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Sphere Rigidity in the Euclidean Space

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Abstract

In this article, we prove new stability results for almost-Einstein hypersurfaces of the Euclidean space, based on previous eigenvalue pinching results. Then, we deduce some comparable results for almost-umbilic hypersurfaces and new characterizations of geodesic spheres.

Keywords: Hypersurfaces, Rigidity, Pinching, Ricci Curvature, Umbilicity Tensor, Higher Order Mean Curvatures.

Mathematical Subject Classification: 53A07, 53A10, 53C20, 53C24.

1 Introduction

The well-known Alexandrov theorem [1] says that embedded hypersurfaces in \mathbb{R}^{n+1} with constant mean curvature are geodesic spheres. This result is not true for only immersed hypersurfaces. For instance, the so-called Wente's tori (see [14]) are examples of compact surfaces with constant mean curvature in \mathbb{R}^3 , which are not geodesic spheres. Other examples of higher genus are known (see [8] for instance).

For immersed hypersurfaces of constant mean curvature, an additional assumption is needed. One condition is given by the Hopf theorem [7], which

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says that constant mean curvature spheres immersed in \mathbb{R}^{n+1} are geodesic spheres.

In this paper, we give a new rigidity theorem for spheres, where we replace the topological assumption by a metric assumption. Precisely, it is easy to see that hypersurfaces of \mathbb{R}^{n+1} with constant mean curvature and constant scalar curvature are geodesic spheres. This result comes from the fact that a hypersurface of constant mean curvature and constant scalar curvature is totally umbilic. Here, we give a new rigidity result with a weaker assumption on the scalar curvature. Namely, we show

Theorem 1. *Let (M^n, g) be a compact, connected and oriented Riemannian manifold without boundary isometrically immersed into \mathbb{R}^{n+1} . Let h be a positive constant. Then, there exists $\varepsilon > 0$ such that if*

$$(1) \quad H = h \text{ and}$$

$$(2) \quad |\text{Scal} - s| \leq \varepsilon,$$

for some constant s , then M is the sphere $\mathbb{S}^n\left(\frac{1}{h}\right)$ with its standard metric.

We derive this theorem from results about almost umbilic hypersurfaces that we prove in Sect. 4, and based on a previous eigenvalue pinching result given in [12].

2 Preliminaries

Let (M^n, g) be a n -dimensional compact, connected, oriented Riemannian manifold without boundary, isometrically immersed into the $(n+1)$ -dimensional Euclidean space $(\mathbb{R}^{n+1}, \text{can})$. The second fundamental form B of the immersion is the bilinear symmetric form defined by

$$B(Y, Z) = -g(\bar{\nabla}_Y \nu, Z),$$

where $\bar{\nabla}$ is the Riemannian connection on \mathbb{R}^{n+1} and ν the outward normal unit vector field on M .

From B , we can define the mean curvature,

$$H = \frac{1}{n} \text{tr}(B),$$

and, more generally, the higher order mean curvatures,

$$H_r = \frac{1}{\binom{n}{r}} \sigma_r(\kappa_1, \dots, \kappa_n),$$

where σ_r is the r -th symmetric polynomial and $\kappa_1, \dots, \kappa_n$ are the principal curvatures of the immersion. By convention, we set $H_0 = 1$.

Note that $H_1 = H$ and from the Gauss equation, H_2 is, up to a multiplicative constant, the scalar curvature. Namely, we have $H_2 = \frac{1}{n(n-1)} \text{Scal}$.

These extrinsic curvatures satisfy the well-known Hsiung-Minkowski formula, for $1 \leq r \leq n$,

$$(1) \quad \int_M (H_{r-1} - H_r \langle X, \nu \rangle) dv_g = 0,$$

where X is the position vector of the immersion. They also satisfy the following inequalities if H_r is a positive function:

$$(2) \quad H_r^{\frac{1}{r}} \leq H_{r-1}^{\frac{1}{r-1}} \leq \dots \leq H_2^{\frac{1}{2}} \leq H.$$

Moreover, Reilly [10] proved some upper bounds for the first eigenvalue of the Laplacian for hypersurfaces of \mathbb{R}^{n+1} in terms of higher order mean curvatures. Precisely, he shows

$$(3) \quad \lambda_1(M) \left(\int_M H_{r-1} dv_g \right)^2 \leq \frac{n}{\text{Vol}(M)} \int_M H_r^2 dv_g,$$

with equality only for the geodesic hyperspheres of \mathbb{R}^{n+1} .

By the Hölder inequality, we obtain for $p \geq 2$,

$$\lambda_1(M) \leq n \frac{\|H_r\|_{2p}^2 \text{Vol}(M)^{2-\frac{1}{p}}}{\left(\int_M H_{r-1} dv_g \right)^2}.$$

Now, for $p \geq 2$ and $1 \leq r \leq n$, we define $k_{p,r} = \frac{\|H_r\|_{2p}^2 \text{Vol}(M)^{2-\frac{1}{p}}}{\left(\int_M H_{r-1} dv_g \right)^2}$, which

are the constants involved in Theorem 2.

