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# A numerical method for spatial diffusion in age-structured populations

Caterina CUSULIN <sup>\*</sup> and Luca GERARDO-GIORDA <sup>†</sup>

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## Abstract

We propose a method to approximate numerically the diffusion of an age-structured population in a spatial environment. We integrate separately the age and time variables by finite differences and we discretize the space variable by finite elements. The method is implicit in time and, inside each time step, implicit in age. We provide stability and convergence results and we illustrate our approach with some numerical result.

## 1 Introduction

Modeling the dynamics of a population involves considerations on a great number of features of the population itself. In particular, empirical evidence suggests that both the spatial diffusion of individuals and the internal heterogeneity of the population have to be taken into account. In this direction, over the last decades models for the diffusion of structured populations have been formulated and analyzed, focusing especially on the case of age-structured populations.

The mathematical problem describing the spreading of an age-structured population in a bounded region  $\Omega \in \mathbb{R}^d$  ( $d = 1, 2, 3$ ) consists in a reaction-diffusion equation for the population density, together with a given initial condition, an integral condition at age  $a = 0$ , giving the newborns rate, and boundary conditions on  $\partial\Omega$  depending on the specific features of the population and of the environment (an homogeneous Neumann boundary condition is used to model  $\Omega$  as an isolated environment, while an homogeneous Dirichlet boundary condition models an hostile habitat at the boundary of  $\Omega$ ). For an almost complete review of the results concerning existence, uniqueness and asymptotic behaviour of the solution of age-structured diffusion models, we refer the interested reader to the book by A. Okubo and S.A. Levin ([9], Sec.10.8).

Classical approaches to the numerical solution of diffusion problems in age-structured population dynamics integrate along characteristics in age and time (see for instance [4, 5, 8]). Such approach relies historically on the fact that the earliest age-structured models did not include a spatial distribution of the population density (see e.g. [3]): under the hypothesis of space homogeneity, indeed, the problem reduces to a pure first

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<sup>\*</sup>Faculty of Mathematics, University of Vienna - Austria. This author's work was supported by the project WWTF Mathematics and Evolution.

<sup>†</sup>Department of Mathematics and Computer Science, Emory University, Atlanta - USA.

order hyperbolic partial differential equation, which is naturally solved by integration along characteristics. The simultaneous discretization in age and time, peculiar of such method, forces the time and age steps to be equal. However, the presence of different time scales in the dynamics (which is typically the case when space is involved) suggests the use of different steps in the discretization of time and age, as done by A. de Roos in [2]. This is also the approach followed by B. Ayati *et al.* in [1], where the time variable is left continuous, the age domain is the positive real axis, and an approximation space in age is built by discontinuous piecewise polynomials subspaces of  $L^2(\mathbb{R}^+)$  moving along characteristic lines.

In this paper we present a method where the age and time variables are decoupled and discretized separately by finite differences, while the space variable is discretized by finite elements. The proposed method is implicit in time and, inside each time step, implicit in age.

The paper is organized as follows. In section 2 we briefly recall the linear model we are dealing with. In section 3 we describe the time discretization and we give an energy estimate for the time discretized solution. In section 4 we introduce the age and space discretization. Section 5 contains the stability and convergence analysis of the method. Finally, in section 6 we outline the algorithmic aspects of the procedure, and we present some numerical results to illustrate our method.

## 2 Model description

We consider an age-structured population diffusing in a bounded spatial domain  $\Omega \subset \mathbb{R}^d$ ,  $d = 1, 2, 3$ , with boundary  $\partial\Omega \in C^2$ . The density per unit space and age of the population at time  $t$  is usually denoted by  $p(t, a, x)$ , where  $a \in [0, a_+]$  and  $x \in \Omega$ , thus the total population at time  $t$  is then given by

$$P(t) = \int_{\Omega} \int_0^{a_+} p(t, a, x) da dx.$$

With these notations, given  $T > 0$ , the population density  $p(t, a, x) \in C(0, T; L^1(0, a_+; H^1(\Omega)))$  satisfies the linear model problem

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} + \mu(a)p - \operatorname{div}(k(a, x)\nabla p) = 0 \quad \text{in } (0, T) \times (0, a_+) \times \Omega, \quad (2.1)$$

$$p(0, a, x) = p_0(a, x) \quad \text{in } (0, a_+) \times \Omega, \quad (2.2)$$

$$p(t, 0, x) = \int_0^{a_+} \beta(a)p(t, a, x) da \quad \text{in } (0, T) \times \Omega, \quad (2.3)$$

$$\mathbf{n} \cdot k(a, x)\nabla p = 0 \quad \text{on } (0, T) \times (0, a_+) \times \partial\Omega, \quad (2.4)$$

where the operators  $\operatorname{div}(\cdot)$  and  $\nabla(\cdot)$  are the standard divergence and gradient operators in  $\Omega$ , and  $\mathbf{n}$  is the unit vector normal to  $\partial\Omega$  pointing outwards, while  $\mu(a)$  and  $\beta(a)$  represent the age-specific mortality and the age-specific fertility, respectively, which are supposed to be non-negative functions of age only. In (2.2)  $p_0$  is the given non-negative initial age distribution. The integral condition (2.3) is the so-called renewal condition, providing the newborns rate. Finally, the zero-flux boundary condition (2.4) reflects the absence of both immigration and emigration.

We assume that the age-specific mortality  $\mu(\cdot)$  is a measurable function, satisfying

$$\int_0^{a_+} \mu(\sigma) d\sigma = +\infty, \quad (2.5)$$

in order to guarantee that the probability for an individual to survive at age  $a$ , which is defined as

$$\pi(a) = e^{-\int_0^a \mu(\sigma) d\sigma}, \quad (2.6)$$

vanishes at the maximum age  $a_+$ .

Concerning the age-specific fertility  $\beta(\cdot)$ , we assume that it is measurable and essentially bounded, namely there exists a constant  $\beta_+$  such that

$$0 \leq \beta(a) \leq \beta_+.$$

Finally, concerning diffusion, we impose the standard conditions on  $k$

$$k \in L^\infty((0, a_+) \times \Omega), \quad 0 < k_0 \leq k(a, x) \leq k_+.$$

We refer again to [9] for issues concerning existence and uniqueness of a nonnegative solution of (2.1)-(2.4). The presence of the unbounded coefficient  $\mu(a)$  entails some difficulties at the numerical level. In order to avoid this major drawback of the model, we can rewrite the problem of finding  $p(t, a, x)$ , the solution of (2.1)-(2.4), by performing a standard change of variable. By taking  $p(t, a, x) = \pi(a)u(t, a, x)$ , where  $\pi(a)$  is the survival probability defined in (2.6), we are led to the problem of finding  $u(t, a, x) \in C(0, T; L^1(0, a_+; H^1(\Omega)))$  such that

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} - \operatorname{div} (k(a, x) \nabla u) = 0 \quad \text{in } (0, T) \times (0, a_+) \times \Omega, \quad (2.7)$$

$$u(0, a, x) = u_0(a, x) \quad \text{in } (0, a_+) \times \Omega, \quad (2.8)$$

$$u(t, 0, x) = \int_0^{a_+} m(a) u(t, a, x) da \quad \text{in } (0, T) \times \Omega, \quad (2.9)$$

$$\mathbf{n} \cdot k(a, x) \nabla u = 0 \quad \text{on } (0, T) \times (0, a_+) \times \partial\Omega, \quad (2.10)$$

where now  $u_0(a, x) = \frac{p_0(a, x)}{\pi(a)}$  and  $m(a) = \beta(a)\pi(a)$  is the so called maternity function. Notice that  $m \in L^\infty(0, a_+)$  since for all  $a \in (0, a_+)$  we have  $m(a) \leq \beta_+$ .

We focus here on the numerical treatment of the problem and we assume throughout the paper existence and uniqueness of smooth, nonnegative solutions. In that order, notice that the assumption on the mortality function (2.5) is satisfied by most applications. However, there are cases in which (2.5) is not satisfied: we address at the end of section 4 how the method presented hereafter can be adapted to such situations.

### 3 Time discretization

As already pointed out in the introduction, the presence of different time scales suggests the use of different steps in the discretization of time and age (see [1]).

In this direction, let

$$t^n = n\Delta t \quad n = 0, 1, \dots, N_t$$

be a partition of the interval  $(0, T)$  into  $N_t$  subintervals of length  $\Delta t = T/N_t$  (for simplicity in presentation we consider an uniform discretization, adaptivity in time being beyond the scope of this paper). We use a modified backward Euler scheme where the initial condition at age  $a = 0$  is treated explicitly. For sake of simplicity in notations, we denote the age-space domain by  $Q = ((0, a_+) \times \Omega) \subset \mathbb{R}^{d+1}$ , and we introduce the

differential operators  $\mathbf{div}_Q$  and  $\nabla_Q$  as being the standard divergence and gradient operators in  $Q$ .

