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LARGE, GLOBAL SOLUTIONS TO THE NAVIER-STOKES EQUATIONS, SLOWLY VARYING IN ONE DIRECTION

JEAN-YVES CHEMIN AND ISABELLE GALLAGHER

Abstract. In [3] and [4] classes of initial data to the three dimensional, incompressible Navier-Stokes equations were presented, generating a global smooth solution although the norm of the initial data may be chosen arbitrarily large. The aim of this article is to provide new examples of arbitrarily large initial data giving rise to global solutions, in the whole space. Contrary to the previous examples, the initial data has no particular oscillatory properties, but varies slowly in one direction. The proof uses the special structure of the nonlinear term of the equation.

1. Introduction

The purpose of this paper is to use the special structure of the tridimensional Navier-Stokes equations to prove the global existence of smooth solutions for a class of (large) initial data which are slowly varying in one direction. Before entering further in the details, let us recall briefly some classical facts on the global wellposedness of the incompressible Navier-Stokes equations in the whole space $\mathbb{R}^3$. The equation itself writes

$$(NS)\left\{\begin{array}{c}\partial_t u + u\cdot\nabla u - \Delta u = -\nabla p\\ \operatorname{div} u = 0\\ u_{|t=0} = u_0\end{array}\right.$$

where $u = (u^1, u^2, u^3) = (u^h, u^3)$ is a time dependent vector field on $\mathbb{R}^3$. The divergence free condition determines p through the relation

$$-\Delta p = \sum_{1\le j,k\le 3} \partial_j\partial_k(u^j u^k).$$

This relation allows to put the system (NS) under the more general form

$$(GNS)\left\{\begin{array}{c}\partial_t u - \Delta u = Q(u,u)\\ u_{|t=0} = u_0\end{array}\right.$$

where $Q(v,w) \stackrel{\text{def}}{=} \sum_{1\le j,k\le 3} Q_{j,k}(D)(v^j w^k)$ and $Q_{j,k}(D)$ are smooth homogeneous Fourier multipliers of order 1.

Moreover, this system has the following scaling invariance: if (u,p) is a solution on the time interval $[0,T)$, then (u_λ, p_λ) defined by

$$u_\lambda(t,x) \stackrel{\text{def}}{=} \lambda u(\lambda^2 t, \lambda x) \quad \text{and} \qquad p_\lambda(t,x) \stackrel{\text{def}}{=} \lambda^2 p(\lambda^2 t, \lambda x)$$

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is a solution on the time interval $[0, \lambda^{-2}T)$. Of course, any smallness condition on the initial data that ensures global solutions, must be invariant under the above scaling transformation. The search of the "best" smallness condition is a long story initiated in the seminal paper of J. Leray (see [12]), continuated in particular by H. Fujita and T. Kato in [6], Y. Giga and T. Miyakawa in [8], and M. Cannone, Y. Meyer and F. Planchon in [1]. This leads to the following theorem proved by H. Koch and D. Tataru in [11]. In the statement of the theorem, $P(x,R)$ stands for the parabolic set $[0,R^2]\times B(x,R)$ where $B(x,R)$ is the ball centered at x, of radius R.

Theorem 1 ([11])**.** *If the initial data u_0 is such that*

$$\|u_0\|^2_{BMO^{-1}} \stackrel{\text{def}}{=} \sup_{t>0} t\|e^{t\Delta}u_0\|^2_{L^\infty} + \sup_{\substack{x\in\mathbb{R}^3\\ R>0}} \frac{1}{R^3}\int_{P(x,R)} |(e^{t\Delta}u_0)(t,y)|^2 dy \tag{1.1}$$

is small enough, then there exists a global smooth solution to (GNS).

A typical example of application of this theorem is the initial data

$$u_0^\varepsilon(x) \stackrel{\text{def}}{=} \cos\Big(\frac{x_3}{\varepsilon}\Big)(\partial_2\phi(x_1,x_2), -\partial_1\phi(x_1,x_2), 0) \quad \text{with} \quad \phi\in\mathcal{S}(\mathbb{R}^2), \tag{1.2}$$

as soon as ε is small enough (see for example [4] for a proof). The above theorem is probably the end point for the following reason, as observed for instance in [4]. If B is a Banach space continuously included in the space $\mathcal{S}'$ of tempered distributions on $\mathbb{R}^3$, such that, for any $(\lambda,a)\in\mathbb{R}^+_\star\times\mathbb{R}^3$, $\|f(\lambda(\cdot-a))\|_B = \lambda^{-1}\|f\|_B$, then $\|\cdot\|_B \le C\sup_{t>0} t^{\frac12}\|e^{t\Delta}u_0\|_{L^\infty}$. The second condition entering in the definition of the BMO^{-1} norm given in (1.1) merely translates the fact that the first Picard iterate should be locally square integrable in space and time.

Those results of global existence under a smallness condition do not use the special structure of the incompressible Navier-Stokes system and are valid for the larger class of systems of the type (GNS). The purpose of this paper is to provide a class of examples of large initial data which give rise to global smooth solutions for the system (NS) itself, and not for the larger class (GNS). In all that follows, an initial data u_0 will be said "large" if

$$\|u_0\|_{\dot B^{-1}_{\infty,\infty}} \stackrel{\text{def}}{=} \sup_{t>0} t^{\frac12}\|e^{t\Delta}u_0\|_{L^\infty} \tag{1.3}$$

is not small.

Such initial data, in the spirit of the example provided by (1.2), are exhibited in [4] (see also [3] for the periodic case). In particular, the following theorem is proved in [4].

Theorem 2 ([4])**.** *Let $\phi\in\mathcal{S}(\mathbb{R}^3)$ be a given function, and consider two real numbers ε and α in $]0,1[$. Define*

$$\varphi^\varepsilon(x) = \frac{(-\log\varepsilon)^{\frac15}}{\varepsilon^{1-\alpha}}\cos\Big(\frac{x_3}{\varepsilon}\Big)\,\phi\Big(x_1, \frac{x_2}{\varepsilon^\alpha}, x_3\Big).$$

Then for ε small enough, the smooth, divergence free vector field

$$u_0^\varepsilon(x) = (\partial_2\varphi^\varepsilon(x), -\partial_1\varphi^\varepsilon(x), 0)$$

satisfies $\lim_{\varepsilon\to0}\|u_0^\varepsilon\|_{\dot B^{-1}_{\infty,\infty}} = \infty$, and generates a unique global solution to (NS).

The proof of that theorem shows that the solution remains close to the solution of the free equation $e^{t\Delta}u_{0,\varepsilon}$. It is important to notice that the proof uses in a crucial way the algebraic structure of the non linear term $u \cdot \nabla u$, but uses neither the energy estimate, nor the fact that the two dimensional, incompressible Navier-Stokes is globally wellposed. Let us give some other results on large initial data giving rise to global solutions: In [13], the initial data is chosen so as to transform the equation into a rotating fluid equation. In [10], [14] the equations are posed in a thin domain (in all those cases the global wellposedness of the two dimensional equation is a crucial ingredient in the proof). In [7] and [9], the case of an initial data close to bidimensional vector field is studied, in the periodic case. Finally in [5] (Remark 7), an arbitrary large initial data is constructed in the periodic case, generating a global solution (which is in fact a solution to the heat equation, as the special dependence on the space variables implies that the nonlinear term cancels).

The class of examples we exhibit here is quite different. They are close to a two dimensional flow in the sense that they are slowly varying in the one direction (the vertical one). More precisely the aim of this paper is the proof of the following theorem.

Theorem 3. *Let $v_0^h = (v_0^1, v_0^2, 0)$ be a horizontal, smooth divergence free vector field on $\mathbb{R}^3$ (i.e. v_0^h is in $L^2(\mathbb{R}^3)$ as well as all its derivatives), belonging, as well as all its derivatives, to $L^2(\mathbb{R}_{x_3}; \dot{H}^{-1}(\mathbb{R}^2))$; let w_0 be a smooth divergence free vector field on $\mathbb{R}^3$. Then if ε is small enough, the initial data*

$$u_0^\varepsilon(x) = (v_0^h + \varepsilon w^h, w_0^3)(x_h, \varepsilon x_3)$$

generates a unique, global solution u^ε of (NS).

Remarks

- A typical example of vector fields v_0^h satisfying the hypothesis is $v_0^h = (-\partial_2\phi, \partial_1\phi, 0)$ where ϕ is a function of the Schwarz class $\mathcal{S}(\mathbb{R}^3)$.
- This class of examples of initial data corresponds to a "well prepared" case. The "ill prepared" case would correspond to the case when the horizontal divergence of the initial data is of size ε^α with α less than 1, and the vertical component of the initial data is of size $\varepsilon^{\alpha-1}$. This case is certainly very interesting to understand, but that goes probably far beyond the methods used in this paper.
- We have to check that the initial data may be large. This is ensured by the following proposition.

