\sum_{\pi\in\mathcal{P}_{n/2}} \epsilon(\pi)\prod_{j=1}^{n/2}\omega_T\left(\varphi(f_{\pi(2j-1)}), \varphi(f_{\pi(2j)})\right) & \text{if } n \text{ is even;} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

For any bounded subset $\mathfrak{f} \subset \mathfrak{h}$ we set

$$M_{\mathfrak{f}} = \sup_{f\in\mathfrak{f}} \|f\|,$$

and

$$C_{\mathfrak{f}} = \max\left(1, \sup_{f,g\in\mathfrak{f}} \frac{2}{\|f\|\,\|g\|}\int_{-\infty}^{\infty}\left|\omega_0\left(\varphi(f)\tau_0^t(\varphi(g))\right)\right| \mathrm{d}t\right),$$

and we denote by $\mathcal{M}(\mathfrak{f})$ the set of monomials with factors from $\{\varphi(f) | f \in \mathfrak{f}\}$. We further say that $A \in \mathcal{M}(\mathfrak{f})$ is of degree at most k if, for some $f_1,\ldots,f_k \in \mathfrak{f}$, one can write $A = \varphi(f_1)\cdots\varphi(f_k)$.

Theorem 3.5 *Suppose that $C_{\mathfrak{f}} < \infty$. Then for any $A_1,\ldots,A_n \in \mathcal{M}(\mathfrak{f})$ of degrees at most $k_1,\ldots,k_n$ the following holds:*

1. $$\sup_{t>0} t^{-n/2}\int_{[0,t]^n}\left|\omega_0\left(\prod_{i=1}^n\left(\tau_0^{t_i}(A_i) - \omega_0(A_i)\right)\right)\right| \mathrm{d}t_1\cdots\mathrm{d}t_n \le \left(2^{7/2}M_{\mathfrak{f}}\right)^{\sum_i k_i} C_{\mathfrak{f}}^n n!.$$

2. *If n is odd,*
$$\lim_{t\to\infty} t^{-n/2}\int_{[0,t]^n}\omega_0\left(\prod_{i=1}^n\left(\tau_0^{t_i}(A_i) - \omega_0(A_i)\right)\right) \mathrm{d}t_1\cdots\mathrm{d}t_n = 0.$$

3. *If n is even,*
$$\lim_{t\to\infty} t^{-n/2}\int_{[0,t]^n}\omega_0\left(\prod_{i=1}^n\left(\tau_0^{t_i}(A_i) - \omega_0(A_i)\right)\right) \mathrm{d}t_1\cdots\mathrm{d}t_n = \sum_{\pi\in\mathcal{P}_{n/2}}\prod_{j=1}^{n/2} L_0(A_{\pi(2j-1)}, A_{\pi(2j)}),$$

   where
$$L_0(A_i, A_j) = \int_{-\infty}^{\infty}\omega_0\left((\tau_0^t(A_i) - \omega_0(A_i))(A_j - \omega_0(A_j))\right) \mathrm{d}t. \tag{3.44}$$

Remark. As in Remark 2 after Theorem 3.1, Part 1 of the previous theorem with $n = 2$ implies that

$$\int_{-\infty}^{\infty} \left|\omega_0\left((\tau_0^t(A_i) - \omega_0(A_i))(A_j - \omega_0(A_j))\right)\right| \, \mathrm{d}t < \infty,$$

and so $L_0(A_i, A_j)$ is well defined.

Theorem 3.5 is in essence the main technical result of our paper. Its proof is given in Section 4.

We have formulated Theorem 3.5 in terms of field operators since that allows for a combinatorially natural approach to its proof. Using the identities (3.43) one effortlessly gets the following reformulation which is more convenient for our application.

Denote by $\tilde{\mathcal{M}}(\mathfrak{f})$ the set of monomials with factors from $\{a^{\#}(f) | f \in \mathfrak{f}\}$. $A \in \mathcal{M}(\mathfrak{f})$ is of degree at most k if, for some $f_1, \ldots, f_k \in \mathfrak{f}$, one can write $A = a^{\#}(f_1) \cdots a^{\#}(f_k)$. Let

$$D_{\mathfrak{f}} = \max\left(1, \sup_{f,g \in \mathfrak{f} \cup \mathrm{i}\mathfrak{f}} \frac{2}{\|f\| \, \|g\|} \int_{-\infty}^{\infty} \left|\omega_0\left(\varphi(f)\tau_0^t(\varphi(g))\right)\right| \, \mathrm{d}t\right),$$

Corollary 3.6 *Suppose that $D_{\mathfrak{f}} < \infty$. Then for any $A_1, \ldots, A_n \in \tilde{\mathcal{M}}(\mathfrak{f})$ of degrees at most $k_1, \ldots, k_n$ the following holds:*

1. $$\sup_{t>0} t^{-n/2} \int_{[0,t]^n} \left|\omega_0\left(\prod_{i=1}^n \left(\tau_0^{t_i}(A_i) - \omega_0(A_i)\right)\right)\right| \, \mathrm{d}t_1 \cdots \mathrm{d}t_n \leq \left(2^4 M_{\mathfrak{f}}\right)^{\sum_i k_i} D_{\mathfrak{f}}^n n!.$$

2. *If n is odd,*
$$\lim_{t\to\infty} t^{-n/2} \int_{[0,t]^n} \omega_0\left(\prod_{i=1}^n \left(\tau_0^{t_i}(A_i) - \omega_0(A_i)\right)\right) \, \mathrm{d}t_1 \cdots \mathrm{d}t_n = 0.$$

3. *If n is even,*
$$\lim_{t\to\infty} t^{-n/2} \int_{[0,t]^n} \omega_0\left(\prod_{i=1}^n \left(\tau_0^{t_i}(A_i) - \omega_0(A_i)\right)\right) \, \mathrm{d}t_1 \cdots \mathrm{d}t_n = \sum_{\pi \in \mathcal{P}_{n/2}} \prod_{j=1}^{n/2} L_0(A_{\pi(2j-1)}, A_{\pi(2j)}),$$

   where $L_0(A_i, A_k)$ is defined by (3.44).

Note that if $\mathfrak{c}$ is as in Subsection 1.2 and $\mathfrak{f}$ is a finite subset of $\mathfrak{c}$, then $C_{\mathfrak{f}} < \infty$ and $D_{\mathfrak{f}} < \infty$.

After this paper was completed we have learned of a beautiful paper [De1] which is perhaps deepest among early works on quantum central limit theorems (Dereziński's work was motivated by [Ha1, Ha2, Ru1, HL1, HL2, HL3, Da2]). In relation to our work, in [De1] Theorem 3.5 was proven in the special case $k_1 = \cdots = k_n = 2$ of quadratic interactions. This suffices for the proof of SQCLT for quasi-free dynamics and for observables which are polynomials in Fermi fields. The proofs of Parts (2) and (3) of Theorem 3.5 are not that much different in the general case $k_j \geq 2$. The key difference is in Part (1) which in the quadratic case follows easily from Stirling's formula. To prove Part (1) for any $k_j \geq 2$ is much more difficult and the bulk of the proof of Theorem 3.5 in Section 4 is devoted to this estimate. The proof of QCLT for locally interacting fermionic systems critically depends on this result.

3.4 Proofs of Theorems 3.1 and 3.2

In this subsection we shall show that Theorems 3.4 and 3.5 imply Theorems 3.1 and 3.2, thereby reducing the proof of Theorem 1.6 to the proof of Theorem 3.5.