With these positions, at time level  $t^n$  ( $n \geq 1$ ), we look for  $u^n \in L^1(0, a_\dagger; H^1(\Omega))$  such that

$$\begin{cases} \frac{u^n - u^{n-1}}{\Delta t} + \mathbf{e}_1 \cdot \nabla_Q u^n - \mathbf{div}_Q \left[ \begin{pmatrix} 0 & 0 \\ 0 & k(a, x) \end{pmatrix} \nabla_Q u^n \right] = 0 & \text{in } Q \\ u^n(0, x) = \int_0^{a_\dagger} m(a) u^{n-1}(a, x) da & \text{in } \Omega \\ \mathbf{n} \cdot k(a, x) \nabla u^n = 0 & \text{on } (0, a_\dagger) \times \partial\Omega, \end{cases} \quad (3.1)$$

where  $u^0(a, x) = u_0(a, x)$  in  $Q$ , and where  $\mathbf{e}_1 = (1, 0, \dots, 0)$  is the first element of the canonical basis in  $\mathbb{R}^{d+1}$ .

The parabolic (in age and space) problem of reaction-diffusion type (3.1) can be recast in variational form by integrating over the spatial domain  $\Omega$  in the following way.

Given  $u^0 \in L^1(0, a_\dagger; H^1(\Omega))$ , for all  $n = 1, \dots, N_t$ , find  $u^n \in L^1(0, a_\dagger; H^1(\Omega))$  such that

$$\begin{cases} \frac{d}{da} \langle u^n, v \rangle + b(a; u^n, v) + \frac{1}{\Delta t} \langle u^n, v \rangle = \frac{1}{\Delta t} \langle u^{n-1}, v \rangle & \forall v \in H^1(\Omega) \\ u^n(0, x) = \int_0^{a_\dagger} m(a) u^{n-1}(a, x) da. \end{cases} \quad (3.2)$$

Here  $\langle \cdot, \cdot \rangle$  is the duality pairing between  $H^1(\Omega)$  and  $H^{-1}(\Omega)$ ,  $(\cdot, \cdot)$  is the inner product in  $L^2(\Omega)$ , and the bilinear form  $b(a; u, v)$  is given by

$$b(a; u, v) = \int_{\Omega} k(a, x) \nabla u \cdot \nabla v dx.$$

In the following we denote by  $\|\cdot\|_0$  and  $\|\cdot\|_1$  the usual  $L^2(\Omega)$  and  $H^1(\Omega)$  norms.

**Remark 3.1** *The coerciveness and the continuity of the bilinear form  $b(a; \cdot, \cdot) + \frac{1}{\Delta t}(\cdot, \cdot)$  are straightforward. Moreover the fact that the maternity function  $m \in L^\infty(0, a_\dagger)$  guarantees that  $u^n(0, x) \in L^2(\Omega)$  as long as  $u^{n-1} \in L^2(Q)$ .*

By standard coerciveness arguments one can prove existence and uniqueness for the solution of (3.2), as stated in the following proposition.

**Proposition 3.1** *If  $u^n(0) \in L^2(\Omega)$ , then for any  $n = 1, \dots, N_t$  there exists a unique solution  $u^n \in L^2(0, a_\dagger; H^1(\Omega)) \cap C^0([0, a_\dagger]; L^2(\Omega))$  to problem (3.2), with  $\frac{\partial u^n}{\partial a} \in L^2(0, a_\dagger; H^{-1}(\Omega))$ . Moreover, for each  $a \in [0, a_\dagger]$ , the following energy estimate holds*

$$\|u^n(a)\|_0^2 + 2\alpha \int_0^a \|u^n(\sigma)\|_1^2 d\sigma + \frac{1}{\Delta t} \int_0^a \|u^n(\sigma)\|_0^2 d\sigma \leq \|u^n(0)\|_0^2 + \frac{1}{\Delta t} \int_0^a \|u^{n-1}(\sigma)\|_0^2 d\sigma, \quad (3.3)$$

where  $\alpha$  is the coerciveness constant of the bilinear form  $b(a; \cdot, \cdot)$ .

If  $u^n(0) \in H^1(\Omega)$  and  $k(a, x) \in C^1(\overline{Q})$ , then  $u^n \in L^\infty(0, a_\dagger; H^1(\Omega)) \cap H^1(0, a_\dagger; L^2(\Omega))$  and the following energy estimate holds

$$\|u^n(a)\|_1^2 + \frac{1}{\Delta t} \|u^n(a)\|_0^2 + \int_0^a \left\| \frac{\partial u^n}{\partial a}(a) \right\|_0^2 da \leq C \left( \|u^n(0)\|_1^2 + \frac{1}{\Delta t} \|u^n(0)\|_0^2 + \frac{1}{(\Delta t)^2} \int_0^a \|u^{n-1}(a)\|_0^2 da \right), \quad (3.4)$$

where the constant  $C_\alpha > 0$  is independent of  $a_\dagger$ .

**Proof** For sake of simplicity, we denote, throughout the proof, by subindex the partial derivative with respect to  $a$ , *i.e* we let  $u_a^n = \frac{\partial u^n}{\partial a}$ . For any fixed  $a$ , we have, for all  $v \in H^1(\Omega)$ ,

$$(u_a^n, v) + b(a; u^n, v) + \frac{1}{\Delta t}(u^n, v) = \frac{1}{\Delta t}(u^{n-1}, v). \quad (3.5)$$

Choosing  $v = u^n$  in (3.5) we get

$$\frac{1}{2} \frac{d}{da} \|u^n\|_0^2 + b(a; u^n, u^n) + \frac{1}{\Delta t} \|u^n\|_0^2 = \frac{1}{\Delta t} (u^{n-1}, u^n),$$

and, owing to the coercivity of  $b(a; \cdot, \cdot)$ , we have

$$\frac{1}{2} \frac{d}{da} \|u^n\|_0^2 + \alpha \|u^n\|_1^2 + \frac{1}{2} \frac{1}{\Delta t} \|u^n\|_0^2 \leq \frac{1}{2} \frac{1}{\Delta t} \|u^{n-1}\|_0^2. \quad (3.6)$$

Integrating (3.6) with respect to  $a$  we easily get (3.3).

We now turn to estimate (3.4), and in that order we introduce the bilinear form

$$b_a(a; \phi, \psi) := \int_{\Omega} k_a(a, x) \nabla \phi \nabla \psi \, dx, \quad \forall \phi, \psi \in H^1(\Omega).$$

From the boundedness of the coefficients and their derivatives, it follows that there exists a positive constant  $\gamma > 0$  such that

$$|b_a(a; \phi, \psi)| \leq \gamma \|\phi\|_1 \|\psi\|_1, \quad \forall \phi, \psi \in H^1(\Omega). \quad (3.7)$$

Choosing  $v = u_a^n$  in (3.5) we get

$$\|u_a^n\|_0^2 + b(a; u^n, u_a^n) + \frac{1}{\Delta t} (u^n, u_a^n) = \frac{1}{\Delta t} (u^{n-1}, u_a^n).$$

Since

$$b(a; u^n, u_a^n) = \frac{1}{2} \frac{d}{da} b(a; u^n, u^n) - \frac{1}{2} b_a(a; u^n, u^n),$$

we have from (3.7), for all  $\varepsilon > 0$ ,

$$\|u_a^n\|_0^2 + \frac{1}{2} \frac{d}{da} b(a; u^n, u^n) + \frac{1}{2} \frac{d}{da} \frac{1}{\Delta t} \|u^n\|_0^2 \leq \frac{1}{\Delta t} \varepsilon \|u_a^n\|_0^2 + \frac{1}{\Delta t} \frac{1}{4\varepsilon} \|u^{n-1}\|_0^2 + \gamma \|u^n\|_1^2.$$

By choosing  $\varepsilon < \Delta t/2$ , we have

$$\frac{1}{2} \|u_a^n\|_0^2 + \frac{1}{2} \frac{d}{da} b(a; u^n, u^n) + \frac{1}{2} \frac{d}{da} \frac{1}{\Delta t} \|u^n\|_0^2 \leq C \left( \|u^n\|_1^2 + \frac{1}{(\Delta t)^2} \|u^{n-1}\|_0^2 \right).$$

Integrating with respect to  $a$ , and using (3.3) we conclude the proof.  $\square$

## 4 Age and space discretization

We use a Galerkin finite element method in space to approximate the solution of (3.2). Let then  $\mathcal{T}_h$  be a regular triangulation of  $\Omega$ , namely  $\Omega = \bigcup_{j=1}^N K_j$ , where each  $K_j = T_{K_j}(E) \in \mathcal{T}_h$ ,  $E$  being the reference element, a simplex (namely the triangle with vertices  $(0, 0)$ ,  $(1, 0)$ , and  $(0, 1)$  when  $d = 2$  or the tetrahedron with vertices  $(0, 0, 0)$ ,  $(1, 0, 0)$ ,  $(0, 1, 0)$ , and  $(0, 0, 1)$  when  $d = 3$ ), or the unit cube  $[0, 1]^d$ , and where  $T_{K_j}$  is an invertible affine map. We define  $h$  as the maximum diameter of the elements of the triangulation, and

we assume the triangulation to be regular. The associated finite element spaces  $X_h$  and  $Y_h$  (see e.g. [10] for an introduction to finite element methods) are defined as

$$X_h = \left\{ \varphi_h \in C^0(\Omega) \mid \varphi_h|_{K_j} \circ T_{K_j} \in \mathbb{P}_1(E) \right\}, \quad Y_h = \left\{ \varphi_h \in C^0(\Omega) \mid \varphi_h|_{K_j} \circ T_{K_j} \in \mathbb{Q}_1(E) \right\},$$

where  $\mathbb{P}_1(E)$  is the space of polynomials of degree at most one on  $E$ , whereas  $\mathbb{Q}_1(E)$  is the space of polynomials of degree at most one with respect to each variable on  $E$ .