Proposition 1.1. *Let (f, g) be in $\mathcal{S}(\mathbb{R}^3)$. Let us define $h^\varepsilon(x_h, x_3) \stackrel{\text{def}}{=} f(x_h)g(\varepsilon x_3)$. We have, if ε is small enough,*

$$\|h^\varepsilon\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)} \geq \frac{1}{4}\|f\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^2)}\|g\|_{L^\infty(\mathbb{R})}.$$

Proof. By the definition of $\|\cdot\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)}$ given by (1.3), we have to bound from below the quantity $\|e^{t\Delta}h^\varepsilon\|_{L^\infty(\mathbb{R}^3)}$. Let us write that

$$(e^{t\Delta}h^\varepsilon)(t, x) = (e^{t\Delta_h}f)(t, x_h)(e^{t\partial_3^2}g)(\varepsilon^2 t, \varepsilon x_3).$$

Let us consider a positive time t_0 such that

$$t_0^{\frac{1}{2}}\|e^{t_0\Delta_h}f\|_{L^\infty(\mathbb{R}^2)} \geq \frac{1}{2}\|f\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^2)}.$$

Then we have

$$\begin{aligned} t_0^{\frac12}\|e^{t_0\Delta}h^\varepsilon\|_{L^\infty(\mathbb{R}^3)} &= t_0^{\frac12}\|e^{t_0\Delta_h}f\|_{L^\infty(\mathbb{R}^2)}\|(e^{t_0\partial_3^2}g)(\varepsilon^2 t_0,\varepsilon\cdot)\|_{L^\infty(\mathbb{R})} \\ &\geq \frac12\|f\|_{\dot B^{-1}_{\infty,\infty}(\mathbb{R}^2)}\|e^{\varepsilon^2 t_0\partial_3^2}g\|_{L^\infty(\mathbb{R})}. \end{aligned}$$

As $\lim_{\varepsilon\to 0} e^{\varepsilon^2 t_0\partial_3^2}g = g$ in $L^\infty(\mathbb{R})$, the proposition is proved. □

Structure of the paper: The proof of Theorem 3 is achieved in the next section, assuming two crucial lemmas. The proof of those lemmas is postponed to Sections 3 and 4 respectively.

Notation: If A and B are two real numbers, we shall write $A \lesssim B$ if there is a universal constant C, which does not depend on varying parameters of the problem, such that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, then we shall write $A \sim B$.

If v_0 is a vector field, then we shall denote by C_{v_0} a constant depending only on norms of v_0. Similarly we shall use the notation C_{v_0,w_0} if the constant depends on norms of two vector fields v_0 and w_0, etc.

A function space with a subscript "h" (for "horizontal") will denote a space defined on $\mathbb{R}^2$, while the subscript "v" (for "vertical") will denote a space defined on $\mathbb{R}$. For instance $L^p_h \stackrel{\text{def}}{=} L^p(\mathbb{R}^2)$, $L^q_v \stackrel{\text{def}}{=} L^q(\mathbb{R})$, and similarly for Sobolev spaces or for mixed spaces such as $L^p_v L^q_h$ or $L^p_v \dot H^\sigma_h$.

2. Proof of the theorem

The proof of Theorem 3 consists in constructing an approximate solution to (NS) as a perturbation to the 2D Navier-Stokes system. Following the idea that we are close to the two dimensional, periodic incompressible Navier-Stokes system, let us define $\underline{v}^h$ as the solution of the following system, where $y_3 \in \mathbb{R}$ is a parameter:

$$(NS2D_3)\left\{\begin{array}{c} \partial_t \underline{v}^h + \underline{v}^h\cdot\nabla_h\underline{v}^h - \Delta_h\underline{v}^h = -\nabla_h\underline{p}_0 \quad \text{in } \mathbb{R}^+\times\mathbb{R}^2 \\ \operatorname{div}_h \underline{v}^h = 0 \\ \underline{v}^h_{|t=0} = v_0^h(\cdot,y_3). \end{array}\right.$$

This system is globally wellposed for any $y_3 \in \mathbb{R}$, and the solution is smooth in (two dimensional) space, and in time. Let us consider the solution $\underline{w}^\varepsilon$ of the linear equation

$$(T^\varepsilon_{\underline{v}})\left\{\begin{array}{c} \partial_t \underline{w}^\varepsilon + \underline{v}^h\cdot\nabla_h\underline{w}^\varepsilon - \Delta_h\underline{w}^\varepsilon - \varepsilon^2\partial_3^2\underline{w}^\varepsilon = -(\nabla^h\underline{p}_1, \varepsilon^2\partial_3\underline{p}_1) \quad \text{in } \mathbb{R}^+\times\mathbb{R}^3 \\ \operatorname{div}\underline{w}^\varepsilon = 0 \\ \underline{w}^\varepsilon_{|t=0} = w_0, \end{array}\right.$$

and let us define the approximate solution

$$\begin{aligned} v^\varepsilon_{app}(t,x) &= ((\underline{v}^h,0) + \varepsilon(\underline{w}^{\varepsilon,h}, \varepsilon^{-1}\underline{w}^{\varepsilon,3}))(t,x_h,\varepsilon x_3) \quad \text{and} \\ p^\varepsilon_{app}(t,x) &= (\underline{p}_0 + \varepsilon\underline{p}_1)(t,x_h,\varepsilon x_3). \end{aligned} \tag{2.1}$$

Finally let us consider the unique smooth solution u^ε of (NS) associated with the initial data u_0^ε on its maximal time interval of existence $[0,T_\varepsilon)$. The proof of Theorem 3 consists in

proving global in time estimates on v^ε_{app}, in order to prove that $R^\varepsilon \stackrel{\text{def}}{=} u^\varepsilon - v^\varepsilon_{app}$ remains small, globally in time; this ensures the global regularity for (NS).

More precisely, the proof of Theorem 3 relies on the following two lemmas, whose proofs are postponed to Sections 3 and 4 respectively.

Lemma 2.1. *The vector field v^ε_{app} defined in (2.1) satisfies the following estimate:*

$$\|v^\varepsilon_{app}\|_{L^2(\mathbb{R}^+;L^\infty(\mathbb{R}^3))} + \|\nabla v^\varepsilon_{app}\|_{L^2(\mathbb{R}^+;L^\infty_v L^2_h)} \leq C_{v_0,w_0}.$$

Lemma 2.2. *The vector field $R^\varepsilon \stackrel{\text{def}}{=} u^\varepsilon - v^\varepsilon_{app}$ satisfies the equation*

$$(E^\varepsilon)\left\{\begin{array}{c}\partial_t R^\varepsilon + R^\varepsilon\cdot\nabla R^\varepsilon - \Delta R^\varepsilon + v^\varepsilon_{app}\cdot\nabla R^\varepsilon + R^\varepsilon\cdot\nabla v^\varepsilon_{app} = F^\varepsilon - \nabla q^\varepsilon\\ \operatorname{div} R^\varepsilon = 0\\ R^\varepsilon_{|t=0} = 0\end{array}\right.$$

with $\|F^\varepsilon\|_{L^2(\mathbb{R}^+;\dot H^{-\frac12}(\mathbb{R}^3))} \leq C_{v_0,w_0}\varepsilon^{\frac13}$.

Let us postpone the proof of those lemmas and conclude the proof of Theorem 3. We denote, for any positive λ,

$$V_\varepsilon(t) \stackrel{\text{def}}{=} \|v^\varepsilon_{app}(t,\cdot)\|^2_{L^\infty(\mathbb{R}^3)} + \|\nabla v^\varepsilon_{app}(t,\cdot)\|^2_{L^\infty_v L^2_h} \quad\text{and}\quad R^\varepsilon_\lambda(t) \stackrel{\text{def}}{=} \exp\left(-\lambda\int_0^t V_\varepsilon(t')dt'\right)R^\varepsilon(t).$$

Lemma 2.1 implies that $I_0 \stackrel{\text{def}}{=} \displaystyle\int_0^\infty V_\varepsilon(t)dt$ is finite. By an $\dot H^{\frac12}$ energy estimate in $\mathbb{R}^3$, we get

$$\begin{aligned}\frac12\frac{d}{dt}\|R^\varepsilon_\lambda(t)\|^2_{\dot H^{\frac12}} + \|\nabla R^\varepsilon_\lambda(t)\|^2_{\dot H^{\frac12}} &\leq -2\lambda V_\varepsilon(t)\|R^\varepsilon_\lambda(t)\|^2_{\dot H^{\frac12}} + e^{\lambda I_0}\left|(R^\varepsilon_\lambda(t)\cdot\nabla R^\varepsilon_\lambda(t)|R^\varepsilon_\lambda(t))_{\dot H^{\frac12}}\right|\\ &+ \left|(R^\varepsilon_\lambda(t)\cdot\nabla v^\varepsilon_{app}(t)|R^\varepsilon_\lambda(t))_{\dot H^{\frac12}}\right| + \left|(v^\varepsilon_{app}(t)\cdot R^\varepsilon_\lambda(t)|R^\varepsilon_\lambda(t))_{\dot H^{\frac12}}\right| + \left|(F^\varepsilon(t)|R^\varepsilon_\lambda(t))_{\dot H^{\frac12}}\right|.\end{aligned}$$

