Existence of solutions to periodic boundary value problems for impulsive nonconvex functional differential inclusions

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EXISTENCE OF SOLUTIONS TO PERIODIC BOUNDARY VALUE PROBLEMS FOR IMPULSIVE NONCONVEX FUNCTIONAL DIFFERENTIAL INCLUSIONS

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Abstract. In this paper we prove an existence theorem for periodic boundary value problems for impulsive first order functional differential inclusions with nonconvex valued right-hand sides. Our approach is based on the nonlinear alternative of Leray-Schauder combined with a continuous selection theorem due to Bressan and Colombo.

1. Introduction

This paper is concerned with the existence of solutions of the following periodic boundary value problem for a first order functional differential inclusion with impulses and nonconvex valued right-hand side

$$\dot{x}(t) - \lambda x(t) \in F(t, x_t), \text{ a.e. } t \in J, \ t \neq t_k, k = 1, \dots, m, \tag{1.1}$$

$$\Delta x|_{t=t_k} = I_k(x(t_k)), \quad k = 1, \dots, m, \tag{1.2}$$

$$x_0 = \phi, \quad x(0) = x(1), \tag{1.3}$$

where $\lambda \in \mathbb{R}$, $\lambda \neq 0$, $J = [0, 1]$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, 2, \dots, m$, $F : J \times \mathcal{D} \to \mathcal{P}(\mathbb{R}^n)$ is a given multivalued function, $\mathcal{D} = \{\psi : J_0 \to \mathbb{R}^n; \psi$ is continuous everywhere except for a finite number of points at which ψ has a discontinuity of the first type and is left continuous$\}$, $J_0 = [-r, 0]$, $\phi \in \mathcal{D}$, $r > 0$, $\mathcal{P}(\mathbb{R}^n)$ is the collection of all subsets of $\mathbb{R}^n$ and $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, $k = 1, 2, \dots, m$.

Our approach is based on the nonlinear alternative of Leray-Schauder [6] combined with a continuous selection theorem due to Bressan and Colombo [5].

In many references (see for example [2, 4, 9, 10, 11] and references therein; see also monographs [1, 3, 8, 12]) authors prove results on the existence of solutions for impulsive differential equations and inclusions under restrictive conditions on the impulses (Lipschitz, boundedness, etc.). In this paper our existence result is presented under fairly general conditions on the multivalued function F and the only continuity condition on the I_k, $k = 1, \dots, m$.

2. Preliminaries

For $\psi \in \mathcal{D}$, the norm of ψ is defined by

$$\|\psi\|_{\mathcal{D}} = \sup\{|\psi(\theta)| : \theta \in J_0\}.$$

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Let $PC(J,\mathbb{R}^n)$ be the space of functions $x : J \to \mathbb{R}^n$ such that x is continuous everywhere except for $t = t_k$, $k = 1,\dots,m$, at which x has a discontinuity of the first type and is left continuous. Set $\Omega = \{x : J_1 \to \mathbb{R}^n \ / \ x \in \mathcal{D} \cap PC(J,\mathbb{R}^n)\}$, where $J_1 = [-r, 1]$. Then Ω is a Banach space with the norm

$$\|x\| = \sup\{|x(t)| : t \in J_1\}, \quad x \in \Omega.$$

Obviously, for any $x \in \Omega$ and any $t \in J$, the history function $x_t : J_0 \to \mathbb{R}^n$ defined by $x_t(\theta) = x(t+\theta)$, for $\theta \in J_0$, belongs to $\mathcal{D}$.

Also, for any $k = 0,\dots,m$, we denote by $AC((t_k, t_{k+1}),\mathbb{R}^n)$ the space of all absolutely continuous functions $x : (t_k, t_{k+1}) \to \mathbb{R}^n$.

Let $L^1(J,\mathbb{R}^n)$ be the Banach space of measurable functions $x : J \to \mathbb{R}^n$ which are Lebesgue integrable, normed by

$$\|x\|_{L^1} = \int_0^1 |x(t)|dt.$$

A nonempty subset A of $L^1(J,\mathbb{R}^n)$ is decomposable if for all $u, v \in A$ and $I \subset J$ measurable, the function $u\chi_I + v\chi_{J\setminus I}$ belongs in A, where χ_I is the characteristic function of the set I.