If η is a state, we shall denote

$$\eta_{\mathrm{T}}(A_1, \ldots, A_n) \equiv \eta\left(\prod_{i=1}^n (A_i - \eta(A_i))\right). \tag{3.45}$$

Let

$$A = \sum_{k=1}^{K_A} A_k, \quad A_k = \prod_{j=1}^{n_k} a^*(f_{kj})a(g_{kj}),$$

be an element of $\mathfrak{C}$. Without loss of generality we may assume that $\max_{f\in\mathcal{F}(A)} \|f\| = 1$. With

$$\mathfrak{f} = \left\{ \mathrm{e}^{\mathrm{i}sh_0} f \;\middle|\; f \in \mathcal{F}(V) \cup \mathcal{F}(A), s \in \mathbb{R} \right\},$$

$$D_{V,A} = \max\left(1, \max_{f,g\in\mathcal{F}(V)\cup\mathcal{F}(A)} \frac{1}{\|f\|\|g\|} \int_{-\infty}^{\infty} \left(2^{-1}|(f, \mathrm{e}^{\mathrm{i}th_0} g)| + |(f, \mathrm{e}^{\mathrm{i}th_0} Tg)|\right) \mathrm{d}t\right),$$

we clearly have $M_{\mathfrak{f}} = 1$ and $D_{\mathfrak{f}} \le D_{V,A}$.

Proof of Theorem 3.1. For $|\lambda| \le \lambda_V$,

$$\omega^+_{\lambda\mathrm{T}}\left(\tau^{t_1}_\lambda(A), \ldots, \tau^{t_n}_\lambda(A)\right) = \sum_{k_1,\ldots,k_n=1}^{K_A} \omega_{0\mathrm{T}}\left(\tau^{t_1}_0 \circ \gamma^+_\lambda(A_{k_1}), \ldots, \tau^{t_n}_0 \circ \gamma^+_\lambda(A_{k_n})\right), \tag{3.46}$$

and the expansion (3.42) yields that

$$\tau^t_0 \circ \gamma^+_\lambda(A_k) - \omega_0 \circ \gamma^+_\lambda(A_k) = \sum_{j\ge 0} (\mathrm{i}\lambda)^j \sum_{q\in\mathcal{Q}_j(A_k)} \int_{\Delta_j} G^{(j)}_{A_k,q}(s) \left(\tau^t_0\left(F^{(j)}_{A_k,q}(s)\right) - \omega_0\left(F^{(j)}_{A_k,q}(s)\right)\right) \mathrm{d}s, \tag{3.47}$$

where Δ_j denotes the simplex $\{s = (s_1, \ldots, s_j) \in \mathbb{R}^j \mid -\infty < s_j < \cdots < s_1 < 0\}$. We have adopted the convention that $Q_0(A_k)$ is a singleton, that $G^{(0)}_{A_k,q} = 1$ and that $F^{(0)}_{A_k,q} = A_k$. Moreover, integration over the empty simplex Δ_0 is interpreted as the identity map. Applying Fubini's theorem we get

$$t^{-n/2} \int_{[0,t]^n} \omega_{0\mathrm{T}}\left(\tau^{t_1}_0 \circ \gamma^+_\lambda(A_{k_1}), \ldots, \tau^{t_n}_0 \circ \gamma^+_\lambda(A_{k_n})\right) \mathrm{d}t_1 \cdots \mathrm{d}t_n = \sum_{j_1,\ldots,j_n\ge 0} (\mathrm{i}\lambda)^{j_1+\cdots+j_n} \sum_{q_1\in Q_{j_1}(A_{k_1}),\ldots,q_n\in Q_{j_n}(A_{k_n})}$$

$$\int_{\Delta_{j_1}} \mathrm{d}s_1 \cdots \int_{\Delta_{j_n}} \mathrm{d}s_n \left(\prod_{l=1}^n G^{(j_l)}_{A_{k_l} q_l}(s_l)\right) C_t(j, q, s; A_{k_1}, \ldots, A_{k_n}), \tag{3.48}$$

where we have set

$$C_t(j, q, s; A_{k_1}, \ldots, A_{k_n}) = t^{-n/2} \int_{[0,t]^n} \omega_{0T}\left(\tau^{t_1}_0\left(F^{(j_1)}_{A_{k_1} q_1}(s_1)\right), \ldots, \tau^{t_n}_0\left(F^{(j_n)}_{A_{k_n} q_n}(s_n)\right)\right) \mathrm{d}t_1 \cdots \mathrm{d}t_n.$$

We derive from Corollary 3.6 and Theorem 3.4 that

$$|C_t(j, q, s; A_{k_1}, \ldots, A_{k_n})| \le 2^{8(\overline{n}_V - 1)\sum_{l=1}^n j_l} \left(2^{8\overline{n}_A} D_{\mathfrak{f}}\right)^n n!, \tag{3.49}$$

holds for $t > 0$. Using this bound we further get from (3.48)

$$\sup_{t>0} t^{-n/2} \int_{[0,t]^n} \left|\omega_{0\mathrm{T}}\left(\tau^{t_1}_0 \circ \gamma^+_\lambda(A_{k_1}), \ldots, \tau^{t_n}_0 \circ \gamma^+_\lambda(A_{k_n})\right)\right| \mathrm{d}t_1 \cdots \mathrm{d}t_n$$

$$\le \prod_{l=1}^n \left(2^{8\overline{n}_A} D_{\mathfrak{f}} \sum_{j_l\ge 0} |2^{8(\overline{n}_V - 1)}\lambda|^{j_l} \sum_{q_l\in Q_{j_l}(A_{k_l})} \int_{\Delta_l} \left|G^{(j_l)}_{A_{k_l} q_l}(s_l)\right| \mathrm{d}s_l\right) n!. \tag{3.50}$$

For $|\lambda| \le \tilde{\lambda}_V$ we have (recall Definitions (1.17) and (3.41)),

$$\sum_{j_l\ge 0} |2^{8(\overline{n}_V - 1)}\lambda|^{j_l} \sum_{q_l\in Q_{j_l}(A_{k_l})} \int_{\Delta_l} \left|G^{(j_l)}_{A_{k_l} q_l}(s_l)\right| \mathrm{d}s_l \le 1 + W_{V,A_{k_l}}.$$

By Theorem 3.4, the right hand side of this inequality is finite. Combining this bound with (3.46) and (3.50) we finally obtain

$$\sup_{|\lambda|<\tilde{\lambda}_V,t>0} t^{-n/2}\int_{[0,t]^n}\left|\omega^+_{\lambda\mathrm{T}}\left(\tau^{t_1}_\lambda(A),\dots,\tau^{t_n}_\lambda(A)\right)\right|\,\mathrm{d}t_1\cdots\mathrm{d}t_n\le\left(2^{8\overline{n}_A}D_{\mathfrak{f}}\sum_{k=1}^{K_A}(1+W_{V,A_k})\right)^n n!,$$

which concludes the proof.□

The above proof gives that in Theorem 3.1 one may take

$$C_{V,A}=2^{8\overline{n}_A}D_{V,A}\sum_{j=1}^{K_A}(1+W_{V,A_k})\,. \tag{3.51}$$

For an explicit estimate on W_{V,A_k} we refer the reader to [JOP4].

Proof of Theorem 3.2. Note that

$$\omega^+_\lambda\left((\tilde{A}_t)^n\right)=\sum_{k_1,\dots,k_n=1}^{K_A}t^{-n/2}\int_{[0,t]^n}\omega_{0\mathrm{T}}\left(\tau^{t_1}_0\circ\gamma^+_\lambda(A_{k_1}),\dots,\tau^{t_n}_0\circ\gamma^+_\lambda(A_{k_n})\right)\,\mathrm{d}t_1\cdots\mathrm{d}t_n. \tag{3.52}$$

In the proof of Theorem 3.1 we have established that the power series (3.48) converges uniformly for $|\lambda|\le\tilde{\lambda}_V$ and $t>0$. Suppose first that n is odd. Corollary 3.6 yields that

$$\lim_{t\to\infty}C_t(j,q,s;A_{k_1},\dots,A_{k_n})=0. \tag{3.53}$$

By (3.49) and Part 3 of Theorem 3.4 we can apply the dominated convergence theorem to the s-integration in (3.48) to conclude that each term of this power series vanishes as $t\to\infty$, and so

$$\lim_{t\to\infty}\omega^+_\lambda\left((\tilde{A}_t)^n\right)=0,$$

for $|\lambda|\le\tilde{\lambda}_V$.