A semi-discrete problem in space is then obtained by applying a Galerkin procedure and choosing the finite dimensional space  $V_h = X_h$  (notice that the choice  $V_h = Y_h$  would lead to the same results we present in the following).

From now on and throughout this section, we omit the dependence on the  $x$  variable for all the functions involved. Since the finite element space does not depend on age, problem (3.2) can be rewritten as follows.

Given  $u_0(a, x)$ , for all  $n = 1, \dots, N_t$ , find  $u_h^n \in L^2(0, a_\dagger; V_h)$  such that

$$\begin{cases} \Delta t \frac{d}{da} (u_h^n(a), v_h) + \Delta t b(a; u_h^n(a), v_h) + (u_h^n(a), v_h) = (u_h^{n-1}(a), v_h) & \forall v_h \in V_h \\ u_h^n(0) = \int_0^{a_\dagger} m(a) u_h^{n-1}(a) da, \end{cases} \quad (4.1)$$

where  $u_h^0 = \pi_h u_0(a, x)$ ,  $\pi_h$  being the interpolation operator on  $V_h$ .

Owing to (4.1), we introduce the elliptic projection  $\Pi_{1,h} : H^1(\Omega) \rightarrow V_h$ , defined, for each  $w \in H^1(\Omega)$ , as

$$\Pi_{1,h} w \in V_h : \quad \Delta t b(a; \Pi_{1,h} w, v_h) + (\Pi_{1,h} w, v_h) = \Delta t b(a; w, v_h) + (w, v_h) \quad \forall v_h \in V_h.$$

As, at each age  $a \in (0, a_\dagger)$ , the bilinear form  $\mathcal{A}(\cdot, \cdot) := \Delta t b(a; \cdot, \cdot) + (\cdot, \cdot)$  is symmetric, the operator  $\Pi_{1,h}$  is actually an orthogonal projection onto  $V_h$  with respect to the scalar product  $\mathcal{A}(\cdot, \cdot)$ , and satisfies

$$\|v - \Pi_{1,h} v\|_0 \leq Ch |v|_1 \quad \forall v \in H^1(\Omega), \quad (4.2)$$

where  $|\cdot|_1$  denotes the  $H^1$  seminorm in  $\Omega$ .

Let  $\{\varphi_j\}_{j=1, \dots, N_h}$  be the nodal basis of the finite element space  $V_h$ . The semi-discrete solution  $u_h^n(a, x)$  is thus given by

$$u_h^n(a, x) = \sum_{j=1}^{N_h} u_j^n(a) \varphi_j(x).$$

Denoting by  $\mathbf{u}^n(a) = (u_1^n(a), \dots, u_{N_h}^n(a))^T$ , equation (4.1) can be rewritten as

$$\Delta t M \frac{d\mathbf{u}^n}{da}(a) + [\Delta t B(a) + M] \mathbf{u}^n(a) = M \mathbf{u}^{n-1}(a),$$

where  $B(a)$  and  $M$  are the stiffness and the mass matrices, defined as

$$B_{ij}(a) = \int_{\Omega} k(a, x) \nabla \varphi_j \nabla \varphi_i dx \quad M_{ij} = \int_{\Omega} \varphi_j \varphi_i dx. \quad (4.3)$$

We advance in age problem (4.1) by means of a backward Euler scheme. Let

$$a^m = m \Delta a \quad m = 0, 1, \dots, N_a$$

be a partition of the age interval  $[0, a_\dagger]$  into  $N_a$  subintervals of amplitude  $\Delta a = a_\dagger / N_a$ . The fully discrete approximation of (2.7)-(2.10) reads as follows.

Given  $u_h^0$ , for  $n = 1, \dots, N_t$ , solve

$$\left\{ \begin{array}{l} u_h^{n,0} = \sum_{m=1}^{N_a} \Delta a [m(a^m) u_h^{n-1,m}] \\ \text{for } m = 1, \dots, N_a, \text{ find } u_h^{n,m} \in V_h \text{ such that} \\ \frac{\Delta t}{\Delta a} (u_h^{n,m} - u_h^{n,m-1}, v_h) + \Delta t b(a^m; u_h^{n,m}, v_h) + (u_h^{n,m}, v_h) = (u_h^{n-1,m}, v_h) \quad \forall v_h \in V_h \end{array} \right. \quad (4.4)$$

**Remark 4.1** *The initial condition in age is computed, at each time step, by a suitable quadrature formula. In the next section we prove stability and convergence of the method using the midpoint rule in (4.4), whereas in the numerical tests we use a Simpson formula over two adjacent intervals.*

**Remark 4.2** *System (4.1) is obtained by advancing in age problem (4.1) by means of a backward Euler scheme. Such choice relies on issues of simplicity in presentation. Indeed, inside each time step, the problem to be solved is parabolic in age and space and second order approximation in age can be achieved by means of a Crank-Nicholson scheme, without additional stability requirements. Moreover, higher order methods in age can be devised choosing to advance the parabolic problem by means of a multistep or a Runge-Kutta method, though these latter could entail some stability concerns.*

**Remark 4.3** *The assumption on the mortality term made in Section 2 is actually satisfied by most applications, but there are important occurrences in which the mortality term can be bounded and depend on the whole population, while the age interval is the positive real axis. In such situations the proposed method cannot be applied as it is, and has to undergo some suitable modification. However, as long as the mortality term is bounded, the problem inside each time step is no more than a (possibly nonlinear) parabolic problem in age and space. If the mortality coefficient is independent of the total population, it can be treated implicitly in age, augmenting the coerciveness of the bilinear form in the variational formulation. If the mortality coefficient depends on the total population, the parabolic problem can be linearized by treating the mortality term in a semi-implicit way, computing the total population at the previous time step. In these cases, as the age variable runs up to infinity, it is usual to truncate numerically the age domain at a maximal value, according to the specific case under investigation.*

## 5 Stability and convergence analysis

In this section we analyse the stability and the convergence of the method. We first prove two intermediate results.

**Lemma 5.1** *Let  $u_h^{n,m}$  be the solution of (4.4). Then, for any  $n, m > 0$ ,*

$$\|u_h^{n,m}\|_0 \leq \frac{\Delta a}{\Delta t + \Delta a} \|u_h^{n-1,m}\|_0 + \frac{\Delta t}{\Delta t + \Delta a} \|u_h^{n,m-1}\|_0. \quad (5.1)$$

**Proof** We can rewrite the equation in (4.4) as

$$\left( \frac{u_h^{n,m} - u_h^{n,m-1}}{\Delta a}, v_h \right) + b(a^m; u_h^{n,m}, v_h) + \frac{1}{\Delta t} (u_h^{n,m}, v_h) = \frac{1}{\Delta t} (u_h^{n-1,m}, v_h), \quad (5.2)$$



and, taking  $v_h = u_h^{n,m}$  in (5.2), we get

$$\left( \frac{1}{\Delta a} + \frac{1}{\Delta t} \right) \|u_h^{n,m}\|_0^2 + b(a^m; u_h^{n,m}, u_h^{n,m}) = \frac{1}{\Delta a} (u_h^{n,m-1}, u_h^{n,m}) + \frac{1}{\Delta t} (u_h^{n-1,m}, u_h^{n,m}).$$

Since  $b(a^m; u_h^{n,m}, u_h^{n,m}) \geq 0$ , we have, owing to the Schwarz inequality,

$$\left( \frac{1}{\Delta a} + \frac{1}{\Delta t} \right) \|u_h^{n,m}\|_0 \leq \frac{1}{\Delta a} \|u_h^{n,m-1}\|_0 + \frac{1}{\Delta t} \|u_h^{n-1,m}\|_0,$$

and (5.1) follows.  $\square$

**Lemma 5.2** *Let  $u_h^{n,m}$  be the solution of (4.4). Then, for any  $n, m > 0$ ,*

$$\Delta t \sum_{p=1}^n \|u_h^{p,m}\|_0 + \Delta a \sum_{q=1}^m \|u_h^{n,q}\|_0 \leq \Delta a \sum_{q=1}^m \|u_h^{0,q}\|_0 + \Delta t \sum_{p=1}^n \|u_h^{p,0}\|_0. \quad (5.3)$$