Let M be a measurable space, X a Banach space and $H : M \to \mathcal{P}(X)$ a multivalued function with nonempty closed values. H is measurable if the set $\{t \in M : H(t) \cap A \neq \emptyset\}$ is measurable for any closed A in X.

Let X_1 and X_2 be two Banach spaces, E a nonempty closed subset of X_1, and $G : E \to \mathcal{P}(X_2)$ a multivalued function with nonempty closed values. G is lower semi-continuous (abbreviated as l.s.c.) (respectively upper semi-continuous (u.s.c.)) if the set $\{x \in E : G(x) \cap A \neq \emptyset\}$ is open (respectively closed) for any open (respectively closed) set A in X_2. If G is l.s.c. and u.s.c., then G is continuous. G is completely continuous if $\overline{G(A)}$ is compact for all bounded sets $A \subset E$.

Next we state a selection theorem [5] due to Bressan and Colombo. Let X be a metric space and $G : X \to \mathcal{P}(L^1(J,\mathbb{R}^n))$ a multivalued function. We say that G has property (BC) if

(i) G is l.s.c.
(ii) G has nonempty closed and decomposable values.

Theorem 2.1. [5] *Let X be a separable metric space and let $G : X \to \mathcal{P}(L^1(J,\mathbb{R}^n))$ be a multivalued function which has property (BC). Then G has a continuous selection, i.e. there exists a continuous function (single valued) $g : X \to L^1(J,\mathbb{R}^n)$ such that $g(x) \in G(x)$ for every $x \in X$.*

Let $F : J \times \mathcal{D} \to \mathcal{P}(\mathbb{R}^n)$ be a multivalued function with nonempty compact values. Throughout this paper, F will satisfy the following conditions:

(C1) (i) $t \mapsto F(t,x)$ is measurable for every $x \in \mathcal{D}$,
(ii) $x \mapsto F(t,x)$ is continuous for a.e. $t \in J$.

(C2) $|F(t,x)| \leq q(t)\psi(\|x\|_{\mathcal{D}})$, for almost all $t \in J$ and all $x \in \mathcal{D}$, for some $q \in L^1(J,\mathbb{R})$ with $q(t) > 0$ for a.e. $t \in J$ and some increasing continuous function $\psi : [0,\infty) \to [0,\infty)$ such that

$$\limsup_{\rho\to\infty} \frac{\rho}{\psi(\rho)} = \infty. \tag{2.1}$$

(C3) Each function $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1,\dots,m$, is continuous.

We denote $S^1_{F(\cdot,x_\cdot)} = \{h \in L^1(J,\mathbb{R}^n) : h(t) \in F(t,x_t) \text{ for } a.e.\ t \in J\}$ the set of selections of F that belong to $L^1(J,\mathbb{R}^n)$. By a solution of (1.1)-(1.3) we mean a function $x \in \Omega \cap AC((t_k,t_{k+1}),\mathbb{R}^n)$, $k = 0,\dots,m$, such that

$$\dot{x}(t) - \lambda x(t) = h(t), \text{ a.e. } t \in J,\ t \neq t_k, k = 1,\dots,m,$$
$$\Delta x|_{t=t_k} = I_k(x(t_k)), \quad k = 1,\dots,m,$$
$$x_0 = \phi, \quad x(0) = x(1),$$

where $h \in S^1_{F(\cdot,x_\cdot)}$.

Note that for a multivalued function $F : J \times \mathcal{D} \to \mathcal{P}(\mathbb{R}^n)$ satisfying conditions (C1) and (C2) above, the set $S^1_{F(\cdot,x_\cdot)}$ is nonempty.

Now consider the multivalued operator $\mathcal{F} : \Omega \to \mathcal{P}(L^1(J,\mathbb{R}^n))$ defined by

$$\mathcal{F}(x) = S^1_{F(\cdot,x_\cdot)}. \tag{2.2}$$

We say that F is of lower semi-continuous type (l.s.c. type) if $\mathcal{F}$, given above, has property (BC).

The following crucial result will be used later.

Theorem 2.2. [7] *Let $F : J \times \mathcal{D} \to \mathcal{P}(\mathbb{R}^n)$ be a multivalued function with nonempty compact values. Assume that (C1) and (C2) hold. Then F is of l.s.c. type.*

Finally we state the following well-known fixed point result often referred to as the nonlinear alternative of Leray-Schauder [6], which will be used to prove the main result of this paper.