The estimate (i) of Lemma 1.1 of [2] claims in particular that, for any $s\in]-d/2,d/2[$, for any divergence free vector field a in d space dimensions and any function b, we have

$$(a\cdot\nabla b|b)_{\dot H^s} \leq C\|\nabla a\|_{\dot H^{\frac d2-1}}\|b\|_{\dot H^s}\|\nabla b\|_{\dot H^s}. \tag{2.2}$$

Applying with $d=3$ and $s=1/2$, this implies that

$$\left|(R^\varepsilon_\lambda(t)\cdot\nabla R^\varepsilon_\lambda(t)|R^\varepsilon_\lambda(t))_{\dot H^{\frac12}}\right| \lesssim \|R^\varepsilon_\lambda(t)\|_{\dot H^{\frac12}}\|\nabla R^\varepsilon_\lambda(t)\|^2_{\dot H^{\frac12}}. \tag{2.3}$$

In order to estimate the other non linear terms, let us establish the following lemma.

Lemma 2.3. *Let a and b be two vector fields. We have*

$$\left|(a\cdot\nabla b|b)_{\dot H^{\frac12}}\right| + \left|(b\cdot\nabla a|b)_{\dot H^{\frac12}}\right| \lesssim \left(\|a\|_{L^\infty} + \|\nabla a\|_{L^\infty_v(L^2_h)}\right)\|b\|_{\dot H^{\frac12}}\|\nabla b\|_{\dot H^{\frac12}}.$$

Proof. By definition of the $\dot H^{\frac12}$ scalar product, we have

$$\begin{aligned}(a\cdot\nabla b|b)_{\dot H^{\frac12}} &\leq \|a\cdot\nabla b\|_{L^2}\|\nabla b\|_{L^2}\\ &\leq \|a\|_{L^\infty}\|\nabla b\|^2_{L^2}.\end{aligned}$$

The interpolation inequality between Sobolev norm gives

$$(a\cdot\nabla b|b)_{\dot H^{\frac12}} \leq \|a\|_{L^\infty}\|b\|_{\dot H^{\frac12}}\|\nabla b\|_{\dot H^{\frac12}}.$$

Now let us estimate $(b\cdot\nabla a|b)_{\dot H^{\frac12}}$. Again we use that

$$(b\cdot\nabla a|b)_{\dot H^{\frac12}}\le\|b\cdot\nabla a\|_{L^2}\|\nabla b\|_{L^2}.$$

Then let us write that

$$\|b\cdot\nabla a\|_{L^2}^2=\int_{\mathbb{R}^3}|b(x_h,x_3)\nabla a(x_h,x_3)|^2dx_hdx_3.$$

Gagliardo-Nirenberg's inequality in the horizontal variable implies that

$$\forall x_3\in\mathbb{R},\quad |b(x_h,x_3)|^2\lesssim\|b(\cdot,x_3)\|_{\dot H_h^{\frac12}}\|\nabla_h b(\cdot,x_3)\|_{\dot H_h^{\frac12}}.$$

Let us use the Cauchy-Schwarz inequality; this gives

$$\begin{aligned}\|b\cdot\nabla a\|_{L^2}^2&\le\int_{\mathbb{R}}\|b(\cdot,x_3)\|_{\dot H_h^{\frac12}}\|\nabla_h b(\cdot,x_3)\|_{\dot H_h^{\frac12}}\|\nabla a(\cdot,x_3)\|_{L_h^2}^2dx_3\\&\le\|\nabla a\|_{L_v^\infty(L_h^2)}^2\int_{\mathbb{R}}\|b(\cdot,x_3)\|_{\dot H_h^{\frac12}}\|\nabla_h b(\cdot,x_3)\|_{\dot H_h^{\frac12}}dx_3\\&\le\|\nabla a\|_{L_v^\infty L_h^2}^2\|b\|_{L_v^2\dot H_h^{\frac12}}\|\nabla_h b\|_{L_v^2\dot H_h^{\frac12}}.\end{aligned}$$

When s is positive, we have, thanks to Fourier-Plancherel in the vertical variable,

$$\begin{aligned}\|b\|_{L_v^2(\dot H_h^s)}&=\int_{\mathbb{R}}\int_{\mathbb{R}^2}|\xi_h|^{2s}|\mathcal{F}_h b(\xi_h,x_3)|^2d\xi_hdx_3\\&\sim\int_{\mathbb{R}}\int_{\mathbb{R}^2}|\xi_h|^{2s}|\widehat{b}(\xi_h,\xi_3)|^2d\xi_hd\xi_3\\&\lesssim\int_{\mathbb{R}^3}|\xi|^{2s}|\widehat{b}(\xi)|^2d\xi\\&\lesssim\|b\|_{\dot H^s}^2.\end{aligned}$$

This concludes the proof of Lemma 2.3. □

Conclusion of the proof of Theorem 3. We infer from the above lemma that

$$\begin{aligned}&\left|(R_\lambda^\varepsilon(t)\cdot\nabla v_{app}^\varepsilon(t)|R_\lambda^\varepsilon(t))_{\dot H^{\frac12}}\right|+\left|(v_{app}^\varepsilon(t)\cdot R_\lambda^\varepsilon(t)|R_\lambda^\varepsilon(t))_{\dot H^{\frac12}}\right|\\&\qquad\le\frac14\|\nabla R_\lambda^\varepsilon(t)\|_{\dot H^{\frac12}}^2+CV_\varepsilon(t)\|R_\lambda^\varepsilon(t)\|_{\dot H^{\frac12}}^2.\end{aligned}$$

Together with (2.3), this gives

$$\begin{aligned}\frac12\frac{d}{dt}\|R_\lambda^\varepsilon(t)\|_{\dot H^{\frac12}}^2+\|\nabla R_\lambda^\varepsilon(t)\|_{\dot H^{\frac12}}^2&\le(C-2\lambda)V_\varepsilon(t)\|R_\lambda^\varepsilon(t)\|_{\dot H^{\frac12}}^2\\&\quad+Ce^{\lambda I_0}\|R_\lambda^\varepsilon(t)\|_{\dot H^{\frac12}}\|\nabla R_\lambda^\varepsilon(t)\|_{\dot H^{\frac12}}^2+C\|F_\varepsilon(t)\|_{\dot H^{\frac12}}^2.\end{aligned}$$

Choosing λ such that $C-2\lambda$ is negative, we infer that

$$\frac{d}{dt}\|R_\lambda^\varepsilon(t)\|_{\dot H^{\frac12}}^2+(1-Ce^{\lambda I_0})\|\nabla R_\lambda^\varepsilon(t)\|_{\dot H^{\frac12}}^2\le C\|F_\varepsilon(t)\|_{\dot H^{\frac12}}^2.$$

Since $R^\varepsilon(0)=0$, we get, as long as $\|R_\lambda^\varepsilon(t)\|_{\dot H^{\frac12}}$ is less or equal to $1/2Ce^{-\lambda I_0}$, that

$$\|R_\lambda^\varepsilon(t)\|_{\dot H^{\frac12}}^2+\frac12\int_0^t\|\nabla R_\lambda^\varepsilon(t')\|_{\dot H^{\frac12}}^2dt'\le C_{v_0,w_0}\varepsilon^{\frac13}.$$

We therefore obtain that R^ε goes to zero in $L^\infty(\mathbb{R}^+;\dot H^{\frac12})\cap L^2(\mathbb{R}^+;\dot H^{\frac32})$. That implies that u^ε remains close for all times to v^ε_{app}, which in particular implies Theorem 3. □

3. Estimates on the approximate solution

In this section we shall prove Lemma 2.1 stated in the previous section. The proof of the lemma is achieved by obtaining estimates on $\underline{v}$, stated in the next lemma, as well as on $\underline{w}^\varepsilon$ (see Lemma 3.2 below).

Lemma 3.1. *Let $\underline{v}^h$ be a solution of the system $(NS2D_3)$. Then, for any s greater than -1 and any $\alpha\in\mathbb{N}^3$, we have, for any y_3 in $\mathbb{R}$ and for any positive t,*

$$\|\partial^\alpha\underline{v}^h(t,\cdot,y_3)\|^2_{\dot H^s_h}+\int_0^t\|\partial^\alpha\nabla_h\underline{v}^h(t',\cdot,y_3)\|^2_{\dot H^s_h}dt'\le C_{v_0}(y_3),$$

where $C_{v_0}(\cdot)$ belongs to $L^1\cap L^\infty(\mathbb{R})$ and its norm is controled by a constant C_{v_0}.