**Proof** For sake of simplicity in notations, let us denote  $\eta^{n,m} = \|u_h^{n,m}\|_0$ .

Owing to (5.1) we have

$$\begin{aligned} \sum_{p=1}^n \sum_{q=1}^m \eta^{p,q} &\leq \frac{\Delta a}{\Delta t + \Delta a} \sum_{p=1}^{n-1} \sum_{q=1}^{m-1} \eta^{p,q} + \frac{\Delta a}{\Delta t + \Delta a} \sum_{q=1}^m \eta^{0,q} + \frac{\Delta t}{\Delta t + \Delta a} \sum_{p=1}^{n-1} \sum_{q=1}^{m-1} \eta^{p,q} \\ &\quad + \frac{\Delta t}{\Delta t + \Delta a} \sum_{p=1}^n \eta^{p,0} + \frac{\Delta a}{\Delta t + \Delta a} \sum_{p=1}^{n-1} \eta^{p,m} + \frac{\Delta t}{\Delta t + \Delta a} \sum_{q=1}^{m-1} \eta^{n,q}, \end{aligned}$$

which, since  $\sum_{p=1}^n \sum_{q=1}^m \eta^{p,q} = \sum_{p=1}^{n-1} \sum_{q=1}^{m-1} \eta^{p,q} + \sum_{p=1}^{n-1} \eta^{p,m} + \sum_{q=1}^{m-1} \eta^{n,q} + \eta^{n,m}$ , is equivalent to

$$\sum_{p=1}^{n-1} \eta^{p,m} + \sum_{q=1}^{m-1} \eta^{n,q} + \eta^{n,m} \leq \frac{\Delta a}{\Delta t + \Delta a} \sum_{p=1}^{n-1} \eta^{p,m} + \frac{\Delta a}{\Delta t + \Delta a} \sum_{q=1}^m \eta^{0,q} + \frac{\Delta t}{\Delta t + \Delta a} \sum_{q=1}^{m-1} \eta^{n,q} + \frac{\Delta t}{\Delta t + \Delta a} \sum_{p=1}^n \eta^{p,0}.$$

This latter inequality can be rearranged to have

$$\frac{\Delta t}{\Delta t + \Delta a} \sum_{p=1}^n \eta^{p,m} + \frac{\Delta a}{\Delta t + \Delta a} \sum_{q=1}^m \eta^{n,q} \leq \frac{\Delta a}{\Delta t + \Delta a} \sum_{q=1}^m \eta^{0,q} + \frac{\Delta t}{\Delta t + \Delta a} \sum_{p=1}^n \eta^{p,0},$$

and (5.3) follows by multiplying by  $(\Delta t + \Delta a)$ .  $\square$

We denote by  $U_h^n = (u_h^{n,0}, u_h^{n,1}, \dots, u_h^{n,N_a})$  the approximate solution at time  $t = t^n$ . The stability of the numerical scheme is guaranteed by the following result.

**Proposition 5.1 (Stability)** *For any  $n = 1, \dots, N_t$ , the following estimate holds:*

$$\|U_h^n\|_{\mathcal{L}^1(0, a_{\dagger}; L^2(\Omega))} \leq \left(1 + e^{a_{\dagger} \beta_{\dagger}^2 T}\right) \|U_h^0\|_{\mathcal{L}^1(0, a_{\dagger}; L^2(\Omega))},$$

where  $\|U_h^n\|_{\mathcal{L}^1(0, a_{\dagger}; L^2(\Omega))} = \sum_{m=0}^{N_a} \Delta a \|u_h^{n,m}\|_0$  denotes the discrete  $L^1(0, a_{\dagger}; L^2(\Omega))$  norm.

**Proof** We have, owing to (5.3),

$$\|U_h^n\|_{\mathcal{L}^1(0, a_{\dagger}; L^2(\Omega))} \leq \Delta a \sum_{q=0}^{N_a} \|u_h^{0,q}\|_0 + \Delta t \sum_{p=1}^n \|u_h^{p,0}\|_0 = \|U_h^0\|_{\mathcal{L}^1(0, a_{\dagger}; L^2(\Omega))} + \Delta t \sum_{p=1}^n \|u_h^{p,0}\|_0.$$

Since  $u_h^{p,0} = \sum_{q=0}^{N_a} \Delta a [m(a^q) u_h^{p-1,q}]$ , the boundedness of the maternity function ( $m(a) \leq \beta_+$ ) entails

$$\|u_h^{p,0}\|_0 \leq a_+ \beta_+^2 \sum_{q=0}^{N_a} \Delta a \|u_h^{p-1,q}\|_0 = a_+ \beta_+^2 \|U_h^{p-1}\|_{\mathcal{L}^1(0, a_+; L^2(\Omega))}.$$

Thus, we have

$$\|U_h^n\|_{\mathcal{L}^1(0, a_+; L^2(\Omega))} \leq \|U_h^0\|_{\mathcal{L}^1(0, a_+; L^2(\Omega))} + \sum_{p=0}^{n-1} a_+ \beta_+^2 \Delta t \|U_h^p\|_{\mathcal{L}^1(0, a_+; L^2(\Omega))},$$

and a direct application of the Discrete Gronwall Lemma<sup>1</sup> concludes the proof, as  $n \leq N_t = \frac{T}{\Delta t}$ .  $\square$

Finally, the convergence of the method is given by the following proposition.

**Proposition 5.2 (Convergence)** *Let  $\mathcal{T}_h$  be a regular family of triangulations on  $\Omega$ . Assume that the solution  $u$  of problem (2.7)-(2.10), is such that, for all  $t \in (0, T)$ ,  $\frac{\partial u}{\partial a}(t, \cdot, \cdot), \frac{\partial u}{\partial t}(t, \cdot, \cdot) \in L^1(0, a_+; H^1(\Omega))$ , and  $\frac{\partial^2 u}{\partial a^2}(t, \cdot, \cdot), \frac{\partial^2 u}{\partial t^2} \in L^1(0, a_+; L^2(\Omega))$ . Then, using linear finite elements, the following estimate holds*

$$\begin{aligned} \|u(t^n, \cdot, \cdot) - U_h^n\|_{\mathcal{L}^1(0, a_+; L^2(\Omega))} &\leq \|U_h^0 - \Pi_h u_0\|_{\mathcal{L}^1(0, a_+; L^2(\Omega))} + Ch \|u(t^n, \cdot, \cdot)\|_{\mathcal{L}^1(0, a_+; H^1(\Omega))} \\ &+ Ch \sum_{p=0}^n \Delta t \left\| \frac{\partial u}{\partial a}(t^p, \cdot, \cdot) \right\|_{L^1(0, a_+; H^1(\Omega))} + Ch \int_0^{t^n} \left\| \frac{\partial u}{\partial t}(t, \cdot, \cdot) \right\|_{L^1(0, a_+; H^1(\Omega))} dt \\ &+ C \Delta t \int_0^{t^n} \left\| \frac{\partial^2 u}{\partial t^2}(t, \cdot, \cdot) \right\|_{L^1(0, a_+; L^2(\Omega))} dt + C \Delta a \sum_{p=0}^n \Delta t \left\| \frac{\partial^2 u}{\partial a^2}(t^p, \cdot, \cdot) \right\|_{L^1(0, a_+; L^2(\Omega))} \end{aligned} \quad (5.4)$$

where the constant  $C > 0$  is independent of  $h$ ,  $\Delta a$ , and  $\Delta t$ .

**Proof** For sake of simplicity, we omit throughout the proof the dependence on the  $x$  variable.

We have

$$\|u(t^n, \cdot) - U_h^n\|_{\mathcal{L}^1(0, a_+; L^2(\Omega))} \leq \|u(t^n, \cdot) - \Pi_{1,h} u(t^n, \cdot)\|_{\mathcal{L}^1(0, a_+; L^2(\Omega))} + \|\Pi_{1,h} u(t^n, \cdot) - U_h^n\|_{\mathcal{L}^1(0, a_+; L^2(\Omega))}. \quad (5.5)$$

The first term in (5.5) can be estimated, by (4.2), as follows

$$\begin{aligned} \|u(t^n, \cdot) - \Pi_{1,h} u(t^n, \cdot)\|_{\mathcal{L}^1(0, a_+; L^2(\Omega))} &= \sum_{m=0}^{N_a} \Delta a \|u(t^n, a^m) - \Pi_{1,h} u(t^n, a^m)\|_0 \\ &\leq Ch \sum_{m=0}^{N_a} \Delta a \|u(t^n, a^m)\|_1 = Ch \|u(t^n, \cdot)\|_{\mathcal{L}^1(0, a_+; H^1(\Omega))}. \end{aligned} \quad (5.6)$$

Concerning the second term in (5.5), owing to the equation in (4.4), we easily see that the difference  $\eta_h^{n,l} := u_h^{n,l} - \Pi_{1,h} u(t^n, a^l)$  satisfies

$$\frac{1}{\Delta t} (\eta_h^{n,l} - \eta_h^{n-1,l}, v_h) + \frac{1}{\Delta a} (\eta_h^{n,l} - \eta_h^{n,l-1}, v_h) + b(a^l; \eta_h^{n,l}, v_h) = (\varepsilon_h^{n-1,l}, v_h), \quad \forall v_h \in V_h \quad (5.7)$$