If n is even, Corollary 3.6 yields

$$\lim_{t\to\infty}C_t(j,q,s;A_{k_1},\dots,A_{k_n})=\sum_{\pi\in\mathcal{P}_{n/2}}\prod_{i=1}^{n/2}L_0\left(F^{(j_{\pi(2i-1)})}_{A_{k_{\pi(2i-1)}},q_{\pi(2i-1)}}(s_{\pi(2i-1)}),F^{(j_{\pi(2i)})}_{A_{k_{\pi(2i)}},q_{\pi(2i)}}(s_{\pi(2i)})\right)$$

$$=\sum_{\pi\in\mathcal{P}_{n/2}}\int_{\mathbb{R}^{n/2}}\prod_{i=1}^{n/2}\omega_{0T}\left(\tau^{t_i}_0\left(F^{(j_{\pi(2i-1)})}_{A_{k_{\pi(2i-1)}},q_{\pi(2i-1)}}(s_{\pi(2i-1)})\right),F^{(j_{\pi(2i)})}_{A_{k_{\pi(2i)}},q_{\pi(2i)}}(s_{\pi(2i)})\right)\,\mathrm{d}t_1\cdots\mathrm{d}t_{n/2}.$$

The estimate (3.49) (applied in the case $n=2$) yields that

$$\int_{\mathbb{R}}\left|\omega_{0T}\left(\tau^t_0\left(F^{(j)}_{A_k,q}(s)\right),F^{(j')}_{A_{k'},q'}(s')\right)\right|\,\mathrm{d}t\le\left(2^{8\overline{n}_A+1/2}D_{\mathfrak{f}}\right)^2 2^{8(\overline{n}_V-1)(j+j')},$$

from which we obtain

$$\left|\lim_{t\to\infty}C_t(j,q,s;A_{k_1},\dots,A_{k_n})\right|\le\left(2^{8\overline{n}_A+1/2}D_{\mathfrak{f}}\right)^n 2^{8(\overline{n}_V-1)\sum_i j_i}.$$

Arguing as in the previous case we get, for $|\lambda|\le\tilde{\lambda}_V$, the expansion

$$\lim_{t\to\infty}t^{-n/2}\int_{[0,t]^n}\omega_{0\mathrm{T}}\left(\tau^{t_1}_0\circ\gamma^+_\lambda(A_{k_1}),\dots,\tau^{t_n}_0\circ\gamma^+_\lambda(A_{k_n})\right)\mathrm{d}t_1\cdots\mathrm{d}t_{n/2}$$

$$=\sum_{j_1,\dots,j_n\ge0}(\mathrm{i}\lambda)^{j_1+\cdots+j_n}\sum_{q_1\in\mathcal{Q}_{j_1}(A_{k_1}),\cdots,q_n\in\mathcal{Q}_{j_n}(A_{k_n})}\int_{\Delta_{j_1}}\mathrm{d}s_1\cdots\int_{\Delta_{j_n}}\mathrm{d}s_n\left(\prod_{l=1}^n G^{(j_l)}_{A_{k_l},q_l}(s_l)\right) \tag{3.54}$$

$$\sum_{\pi\in\mathcal{P}_{n/2}}\int_{\mathbb{R}^{n/2}}\prod_{i=1}^{n/2}\omega_{0T}\left(\tau^{t_i}_0\left(F^{(j_{\pi(2i-1)})}_{A_{k_{\pi(2i-1)}},q_{\pi(2i-1)}}(s_{\pi(2i-1)})\right),F^{(j_{\pi(2i)})}_{A_{k_{\pi(2i)}},q_{\pi(2i)}}(s_{\pi(2i)})\right)\,\mathrm{d}t_1\cdots\mathrm{d}t_{n/2}.$$

By Fubini's theorem, this can be rewritten as

$$\sum_{\pi\in\mathcal{P}_{n/2}}\int_{\mathbb{R}^{n/2}}\Bigg[\sum_{j_1,\dots,j_n\geq 0}(\mathrm{i}\lambda)^{j_1+\cdots+j_n}\sum_{q_1\in\mathcal{Q}_{j_1}(A_{k_1}),\cdots,q_n\in\mathcal{Q}_{j_n}(A_{k_n})}\int_{\Delta_{j_1}}\mathrm{d}s_1\cdots\int_{\Delta_{j_n}}\mathrm{d}s_n$$
$$\left(\prod_{l=1}^{n}G^{(j_l)}_{A_{k_l},q_l}(s_l)\right)\prod_{i=1}^{n/2}\omega_{0T}\left(\tau_0^{t_i}\left(F^{(j_{\pi(2i-1)})}_{A_{k_{\pi(2i-1)}},q_{\pi(2i-1)}}(s_{\pi(2i-1)})\right),F^{(j_{\pi(2i)})}_{A_{k_{\pi(2i)}},q_{\pi(2i)}}(s_{\pi(2i)})\right)\Bigg]\,\mathrm{d}t_1\cdots\mathrm{d}t_{n/2}.$$

By Expansion (3.42), the expression inside the square brackets is

$$\prod_{i=1}^{n/2}\omega_{0T}\left(\tau_0^{t_i}\circ\gamma_\lambda^+\left(A_{k_{\pi(2i-1)}}\right),\gamma_\lambda^+(A_{k_{\pi(2i)}})\right)=\prod_{i=1}^{n/2}\omega_{\lambda T}^+\left(\tau^{t_i}\left(A_{k_{\pi(2i-1)}}\right),A_{k_{\pi(2i)}}\right),$$

so that, by (3.52),

$$\begin{aligned}\lim_{t\to\infty}\omega_\lambda^+\left((\tilde{A}_t)^n\right)&=\sum_{k_1,\dots,k_n=1}^{K_A}\sum_{\pi\in\mathcal{P}_{n/2}}\prod_{i=1}^{n/2}\left(\int_{\mathbb{R}}\omega_{\lambda T}^+\left(\tau^t\left(A_{k_{\pi(2i-1)}}\right),A_{k_{\pi(2i)}}\right)\mathrm{d}t\right)\\&=\sum_{\pi\in\mathcal{P}_{n/2}}\prod_{i=1}^{n/2}\left(\int_{\mathbb{R}}\omega_{\lambda T}^+\left(\tau^t\left(A\right),A\right)\,\mathrm{d}t\right)\\&=\frac{n!}{2^{n/2}(n/2)!}L(A,A)^{n/2}.\end{aligned}$$

□

4 Proof of Theorem 3.5

For notational simplicity throughout this section we shall drop the subscript 0 and write h for h_0, τ for τ_0, ω for ω_0. We shall also use the shorthand (3.45).

4.1 Graphs, pairings and Pfaffians

An graph is a pair of sets $g=(V,E)$ where E is a set of 2-elements subsets of V. The elements of V are called points or vertices of g, those of E are its edges. Abusing notation, we shall write $v\in g$ for vertices of g and $e\in g$ for its edges. If $v\in e\in g$ we say that the edge e is incident to the vertex v. If the edge e is incident to the vertices u and v we write $e=uv$ and say that the edge e connects u to v. The degree of a vertex $v\in g$ is the number of distinct edges $e\in g$ incident to v. A graph is k-regular if all its vertices share the same degree k. A vertex $v\in g$ of degree 0 is said to be isolated. A path on g is a sequence $(v_0,e_1,v_1,e_2,\dots,e_n,v_n)$ where $v_i\in V$, $e_i\in E$ and $e_i=v_{i-1}v_i$. We say that such a path connects the vertices v_0 and v_n. If $v_0=v_n$ the path is closed and is called a loop. The graph g is connected if, given any pair $v,v'\in V$ there is a path on g which connects v and v'. A connected graph without loops is a tree.

A graph $g'=(V',E')$ is a subgraph of the graph $g=(V,E)$ if $V'\subset V$ and $E'\subset E$. A subgraph g' of g is said to be spanning g if $V'=V$. A connected graph g has a spanning tree *i.e.,* a subgraph which is spanning and is a tree.