Proof. For $s=0$ and $\alpha=0$, the estimate is simply the classical L^2-energy estimate with y_3 as a parameter: writing $\underline{v}=(\underline{v}^h,0)$ we have

$$\|\underline{v}(t,\cdot,y_3)\|^2_{L^2_h}+2\int_0^t\|\nabla_h\underline{v}(t',\cdot,y_3)\|^2_{L^2_h}dt'=\|\underline{v}_0(\cdot,y_3)\|^2_{L^2_h}. \tag{3.1}$$

In the case when $\alpha=0$, the estimate (i) in Lemma 1.1 of [2] gives, for any s greater than -1,

$$\frac12\frac{d}{dt}\|\underline{v}(t,\cdot,y_3)\|^2_{\dot H^s_h}+\|\nabla_h\underline{v}(t,\cdot,y_3)\|^2_{\dot H^s_h}\le C\|\nabla_h\underline{v}(t,\cdot,y_3)\|_{L^2_h}\|\underline{v}(t,\cdot,y_3)\|_{\dot H^s_h}\|\nabla_h\underline{v}(t,\cdot,y_3)\|_{\dot H^s_h}.$$

We infer that

$$\frac{d}{dt}\|\underline{v}(t,\cdot,y_3)\|^2_{\dot H^s_h}+\|\nabla_h\underline{v}(t,\cdot,y_3)\|^2_{\dot H^s_h}\le C\|\nabla_h\underline{v}(t,\cdot,y_3)\|^2_{L^2_h}\|\underline{v}(t,\cdot,y_3)\|^2_{\dot H^s_h}.$$

Gronwall's lemma ensures that

$$\|\underline{v}(t,\cdot,y_3)\|^2_{\dot H^s_h}+\int_0^t\|\nabla_h\underline{v}(t',\cdot,y_3)\|^2_{\dot H^s_h}dt'\le\|\underline{v}_0(\cdot,y_3)\|^2_{\dot H^s_h}\exp\Big(C\int_0^t\|\nabla_h\underline{v}(t',\cdot,y_3)\|^2_{L^2_h}dt'\Big).$$

The energy estimate (3.1) implies that

$$\|\underline{v}(t,\cdot,y_3)\|^2_{\dot H^s_h}+\int_0^t\|\nabla_h\underline{v}(t',\cdot,y_3)\|^2_{\dot H^s_h}dt'\le\|\underline{v}_0(\cdot,y_3)\|^2_{\dot H^s_h}\exp\left(C\|\underline{v}_0\|^2_{L^\infty_v L^2_h}\right).$$

This proves the lemma in the case when $\alpha=0$. Let us now turn to the general case, by induction on the length of α. It is clear that in the proof, we can restrict ourselves to the case when $s\in]-1,1[$.

Let us assume that, for some $k\in\mathbb{N}$,

$$\forall s\in]-1,1[,\quad\sum_{|\alpha|\le k}\left(\|\partial^\alpha\underline{v}(t,\cdot,y_3)\|^2_{\dot H^s_h}+\int_0^t\|\partial^\alpha\nabla_h\underline{v}(t',\cdot,y_3)\|^2_{\dot H^s_h}dt'\right)\le C_{k,v_0}(y_3), \tag{3.2}$$

with $C_{k,v_0}(\cdot)\in L^1\cap L^\infty(\mathbb{R})$.

Thanks to the Leibnitz formula we have, for $|\alpha| \leq k+1$,

$$\partial_t \partial^\alpha \underline{v}^h + \underline{v}^h \cdot \nabla_h \partial^\alpha \underline{v}^h - \Delta_h \partial^\alpha \underline{v}^h = -\nabla_h p_\alpha - \sum_{\substack{\beta \leq \alpha \\ \beta \neq \alpha}} C_\alpha^\beta \partial^{\alpha-\beta} \underline{v}^h \cdot \nabla_h \, \partial^\beta \underline{v}^h.$$

Performing a $\dot{H}^s_h$ energy estimate in the horizontal variable and using the estimate (2.2) in the case when $d = 2$ gives

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\partial^\alpha \underline{v}(t,\cdot,y_3)\|^2_{\dot{H}^s_h} + \|\nabla_h \partial^\alpha \underline{v}(t,\cdot,y_3)\|^2_{\dot{H}^s_h} \\ &\leq C\|\nabla_h \underline{v}(t,\cdot,y_3)\|_{L^2_h}\|\partial^\alpha \underline{v}(t,\cdot,y_3)\|_{\dot{H}^s_h}\|\nabla_h \partial^\alpha \underline{v}(t,\cdot,y_3)\|_{\dot{H}^s_h} \\ &+ C_\alpha \sum_{\substack{\beta \leq \alpha \\ \beta \neq \alpha}} \Big| \Big(\partial^{\alpha-\beta} \underline{v}^h(t,\cdot,y_3) \cdot \nabla_h \, \partial^\beta \underline{v}^h(t,\cdot,y_3) \big| \partial^\alpha \underline{v} h(t,\cdot,y_3) \Big)_{\dot{H}^s_h} \Big|. \end{aligned}$$

To estimate the last term, we shall treat differently the case $|\beta| = 0$ and $|\beta| \neq 0$. In the first case, we notice first that when $s = 0$, laws of product for Sobolev spaces in $\mathbb{R}^2$ give

$$\begin{aligned} \Big(\partial^\alpha \underline{v}^h(t,\cdot,y_3) \cdot \nabla_h \underline{v}^h(t,\cdot,y_3) \big| \partial^\alpha \underline{v}^h(t,\cdot,y_3) \Big)_{L^2_h} &\lesssim \|\partial^\alpha \underline{v}^h(t,\cdot,y_3)\|^2_{\dot{H}^{\frac{1}{2}}} \|\nabla_h \underline{v}^h(t,\cdot,y_3)\|_{L^2} \\ &\lesssim \|\partial^\alpha \underline{v}^h(t,\cdot,y_3)\|_{L^2_h} \|\nabla_h \partial^\alpha \underline{v}^h(t,\cdot,y_3)\|_{L^2_h} \|\nabla_h \underline{v}^h(t,\cdot,y_3)\|_{L^2_h}. \end{aligned}$$

If $s > 0$, then again laws of product for Sobolev spaces in $\mathbb{R}^2$ give, for $s \in]0,1[$,

$$\Big| \Big(\partial^\alpha \underline{v}^h(t,\cdot,y_3) \cdot \nabla_h \underline{v}^h(t,\cdot,y_3) \big| \partial^\alpha \underline{v}(t,\cdot,y_3) \Big)_{\dot{H}^s_h} \Big| \lesssim \|\partial^\alpha \underline{v}^h\|_{\dot{H}^s_h} \|\nabla_h \underline{v}^h\|_{L^2_h} \|\nabla_h \partial^\alpha \underline{v}^h\|_{\dot{H}^s_h},$$

whereas if $-1 < s < 0$,

$$\Big| \Big(\partial^\alpha \underline{v}^h(t,\cdot,y_3) \cdot \nabla_h \underline{v}^h(t,\cdot,y_3) \big| \partial^\alpha \underline{v}(t,\cdot,y_3) \Big)_{\dot{H}^s_h} \Big| \lesssim \|\nabla_h \partial^\alpha \underline{v}^h\|_{\dot{H}^s_h} \|\nabla_h \underline{v}^h\|_{L^2_h} \|\partial^\alpha \underline{v}^h\|_{\dot{H}^s_h}.$$

So in any case we have

$$\Big| \Big(\partial^\alpha \underline{v}^h(t,\cdot,y_3) \cdot \nabla_h \underline{v}^h(t,\cdot,y_3) \big| \partial^\alpha \underline{v}(t,\cdot,y_3) \Big)_{\dot{H}^s_h} \Big| \leq \frac{1}{4}\|\nabla_h \partial^\alpha \underline{v}^h\|^2_{\dot{H}^s_h} + C\|\nabla_h \underline{v}^h\|^2_{L^2_h}\|\partial^\alpha \underline{v}^h\|^2_{\dot{H}^s_h}.$$

Now let us consider the case when $|\beta| \neq 0$. As the horizontal divergence of $\underline{v}$ is identically 0, we have

$$\Big| \Big(\partial^{\alpha-\beta} \underline{v}^h \cdot \nabla_h \, \partial^\beta \underline{v}^h \big| \partial^\alpha \underline{v}(t,\cdot,y_3) \Big)_{\dot{H}^s_h} \Big| \leq \|\partial^{\alpha-\beta} \underline{v}^h(t,\cdot,y_3) \otimes \partial^\beta \underline{v}^h(t,\cdot,y_3)\|_{\dot{H}^s_h} \|\nabla_h \partial^\alpha \underline{v}(t,\cdot,y_3)\|_{\dot{H}^s_h}.$$

Laws of product for Sobolev spaces in $\mathbb{R}^2$ give, for $s \in]-1,1[$,

$$\|\partial^{\alpha-\beta} \underline{v}^h \otimes \partial^\beta \underline{v}^h\|_{\dot{H}^s_h} \leq C\|\partial^{\alpha-\beta} \underline{v}^h\|_{\dot{H}^{s'}_h} \|\partial^\beta \nabla_h \underline{v}^h\|_{\dot{H}^{s-s'}_h},$$

where s' is chosen so that $s < s' < 1$.