---

<sup>1</sup>Let  $k_n$  be a non-negative sequence, and let  $\phi_n$  be a sequence that satisfies

$$\begin{cases} \phi_0 \leq g_0, \\ \phi_n \leq g_0 + \sum_{s=0}^{n-1} p_s + \sum_{s=0}^{n-1} k_s \phi_s \end{cases} \quad (n \geq 1)$$

Then, if  $g_0 \geq 0$ , and  $p_m \geq 0$  for  $m \geq 0$ , it follows, for  $n \geq 1$ ,

$$\phi_n \leq \left( g_0 + \sum_{s=0}^{n-1} p_s \right) \exp \left( \sum_{s=0}^{n-1} k_s \right).$$

where the quantity  $\varepsilon_h^{n,l-1} \in V_h$  is defined by the relation

$$\begin{aligned} (\varepsilon_h^{n-1,l}, v_h) &= -\frac{1}{\Delta t} (\Pi_{1,h} u(t^n, a^l) - \Pi_{1,h} u(t^{n-1}, a^l), v_h) - b(a^l; \Pi_{1,h} u(t^n, a^l), v_h) \\ &\quad - \frac{1}{\Delta a} (\Pi_{1,h} u(t^n, a^l) - \Pi_{1,h} u(t^n, a^{l-1}), v_h) \quad \forall v_h \in V_h. \end{aligned} \quad (5.8)$$

Now, since the bilinear form  $b(a^l; \cdot, \cdot)$  is continuous, coercive, and symmetric in  $V_h$ , there exists a non-decreasing sequence of eigenvalues  $0 < \alpha \leq \mu_{1,h} \leq \mu_{2,h} \leq \dots \leq \mu_{N_h,h}$  and a  $L^2(\Omega)$ -orthonormal basis of eigenvectors  $\{\omega_{i,h} \in V_h, i = 1, \dots, N_h\}$  such that

$$b(a^l; \omega_{i,h}, v_h) = \mu_{i,h} (\omega_{i,h}, v_h) \quad \forall v_h \in V_h.$$

Any function  $v_h \in V_h$  can then be expanded with respect to the system  $\{\omega_{i,h}\}$ ,

$$v_h = \sum_{i=1}^{N_h} (v_h, \omega_{i,h}) \omega_{i,h} \quad \|v_h\|_0^2 = \sum_{i=1}^{N_h} |(v_h, \omega_{i,h})|^2, \quad (5.9)$$

and in particular we have

$$\eta_h^{n,l} = \sum_{i=1}^{N_h} \eta_i^{n,l} \omega_{i,h} \quad \eta_h^{n-1,l} = \sum_{i=1}^{N_h} \eta_i^{n-1,l} \omega_{i,h} \quad \eta_h^{n,l-1} = \sum_{i=1}^{N_h} \eta_i^{n,l-1} \omega_{i,h} \quad \varepsilon_h^{n-1,l} = \sum_{i=1}^{N_h} \varepsilon_i^{n-1,l} \omega_{i,h},$$

where we have set  $\eta_i^{n,l} = (\eta_h^{n,l}, \omega_{i,h})$ ,  $\eta_i^{n-1,l} = (\eta_h^{n-1,l}, \omega_{i,h})$ ,  $\eta_i^{n,l-1} = (\eta_h^{n,l-1}, \omega_{i,h})$ , and  $\varepsilon_i^{n-1,l} = (\varepsilon_h^{n-1,l}, \omega_{i,h})$ .

With these positions equation (5.7) is equivalent to

$$\frac{1}{\Delta t} \eta_i^{n,l} - \frac{1}{\Delta t} \eta_i^{n-1,l} + \frac{1}{\Delta a} \eta_i^{n,l} - \frac{1}{\Delta a} \eta_i^{n,l-1} + \mu_{i,h} \eta_i^{n,l} = \varepsilon_i^{n-1,l}$$

for each  $i = 1, \dots, N_h$ . We can rewrite the above expression as

$$\eta_i^{n,l} = \frac{\Delta a}{\Delta a + \Delta t + \mu_{i,h} \Delta a \Delta t} \eta_i^{n-1,l} + \frac{\Delta t}{\Delta a + \Delta t + \mu_{i,h} \Delta a \Delta t} \eta_i^{n,l-1} + \frac{\Delta a \Delta t}{\Delta a + \Delta t + \mu_{i,h} \Delta a \Delta t} \varepsilon_i^{n-1,l}. \quad (5.10)$$

By taking the absolute value in (5.10), we obtain, from (5.9) and Minkowski inequality,

$$\|\eta_h^{n,l}\|_0 \leq \frac{\Delta a}{\Delta a + \Delta t} \|\eta_h^{n-1,l}\|_0 + \frac{\Delta t}{\Delta a + \Delta t} \|\eta_h^{n,l-1}\|_0 + \frac{\Delta t \Delta a}{\Delta a + \Delta t} \|\varepsilon_h^{n-1,l}\|_0. \quad (5.11)$$

Summing (5.11) on  $l$ , and multiplying the result by  $(\Delta a + \Delta t)$

$$\sum_{l=0}^{N_a} \Delta a \|\eta_h^{n,l}\|_0 \leq \sum_{l=0}^{N_a} \Delta a \|\eta_h^{n-1,l}\|_0 + \Delta t \sum_{l=0}^{N_a} \Delta a \|\varepsilon_h^{n-1,l}\|_0 \quad (5.12)$$

Denoting by  $\boldsymbol{\eta}^n = (\eta_h^{n,0}, \dots, \eta_h^{n,N_a})$ , and  $\boldsymbol{\varepsilon}^n = (\varepsilon_h^{n,0}, \dots, \varepsilon_h^{n,N_a})$ , equation (5.12) is equivalent to

$$\|\boldsymbol{\eta}^n\|_{\mathcal{L}^1(0, a_+; L^2(\Omega))} \leq \|\boldsymbol{\eta}^{n-1}\|_{\mathcal{L}^1(0, a_+; L^2(\Omega))} + \Delta t \|\boldsymbol{\varepsilon}^{n-1}\|_{\mathcal{L}^1(0, a_+; L^2(\Omega))},$$

and, by iteration, we have

$$\|\boldsymbol{\eta}^n\|_{\mathcal{L}^1(0, a_+; L^2(\Omega))} \leq \|\boldsymbol{\eta}^0\|_{\mathcal{L}^1(0, a_+; L^2(\Omega))} + \Delta t \sum_{p=0}^{n-1} \|\boldsymbol{\varepsilon}^p\|_{\mathcal{L}^1(0, a_+; L^2(\Omega))}. \quad (5.13)$$

We now turn to estimate  $\|\boldsymbol{\varepsilon}^n\|_{\mathcal{L}^1(0, a_+; L^2(\Omega))}$ . Owing to (2.7), we have

$$\left( \frac{\partial u}{\partial t}(t^n, a^l), v_h \right) + \left( \frac{\partial u}{\partial a}(t^n, a^l), v_h \right) = -b(a^l; u(t^n, a^l), v_h),$$

and, since the operator  $\Pi_{1,h}$  commutes with both  $\frac{\partial}{\partial t}$  and  $\frac{\partial}{\partial a}$ , equation (5.8) can be rewritten as

$$\begin{aligned}
(\varepsilon_h^{n-1,l}, v_h) &= \left( \frac{\partial u}{\partial t}(t^n, a^l), v_h \right) - \frac{1}{\Delta t} (\Pi_{1,h} u(t^n, a^l) - \Pi_{1,h} u(t^{n-1}, a^l), v_h) \\
&\quad + \left( \frac{\partial u}{\partial a}(t^n, a^l), v_h \right) - \frac{1}{\Delta a} (\Pi_{1,h} u(t^n, a^l) - \Pi_{1,h} u(t^n, a^{l-1}), v_h) \\
&= \left( \frac{\partial u}{\partial t}(t^n, a^l) - \frac{u(t^n, a^l) - u(t^{n-1}, a^l)}{\Delta t}, v_h \right) + \frac{1}{\Delta t} \left( \int_{t^{n-1}}^{t^n} (I - \Pi_{1,h}) \frac{\partial u}{\partial t}(\tau, a^l) d\tau, v_h \right) \\
&\quad \left( \frac{\partial u}{\partial a}(t^n, a^l) - \frac{u(t^n, a^l) - u(t^n, a^{l-1})}{\Delta a}, v_h \right) + \frac{1}{\Delta a} \left( \int_{a^{l-1}}^{a^l} (I - \Pi_{1,h}) \frac{\partial u}{\partial t}(t^n, \sigma) d\sigma, v_h \right) \\
&= \frac{1}{\Delta t} \left( \int_{t^{n-1}}^{t^n} (I - \Pi_{1,h}) \frac{\partial u}{\partial t}(\tau, a^l) d\tau, v_h \right) + \frac{1}{\Delta t} \left( \int_{t^{n-1}}^{t^n} (\tau - t^{n-1}) \frac{\partial^2 u}{\partial t^2}(\tau, a^l) d\tau, v_h \right) \\
&\quad \frac{1}{\Delta a} \left( \int_{a^{l-1}}^{a^l} (I - \Pi_{1,h}) \frac{\partial u}{\partial t}(t^n, \sigma) d\sigma, v_h \right) + \frac{1}{\Delta a} \left( \int_{a^{l-1}}^{a^l} (\sigma - a^{l-1}) \frac{\partial^2 u}{\partial a^2}(t^n, \sigma) d\sigma, v_h \right).
\end{aligned} \tag{5.14}$$