Let $g=(V,E)$ be a graph. To a subset $W\subset V$ we associate a subgraph $g_{|W}=(W,E_{|W})$ of g by setting $E_{|W}=\{e=uv\in E\,|\,u,v\in W\}$. Given two graphs $g_1=(V_1,E_1)$ and $g_2=(V_2,E_2)$ such that V_1 and V_2 are disjoint we denote by $g_1\vee g_2$ the joint graph $(V_1\cup V_2,E_1\cup E_2)$.

Let $g=(V,E)$ be a graph and $\Pi=\{V_1,\dots,V_n\}$ a partition of V. The set

$$E/\Pi=\{V_iV_j\,|\,\text{there are } u\in V_i,v\in V_j \text{ such that } uv\in E\}.$$

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 4 & 2 & 6 & 3 & 5 & 7 & 8 \end{pmatrix}$$

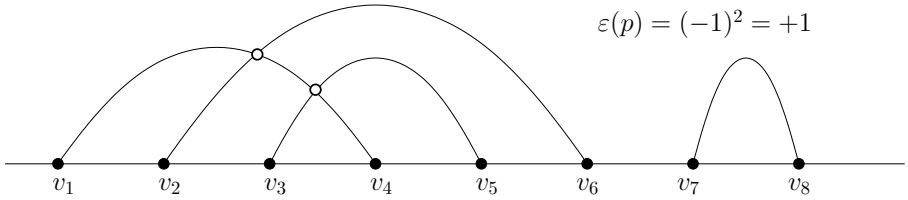

Figure 1: Diagrammatic representation of a pairing p

defines a graph $g/\Pi = (\Pi, E/\Pi)$. We say that g/Π is the Π-skeleton of g.

A graph $g = (V, E)$ is said to be (V_1, V_2)-bipartite if there is a partition $V = V_1 \cup V_2$ such that all edges $e \in E$ connect a vertex of V_1 to a vertex of V_2.

A pairing on a set V is a graph $p = (V, E)$ such that every vertex $v \in V$ belongs to exactly one edge $e \in E$. Equivalently, $p = (V, E)$ is a pairing if E is a partition of V or if it is 1-regular. We denote by $\mathcal{P}(V)$ the set of all pairings on V. Clearly, only sets V of even parity $|V| = 2n$ admit pairings and in this case one has

$$|\mathcal{P}(V)| = \frac{(2n)!}{2^n n!} = (2n-1)!!.$$

If the set $V = \{v_1, \ldots, v_{2n}\}$ is completely ordered, $v_1 < v_2 < \cdots < v_{2n}$, writing

$$E = \{\pi(v_1)\pi(v_2), \pi(v_3)\pi(v_4), \ldots, \pi(v_{2n-1})\pi(v_{2n})\},$$

sets a one-to-one correspondence between pairings $p = (V, E)$ and permutations $\pi \in S_V$ such that $\pi(v_{2i-1}) < \pi(v_{2i})$ and $\pi(v_{2i-1}) < \pi(v_{2i+1})$ for $i = 1, \ldots, n$ (compare with (1.5)). In the sequel we will identify the two pictures and denote by p the permutation of V associated to the pairing p. In particular, the signature $\varepsilon(p)$ of a pairing p is given by the signature of the corresponding permutation. A diagrammatic representation of a pairing $p \in \mathcal{P}(V)$ is obtained by drawing the vertices $v_1, \ldots, v_{2n}$ as $2n$ consecutive points on a line. Each edge $e \in p$ is drawn as an arc connecting the corresponding points above this line (see Figure 1). It is well known that the signature of p is then given by $\varepsilon(p) = (-1)^k$ where k is the total number of intersection points of these arcs.

If $V = V_1 \cup V_2$ is a partition of V into two equipotent subsets we denote by $\mathcal{P}(V_1, V_2) \subset \mathcal{P}(V)$ the corresponding set of (V_1, V_2)-bipartite pairings and note that

$$|\mathcal{P}(V_1, V_2)| = n!.$$

If $V_1 = \{v_1, \ldots, v_n\}$ and $V_2 = \{v_{n+1}, \ldots, v_{2n}\}$ are completely ordered by $v_1 < \cdots < v_n < \cdots < v_{2n}$ then $p(v_{2i-1}) = v_i$ and $\sigma(v_{n+i}) = p(v_{2i})$ for $1 \leq i \leq n$ defines a one-to-one correspondence between bipartite pairings $p \in \mathcal{P}(V_1, V_2)$ and permutations $\sigma \in S_{V_2}$. A simple calculation shows that $\varepsilon(p) = (-1)^{n(n-1)/2}\varepsilon(\sigma)$.

In the special case $V = \{1, \ldots, 2n\}$, $V_1 = \{1, \ldots, n\}$ and $V_2 = \{n+1, \ldots, 2n\}$ we shall set $\mathcal{P}(V) = \mathcal{P}_n$ and $\mathcal{P}(V_1, V_2) = \widetilde{\mathcal{P}}_n$.

The Pfaffian of a $2n \times 2n$ skew-symmetric matrix M is defined by

$$\mathrm{Pf}(M) = \sum_{p \in \mathcal{P}_n} \varepsilon(p) \prod_{i=1}^{n} M_{p(2i-1)p(2i)}.$$

If B is a $n \times n$ matrix and

$$M = \begin{bmatrix} 0 & B \\ -B^T & 0 \end{bmatrix},$$

then only bipartite pairings $p \in \widetilde{\mathcal{P}}_n$ contribute to the Pfaffian of M which reduces to

$$\begin{aligned} \mathrm{Pf}(M) &= \sum_{p\in\widetilde{\mathcal{P}}_n} \varepsilon(p) \prod_{i=1}^{n} B_{p(2i-1)p(2i)} \\ &= \sum_{\sigma\in S_n} (-1)^{n(n-1)/2} \varepsilon(\sigma) \prod_{i=1}^{n} B_{i\sigma(i)} \\ &= (-1)^{n(n-1)/2} \det(B). \end{aligned} \tag{4.55}$$

4.2 Truncating quasi-free expectations

Let $V \subset \mathfrak{h}$ be finite and totally ordered. To any subset $W \subset V$ we assign the monomial

$$\Phi(W) \equiv \prod_{u\in W} \varphi(u),$$

where the product is ordered from left to right in increasing order of the index u.

Let ω be a gauge invariant quasi-free state on $\mathrm{CAR}(\mathfrak{h})$. We define a $|V| \times |V|$ skew-symmetric matrix Ω by setting

$$\Omega_{uv} \equiv \omega(\varphi(u)\varphi(v)),$$

for $u, v \in V$ and $u < v$. We also denote by Ω^W the sub-matrix of Ω whose row and column indices belong to W. Then we have

$$\omega(\Phi(W)) = \begin{cases} \mathrm{Pf}(\Omega^W) & \text{if } |W| \text{ is even,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.56}$$

If $|W|$ is even, assigning to any pairing $p \in \mathcal{P}(W)$ the weight

$$\Omega(p) \equiv \prod_{\substack{uv\in p \\ u<v}} \Omega_{uv},$$

we can rewrite Equ. (4.56) as

$$\omega(\Phi(W)) = \sum_{p\in\mathcal{P}(W)} \varepsilon(p)\Omega(p). \tag{4.57}$$

The following simple lemma is our fundamental tool when dealing with such expansions.