Theorem 2.3. *Let X be a Banach space and $K \subset X$ be convex. Assume that U is an open subset of K with $0 \in U$. Let $\Gamma : \overline{U} \to K$ be continuous and completely continuous. Then, either*

(i) *Γ has a fixed point in $\overline{U}$; or*
(ii) *there is a $x \in \partial U$ (the boundary of U) and a $\lambda \in (0,1)$ such that $x = \lambda \Gamma x$.*

3. Existence result

In this section we state and prove our existence result for problem (1.1)-(1.3).

Theorem 3.1. *Let $F : J \times \mathcal{D} \to \mathcal{P}(\mathbb{R}^n)$ be a multivalued function with nonempty compact values. Suppose (C1), (C2) and (C3) are satisfied. Then (1.1)-(1.3) has a solution.*

Before proving Theorem 3.1 we need to prove the following lemma.

Lemma 3.2 (A priori bounds on solutions)**.** *If the conditions of Theorem 3.1 hold, then there exists a constant $\rho > 0$ such that every solution x of the problem (1.1)-(1.3) satisfies $\|x\| \leq \rho$.*

Proof. Let x be a possible solution of problem (1.1)-(1.3). It follows from the definition of solutions that

$$\dot{x}(t) - \lambda x(t) = h(t), \text{ a.e. } t \in J,\ t \neq t_k, k = 1,\dots,m,$$
$$\Delta x|_{t=t_k} = I_k(x(t_k)), \quad k = 1,\dots,m,$$
$$x_0 = \phi, \quad x(0) = x(1)$$

for some $h \in S^1_{F(\cdot,x.)}$. It is easy to verify that x is given by

$$x(t) = \begin{cases} \phi(t) \text{ for } t \in J_0, \\ \displaystyle\int_0^1 g(t,s)h(s)ds + \sum_{k=1}^{m} g(t,t_k)I_k(x(t_k)) \text{ for } t \in J \end{cases}$$

where

$$g(t,s) = \frac{1}{e^{-\lambda}-1} \begin{cases} e^{-\lambda(1+s-t)}, & 0 \le s \le t \le 1 \\ e^{-\lambda(s-t)}, & 0 \le t < s \le 1. \end{cases} \tag{3.1}$$

We note that

$$|g(t,s)| \le \frac{e^{|\lambda|}}{|e^{-\lambda}-1|}, \text{ for all } t \in J \text{ and all } s \in [0,t].$$

First we consider $x|_{[-r,t_1]}$. It satisfies, for $t \in [0,t_1]$,

$$x(t) = \int_0^1 g(t,s)h(s)ds.$$

Condition (C2) yields

$$|x(t)| \quad \le \frac{e^{|\lambda|}}{|e^{-\lambda}-1|} \int_0^1 q(s)\psi(\|x_s\|_{\mathcal{D}})ds. \tag{3.2}$$

Let $\varrho_1 = \max\{|x(t)| : t \in [-r,t_1]\}$. We have $\|x_t\|_{\mathcal{D}} \le \varrho_1$ for all $t \in [0,t_1]$ and there is $t^* \in [-r,t_1]$ such that $\varrho_1 = |x(t^*)|$. If $t^* < 0$, we have $\varrho_1 \le \|\phi\|_{\mathcal{D}}$. If $t^* \ge 0$, from (3.2) it follows that

$$\varrho_1 \le \frac{e^{|\lambda|}}{|e^{-\lambda}-1|}\psi(\varrho_1)\|q\|_{L^1}. \tag{3.3}$$

Now the condition (2.1) on ψ shows that there exists $\rho^* > 0$ such that, for $\rho > \rho^*$,

$$\rho > \frac{e^{|\lambda|}}{|e^{-\lambda}-1|}\psi(\rho)\|q\|_{L^1}. \tag{3.4}$$

Comparing these last two inequalities (3.3) and (3.4) we see that $\varrho_1 \le \rho^*$.