So far, taking  $v_h = \varepsilon_h^{n-1,l}$  in (5.14), and applying Schwarz inequality, we have

$$\begin{aligned}
\|\varepsilon_h^{n-1,l}\|_0 &\leq \frac{1}{\Delta t} \int_{t^{n-1}}^{t^n} \left\| (I - \Pi_{1,h}) \frac{\partial u}{\partial t}(t, a^l) \right\|_0 dt + \int_{t^{n-1}}^{t^n} \left\| \frac{\partial^2 u}{\partial t^2}(t, a^l) \right\|_0 dt \\
&\quad + \frac{1}{\Delta a} \int_{a^{l-1}}^{a^l} \left\| (I - \Pi_{1,h}) \frac{\partial u}{\partial a}(t^n, a) \right\|_0 da + \int_{a^{l-1}}^{a^l} \left\| \frac{\partial^2 u}{\partial a^2}(t^n, a) \right\|_0 da,
\end{aligned}$$

and, consequently, from (4.2),

$$\begin{aligned}
\|\boldsymbol{\varepsilon}^p\|_{\mathcal{L}^1(0, a_\dagger; L^2(\Omega))} &\leq \frac{1}{\Delta t} \sum_{l=0}^{N_a} \Delta a \int_{t^{p-1}}^{t^p} \left\| (I - \Pi_{1,h}) \frac{\partial u}{\partial t}(t, a^l) \right\|_0 dt + \sum_{l=0}^{N_a} \Delta a \int_{t^{p-1}}^{t^p} \left\| \frac{\partial^2 u}{\partial t^2}(t, a^l) \right\|_0 dt \\
&\quad + \frac{1}{\Delta a} \sum_{l=0}^{N_a} \Delta a \int_{a^{l-1}}^{a^l} \left\| (I - \Pi_{1,h}) \frac{\partial u}{\partial a}(t^p, a) \right\|_0 da + \sum_{l=0}^{N_a} \Delta a \int_{a^{l-1}}^{a^l} \left\| \frac{\partial^2 u}{\partial a^2}(t^p, a) \right\|_0 da \\
&\leq \frac{Ch}{\Delta t} \int_{t^{p-1}}^{t^p} \left( \sum_{l=0}^{N_a} \Delta a \left\| \frac{\partial u}{\partial t}(t, a^l) \right\|_1 \right) dt + \int_{t^{p-1}}^{t^p} \left( \sum_{l=0}^{N_a} \Delta a \left\| \frac{\partial^2 u}{\partial t^2}(t, a^l) \right\|_0 \right) dt \\
&\quad + Ch \int_0^{a_\dagger} \left\| \frac{\partial u}{\partial a}(t^p, a) \right\|_1 da + \Delta a \int_0^{a_\dagger} \left\| \frac{\partial^2 u}{\partial a^2}(t^p, a) \right\|_0 da \\
&= \frac{Ch}{\Delta t} \int_{t^{p-1}}^{t^p} \left\| \frac{\partial u}{\partial t}(t, a^l) \right\|_{\mathcal{L}^1(0, a_\dagger; H^1(\Omega))} dt + \int_{t^{p-1}}^{t^p} \left\| \frac{\partial^2 u}{\partial t^2}(t, a^l) \right\|_{\mathcal{L}^1(0, a_\dagger; L^2(\Omega))} dt \\
&\quad + Ch \left\| \frac{\partial u}{\partial a}(t^p, a) \right\|_{L^1(0, a_\dagger; H^1(\Omega))} + \Delta a \left\| \frac{\partial^2 u}{\partial a^2}(t^p, a) \right\|_{L^1(0, a_\dagger; L^2(\Omega))}.
\end{aligned}$$

Thus, using this in (5.13), we get

$$\begin{aligned}
\|\boldsymbol{\eta}^n\|_{\mathcal{L}^1(0, a_\dagger; L^2(\Omega))} &\leq \|\boldsymbol{\eta}^0\|_{\mathcal{L}^1(0, a_\dagger; L^2(\Omega))} + \Delta t \sum_{p=0}^n \frac{Ch}{\Delta t} \int_{t^{p-1}}^{t^p} \left\| \frac{\partial u}{\partial t}(t, \cdot) \right\|_{\mathcal{L}^1(0, a_\dagger; H^1(\Omega))} dt \\
&\quad + \Delta t \sum_{p=0}^n \int_{t^{p-1}}^{t^p} \left\| \frac{\partial^2 u}{\partial t^2}(t, \cdot) \right\|_{\mathcal{L}^1(0, a_\dagger; L^2(\Omega))} dt + \Delta t \sum_{p=0}^n Ch \left\| \frac{\partial u}{\partial a}(t^p, \cdot) \right\|_{L^1(0, a_\dagger; H^1(\Omega))} \\
&\quad + \Delta t \sum_{p=0}^n \Delta a \left\| \frac{\partial^2 u}{\partial a^2}(t^p, \cdot) \right\|_{L^1(0, a_\dagger; L^2(\Omega))} \\
&= \|\boldsymbol{\eta}^0\|_{\mathcal{L}^1(0, a_\dagger; L^2(\Omega))} + Ch \int_0^{t^n} \left\| \frac{\partial u}{\partial t}(t, \cdot) \right\|_{\mathcal{L}^1(0, a_\dagger; L^2(\Omega))} dt \\
&\quad + \Delta t \int_0^{t^n} \left\| \frac{\partial^2 u}{\partial t^2}(t, \cdot) \right\|_{\mathcal{L}^1(0, a_\dagger; L^2(\Omega))} dt + Ch \sum_{p=0}^n \Delta t \left\| \frac{\partial u}{\partial a}(t^p, \cdot) \right\|_{L^1(0, a_\dagger; H^1(\Omega))} \\
&\quad + \Delta a \sum_{p=0}^n \Delta t \left\| \frac{\partial^2 u}{\partial a^2}(t^p, \cdot) \right\|_{L^1(0, a_\dagger; L^2(\Omega))}.
\end{aligned} \tag{5.15}$$

Since  $\boldsymbol{\eta}^n = \Pi_{1,h}u(t^n, \cdot) - U_h^n$  ( $n = 1, \dots, N_t$ ), we eventually obtain (5.4) from (5.5), (5.6), and (5.15).  $\square$

An immediate consequence of (5.4) is given by the following corollary, that provides a uniform estimate with respect to time.

**Corollary 5.3** *Let the solution  $u$  of problem (2.7)-(2.10) satisfy the hypotheses of Proposition 5.2. If, in addition,  $u \in L^\infty(0, T; L^1(0, a_\dagger; H^1(\Omega)))$ , the following estimate holds for any  $n = 1, \dots, N_t$*

$$\begin{aligned} \|u(t^n, \cdot, \cdot) - U_h^n\|_{\mathcal{L}^1(0, a_\dagger; L^2(\Omega))} &\leq \|U_h^0 - \Pi_h u_0\|_{\mathcal{L}^1(0, a_\dagger; L^2(\Omega))} + Ch \sup_{t \in (0, T)} \|u(t, \cdot, \cdot)\|_{\mathcal{L}^1(0, a_\dagger; H^1(\Omega))} \\ &+ Ch \sum_{p=0}^{N_t} \Delta t \left\| \frac{\partial u}{\partial a}(t^p, \cdot, \cdot) \right\|_{L^1(0, a_\dagger; H^1(\Omega))} + Ch \int_0^T \left\| \frac{\partial u}{\partial t}(t, \cdot, \cdot) \right\|_{\mathcal{L}^1(0, a_\dagger; H^1(\Omega))} dt \\ &+ C \Delta t \int_0^T \left\| \frac{\partial^2 u}{\partial t^2}(t, \cdot, \cdot) \right\|_{\mathcal{L}^1(0, a_\dagger; L^2(\Omega))} dt + C \Delta a \sum_{p=0}^{N_t} \Delta t \left\| \frac{\partial^2 u}{\partial a^2}(t^p, \cdot, \cdot) \right\|_{L^1(0, a_\dagger; L^2(\Omega))} \end{aligned} \quad (5.16)$$

where the constant  $C > 0$  is independent of  $h$ ,  $\Delta a$ , and  $\Delta t$ .

## 6 Numerical results

We present in this section some numerical results to show the effectiveness of the method, and we first outline some algorithmic aspects of the proposed method in a slightly more general setting, where the discretization steps in age and time are not necessarily uniform.