Lemma 4.1 *Let $W_1 = \{u_1, \ldots, u_r\}$ and $W_2 = \{v_1, \ldots, v_s\}$ be disjoint even subsets of V such that $u_1 < u_2 < \cdots < u_r$ and $v_1 < v_2 < \cdots < v_s$. Denote by $\varepsilon(W_1, W_2)$ the signature of the permutation of $W_1 \cup W_2$ which "orders" the sequence W_1W_2 i.e., which maps the sequence $u_1, u_2, \cdots u_r, v_1, v_2, \cdots, v_s$ into the ordered sequence of elements of $W_1 \cup W_2$. Then, for any $p_1 \in \mathcal{P}(W_1)$ and $p_2 \in \mathcal{P}(W_2)$ one has*

$$\Omega(p_1 \vee p_2) = \Omega(p_1)\Omega(p_2), \quad \varepsilon(p_1 \vee p_2) = \varepsilon(W_1, W_2)\varepsilon(p_1)\varepsilon(p_2). \tag{4.58}$$

Proof. The statement about $\Omega(p_1 \vee p_2)$ is obvious. To prove the statement about signatures we draw the following diagram (see Figure 2). Draw two parallel lines and on the top one the two diagrams corresponding to the pairings p_1 and p_2, one next to the other. On the bottom line draw the diagram representing the pairing $p_1 \vee p_2$ but with the edges drawn below the baseline. Now draw segments connecting each point of the top line with its representant on the bottom line. These segments represent the permutation referred to in the Lemma. Thus, if there are q intersection points of these segments then $\varepsilon(W_1, W_2) = (-1)^q$. Denote by j the number of intersection points in our diagram lying above the top line and by j' the number of those intersections points lying below the bottom line. Then, we have $\varepsilon(p_1)\varepsilon(p_2) = (-1)^j$ and $\varepsilon(p_1 \vee p_2) = (-1)^{j'}$. Now observe that our diagram is a disjoint union of closed loops. Thus, it has an even number of intersection points *i.e.,*

$$(-1)^{j+q+j'} = 1,$$

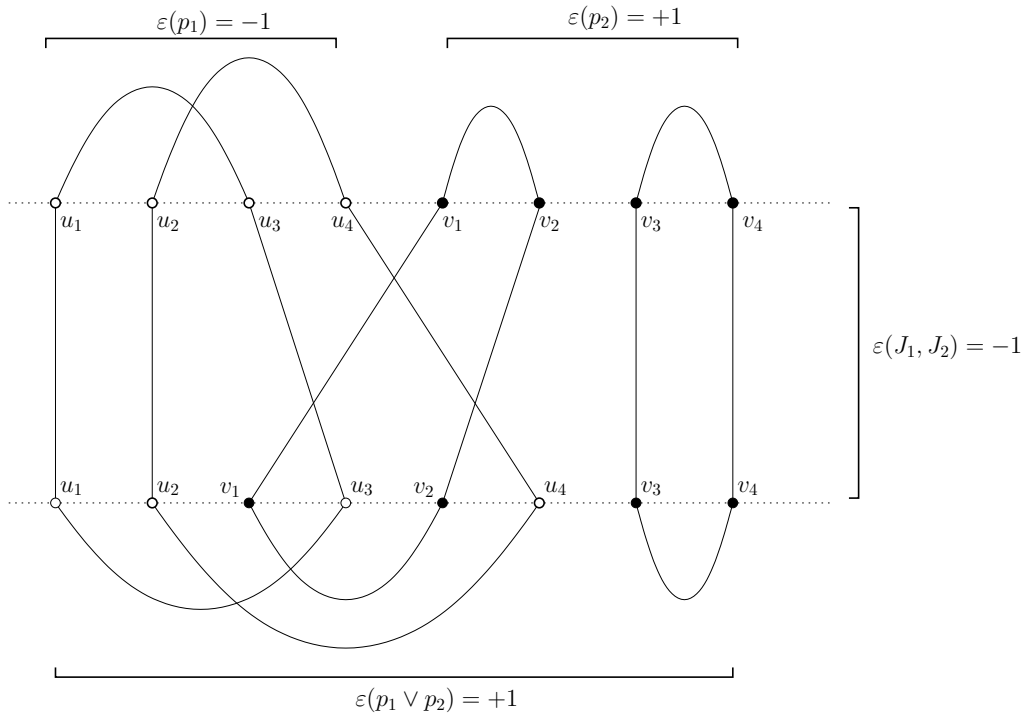

Figure 2: Proof of Lemma 4.1

from which the result follows.□

Iterating Equ. (4.58) we obtain, for disjoint even subsets $W_1, \ldots, W_k \subset V$ and arbitrary pairings $p_i \in \mathcal{P}(W_i)$, the formulas

$$\Omega\left(\bigvee_{i=1}^{k} p_i\right) = \prod_{i=1}^{k} \Omega(p_i), \quad \varepsilon\left(\bigvee_{i=1}^{k} p_i\right) = \varepsilon(W_1, \ldots, W_k) \prod_{i=1}^{k} \varepsilon(p_i),$$

where $\varepsilon(W_1, \ldots, W_k)$ denotes the signature of the permutation which "orders" the sequence $W_1, \ldots, W_k$. Moreover, the recurrence relation

$$\varepsilon(W_1, \ldots, W_k) = \varepsilon(W_1 \cup \cdots \cup W_{k-1}, W_k)\varepsilon(W_1, \ldots, W_{k-1}),$$

holds.

If X, Y are subsets of V we write $X < Y$ whenever $\max(X) < \min(Y)$.

Remark. If $W_1 < W_2 < \cdots < W_k$ it immediately follows from the fact that the W_i are even that $\varepsilon(W_{\pi(1)}, \ldots, W_{\pi(k)}) = 1$ for any permutation $\pi \in S_k$.

Lemma 4.2 *Let $\Pi = (V_1, \ldots, V_n)$ be an ordered partition of V by even subsets i.e.,*

$$V_1 < V_2 < \cdots < V_n, \quad |V_i| \text{ even}.$$

and set $A_i = \Phi(V_i)$. Then one has

$$\omega_T(A_1, \ldots, A_n) = \sum_{p \in \mathcal{P}(\Pi)} \varepsilon(p)\Omega(p), \tag{4.59}$$

where $\mathcal{P}(\Pi)$ denotes the set of pairings $p \in \mathcal{P}(V)$ which have a Π-skeleton p/Π without isolated vertex.

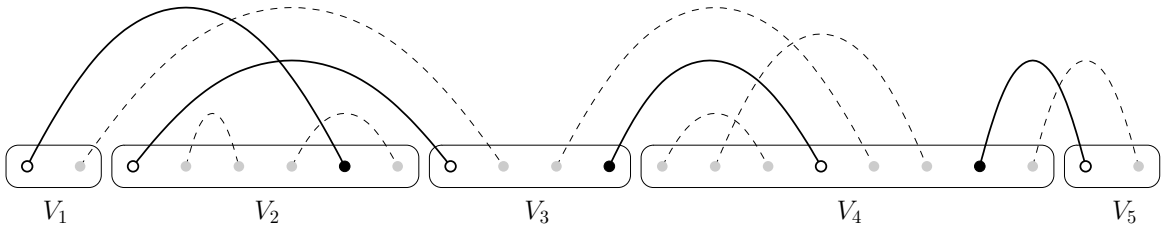

Figure 3: The exit graph $\mathrm{ex}(p)$ (solid lines) for $p \in \mathcal{P}(\Pi)$

Proof. Expanding the left hand side of Equ. (4.59) we get

$$\sum_{K\subset I}(-1)^{|K|}\,\omega(\Phi(\cup_{i\in I\setminus K}V_i))\prod_{i\in K}\omega(\Phi(V_i)),$$

where $I=\{1,\dots,n\}$. Using Lemma 4.1 and the remarks following it, we can rewrite this expression as

$$\sum_{K\subset I}(-1)^{|K|}\sum_{\xi\in\Xi_K}\varepsilon(p(\xi))\,\Omega(p(\xi)), \tag{4.60}$$

where we sum over the sets

$$\Xi_K\equiv\mathcal{P}(\cup_{i\in I\setminus K}V_i)\times\left(\prod_{i\in K}\mathcal{P}(V_i)\right),$$

and, for $\xi=(q,(p_i)_{i\in K})\in\Xi_K$ we have set $p(\xi)=q\vee(\vee_{i\in K}p_i)$. Let us define

$$\mathrm{Is}(p)\equiv\{i\in I\,|\,V_i \text{ is an isolated vertex of } p/\Pi\}.$$

Clearly, if $K\subset I$ and $\xi\in\Xi_K$ then $p(\xi)\in\mathcal{P}(V)$ and $K\subset\mathrm{Is}(p(\xi))$. Reciprocally, suppose that $p\in\mathcal{P}(V)$ and $K\subset\mathrm{Is}(p)$. Then the restricted graphs $q=p_{|W}$ with $W=\cup_{i\in I\setminus K}V_i$ and $p_i=p_{|V_i}$ for $i\in K$ satisfy $\xi=(q,(p_i)_{i\in K})\in\Xi_K$ and $p(\xi)=p$. We conclude that

$$\{(K,p(\xi))\,|\,K\subset I,\xi\in\Xi_K\}=\{(K,p)\,|\,p\in\mathcal{P}(V),K\subset\mathrm{Is}(p)\},$$

and since the map $\xi\mapsto p(\xi)$ is clearly injective we can rewrite the sum (4.60) as

$$\sum_{p\in\mathcal{P}(V)}\varepsilon(p)\Omega(p)\sum_{K\subset\mathrm{Is}(p)}(-1)^{|K|}.$$

The result follows from the fact that the second sum in the last expression vanishes unless $\mathrm{Is}(p)$ is empty.□

4.3 Resummation

The setup in this subsection is the same as in the previous one. We consider a fixed ordered partition $\Pi=(V_1,\dots,V_n)$ of V by even subsets as in Lemma 4.2 and fix our attention on the expansion (4.59) of the truncated correlation.