Finally we deduce that

$$\begin{aligned} \frac{d}{dt}\|\partial^\alpha \underline{v}(t,\cdot,y_3)\|^2_{\dot{H}^s_h} + \|\nabla_h \partial^\alpha \underline{v}(t,\cdot,y_3)\|^2_{\dot{H}^s_h} &\leq C\|\nabla_h \underline{v}\|^2_{L^2_h}\|\partial^\alpha \underline{v}\|^2_{\dot{H}^s_h} \\ &+ C_\alpha \sum_{\substack{\beta \leq \alpha \\ \beta \notin \{0,\alpha\}}} \|\partial^{\alpha-\beta} \underline{v}(t,\cdot,y_3)\|_{\dot{H}^{s'}_h} \|\partial^\beta \nabla_h \underline{v}(t,\cdot,y_3)\|_{\dot{H}^{s-s'}_h} \|\nabla_h \partial^\alpha \underline{v}(t,\cdot,y_3)\|_{\dot{H}^s_h}. \end{aligned}$$

Gronwall's lemma together with the induction hypothesis (3.2) implies that

$$\sum_{|\alpha|=k+1}\Big(\|\partial^\alpha \underline{v}(t,\cdot,y_3)\|^2_{\dot H^s_h}+\int_0^t\|\partial^\alpha\nabla_h\underline{v}(t',\cdot,y_3)\|^2_{\dot H^s_h}dt'\Big)$$
$$\lesssim\Big(\sum_{|\alpha|=k+1}\|\partial^\alpha\underline{v}_0(\cdot,y_3)\|^2_{\dot H^s_h}+C_{k,v_0}(y_3)\Big)\exp\Big(C_k\int_0^t\|\nabla_h\underline{v}(t',\cdot,y_3)\|^2_{L^2_h}dt'\Big).$$

The L^2 energy estimate (3.1) allows to conclude the proof of Lemma 3.1. □

From this lemma, we deduce the following corollary.

Corollary 3.1. *Let $\underline{v}^h$ be a solution of the system $(NS2D_3)$. Then, for any non negative σ, we have*

$$\|\underline{v}^h\|_{L^2(\mathbb{R}^+;\dot H^\sigma(\mathbb{R}^3))}\le C_{v_0}\quad\text{and}\quad\|\partial^\alpha\underline{v}^h\|_{L^2(\mathbb{R}^+;L^\infty_v\dot H^\sigma_h)}\le C_{v_0}.$$

Proof. To start with, let us assume $\sigma>0$. Lemma 3.1 applied with $s=\sigma-1$ implies that

$$\forall\sigma>0\,,\ \forall\alpha\in\mathbb{N}\,,\ \|\partial^\alpha\underline{v}^h\|_{L^2(\mathbb{R}^+;L^2_v\dot H^\sigma_h)}\le C_{v_0}. \tag{3.3}$$

Then, for any non negative σ, we have

$$\begin{aligned}\|\partial^\alpha\underline{v}(t,\cdot,y_3)\|^2_{\dot H^\sigma_h}&=2\int_{-\infty}^{y_3}(\partial_3\partial^\alpha\underline{v}(t,\cdot,y_3')|\partial^\alpha\underline{v}(t,\cdot,y_3')_{\dot H^\sigma_h}dy_3'\\&\le 2\|\partial_3\partial^\alpha\underline{v}^h(t,\cdot)\|_{L^2_v\dot H^\sigma_h}\|\partial^\alpha\underline{v}^h(t,\cdot)\|_{L^2_v\dot H^\sigma_h}\end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$\forall\sigma\ge0\,,\ \forall\alpha\in\mathbb{N}\,,\ \|\partial^\alpha\underline{v}^h\|_{L^2(\mathbb{R}^+;L^\infty_v\dot H^\sigma_h)}\le\|\partial_3\partial^\alpha\underline{v}^h\|^{\frac12}_{L^2(\mathbb{R}^+;L^\infty_v\dot H^\sigma_h)}\|\partial^\alpha\underline{v}^h\|^{\frac12}_{L^2(\mathbb{R}^+;L^\infty_v\dot H^\sigma_h)}.$$

From (3.3), we infer

$$\forall\sigma>0\,,\ \forall\alpha\in\mathbb{N}\,,\ \|\partial^\alpha\underline{v}^h\|_{L^2(\mathbb{R}^+;L^\infty_v\dot H^\sigma_h)}\le C_{v_0}. \tag{3.4}$$

Now, by interpolation, it is enough to prove the first inequality with $\sigma=0$. The system $(NS2D_3)$ can be written

$$\begin{cases}\partial_t\underline{v}-\Delta_h\underline{v}=f\\ \underline{v}_{|t=0}=\underline{v}_0(\cdot,y_3)\end{cases}\quad\text{with}\quad f\stackrel{\text{def}}{=}\sum_{1\le j,k\le2}Q_{j,k}(D)(\underline{v}^j\underline{v}^k).$$

where $Q_{j,k}$ are homogenenous smooth Fourier multipliers of order 1. By Sobolev embeddings in $\mathbb{R}^2$, we get, for any y_3 in $\mathbb{R}$,

$$\begin{aligned}\|\underline{v}(\cdot,y_3)\|_{L^2(\mathbb{R}^+\times\mathbb{R}^2)}&\le\|\underline{v}_0(\cdot,y_3)\|_{\dot H^{-1}_h}+\|f(\cdot,y_3)\|_{L^1(\mathbb{R}^+;\dot H^{-1}_h)}\\&\le\|\underline{v}_0(\cdot,y_3)\|_{\dot H^{-1}_h}+C\|\underline{v}(\cdot,y_3)\|^2_{L^2(\mathbb{R}^+;\dot H^{\frac12}_h)}\\&\le\|\underline{v}_0(\cdot,y_3)\|_{\dot H^{-1}_h}+C\|\underline{v}(\cdot,y_3)\|_{L^2(\mathbb{R}^+;\dot H^{\frac12}_h)}\sup_{y_3}\|\underline{v}(\cdot,y_3)\|_{L^2(\mathbb{R}^+;\dot H^{\frac12}_h)}.\end{aligned}$$

As $\sup_{y_3}\|\underline{v}(\cdot,y_3)\|_{L^2(\mathbb{R}^+;\dot H^{\frac12}_h)}\le\|\underline{v}\|_{L^2(\mathbb{R}^+;L^\infty_v\dot H^{\frac12}_h)}$, we infer from (3.4)

$$\|\underline{v}\|^2_{L^2(\mathbb{R}^+\times\mathbb{R}^3)}\lesssim\|\underline{v}_0\|^2_{L^2_v\dot H^{-1}_h}+C_{v_0}\|\underline{v}\|^2_{L^2(\mathbb{R}^+;L^2_v\dot H^{\frac12}_h)}\le C_{v_0}.$$

The corollary is proved. □

Finally we have the following estimate on $\underline{w}^\varepsilon$.

Lemma 3.2. *Let $\underline{w}^\varepsilon$ be a solution of the system $(T^\varepsilon_{\underline{v}})$. Then, for any s greater than -1 and any $\alpha \in \mathbb{N}^3$ and for any positive t, we have*

$$\|\partial^\alpha \underline{w}^\varepsilon(t,\cdot)\|^2_{L^2_v \dot{H}^s_h} + \int_0^t \|\partial^\alpha \nabla_h \underline{w}^\varepsilon(t',\cdot)\|^2_{L^2_v \dot{H}^s_h} dt' \leq C_{v_0,w_0}.$$

Proof. We shall only sketch the proof, as it is very close to the proof of Lemma 3.1 which was carried out above. The only difference is that the horizontal divergence of $\underline{w}$ does not vanish identically, but that will not change very much the estimates. We shall only write the proof in the case when $\alpha = 0$ and $-1 < s < 1$, and leave the general case to the reader. Using Lemma 1.1 of [2] we have, for any y_3 in $\mathbb{R}$,

$$\Big(\underline{v}^h(t,\cdot,y_3)\cdot\nabla_h \underline{w}^\varepsilon(t,\cdot,y_3)|\underline{w}^\varepsilon(t,\cdot,y_3)\Big)_{\dot{H}^s_h} \leq C\|\nabla_h \underline{v}(t,\cdot,y_3)\|_{L^2_h}\|\nabla_h \underline{w}^\varepsilon(t,\cdot,y_3)\|_{\dot{H}^s_h}\|\underline{w}^\varepsilon(t,\cdot,y_3)\|_{\dot{H}^s_h}.$$