Let  $\{0 = t^0 < t^1 < \dots < t^{N_t} = T\}$  and  $\{0 = a^0 < a^1 < \dots < a^{N_a} = a_\dagger\}$  be suitable discretizations of the intervals  $(0, T)$  and  $(0, a_\dagger)$ , respectively, and let

$$\Delta t^n = t^n - t^{n-1} \quad \Delta a^m = a^m - a^{m-1}.$$

The solution  $u_h^{n,m}(x)$  to problem (4.4) is given by

$$u_h^{n,m}(x) = \sum_{j=1}^{N_h} u_j^{n,m} \varphi_j(x).$$

We denote by  $\mathbf{u}^{n,m} = (u_1^{n,m}, \dots, u_{N_h}^{n,m})^T$  the unknown vector at time  $t^n$  and age  $a^m$ .

At time step  $t^n$ , given  $\mathbf{u}^{n-1,l}$ , the solution, for any age level  $a^l$  ( $l = 0, 1, \dots, N_a$ ), at time step  $t^{n-1}$ :

- Compute the initial value  $\mathbf{u}^{n,0}$  from the previous time step via a Simpson quadrature rule over two adjacent age intervals, *i.e.* for  $j = 1, \dots, N_h$

$$\mathbf{u}_j^{n,0} = \sum_{l=1}^{N_a/2} \frac{\Delta a^{2l-1} + \Delta a^{2l}}{6} \left[ m(a^{2(l-1)}) \mathbf{u}_j^{n-1,2(l-1)} + 4m(a^{2l-1}) \mathbf{u}_j^{n-1,2l-1} + m(a^{2l}) \mathbf{u}_j^{n-1,2l} \right].$$

- For  $l = 1, \dots, N_a$  solve

$$\left[ \Delta t^n \Delta a^l B^l + \left( \Delta t^n + \Delta a^l \right) M \right] \mathbf{u}^{n,l} = \Delta a^l M \mathbf{u}^{n-1,l} + \Delta t^n M \mathbf{u}^{n,l-1}$$

where  $B^l = B(a^l)$  and  $M$  are the stiffness and the mass matrices, defined in (4.3).

**Remark 6.1** Notice that if the age and time discretization grids are uniform,  $\Delta a^l$  and  $\Delta t^n$  are fixed and the matrices  $A^l = [\Delta t \Delta a B^l + (\Delta t + \Delta a) M]$  ( $l = 0, \dots, N_a$ ) can be computed once for all. Moreover, if the diffusion coefficient does not depend on age, the stiffness matrix stays unchanged throughout the computation.

We first consider a one dimensional problem in space. The spatial domain is  $\Omega = (0, 1)$ , the age interval is  $[0, 100]$ , and we choose as maximal time  $T = 30$ . The numerical simulations are run on Matlab<sup>®</sup> 6.5.

We consider a non-symmetric initial distribution of population (with respect to both space and age) given by

$$u_0(x, a) = e^{-\left(\frac{(a-30)^2}{200} + 100(x-0.4)^2\right)},$$

and we take the mortality and fertility function as

$$\mu(a) = \frac{1}{a_{\dagger} - a}, \quad \beta(a) = \begin{cases} 0 & \text{if } a \leq a_1 \\ \frac{\beta(a - a_1)^{\alpha-1} e^{-\frac{(a-a_1)}{\vartheta}}}{\vartheta^{\alpha} \Gamma(\alpha)} & \text{if } a_1 < a < a_2 \\ 0 & \text{if } a \geq a_2, \end{cases}$$

where we set  $a_1 = 17$ ,  $a_2 = 70$ ,  $\beta = 7$ ,  $\alpha = 5$ , and  $\vartheta = 3$ .

We plot in figure 1 the resulting maternity function and the initial profile of the problem. We use an uniform mesh in space, and, at each time level, we use a Simpson quadrature rule over two adjacent subintervals to compute the initial value for the parabolic (in age and space) problem.

In the first test, we choose a constant diffusion coefficient  $k(a, x) = 10^{-3}$ , and we investigate numerically

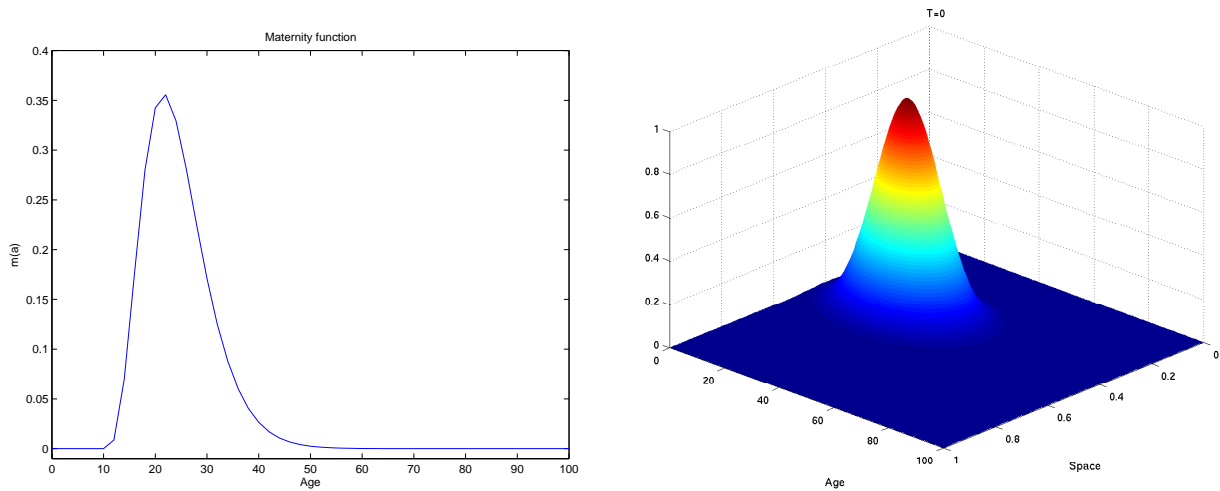


Figure 1: Maternity function (left) and initial age-space profile (right) for the numerical tests

the convergence of the method. In that order, we first vary the mesh size  $h$  having fixed  $\Delta a$  and  $\Delta t$ , then we vary  $\Delta a$  and  $\Delta t$ , having fixed  $h$ . Moreover, in order to investigate the robustness of the method with respect to the age and time discretization step we vary not only  $\Delta a$  and  $\Delta t$ , but also their ratio. We analyze the relative error  $\frac{\|u(t^n, \cdot, \cdot) - U_h^n\|}{\|u(t^n, \cdot, \cdot)\|}$  in the discrete  $\mathcal{L}^1(0, a_{\dagger}; L^2(\Omega))$  norm, with respect to a reference solution computed using a very fine grid in both age and time with  $\Delta a = \Delta t = 0.05$  and  $h = 1/1000$ . In figure 2

we show the work precision, in  $h$  for a uniform grid in age and time with  $\Delta a = \Delta t = 0.1$  (left picture, at time  $T = 5$  and  $T = 10$ ), and in  $\Delta t$  and  $\Delta a$  (right picture, at time  $T = 5$ ). For the latter case, we choose  $h = 1/100$  and we consider different discretization steps in age and time, choosing  $\Delta a/\Delta t$  ranging from  $1/4$  to  $4$ . The method shows the predicted order of convergence. Moreover, it appears to be robust with respect to the ratio between the age and time discretization (the best choice seems to be  $\Delta t = \Delta a/2$ ), and such feature is quite promising in view of adaptivity in the discretization of these variables.

In the second test, we use a uniform mesh in time and age, with  $\Delta a = 2$  and  $\Delta t = 1$ , and we give in

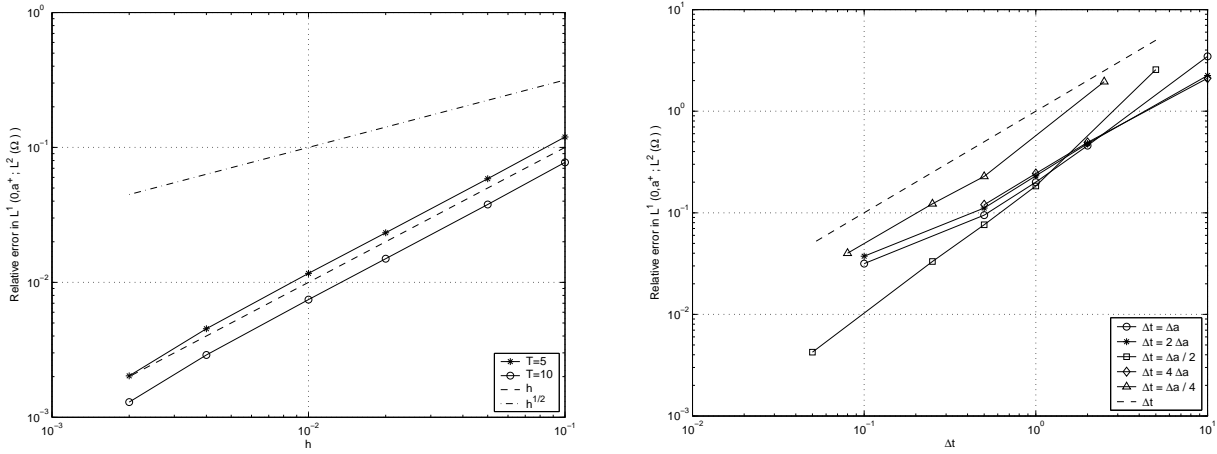


Figure 2: Convergence in  $\mathcal{L}^1(0, a_+; L^2(\Omega))$  norm: relative error in  $h$  (left) and  $\Delta t$  and  $\Delta a$  (right)