Consider a fixed term in this expansion *i.e.,* a pairing $p\in\mathcal{P}(\Pi)$. Since its skeleton p/Π has no isolated point, for each $i\in I=\{1,\dots,n\}$ the set of edges of p which connect a vertex in V_i to a vertex outside V_i is not empty. We call exit edge of p from V_i the element of this set which contains the smallest vertex in V_i. The set of all exit edges of p defines a subgraph of p which we denote by $\mathrm{ex}(p)$ (see Figure 3). We also denote by $\mathrm{Ex}(\Pi)=\{\mathrm{ex}(p)\,|\,p\in\mathcal{P}(\Pi)\}$ the set of all exit graphs. We can rewrite expansion (4.59) as

$$\omega_T(A_1,\dots,A_n)=\sum_{g\in\mathrm{Ex}(\Pi)}\sum_{p\in\mathrm{ex}^{-1}(\{g\})}\varepsilon(p)\Omega(p).$$

A given exit graph g can be seen as a pairing on $X(g) = \{u \in V \mid uv \in g \text{ for some } v \in V\}$. Setting $V(g) = V \setminus X(g)$ and applying Lemma 4.1 we get

$$\omega_T(A_1, \ldots, A_n) = \sum_{g \in \mathrm{Ex}(\Pi)} \Omega(g) S(g), \tag{4.61}$$

where

$$S(g) \equiv \sum_{p \in \mathrm{ex}^{-1}(\{g\})} \varepsilon(p) \Omega(p_{|V(g)}). \tag{4.62}$$

Our next result is a partial resummation formula for $S(g)$.

Define the exit point from V_i by $x_i(g) \equiv \min(X(g) \cap V_i)$. We say that

$$\theta = (X, L, M, M', R),$$

is a g-admissible partition of V if X, L, M, M' and R are disjoint subsets of V such that

$$X = X(g), \quad V = X \cup L \cup M \cup M' \cup R,$$

and which, for all $i \in I$, satisfy the following conditions

(1) $\max((L \cup M) \cap V_i) < x_i(g)$;
(2) $\min((R \cup M') \cap V_i) > x_i(g)$;
(3) $|L \cap V_i|$ is even;
(4) $|M \cap V_i| = |M' \cap V_i|$.

If X, Y are two subsets of V denote by $\Omega^{X,Y}$ the sub-matrix of Ω with row (resp. column) indices in X (resp. Y).

Lemma 4.3 *For $g \in \mathrm{Ex}(\Pi)$ one has*

$$S(g) = \sum_{\theta = (X,L,M,M',R) \in \Theta(g)} \varepsilon(\theta) \omega(\Phi(R)) \prod_{i \in I} \Big(\omega(\Phi(L \cap V_i)) \det(\Omega^{M \cap V_i, M' \cap V_i}) \Big), \tag{4.63}$$

where $\Theta(g)$ denotes the set of g-admissible partitions of V and

$$\varepsilon(\theta) \quad \equiv \quad \varepsilon(X, L \cap V_1, \ldots, L \cap V_n, (M \cup M') \cap V_1, \ldots, (M \cup M') \cap V_n, R) \varepsilon(g_{|X}) \prod_{i \in I} (-1)^{|M \cap V_i|(|M \cap V_i| - 1)/2}.$$

Proof. Let us have a closer look at a pairing p whose exit graph is g. What happens in $X_i(g) \equiv V_i \cap X(g)$ is completely determined by g. However, the structure of $p_{|V_i(g)}$ where $V_i(g) \equiv V_i \cap V(g)$ depends on finer details of p. Edges of p which are incident to a vertex in $V_i(g)$ located to the left of the exit point $x_i(g)$ must connect this vertex to another vertex in $V_i(g)$. These edges split in two categories: the ones which connect two vertices on the left of the exit point and the ones which connect a vertex on the left to a vertex on the right. We denote by $L_i(p)$ the set of vertices which belong to an edge of the first category, and by $M_i(p)$ the vertices located to the left of $x_i(g)$ and belonging to an edge of the second one. By $M'_i(p)$ we denote the set of vertices which are connected to elements of $M_i(p)$. This subset of $V_i(g)$ is located on the right of the exit point. We group the remaining vertices of $V_i(g)$, which are all on the right of the exit point, into a fourth set $R_i(p)$. Elements of this set connect among themselves or with elements of $R_j(p)$ for some $j \neq i$ (see Figure 4). Setting

$$L(p) \equiv \bigcup_{i \in I} L_i(p), \quad M(p) \equiv \bigcup_{i \in I} M_i(p), \quad M'(p) \equiv \bigcup_{i \in I} M'_i(p), \quad R(p) \equiv \bigcup_{i \in I} R_i(p),$$

we obtain a partition

$$\theta(p) \equiv (X(g), L(p), M(p), M'(p), R(p)),$$

of V which is clearly g-admissible. Moreover, setting

$$\begin{aligned} l_i(p) \equiv p_{|L(p) \cap V_i} \quad &\in \quad \mathcal{P}(L(p) \cap V_i), \\ m_i(p) \equiv p_{|(M(p) \cup M'(p)) \cap V_i} \quad &\in \quad \mathcal{P}(M(p) \cap V_i, M'(p) \cap V_i), \\ r(p) \equiv p_{|R(p)} \quad &\in \quad \mathcal{P}(R(p)). \end{aligned}$$

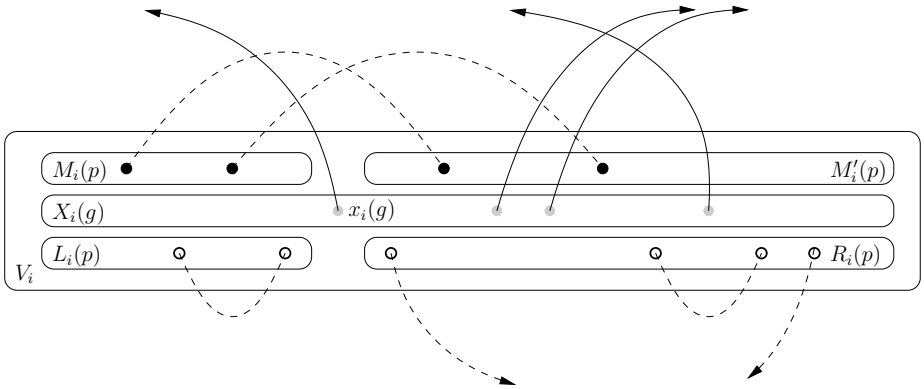

Figure 4: The partition of V_i induced by a pairing p. Solid lines belong to the exit graph $\mathrm{ex}(p)$.

we obtain a map Ψ from $\mathrm{ex}^{-1}(\{g\})$ to the set

$$\bigcup_{\theta=(X,L,M,M',R)\in\Theta(g)} \left[\{\theta\} \times \left(\prod_{i\in I} \mathcal{P}(L\cap V_i)\right) \times \left(\prod_{i\in I} \mathcal{P}(M\cap V_i, M'\cap V_i)\right) \times \mathcal{P}(R)\right].$$