Thus we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\underline{w}^\varepsilon(t)\|^2_{L^2_v \dot{H}^s_h} + \|\nabla_h \underline{w}^\varepsilon(t)\|^2_{L^2_v \dot{H}^s_h} &\leq \frac{1}{4}\|\nabla_h \underline{w}^\varepsilon(t)\|^2_{L^2_v \dot{H}^s_h} + C\|\nabla_h \underline{v}^h(t)\|^2_{L^\infty_v L^2_h}\|\underline{w}^\varepsilon(t)\|^2_{L^2_v \dot{H}^s_h}\\ &\quad - \varepsilon^2\int_{\mathbb{R}}(\partial_3 \underline{p}_1(t,\cdot,y_3)|\underline{w}^{\varepsilon,3}(t,\cdot,y_3))_{\dot{H}^s_h}dy_3 - \int_{\mathbb{R}}(\nabla^h \underline{p}_1(t,\cdot,y_3)|\underline{w}^{\varepsilon,h}(t,\cdot,y_3))_{\dot{H}^s_h}dy_3.\end{aligned}$$

By integration by parts we have, thanks to the divergence free condition on $\underline{w}^\varepsilon$,

$$\begin{aligned}-\int_{\mathbb{R}}(\partial_3 \underline{p}_1(t,\cdot,y_3)|\underline{w}^{\varepsilon,3}(t,\cdot,y_3))_{\dot{H}^s_h}dy_3 &= \int_{\mathbb{R}}(\underline{p}_1(t,\cdot,y_3)|\partial_3 \underline{w}^{\varepsilon,3}(t,\cdot,y_3))_{\dot{H}^s_h}dy_3\\ &= -\int_{\mathbb{R}}(\underline{p}_1(t,\cdot,y_3)|\operatorname{div}_h \underline{w}^{\varepsilon,h}(t,\cdot,y_3))_{\dot{H}^s_h}dy_3.\end{aligned}$$

By definition of the inner product of $\dot{H}^s_h$, we get

$$-\int_{\mathbb{R}}(\partial_3 \underline{p}_1(t,\cdot,y_3)|\underline{w}^{\varepsilon,3}(t,\cdot,y_3))_{\dot{H}^s_h}dy_3 = \int_{\mathbb{R}}(\nabla_h \underline{p}_1(t,\cdot,y_3)|\underline{w}^{\varepsilon,h}(t,\cdot,y_3))_{\dot{H}^s_h}dy_3.$$

Thus we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\underline{w}^\varepsilon(t)\|^2_{L^2_v \dot{H}^s_h} + \|\nabla_h \underline{w}^\varepsilon(t)\|^2_{L^2_v \dot{H}^s_h} &\leq \frac{1}{4}\|\nabla_h \underline{w}^\varepsilon(t)\|^2_{L^2_v \dot{H}^s_h}\\ &\quad + C\|\nabla_h \underline{v}^h(t)\|^2_{L^\infty_v L^2_h}\|\underline{w}^\varepsilon(t)\|^2_{L^2_v \dot{H}^s_h} - (1-\varepsilon^2)\int_{\mathbb{R}}(\nabla_h \underline{p}_1(t,\cdot,y_3)|\underline{w}^{\varepsilon,h}(t,\cdot,y_3))_{\dot{H}^s_h}dy_3.\end{aligned}$$

Now we notice that

$$-(\varepsilon^2\partial_3^2 + \Delta_h)\underline{p}_1 = \operatorname{div}(\underline{v}^h\cdot\nabla_h \underline{w}^\varepsilon) = \operatorname{div}\sum_{j=1}^2 \partial_j(\underline{v}^j \underline{w}^\varepsilon),$$

which can be written in the simpler way

$$-(\varepsilon^2\partial_3^2 + \Delta_h)\underline{p}_1 = \operatorname{div}_h N^h$$

with $N^h = \underline{v}^h\cdot\nabla_h \underline{w}^{\varepsilon,h} + \partial_3(\underline{w}^{\varepsilon,3}\underline{v}^h)$. It is easy to check that for any $\sigma \in \mathbb{R}$,

$$\|\nabla_h \underline{p}_1\|_{L^2_v \dot{H}^\sigma_h} \lesssim \|N^h\|_{L^2_v \dot{H}^\sigma_h}, \tag{3.5}$$

simply by noticing that

$$\begin{aligned}\|\nabla_h \underline{p}_1\|^2_{L^2_v \dot{H}^\sigma_h} &\sim \int |\xi_h|^{2\sigma+2} |\widehat{\underline{p}_1}(\xi)|^2 \, d\xi \\ &\sim \int |\xi_h|^{2\sigma+4} |\widehat{N}^h(\xi)|^2 \, \frac{d\xi}{(|\xi_h|^2+\varepsilon^2|\xi_3|^2)^2} \\ &\lesssim \|N^h\|^2_{L^2_v \dot{H}^\sigma_h}.\end{aligned}$$

We infer from (3.5) that

$$\begin{aligned}\|\nabla_h \underline{p}_1\|_{L^2_v \dot{H}^\sigma_h} &\leq \|\underline{v}^h \cdot \nabla_h \underline{w}^{\varepsilon,h}\|_{L^2_v \dot{H}^\sigma_h} + \|\partial_3(\underline{w}^{\varepsilon,3}\underline{v}^h)\|_{L^2_v \dot{H}^\sigma_h} \\ &\leq \|\underline{v}^h \cdot \nabla_h \underline{w}^{\varepsilon}\|_{L^2_v \dot{H}^\sigma_h} + \|\underline{w}^{\varepsilon,3}\partial_3 \underline{v}\|_{L^2_v \dot{H}^\sigma_h} + \|\underline{v}\,\mathrm{div}_h\, \underline{w}^{\varepsilon,h}\|_{L^2_v \dot{H}^\sigma_h}.\end{aligned}$$

We claim that for all $-1 < s < 1$,

$$\begin{aligned}\mathcal{I}_h &\stackrel{\text{def}}{=} \int_{\mathbb{R}} |(\nabla_h \underline{p}_1(t,\cdot,y_3)|\underline{w}^\varepsilon(t,\cdot,y_3))|_{\dot{H}^s_h} dy_3 \\ &\leq \frac{1}{4}\|\nabla_h \underline{w}^\varepsilon\|^2_{L^2_v \dot{H}^s_h} + C\|\underline{w}^\varepsilon\|^2_{L^2_v \dot{H}^s_h}\|\nabla \underline{v}^h\|^2_{L^\infty_v L^2_h}(1+\|\underline{v}^h\|^2_{L^\infty_v L^2_h}).\end{aligned} \tag{3.6}$$

Let us prove the claim. Suppose first that $s = 0$. Then a product law gives

$$\begin{aligned}\mathcal{I}_h &\leq \|\underline{w}^\varepsilon\|_{L^2_v \dot{H}^{\frac12}_h}\|\nabla_h \underline{p}_1\|_{L^2_v \dot{H}^{-\frac12}_h} \\ &\lesssim \|\underline{w}^\varepsilon\|_{L^2_v \dot{H}^{\frac12}_h}\|\underline{v}^h\|_{L^\infty_v \dot{H}^{\frac12}_h}\|\nabla_h \underline{w}^\varepsilon\|_{L^2(\mathbb{R}^3)} + \|\underline{w}^\varepsilon\|^2_{L^2_v \dot{H}^{\frac12}_h}\|\partial_3 \underline{v}\|_{L^\infty_v L^2_h}.\end{aligned}$$

By interpolation, we get

$$\mathcal{I}_h \leq \|\nabla_h \underline{w}^\varepsilon\|^{\frac32}_{L^2(\mathbb{R}^3)}\|\underline{w}^\varepsilon\|^{\frac12}_{L^2(\mathbb{R}^3)}\|\underline{v}^h\|^{\frac12}_{L^\infty_v L^2_h}\|\nabla_h \underline{v}^h\|^{\frac12}_{L^\infty_v L^2_h} + \|\underline{w}^\varepsilon\|_{L^2(\mathbb{R}^3)}\|\nabla_h \underline{w}^\varepsilon\|_{L^2(\mathbb{R}^3)}\|\partial_3 \underline{v}\|_{L^\infty_v L^2_h}.$$

The claim in the case $s = 0$ follows from a convexity inequality, which gives

$$\mathcal{I}_h \leq \frac{1}{4}\|\nabla_h \underline{w}^\varepsilon\|^2_{L^2(\mathbb{R}^3)} + C\|\underline{w}^\varepsilon\|^2_{L^2(\mathbb{R}^3)}(1+\|\underline{v}\|^2_{L^\infty_v L^2_h})\|\nabla \underline{v}\|^2_{L^\infty_v L^2_h}.$$