The main tool in the proof of Theorem 1 is the following pinching result, associated with these inequalities, that we proved in [12].

Theorem A (Roth [12]). *Let (M^n, g) be a compact, connected, oriented Riemannian manifold without boundary isometrically immersed in \mathbb{R}^{n+1} . Assume that $\text{Vol}(M) = 1$ and let $r \in \{1, \dots, n\}$ such that $H_r > 0$. Then for any $p \geq 2$ and any $\theta \in]0, 1[$, there exists a constant K_θ depending only on n , $\|H\|_\infty$, $\|H_r\|_{2p}$ and θ such that if the pinching condition*

$$(P_{K_\theta}) \quad 0 \geq \lambda_1(M) \left(\int_M H_{r-1} dv_g \right)^2 - n \text{Vol}(M)^{2-1/p} \|H_r\|_{2p}^2 > -K_\theta$$

is satisfied, then M is diffeomorphic and θ -quasi-isometric to $\mathbb{S}^n \left(\sqrt{\frac{n}{\lambda_1}} \right)$.

Remark 1. *Note that if $p \geq \frac{n}{2r}$; then K_θ does not depend on $\|H_r\|_{2p}$.*

3 Almost Einstein hypersurfaces

It is a well-known fact that an Einstein ($\text{Scal} > 0$) hypersurface of the Euclidean space \mathbb{R}^{n+1} is a round sphere. This was proved by Thomas [13] and independently by Fialkow [4] in the 30's. Recently, Grosjean [6] gave a new proof based on an upper bound of the first eigenvalue of the Laplacian involving the scalar curvature.

From this approach, we showed in [12] that almost-Einstein hypersurfaces of \mathbb{R}^{n+1} are close to round spheres. Namely,

Theorem B (Roth [12]). *Let (M^n, g) be a compact, connected, oriented Riemannian manifold without boundary isometrically immersed in \mathbb{R}^{n+1} , $n \geq 3$. Let $\theta \in]0, 1[$. If (M^n, g) is almost-Einstein, that is, $\|\text{Ric} - (n-1)kg\|_\infty \leq \varepsilon$ for a positive constant k , with ε small enough depending on n , k , $\|H\|_\infty$ and θ , then M is diffeomorphic and θ -quasi-isometric to $\mathbb{S}^n \left(\sqrt{\frac{1}{k}} \right)$*

By θ -quasi-isometric, we understand that there exists a diffeomorphism F from M into $\mathbb{S}^n \left(\sqrt{\frac{1}{k}} \right)$ such that, for any $x \in M$ and for any unitary vector $u \in T_x M$, we have

$$\left| |d_x F(u)|^2 - 1 \right| \leq \theta.$$

This theorem is a corollary of our pinching result for the first eigenvalue of the Laplacian (Theorem A).

In this article, we consider almost-Einstein hypersurfaces of \mathbb{R}^{n+1} in a weaker sense, namely for the L^q -norm, that is, $\|\text{Ric} - (n-1)kg\|_q \leq \varepsilon$ for some positive constant k and a sufficiently small ε . We prove that for some suitable constants k , such manifolds are close to round spheres. Precisely, we prove the following

Theorem 2. *Let (M^n, g) be a compact, connected, oriented Riemannian manifold without boundary isometrically immersed in \mathbb{R}^{n+1} . Let $\theta \in]0, 1[$, if (M^n, g) satisfies $\|\text{Ric} - (n-1)k_{p,r}g\|_q \leq \varepsilon$ for some sufficiently small ε depending on n , q , $\|H\|_\infty$ and θ , then M is diffeomorphic and θ -quasi-isometric to $\mathbb{S}^n \left(\sqrt{\frac{1}{k_{p,r}}} \right)$.*

The constants $k_{p,r}$ in the theorem are defined from the higher order mean curvature (see Sect. 2).

After giving the proof of this theorem, we will give some applications to almost-umbilic hypersurfaces. Finally, we derive some applications to almost constant mean curvature and almost constant scalar curvature, and then conclude with the proof of theorem 1

The proof of Theorem 2 is based on the above pinching result combined with a lower bound for the first eigenvalue of the Laplacian due to Aubry [2]. Assume that $\|\text{Ric} - (n-1)kg\|_q \leq \varepsilon(n, q, k)$ for a positive constant k , $q > \frac{n}{2}$ and ε small enough, then from Theorem 1.1 of [2], we deduce that $\lambda_1(\Delta)$ satisfies

$$(4) \quad \lambda_1(\Delta) \geq nk(1 - C_\varepsilon),$$

where C_ε is an explicit constant such that $C_\varepsilon \rightarrow 0$ when $\varepsilon \rightarrow 0$.