Since

$$p = g \vee \left(\bigvee_{i\in I} l_i(p)\right) \vee \left(\bigvee_{i\in I} m_i(p)\right) \vee r(p),$$

Ψ is injective. For any g-admissible partition $\theta = (X, L, M, M', R)$ and any

$$l_i \in \mathcal{P}(L\cap V_i), \quad m_i \in \mathcal{P}(M\cap V_i, M'\cap V_i), \quad r \in \mathcal{P}(R), \tag{4.64}$$

the pairing

$$p = g \vee \left(\bigvee_{i\in I} l_i\right) \vee \left(\bigvee_{i\in I} m_i\right) \vee r, \tag{4.65}$$

satisfies

$$\mathrm{ex}(p) = g, \quad \theta(p) = \theta, \quad l_i(p) = l_i, \quad m_i(p) = m_i, \quad r(p) = r.$$

We conclude that Ψ is bijective. Thus, using Lemma 4.1, we can rewrite the sum $S(g)$ as

$$\sum_{\theta=(X,L,M,M',R)\in\Theta(g)} \varepsilon(g_{|X})\varepsilon(X, L\cap V_1, \ldots, L\cap V_n, (M\cup M')\cap V_1, \ldots, (M\cup M')\cap V_n, R)$$
$$\prod_{i\in I}\left(\sum_{l_i\in\mathcal{P}(L\cap V_i)} \varepsilon(l_i)\Omega(l_i)\right) \prod_{i\in I}\left(\sum_{m_i\in\mathcal{P}(M\cap V_i, M'\cap V_i)} \varepsilon(m_i)\Omega(m_i)\right) \sum_{r\in\mathcal{P}(R)} \varepsilon(r)\Omega(r).$$

The result now follows from Equ. (4.55) and (4.57). □

4.4 Estimating truncated expectations

Apart from the entropic factor $|\Theta(g)|$, the following Lemma controls the partial sum $S(g)$.

Lemma 4.4 *For $g \in \mathrm{Ex}(\Pi)$ one has*

$$|S(g)| \leq 2^{-|V(g)|/2} |\,\Theta(g)| \prod_{v\in V(g)} \|v\|.$$

Proof. Since

$$\varphi(f)^2 = \frac{1}{2}\{a^*(f), a(f)\} = \frac{1}{2}\|f\|^2,$$

we have, for any $X \subset V$, the simple bound

$$|\omega(\Phi(X))| \leq 2^{-|X|/2} \prod_{v \in X} \|v\|.$$

Combining this estimate with the following Lemma, the result is an immediate consequence of Formula (4.63). □

Lemma 4.5 *Let B be the $k \times k$ matrix defined by $B_{ij} = \omega(\varphi(u_i)\varphi(v_j))$. Then, the estimate*

$$|\det(B)| \leq 2^{-k} \prod_{i=1}^{k} \Big(\|u_i\| \, \|v_i\| \Big),$$

holds.

Proof. Let $\overline{\cdot}$ be a complex conjugation on $\mathfrak{h}$. The real-linear map

$$\begin{array}{rccl} Q\colon & \mathfrak{h} & \to & \mathfrak{h} \oplus \mathfrak{h} \\ & f & \mapsto & (1-T)^{1/2} f \oplus \overline{T}^{1/2} \overline{f}, \end{array}$$

is isometric and such that

$$\omega(\varphi(u_i)\varphi(v_j)) = \frac{1}{2}\Big((u_i, v_j) - (u_i, Tv_j) + \overline{(u_i, Tv_j)} \Big) = \frac{1}{2}(Qu_i, Qv_j).$$

It immediately follows that

$$\det(B) = 2^{-k}\, \omega_{\text{Fock}}(a(Qu_1) \cdots a(Qu_k) a^*(Qv_k) \cdots a^*(Qv_1)),$$

where ω_{Fock} denotes the Fock-vacuum state on $\mathrm{CAR}(\mathfrak{h} \oplus \mathfrak{h})$. The fact that

$$\|a(Qu)\| = \|a^*(Qu)\| = \|Qu\| = \|u\|,$$

for any $u \in \mathfrak{h}$ yields the result.□

For $u, v \in V$ such that $u < v$ set

$$\Delta_{uv} \equiv 2\, \frac{|\omega(\varphi(u)\varphi(v))|}{\|u\| \, \|v\|} = 2\, \frac{|\Omega_{uv}|}{\|u\| \, \|v\|},$$

and for any graph p on V set

$$\Delta(p) \equiv \prod_{\substack{uv \in p \\ u < v}} \Delta_{uv}.$$

Note that Δ_{uv} and hence $\Delta(p)$ take values in the interval $[0, 1]$. The following lemma, which controls the contribution of the exit graph g to the sum (4.61) is immediate.

Lemma 4.6 *For any $p \in \mathcal{P}(W)$*

$$|\Omega(p)| \leq 2^{-|W|/2} \Delta(p) \left(\prod_{w \in W} \|w\| \right).$$

Applying this bound to $g \in \mathrm{Ex}(\Pi)$ and using Lemma 4.4 we finally get from Formula (4.61):

Lemma 4.7 *Under the hypotheses of Lemma 4.2 the following estimate holds*

$$|\omega_T(A_1, \ldots, A_n)| \leq 2^{-|V|/2} \left(\prod_{v \in V} \|v\| \right) \sum_{g \in \mathrm{Ex}(\Pi)} |\Theta(g)| \Delta(g).$$

4.5 Counting exit graphs and their admissible partitions

Lemma 4.8 *For any ordered partition* Π *of* V *one has*

$$|\mathrm{Ex}(\Pi)| \leq 4^{|V|}\,|\Pi|!$$

and for any $g \in \mathrm{Ex}(\Pi)$

$$|\Theta(g)| \leq 4^{|V|}.$$

Proof. We set $|V| = 2N$, $\Pi = (V_1, \ldots, V_n)$ and $|V_i| = k_i$. To construct an exit graph we must first select n exit points $x_i \in V_i$. Thus, there are $k_1 k_2 \cdots k_n$ exit points configurations. Each exit point x_i has now to be paired with a different vertex $y_i \in V$, subject to some constraints. Releasing these constraints we obtain the upper bound $2N(2N-1)\cdots(2N-n+1)$ on the number of such pairings. Thus,

$$|\mathrm{Ex}(\Pi)| \leq 2N(2N-1)\cdots(2N-n+1)k_1\cdots k_n = \binom{2N}{n} k_1 \cdots k_n n!.$$

The result follows from the facts that the binomial coefficient is bounded by 2^{2N} and $k_i \leq 2^{k_i}$.

A g-admissible partition is a partition of $V(g)$ into four sets. Since there are $4^{|V(g)|}$ such partitions the second estimate follows.□

4.6 Proof of Theorem 3.5

To prove Theorem 3.5 we set $A_i = \varphi(\mathrm{e}^{\mathrm{i}t_i h} f_{i1}) \cdots \varphi(\mathrm{e}^{\mathrm{i}t_i h} f_{ik_i})$ and apply Lemma 4.7 to the case

$$V_i \equiv \{\mathrm{e}^{\mathrm{i}t_i h} f_{i1}, \ldots, \mathrm{e}^{\mathrm{i}t_i h} f_{ik_i}\}, \quad i \in I \equiv \{1, \ldots, n\}.$$