In the case when $0 < s < 1$, then we use the product rule

$$\|\nabla_h \underline{p}_1\|_{L^2_v \dot{H}^{s-1}_h} \lesssim \|\underline{v}^h\|_{L^\infty_v \dot{H}^{\frac12}_h}\|\nabla_h \underline{w}^\varepsilon\|_{L^2_v \dot{H}^{s-\frac12}_h} + \|\underline{w}^\varepsilon\|_{L^2_v \dot{H}^s_h}\|\nabla \underline{v}\|_{L^\infty_v L^2_h}$$

along with the fact that

$$\mathcal{I}_h \leq \|\nabla_h \underline{p}_1\|_{L^2_v \dot{H}^{s-1}_h}\|\nabla_h \underline{w}^\varepsilon\|_{L^2_v \dot{H}^s_h}.$$

Finally in the case when $-1 < s < 0$, we write

$$\begin{aligned}\mathcal{I}_h &\leq \|\underline{w}^\varepsilon \cdot \nabla \underline{v}^h\|_{L^2_v \dot{H}^s_h}\|\underline{w}^\varepsilon\|_{L^2_v \dot{H}^s_h} + \|\underline{w}^\varepsilon \cdot \underline{v}^h\|_{L^2_v \dot{H}^s_h}\|\nabla_h \underline{w}^\varepsilon\|_{L^2_v \dot{H}^s_h} \\ &\lesssim \|\nabla \underline{v}^h\|_{L^\infty_v L^2_h}\|\underline{w}^\varepsilon\|_{L^2_v \dot{H}^s_h}\|\nabla_h \underline{w}^\varepsilon\|_{L^2_v \dot{H}^s_h} + \|\underline{v}^h\|_{L^\infty_v \dot{H}^{\frac12}_h}\|\underline{w}^\varepsilon\|_{L^2_v \dot{H}^{s+\frac12}_h}\|\nabla_h \underline{w}^\varepsilon\|_{L^2_v \dot{H}^s_h}.\end{aligned}$$

The claim (3.6) follows by interpolation.

Using that result we obtain that

$$\frac{d}{dt}\|\underline{w}^\varepsilon(t)\|^2_{L^2_v \dot{H}^s_h} + \|\nabla_h \underline{w}^\varepsilon(t)\|^2_{L^2_v \dot{H}^s_h} \lesssim \|\nabla \underline{v}^h(t)\|^2_{L^\infty_v L^2_h}\|\underline{w}^\varepsilon(t)\|^2_{L^2_v \dot{H}^s_h}(1+\|\underline{v}^h(t)\|^2_{L^\infty_v L^2_h})$$

and we conclude by a Gronwall lemma. Indeed we get that

$$\|\underline{w}^\varepsilon(t)\|^2_{L^2_v\dot H^s_h} + \int_0^t \|\nabla_h \underline{w}^\varepsilon(t')\|^2_{L^2_v\dot H^s_h}\,dt' \le \|w_0\|^2_{L^2_v\dot H^s_h} \\ \times \exp\left(C\int_0^t \|\nabla \underline{v}^h(t')\|^2_{L^\infty_v L^2_h}(1+\|\underline{v}^h(t')\|^2_{L^\infty_v L^2_h})\,dt'\right).$$

But by the basic energy estimate (3.1), we have

$$\|\underline{v}^h(t)\|_{L^\infty_v L^2_h} \le \|v_0\|_{L^\infty_v L^2_h}.$$

Moreover, Corollary 3.1 implies that

$$\nabla \underline{v}^h \in L^2(\mathbb{R}^+; L^\infty_v L^2_h),$$

so Lemma 3.2 is proved in the case when $\alpha = 0$. The case when α is positive is an easy adaptation of the proof of Lemma 3.1; it is left to the reader. □

Clearly Lemmas 3.1 and 3.2 allow to obtain Lemma 2.1 stated in the previous section.

4. The estimate of the error term

In this section we shall prove Lemma 2.2 stated above. We need to write down precisely the equation satisfied by the remainder term R^ε, and to check that the forcing terms appearing in the equation can be made small.

Let us recall that

$$\begin{aligned} v^\varepsilon_{app}(t,x) &= ((\underline{v}^h,0)+\varepsilon(\underline{w}^{\varepsilon,h},\varepsilon^{-1}\underline{w}^{\varepsilon,3}))(t,x_h,\varepsilon x_3) \quad \text{and} \\ p^\varepsilon_{app}(t,x) &= (\underline{p}_0+\varepsilon\underline{p}_1)(t,x_h,\varepsilon x_3). \end{aligned}$$

It is an easy computation to see that

$$\begin{aligned} &\left(\partial_t v^\varepsilon_{app} + v^\varepsilon_{app}\cdot\nabla v^\varepsilon_{app} - \Delta v^\varepsilon_{app}\right)(t,x_h,x_3) = (\partial_t\underline{v}^h + \underline{v}^h\cdot\nabla_h\underline{v}^h - \Delta_h\underline{v}^h,0)(t,x_h,\varepsilon x_3) \\ + \;& \varepsilon\left(\partial_t\underline{w}^{\varepsilon,h} + \underline{v}^h\cdot\nabla_h\underline{w}^{\varepsilon,h} - \Delta_h\underline{w}^{\varepsilon,h} - \varepsilon^2\partial_3^2\underline{w}^{\varepsilon,h},0\right)(t,x_h,\varepsilon x_3) \\ + \;& \left(0,\partial_t\underline{w}^{\varepsilon,3} + \underline{v}^h\cdot\nabla_h\underline{w}^{\varepsilon,3} - \Delta_h\underline{w}^{\varepsilon,3} - \varepsilon^2\partial_3^2\underline{w}^{\varepsilon,3}\right)(t,x_h,\varepsilon x_3) + \varepsilon\widetilde{F}^\varepsilon(t,x_h,\varepsilon x_3) \end{aligned}$$

where

$$\widetilde{F}^\varepsilon(t,x_h,y_3) \stackrel{\text{def}}{=} \left((\varepsilon\underline{w}^\varepsilon\cdot\nabla\underline{w}^{\varepsilon,h},\underline{w}^\varepsilon\cdot\nabla\underline{w}^{\varepsilon,3}) + (\underline{w}^\varepsilon\cdot\nabla\underline{v}^h,0) + \varepsilon(\partial_3^2\underline{v}^h,0)\right)(t,x_h,y_3).$$

In order to simplify the notation, let us write $\widetilde{F}^\varepsilon = \widetilde{F}^{\varepsilon,1} + \widetilde{F}^{\varepsilon,2}$ with

$$\begin{aligned} \widetilde{F}^{\varepsilon,1} &\stackrel{\text{def}}{=} (\varepsilon\underline{w}^\varepsilon\cdot\nabla\underline{w}^{\varepsilon,h},\underline{w}^\varepsilon\cdot\nabla\underline{w}^{\varepsilon,3}) + (\underline{w}^\varepsilon\cdot\nabla\underline{v}^h,0) \quad \text{and} \\ \widetilde{F}^{\varepsilon,2} &\stackrel{\text{def}}{=} \varepsilon(\partial_3^2\underline{v}^h,0). \end{aligned}$$

Recalling the equations satisfied by $\underline{v}^h$ and $\underline{w}^\varepsilon$, we infer that

$$\left(\partial_t v^\varepsilon_{app} + v^\varepsilon_{app}\cdot\nabla v^\varepsilon_{app} - \Delta v^\varepsilon_{app}\right)(t,x_h,x_3) = -\nabla p^\varepsilon_{app} + \varepsilon G^\varepsilon(t,x_h,\varepsilon x_3)$$

with $G^\varepsilon(t,x_h,y_3) \stackrel{\text{def}}{=} \left(\widetilde{F}^\varepsilon + (0,\partial_3\underline{p}_0)\right)(t,x_h,y_3)$ and $F^\varepsilon(t,x_h,x_3) \stackrel{\text{def}}{=} \varepsilon G^\varepsilon(t,x_h,\varepsilon x_3)$. Denoting $q^\varepsilon = p^\varepsilon - p^\varepsilon_{app}$, we infer that

$$\partial_t R^\varepsilon + R^\varepsilon\cdot\nabla R^\varepsilon + v^\varepsilon_{app}\cdot\nabla R^\varepsilon + R^\varepsilon\cdot\nabla v^\varepsilon_{app} - \Delta R^\varepsilon = -\nabla q^\varepsilon + F^\varepsilon.$$

So Lemma 2.2 will be established as soon as we prove that $\|F^\varepsilon\|_{L^2(\mathbb{R}^+;\dot{H}^{-\frac12}(\mathbb{R}^3))} \le C_{v_0,w_0}\varepsilon^{\frac13}$.