Thus, we have

$$\max\{|x(t)| : t \in [-r,t_1]\} = \varrho_1 \le \max\{\rho^*, \|\phi\|_{\mathcal{D}}\} := \rho_1.$$

Next we consider $x|_{[-r,t_2]}$. Then, for $t \in [0,t_2]$, we have

$$x(t) = \int_0^1 g(t,s)h(s)ds + g(t,t_1)I_1(x(t_1)).$$

Hence

$$|x(t)| \le \frac{e^{|\lambda|}}{|e^{-\lambda}-1|} \left\{ \int_0^1 q(s)\psi(\|x_s\|_{\mathcal{D}})ds + \sup\{|I_1(u)| : |u| \le \rho_1\} \right\}. \tag{3.5}$$

Denote $\varrho_2 = \max\{|x(t)| : t \in [-r,t_2]\}$. Then, for $t \in [0,t_2]$, we have $\|x_t\|_{\mathcal{D}} \le \varrho_2$. Let $t^* \in [-r,t_2]$ be such that $\varrho_2 = |x(t^*)|$. In case $t^* < 0$ we have $\varrho_2 \le \|\phi\|_{\mathcal{D}}$. But, if $t^* \ge 0$, (3.5) yields

$$\varrho_2 \le \frac{e^{|\lambda|}}{|e^{-\lambda}-1|} \big\{ \psi(\varrho_2)\|q\|_{L^1} + \sup\{|I_1(u)| : |u| \le \rho_1\} \big\}$$

which implies

$$\frac{|e^{-\lambda}-1|}{e^{|\lambda|}\|q\|_{L^1}}\cdot\frac{\varrho_2}{\psi(\varrho_2)}\leq 1+\frac{\sup\{|I_1(u)|:|u|\leq\rho_1\}}{\|q\|_{L^1}\psi(\varrho_2)}. \tag{3.6}$$

Again by condition (2.1) there exists $\rho_1^*>0$ such that, for $\rho>\rho_1^*$,

$$\frac{|e^{-\lambda}-1|}{e^{|\lambda|}\|q\|_{L^1}}\cdot\frac{\rho}{\psi(\rho)}>2. \tag{3.7}$$

On the other hand, since ψ is increasing, there exists $\rho_2^*>0$ such that, for $\rho>\rho_2^*$,

$$\frac{\sup\{|I_1(u)|:|u|\leq\rho_1\}}{\|q\|_{L^1}\psi(\rho)}<1. \tag{3.8}$$

Combining (3.7) and (3.8), we deduce that, for $\rho>\max\{\rho_1^*,\rho_2^*\}:=\rho^{**}$,

$$\frac{|e^{-\lambda}-1|}{e^{|\lambda|}\|q\|_{L^1}}\cdot\frac{\rho}{\psi(\rho)}>1+\frac{\sup\{|I_1(u)|:|u|\leq\rho_1\}}{\|q\|_{L^1}\psi(\varrho)}. \tag{3.9}$$

Comparing inequalities (3.6) and (3.9) we see that $\varrho_2\leq\rho^{**}$.

Finally, we obtain

$$\max\{|x(t)|:t\in[-r,t_2]\}=\varrho_2\leq\max\{\rho^{**},\|\phi\|_{\mathcal{D}}\}:=\rho_2.$$

We reiterate this process for $x|_{[-r,t_3]},\dots,x|_{J_1}$, to complete the proof as

$$\|x\|\leq\rho$$

for some constant $\rho:=\rho_{m+1}>0$. □

We now turn to the proof of Theorem 3.1.

First, by Theorem 2.2 together with Theorem 2.1, the multivalued function $\mathcal{F}$, given by (2.2), has a continuous selection $f_0:\Omega\to L^1(J,\mathbb{R}^n)$ such that $f_0(x)\in\mathcal{F}(x)$ for all $x\in\Omega$. By analogy with the single valued case, we denote $f(\cdot,x_\cdot)=f_0(x)(\cdot)$, for $x\in\Omega$.

Consider then the problem

$$\dot{x}(t)-\lambda x(t)=f(t,x_t),\ \text{a.e. } t\in J,\ t\neq t_k, k=1,\dots,m, \tag{3.10}$$

$$\Delta x|_{t=t_k}=I_k(x(t_k)),\quad k=1,\dots,m, \tag{3.11}$$

$$x_0=\phi,\quad x(0)=x(1). \tag{3.12}$$

Clearly any solution of problem (3.10)-(3.12) is a solution of problem (1.1)-(1.3).

To transform the problem above into a fixed point problem, we define the operator $\Gamma:\Omega\to\Omega$ by

$$(\Gamma x)(t)=\begin{cases}\phi(t)\text{ for } t\in J_0,\\ \displaystyle\int_0^1 g(t,s)f(s,x_s)ds+\sum_{k=1}^{m}g(t,t_k)I_k(x(t_k))\text{ for } t\in J\end{cases}$$

where g is given by (3.1). Our purpose is to use Theorem 2.3 to prove that Γ has a fixed point $x\in\Omega$. Then x is a solution of problem (3.10)-(3.12).