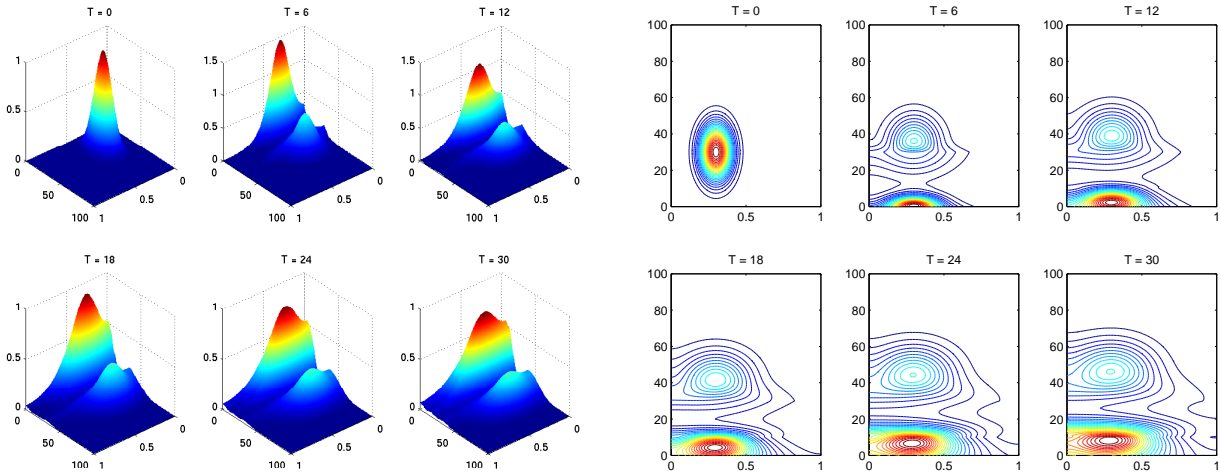


Figure 3: Age-space profiles and contour isolines at different time levels for Test 2, with  $k(a, x) = 10^{-3}$

figure 3 the age-space profiles and contour isolines of the solution at different time levels, with a diffusion



coefficient homogeneous in space but discontinuous in age, given by

$$k(a, x) = \begin{cases} 10^{-3} & \text{if } a < 20 \\ 5 \cdot 10^{-3} & \text{if } 20 \leq a \leq 40 \\ 10^{-3} & \text{if } a > 40. \end{cases}$$

In figure 4 (left) we give the age-space profiles of the solution at different time levels with  $k(a, x) = 1$ . In the presence of strong diffusion the solution tends rapidly to a spatially homogeneous distribution, showing a typical age profile, as portrayed in figure 4 (right).

Finally, in the last test, we consider a two-dimensional spatial domain  $\Omega = [0, 1] \times [0, 1]$ , with an anisotropic diffusion coefficient

$$k(a, x) = \begin{pmatrix} k_x(a, x) & 0 \\ 0 & k_y(a, x) \end{pmatrix},$$

where  $k_x(a, x) = 2 \cdot 10^{-3}$ , and  $k_y(a, x) = 10^{-3}$ . We use the same mortality and fertility functions of the previous tests and an initial profile given by

$$u_0(x, a) = e^{-\left(\frac{(a-30)^2}{200} + 1000[(x-0.5)^2 + (y-0.75)^2]\right)}.$$

We discretize  $\Omega$  by an unstructured triangular grid consisting of 2601 nodes and 5000 elements, and we choose  $\Delta t = \Delta a = 1$ . We report in figure 5 the results: on the left we show the time evolution of the total population, and on the right we show the time evolution of the age profile at point  $(0.5, 0.74)$ .

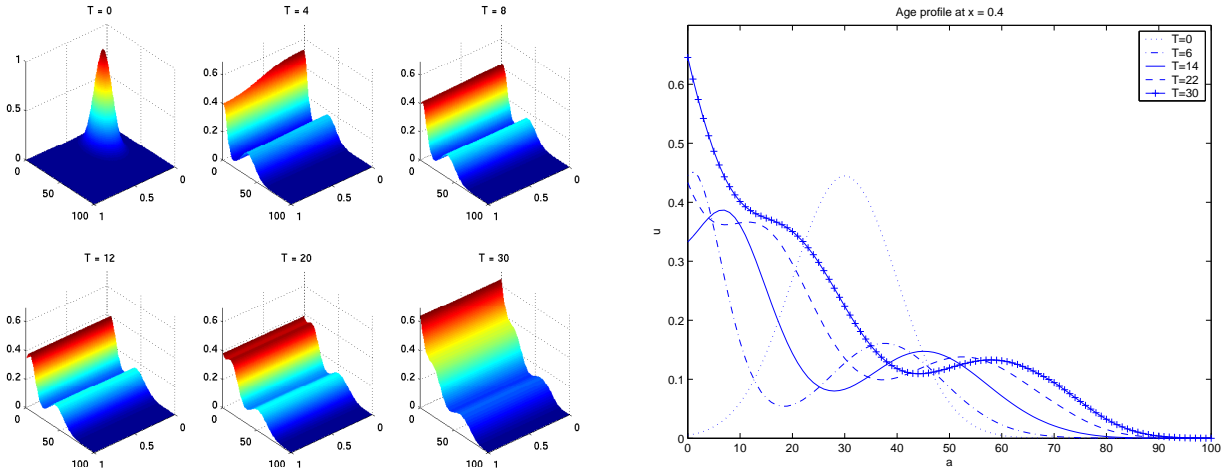


Figure 4: Age-space profiles (left) and age profiles in  $x = 0.4$  (right) at different time levels for Test 2, with  $k(a, x) = 10^{-1}$

## 7 Conclusions

We proposed a Galerkin type method for the numerical approximation of the diffusion of an age-structured population. The method is based on a finite elements discretization in space, and on implicit discretizations

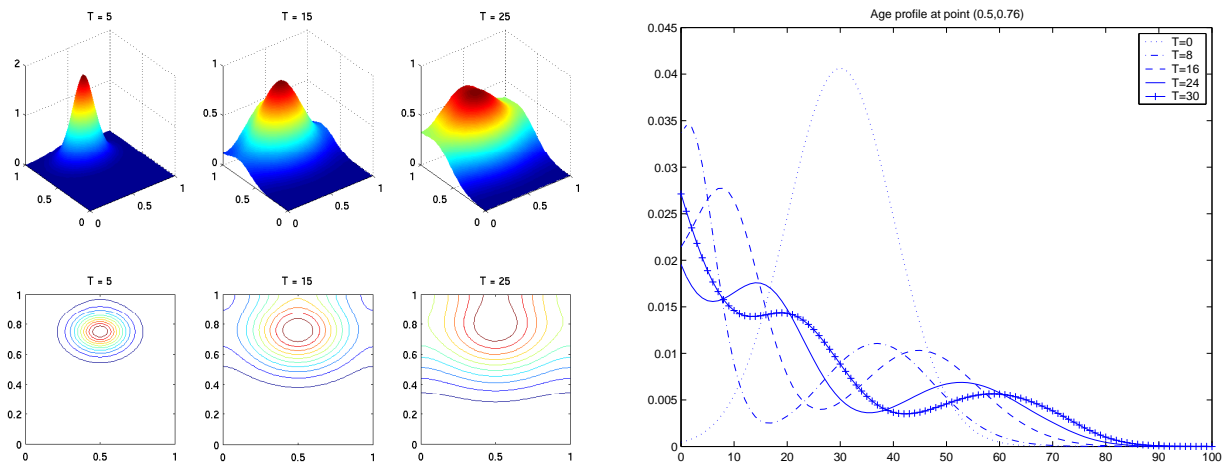


Figure 5: Space profiles and contour isolines of the total population (left) and age profiles in  $(0.5, 0.76)$  (right) at different time levels for the two-dimensional test

in time and age. The key feature of the method is the separate discretization of time and age. We proved stability and convergence, and we presented some numerical result to validate the proposed method.

Further directions of research will be the study of separate adaptivity in age and time and the analysis of a method for a nonlinear problem, where the nonlinearity is not only located in the diffusion coefficient (quite a straightforward extension) but also in fertility or mortality functions, that can depend on the total population or on some weighted means of the population itself (sizes).

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