The forcing term F^ε consists in three different types of terms: a pressure term involving $\underline{p}_0$, a linear term $\varepsilon^2\partial_3^2\underline{v}^h(t,x_h,\varepsilon x_3)$, and finally a number of nonlinear terms, defined as $\varepsilon\widetilde{F}^{\varepsilon,1}(t,x_h,\varepsilon x_3)$ above. Each of these contributions will be dealt with separately. Let us start by the pressure term.

Lemma 4.1. *The following estimate holds:*

$$\varepsilon\|(\partial_3\underline{p}_0)(t,x_h,\varepsilon x_3)\|_{L^2(\mathbb{R}^+;\dot{H}^{-\frac12}(\mathbb{R}^3))} \le C_{v_0,w_0}\varepsilon^{\frac13}.$$

Proof. We define $P^\varepsilon_0(t,x_h,x_3) \stackrel{\text{def}}{=} (\partial_3\underline{p}_0)(t,x_h,\varepsilon x_3)$. Sobolev embeddings enable us to write

$$\begin{aligned}\|P^\varepsilon_0\|_{L^2(\mathbb{R}^+;\dot{H}^{-\frac12}(\mathbb{R}^3))} &\lesssim \|P^\varepsilon_0\|_{L^2(\mathbb{R}^+;L^{\frac32}(\mathbb{R}^3))}\\ &\lesssim \varepsilon^{-\frac23}\|\partial_3\underline{p}_0\|_{L^2(\mathbb{R}^+;L^{\frac32}(\mathbb{R}^3))}.\end{aligned}$$

Recalling that

$$\underline{p}_0 = (-\Delta_h)^{-1}\sum_{j,k=1}^{2}\partial_j\partial_k(\underline{v}^j\underline{v}^k),$$

we have by Sobolev embeddings,

$$\begin{aligned}\|\partial_3\underline{p}_0\|_{L^2(\mathbb{R}^+;L^{\frac32}(\mathbb{R}^3))} &\lesssim \sum_{j,k=1}^{2}\|\underline{v}^j\partial_3\underline{v}^k\|_{L^2(\mathbb{R}^+;L^{\frac32}(\mathbb{R}^3))}\\ &\lesssim \|\underline{v}\|_{L^\infty(\mathbb{R}^+;L^3(\mathbb{R}^3))}\|\partial_3\underline{v}\|_{L^2(\mathbb{R}^+;L^3(\mathbb{R}^3))}\\ &\lesssim \|\underline{v}\|_{L^2(\mathbb{R}^+;H^{\frac12}(\mathbb{R}^3))}\|\partial_3\underline{v}\|_{L^\infty(\mathbb{R}^+;H^{\frac12}(\mathbb{R}^3))},\end{aligned}$$

so we can conclude by Lemma 3.1. This proves Lemma 4.1. □

Now let us consider the linear term $\varepsilon^2\partial_3^2\underline{v}^h(t,x_h,\varepsilon x_3)$. The statement is the following.

Lemma 4.2. *The following estimate holds:*

$$\varepsilon^2\|(\partial_3^2\underline{v}^h)(t,x_h,\varepsilon x_3)\|_{L^2(\mathbb{R}^+;\dot{H}^{-\frac12}(\mathbb{R}^3))} \le C_{v_0}\varepsilon^{\frac12}.$$

Proof. We have

$$\begin{aligned}\varepsilon^2\|(\partial_3^2\underline{v}^h)(t,x_h,\varepsilon x_3)\|_{L^2(\mathbb{R}^+;\dot{H}^{-\frac12}(\mathbb{R}^3))} &\lesssim \varepsilon\Big\|\partial_3\big(\partial_3\underline{v}^h(t,x_h,\varepsilon x_3)\big)\Big\|_{L^2(\mathbb{R}^+;\dot{H}^{-\frac12}(\mathbb{R}^3))}\\ &\lesssim \varepsilon\|(\partial_3\underline{v}^h)(t,x_h,x_3)\|_{L^2(\mathbb{R}^+;\dot{H}^{\frac12}(\mathbb{R}^3))},\end{aligned}$$

A computation in Fourier variables shows that, for any function a on $\mathbb{R}^3$, we have

$$\begin{aligned}\|a(x_h,\varepsilon x_3)\|^2_{\dot H^{\frac12}(\mathbb{R}^3)} &= \frac{1}{\varepsilon^2}\int_{\mathbb{R}^3}\Big|\widehat{a}\Big(\xi_h,\frac{\xi_3}{\varepsilon}\Big)\Big|^2|\xi|\,d\xi\\ &\le \frac{1}{\varepsilon}\int_{\mathbb{R}^3}\Big|\widehat{a}\Big(\xi_h,\frac{\xi_3}{\varepsilon}\Big)\Big|^2|\xi_h|\,d\xi_h\frac{d\xi_3}{\varepsilon}+\int_{\mathbb{R}^3}\Big|\widehat{a}\Big(\xi_h,\frac{\xi_3}{\varepsilon}\Big)\Big|^2\frac{|\xi_3|}{\varepsilon}\,d\xi_h\frac{d\xi_3}{\varepsilon}\,.\end{aligned}$$

By interpolation, we deduce that

$$\|a(x_h,\varepsilon x_3)\|^2_{\dot H^{\frac12}(\mathbb{R}^3)}\le\frac{1}{\varepsilon}\|a\|_{L^2(\mathbb{R}^3)}\|\nabla_h a\|_{L^2(\mathbb{R}^3)}+\|a\|_{L^2(\mathbb{R}^3)}\|\partial_3 a\|_{L^2(\mathbb{R}^3)}.$$

Applying this inequality with $a=\partial_3\underline{v}$, we get

$$\begin{aligned}\varepsilon^2\|(\partial_3^2\underline{v}^h)(t,x_h,\varepsilon x_3)\|_{L^2(\mathbb{R}^+;\dot H^{-\frac12}(\mathbb{R}^3))} &\lesssim \varepsilon^{\frac12}\|\partial_3\underline{v}(t)\|^{\frac12}_{L^2(\mathbb{R}^+;L^2(\mathbb{R}^3))}\|\partial_3\nabla_h\underline{v}(t)\|^{\frac12}_{L^2(\mathbb{R}^+;L^2(\mathbb{R}^3))}\\ &\quad+\varepsilon\|\partial_3\underline{v}(t)\|^{\frac12}_{L^2(\mathbb{R}^+;L^2(\mathbb{R}^3))}\|\partial_3^2\underline{v}(t)\|^{\frac12}_{L^2(\mathbb{R}^+;L^2(\mathbb{R}^3))}\\ &\le C_{v_0}\varepsilon^{\frac12}\end{aligned}$$

by Lemma 3.1. This proves Lemma 4.2. □

Now let us turn to the nonlinear terms composing F^ε, which we denoted above $\varepsilon\widetilde{F}^{\varepsilon,1}$.

Lemma 4.3. *The following estimate holds:*

$$\varepsilon\|\widetilde{F}^{\varepsilon,1}(t,x_h,\varepsilon x_3)\|_{L^2(\mathbb{R}^+;\dot H^{-\frac12}(\mathbb{R}^3))}\le C_{v_0,w_0}\varepsilon^{\frac13}.$$

Proof. We recall that

$$\widetilde{F}^{\varepsilon,1}=(\varepsilon\underline{w}^\varepsilon\cdot\nabla\underline{w}^{\varepsilon,h},\underline{w}^\varepsilon\cdot\nabla\underline{w}^{\varepsilon,3})+(\underline{w}^\varepsilon\cdot\nabla\underline{v}^h,0).$$

Notice that for all functions a and b and any $1\le j\le 3$,

$$\begin{aligned}\|a\partial_j b\|_{L^2(\mathbb{R}^+;\dot H^{-\frac12}(\mathbb{R}^3))} &\lesssim \|a\partial_j b\|_{L^2(\mathbb{R}^+;L^{\frac32}(\mathbb{R}^3))}\\ &\lesssim \|a\|_{L^\infty(\mathbb{R}^+;L^3(\mathbb{R}^3))}\|\partial_j b\|_{L^2(\mathbb{R}^+;L^3(\mathbb{R}^3))}.\end{aligned}$$

Defining $c^\varepsilon(t,x_h,x_3)=(a\partial_j b)(t,x_h,\varepsilon x_3)$ this implies that

$$\|c^\varepsilon\|_{L^2(\mathbb{R}^+;\dot H^{-\frac12}(\mathbb{R}^3))}\lesssim\varepsilon^{-\frac23}\|a\|_{L^\infty(\mathbb{R}^+;\dot H^{\frac12}(\mathbb{R}^3))}\|\partial_j b\|_{L^2(\mathbb{R}^+;\dot H^{\frac12}(\mathbb{R}^3))}.$$

We can apply that inequality to a and b equal to $\underline{v}$ or $\underline{w}^\varepsilon$, due to the results proved in Section 3, and the lemma follows. □

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