Now, with the particular choice of $k = k_{p,r}$, we get:

$$\lambda_1(M) \left(\int_M H_{r-1} dv_g \right)^2 - n \text{Vol}(M)^{2-1/p} \|H_r\|_{2p}^2 > -K_\varepsilon$$

for some constant K_ε such that $K_\varepsilon \rightarrow 0$ when $\varepsilon \rightarrow 0$.

Let $\theta \in]0, 1[$, we choose $\varepsilon(n, q, k, \theta)$ small enough such that K_ε is small enough in Theorem A to obtain a diffeomorphism and θ -quasi-isometry between M and $\mathbb{S}^n \left(\sqrt{\frac{1}{k_{p,r}}} \right)$. \square

Now, we will deduce from Theorem 2 some Corollaries for almost-umbilic hypersurfaces of \mathbb{R}^{n+1} .

4 Almost umbilic hypersurfaces

First, we give the following theorem, which is a direct application of Theorem B.

Theorem 3. *Let (M^n, g) be a compact, connected, oriented Riemannian manifold without boundary isometrically immersed in \mathbb{R}^{n+1} . Let $\theta \in]0, 1[$. If (M^n, g) is almost-umbilic, that is, $\|B - kg\|_\infty \leq \varepsilon$ for a positive constant k , with ε small enough depending on n , k and θ then M is diffeomorphic and θ -quasi-isometric to $\mathbb{S}^n \left(\frac{1}{k} \right)$.*

Proof : We recall the once traced Gauss formula

$$(5) \quad \text{Ric}(Y, Y) = nH \langle B(Y), Y \rangle - \langle B(Y), B(Y) \rangle,$$

for a tangent vector field Y . From (5) and $\|B - kg\|_\infty \leq \varepsilon$, we deduce

$$\begin{aligned} \text{Ric}(Y, Y) &\geq nk^2 \|Y\|^2 (1 - \varepsilon)^2 - k^2 \|Y\|^2 (1 + \varepsilon)^2 \\ &\geq (n - 1)k^2 \|Y\|^2 - \alpha_n(\varepsilon) \|Y\|^2, \end{aligned}$$

where α_n is a positive function such that $\alpha_n(\varepsilon) \rightarrow 0$ when $\varepsilon \rightarrow 0$. Similarly, we get

$$\text{Ric}(Y, Y) \leq (n-1)k^2\|Y\|^2 + \alpha_n(\varepsilon)\|Y\|^2.$$

Finally, we have

$$\|\text{Ric} - (n-1)k^2g\|_\infty \leq \alpha_n(\varepsilon),$$

which implies, by Theorem B, that for ε small enough, M is diffeomorphic and θ -quasi-isometric to $\mathbb{S}^n(\frac{1}{k})$. \square

Now, from Theorem 2, it is possible to obtain, in some particular cases, comparable results for almost-umbilic hypersurfaces in an L^q -sense. We recall that the umbilicity tensor is defined by

$$\tau = B - H\text{Id}.$$

It is a well-known fact that if M is umbilic, *i.e.*, $\tau = 0$, and if M is compact, then M is a geodesic sphere. Here, we prove the following stability result for almost umbilic hypersurfaces.

Theorem 4. *Let (M^n, g) be a compact, connected, oriented Riemannian manifold without boundary isometrically immersed in \mathbb{R}^{n+1} . Let $q > \frac{n}{2}$. For any $\theta \in]0, 1[$, there exists two constants $\varepsilon_i(\theta, n, \|H\|_\infty)$, $i = 1, 2$, such that if*

1. $\|\tau\|_{2q} \leq \varepsilon_1$,
2. $\|H^2 - k_{p,r}\|_q \leq \varepsilon_2$, for $p \geq 4$ and $1 \leq r \leq n$,

then M is diffeomorphic and θ -quasi-isometric to $\mathbb{S}^n\left(\frac{1}{\sqrt{k_{p,r}}}\right)$

Remark 2. *Note that for $r = 1$, the result is due to Grosjean (see [5]), using a pinching result which involves only H_1 (see [3]).*

Proof. We recall the Gauss formula for hypersurfaces of \mathbb{R}^{n+1} :

$$\text{Ric} = nHB - B^2.$$

From this, we deduce that

$$\begin{aligned} \text{Ric} - (n-1)H^2g &= nHB - B^2 - (n-1)H^2g \\ &= (n-2)H\tau - \tau^2, \end{aligned}$$

which implies

$$\begin{aligned}
\|\text{Ric} - (n-1)kg\|_q &\leq \|\text{Ric} - (n-1)H^2g\|_q + (n-1)\sqrt{n}\|(H^2 - k)\|_q \\
&\leq (n-2)\|H\|_\infty\|\tau\|_{2q} + \|\tau\|_{2q}^2 + (n-1)\sqrt{n}\|(H^2 - k)\|_q \\
&\leq (n-2)\|H\|_\infty\varepsilon_1 + \varepsilon_1^2 + (n-1)\sqrt{n}\varepsilon_2
\end{aligned}$$

Now, we conclude by taking ε_1 and ε_2 small enough depending on n , $\|H\|_\infty$ and θ in order to apply Theorem 2 and obtain the θ -quasi-isometry. \square

Then, we deduce the following corollary which is to compare with Theorem 3.