We shall show that Γ is continuous and completely continuous. The proof will be given in several steps.

Step 1. Γ is continuous.

Let $\{x_n\}$ with $x_n \to x$ in Ω. For $t \in J$, we obtain

$$\begin{aligned} &|(\Gamma x_n)(t) - (\Gamma x)(t)| \\ &\leq \frac{e^{|\lambda|}}{|e^{-\lambda}-1|}\left\{\int_0^1 |f(s,x_{ns}) - f(s,x_s)|ds + \sum_{k=1}^m |I_k(x_n(t_k)) - I_k(x(t_k))|\right\}. \end{aligned} \tag{3.13}$$

Now, for each $s \in J$, $x_{ns} \to x_s$, and by the continuity of $f(t,x)$ in x, we have $f(s,x_{ns}) \to f(s,x_s)$ as $n \to \infty$. Let $B = \{u \in \Omega : \|u\| \leq \rho\}$ for some $\rho > 0$ such that $\|x_n\|, \|x\| \leq \rho$, for all n. Since, by (C2),

$$|f(s,x_{ns}) - f(s,x_s)| \leq 2\psi(\rho)q(s), \text{ a.e. on } J$$

then by the continuity of I_k, $k = 1, \ldots, m$, and the Lebesgue's convergence theorem, from (3.13) we deduce that $\Gamma x_n \to \Gamma x$; which completes the proof that Γ is continuous.

Step 2. Γ maps bounded sets into bounded sets.

To show this, let B be a bounded set in Ω. Then there exists a real number $\rho > 0$ such that $\|x\| \leq \rho$, for all $x \in B$. Let $x \in B$ and $t \in J$. We have

$$\begin{aligned} |(\Gamma x)(t)| &\leq \int_0^1 |g(t,s)||f(s,x_s)|ds + \sum_{k=1}^m |g(t,t_k)||I_k(x(t_k))| \\ &\leq \frac{e^{|\lambda|}}{|e^{-\lambda}-1|}\left\{\psi(\rho)\|q\|_{L^1} + \sum_{k=1}^m \sup\{|I_k(u)| : |u| \leq \rho\}\right\} := \eta_0. \end{aligned}$$

If $t \in J_0$, then $|(\Gamma x)(t)| \leq \|\phi\|_{\mathcal{D}}$. Thus

$$\|\Gamma x\| \leq \eta = \max\{\|\phi\|_{\mathcal{D}}, \eta_0\}, \quad \text{for all} \quad x \in B,$$

that is, Γ is bounded on bounded subsets of Ω.

Step 3. Γ maps bounded sets into equicontinuous sets.

Let B be, as in Step 2, a bounded set and $x \in B$. Let $\bar{t}, \underline{t} \in (t_k, t_{k+1}] \cap J$, for some $k = 0, \ldots, m$. After some standard calculations we get

$$\begin{aligned} &|(\Gamma x)(\bar{t}) - (\Gamma x)(\underline{t})| \\ &\leq \frac{e^{|\lambda|}}{|e^{-\lambda}-1|}\left|\int_0^{\bar{t}} f(s,x_s)e^{-\lambda(s-\bar{t})}ds - \int_0^{\underline{t}} f(s,x_s)e^{-\lambda(s-\underline{t})}ds\right| \\ &+\frac{e^{|\lambda|}}{|e^{-\lambda}-1|}\left|\sum_{0<t_k<\bar{t}} I_k(x(t_k))e^{-\lambda(t_k-\bar{t})} - \sum_{0<t_k<\underline{t}} I_k(x(t_k))e^{-\lambda(t_k-\underline{t})}\right| \\ &+\frac{1}{|e^{-\lambda}-1|}\int_0^1 |f(s,x_s)||e^{-\lambda(s-\bar{t})} - e^{-\lambda(s-\underline{t})}|ds \\ &+\frac{1}{|e^{-\lambda}-1|}\left|\sum_{k=1}^m I_k(x(t_k))(e^{-\lambda(t_k-\bar{t})} - e^{-\lambda(t_k-\underline{t})})\right|. \end{aligned} \tag{3.14}$$