We set $2N = |V| = \sum_i k_i$ and obtain

$$\int_{[0,t]^n} |\omega_T(A_1, \ldots, A_n)|\, \mathrm{d}t_1 \cdots \mathrm{d}t_n \leq 2^{-N} \left(\prod_{v \in V} \|v\| \right) \sum_{g \in \mathrm{Ex}(\Pi)} |\Theta(g)|\, C(g), \tag{4.66}$$

where

$$C(g) \equiv \int_{[0,t]^n} \Delta(g)\, \mathrm{d}t_1 \cdots \mathrm{d}t_n.$$

Lemma 4.9 *Let* g *be a graph with vertex set* V*. Denote by* $N_c(g)$ *the number of connected components of its skeleton* g/Π*. Then one has*

$$\int_{[0,t]^n} \Delta(g)\, \mathrm{d}t_1 \cdots \mathrm{d}t_n \leq C^{n - N_c(g)} t^{N_c(g)}, \tag{4.67}$$

with

$$C \equiv \max\left(1, \max_{ijkl} \frac{2}{\|f_{ij}\|\,\|f_{kl}\|} \int_{-\infty}^{\infty} |\omega(\varphi(f_{ij})\tau^t(\varphi(f_{kl})))|\, \mathrm{d}t\right).$$

Proof. Assume first that the skeleton g/Π is connected. Then it has a spanning tree (Π, T). Fix a root V_r in T and for $j \in I \setminus \{r\}$ let $V_{l(j)}$ be the parent of V_j in T. Let $\pi \in S_n$ be a relabeling of the vertices of T such that $\pi(r) = 1$ and $\pi(l(j)) < \pi(j)$ for $j \in I \setminus \{r\}$. Define new variables by $s_j = t_j - t_{l(j)}$ for $j \in I \setminus \{r\}$ and $s_r = t_r$. The corresponding Jacobian matrix is $J_{ij} = \delta_{ij} - (1 - \delta_{ir})\delta_{l(i)j}$. By our choice of the relabeling π the reordered matrix

$$J'_{ij} = J_{\pi^{-1}(i)\pi^{-1}(j)} = \delta_{ij} - (1 - \delta_{i1})\delta_{\pi(l(\pi^{-1}(i)))j},$$

is lower triangular with ones on the diagonal. Thus the Jacobian determinant is given by $|\det J| = |\det J'| = 1$.

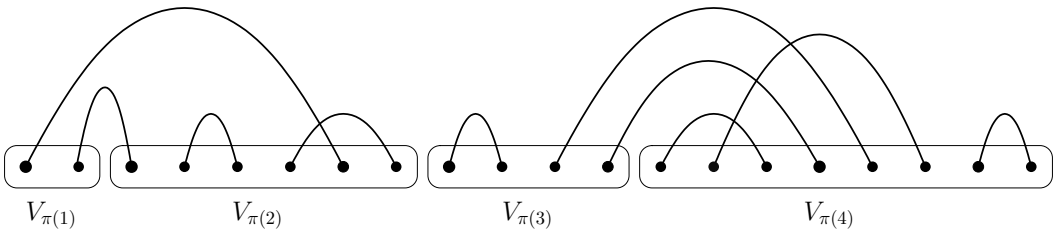

Figure 5: The pairing π induced by a maximally disconnected pairing p.

For each edge $V_j V_{l(j)} \in T$ there is a corresponding edge $e_j = u_j v_j \in g$ with $u_j = \mathrm{e}^{\mathrm{i}t_j h} f_{j,a_j} \in V_j$ and $v_j = \mathrm{e}^{\mathrm{i}t_{l(j)} h} f_{l(j)b_j} \in V_{l(j)}$ and therefore a factor

$$\Delta_{e_j}(s_j) = \frac{2}{\|f_{ja_j}\| \, \|f_{l(j)b_j}\|} \begin{cases} |\omega(\varphi(f_{ja_j})\tau^{-s_j}(\varphi(f_{l(j)b_j})))| & \text{for } j < l(j), \\ |\omega(\varphi(f_{l(j)b_j})\tau^{s_j}(\varphi(f_{ja_j})))| & \text{for } j > l(j), \end{cases},$$

in $\Delta(g)$. It follows that

$$\Delta(g) \leq \prod_{j \in I \setminus \{r\}} \Delta_{e_j}(s_j).$$

and hence

$$\int_{[0,t]^n} \Delta(g) \, \mathrm{d}t_1 \cdots \mathrm{d}t_n \leq \int_0^t \left(\prod_{j \in I \setminus \{r\}} \int_{-t}^t \Delta_{e_j}(s_j) \mathrm{d}s_j \right) \mathrm{d}s_r \leq C^{n-1} t.$$

In the general case, g/Π is the disjoint union of $N_c(g)$ connected subgraphs. Applying the above estimate to each of them yields the result.□

Inserting the estimate (4.67) into Equ. (4.66) and using Lemma 4.8 we finally obtain, taking into account the fact that the skeleton of an exit graph can have at most $n/2$ connected components

$$\int_{[0,t]^n} |\omega_T(A_1, \ldots, A_n)| \, \mathrm{d}t_1 \cdots \mathrm{d}t_n \leq \left(8\sqrt{2} \max_{ij} \|f_{ij}\| \right)^{2N} C^n t^{n/2} n!,$$

which concludes the proof of Part 1.

To prove part 2 it suffices to notice that if n is odd then the skeleton of an exit graph can have at most $(n-1)/2$ connected components.

To prove part 3, we go back to Formula (4.59) and write

$$t^{-n/2} \int_{[0,t]^n} \omega_T(A_1, \ldots, A_n) \, \mathrm{d}t_1 \cdots \mathrm{d}t_n = \sum_{p \in \mathcal{P}(\Pi)} \varepsilon(p) \, t^{-n/2} \int_{[0,t]^n} \Omega(p) \, \mathrm{d}t_1 \cdots \mathrm{d}t_n. \tag{4.68}$$

By Lemmata 4.6 and 4.9 one has, as $t \to \infty$,

$$t^{-n/2} \int_{[0,t]^n} \Omega(p) \, \mathrm{d}t_1 \cdots \mathrm{d}t_n = O(t^{N_c(p) - n/2}).$$

Thus, the pairings $p \in \mathcal{P}(\Pi)$ which contribute to the limit $t \to \infty$ are maximally disconnected in the sense that their skeleton have exactly $n/2$ connected components. The skeleton p/Π of such a pairing induces a pairing $\pi \in \mathcal{P}_{n/2}$ such that

$$p = p_1 \vee \cdots \vee p_{n/2}, \quad p_j \in \mathcal{P}_0(V_{\pi(2j-1)}, V_{\pi(2j)}),$$

where $\mathcal{P}_0(V_i, V_j)$ denotes the set of pairings on $V_i \cup V_j$ whose skeleton w.r.t. the partition (V_i, V_j) has no isolated vertex (see Figure 5). Since the map $p \mapsto (\pi, p_1, \ldots, p_{n/2})$ is clearly bijective we can, for the purpose of computing the limit of (4.68) as $t \to \infty$, replace $\omega_T(A_1, \ldots, A_n)$ by

$$\sum_{\pi \in \mathcal{P}_{n/2}} \sum_{p_j \in \mathcal{P}_0(V_{\pi(2j-1)}, V_{\pi(2j)})} \varepsilon(p_1 \vee \cdots \vee p_{n/2})\, \Omega(p_1 \vee \cdots \vee p_{n/2}).$$

By Lemma 4.1 we have

$$\varepsilon(p_1 \vee \cdots \vee p_{n/2}) = \varepsilon(V_{\pi(1)}, \ldots, V_{\pi(n)})\varepsilon(p_1) \cdots \varepsilon(p_{n/2}), \qquad \Omega(p_1 \vee \cdots \vee p_{n/2}) = \Omega(p_1) \cdots \Omega(p_{n/2}),$$

and by the remark following it $\varepsilon(V_{\pi(1)}, \ldots, V_{\pi(n)}) = 1$. Thus, the last expression can be rewritten as

$$\sum_{\pi \in \mathcal{P}_{n/2}} \prod_{j=1}^{n/2} \left(\sum_{p_j \in \mathcal{P}_0(V_{\pi(2j-1)}, V_{\pi(2j)})} \varepsilon(p_j)\Omega(p_j) \right).$$

Finally observe that, by Lemma 4.2,

$$\sum_{p \in \mathcal{P}_0(V_i, V_j)} \varepsilon(p)\Omega(p) = \omega_T(A_i, A_j).$$

One easily concludes the proof by the remark following Theorem 3.5 and the dominated convergence theorem.

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