Now we have

$$\begin{aligned}
&\left|\int_0^{\bar{t}} f(s,x_s)e^{-\lambda(s-\bar{t})}ds - \int_0^{\underline{t}} f(s,x_s)e^{-\lambda(s-\underline{t})}ds\right| \\
&\quad = \left|\int_0^{\bar{t}} f(s,x_s)(e^{-\lambda(s-\bar{t})} - e^{-\lambda(s-\underline{t})})ds + \int_{\bar{t}}^{\underline{t}} f(s,x_s)e^{-\lambda(s-\underline{t})}ds\right| \\
&\quad \leq e^{|\lambda|}|e^{\lambda\bar{t}} - e^{\lambda\underline{t}}| \int_0^1 |f(s,x_s)|ds + e^{2|\lambda|}\int_{\bar{t}}^{\underline{t}} |f(s,x_s)|ds
\end{aligned} \tag{3.15}$$

and

$$\int_0^1 |f(s,x_s)||e^{-\lambda(s-\bar{t})} - e^{-\lambda(s-\underline{t})}|ds \leq e^{|\lambda|}|e^{\lambda\bar{t}} - e^{\lambda\underline{t}}| \int_0^1 |f(s,x_s)|ds. \tag{3.16}$$

Using inequalities (3.15) and (3.16) in (3.14), we get

$$\begin{aligned}
&|(\Gamma x)(\bar{t}) - (\Gamma x)(\underline{t})| \\
&\quad \leq \frac{(e^{2|\lambda|} + e^{|\lambda|})|e^{\lambda\bar{t}} - e^{\lambda\underline{t}}|}{|e^{-\lambda} - 1|} \int_0^1 |f(s,x_s)|ds + \frac{e^{3|\lambda|}}{|e^{-\lambda} - 1|}\int_{\bar{t}}^{\underline{t}} |f(s,x_s)|ds \\
&\quad + \frac{e^{|\lambda|}}{|e^{-\lambda} - 1|}\left|\sum_{0<t_k<\bar{t}} I_k(x(t_k))e^{-\lambda(t_k-\bar{t})} - \sum_{0<t_k<\underline{t}} I_k(x(t_k))e^{-\lambda(t_k-\underline{t})}\right| \\
&\quad + \frac{1}{|e^{-\lambda} - 1|}\left|\sum_{k=1}^{m} I_k(x(t_k))(e^{-\lambda(t_k-\bar{t})} - e^{-\lambda(t_k-\underline{t})})\right|
\end{aligned}$$

which implies

$$\begin{aligned}
&|(\Gamma x)(\bar{t}) - (\Gamma x)(\underline{t})| \\
&\quad \leq \frac{(e^{2|\lambda|} + e^{|\lambda|})|e^{\lambda\bar{t}} - e^{\lambda\underline{t}}|}{|e^{-\lambda} - 1|}\psi(\rho)\|q\|_{L^1} + \frac{e^{3|\lambda|}}{|e^{-\lambda} - 1|}\psi(\rho)\int_{\bar{t}}^{\underline{t}} q(s)ds \\
&\quad + \frac{e^{|\lambda|}}{|e^{-\lambda} - 1|}\left|\sum_{0<t_k<\bar{t}} I_k(x(t_k))e^{-\lambda(t_k-\bar{t})} - \sum_{0<t_k<\underline{t}} I_k(x(t_k))e^{-\lambda(t_k-\underline{t})}\right| \\
&\quad + \frac{1}{|e^{-\lambda} - 1|}\left|\sum_{k=1}^{m} I_k(x(t_k))(e^{-\lambda(t_k-\bar{t})} - e^{-\lambda(t_k-\underline{t})})\right|.
\end{aligned}$$

Hence $|(\Gamma x)(\bar{t}) - (\Gamma x)(\underline{t})|$ tends to zero as $|\bar{t} - \underline{t}| \to 0$, that is Γ is equicontinuous on J.

The equicontinuity on J_0 follows from the uniform continuity of ϕ on this interval.

As a consequence of Steps 2 and 3, together with the Arzelá-Ascoli theorem, we conclude that Γ is completely continuous.

Finally, let ρ as given in Lemma 3.2 and set $X = K = \Omega$,

$$U = \{x \in \Omega : \|x\| < \rho + 1\}$$

and apply Theorem 2.3 to deduce that Γ has a fixed point $x \in \overline{U}$ which is a solution of problem (1.1)-(1.3). This completes the proof of Theorem 3.1.

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