Corollary 1. *Let (M^n, g) be a compact, connected, oriented Riemannian manifold without boundary isometrically immersed in \mathbb{R}^{n+1} . Let $\theta \in]0, 1[$. If (M^n, g) is almost-umbilic, that is, $\|B - \sqrt{k_{p,r}}g\|_{2q} \leq \varepsilon$, for $q > \frac{n}{2}$, with ε small enough depending on n , $\|H\|_\infty$ and θ then M is diffeomorphic and θ -quasi-isometric to $\mathbb{S}^n \left(\frac{1}{\sqrt{k_{p,r}}} \right)$.*

Proof: A simple computation shows that

$$\|H^2 - k_{p,r}\|_{2q} \leq \alpha_1 \|B - \sqrt{k_{p,r}}g\|_{2q}, \text{ and}$$

$$\|\tau\|_{2q} \leq \alpha_2 \|B - \sqrt{k_{p,r}}g\|_{2q},$$

for two constants α_1 and α_2 depending on n and $\|H\|_\infty$. Since we assume that $\|B - \sqrt{k_{p,r}}g\|_{2q} \leq \varepsilon$, we get

1. $\|H^2 - k_{p,r}\|_{2q} \leq \alpha_1\varepsilon$,
2. $\|\tau\|_{2q} \leq \alpha_2\varepsilon$.

For ε small enough, the assumptions of Theorem 4 are satisfied and we can conclude that M is diffeomorphic and quasi-isometric to $\mathbb{S}^n \left(\frac{1}{\sqrt{k_{p,r}}} \right)$. \square

Remark 3. *We want to point out that this corollary is an improvement of Theorem 3 only in some sense. Indeed, we improve the L^∞ -proximity to an L^{2q} -proximity, but this corollary is valid only for some special constants $k_{p,r}$ and not for any positive constant as in Theorem 3. Nevertheless, this result is sufficient to deduce an interesting application for $r = 2$.*

5 Proof of Theorem 1

In this section, we will give the proof of Theorem 1. First, using Corollary 1, we will show that hypersurfaces of \mathbb{R}^{n+1} with almost constant mean and scalar curvatures are close to a geodesic sphere. Precisely, we show the following

Theorem 5. *Let (M^n, g) be a compact, connected and oriented Riemannian manifold without boundary isometrically immersed into \mathbb{R}^{n+1} . Let $h > 0$ and $\theta \in]0, 1[$. Then, there exists $\varepsilon(n, h, \theta) > 0$ so that if*

$$(1) |H - h| \leq \varepsilon, \text{ and}$$

$$(2) |\text{Scal} - s| \leq \varepsilon,$$

for some constant s , then $|s - n(n-1)h^2| \leq A(n, h)\varepsilon$ and M is diffeomorphic and θ -quasi-isometric to $\mathbb{S}^n(\frac{1}{h})$.

Proof : First, we show that h and s are related.

Lemma 1. *The two constants h and s satisfy*

$$h = \frac{1}{n(n-1)}s + A\varepsilon,$$

where A is constant depending only on n and h .

Proof : We recall the so-called Hsiung-Minkowski formula:

$$(6) \quad \int_M \left(H - \frac{1}{n(n-1)} \text{Scal} \langle X, \nu \rangle \right) dv_g = 0.$$

By assumption, we have $H(x) = h + f_1(x)\varepsilon$ and $\text{Scal}(x) = h + f_2(x)\varepsilon$, with f_1 and f_2 two functions satisfying $|f_1(x)| \leq 1$ and $|f_2(x)| \leq 1$. Now, by (6), we have

$$\begin{aligned} 0 &= \int_M \left(h + \varepsilon f_1(x) - \left(\frac{1}{n(n-1)}s + \varepsilon f_2(x) \right) \langle X, \nu \rangle \right) dv_g \\ &= h \text{Vol}(M) + \varepsilon \int_M f_1(x) dv_g - \frac{1}{n(n-1)} \int_M s \langle X, \nu \rangle dv_g \\ &\quad - \frac{\varepsilon}{n(n-1)} \int_M f_2(x) \langle X, \nu \rangle dv_g \\ &= A_1 \varepsilon + h \text{Vol}(M) - \frac{s}{n(n-1)h} \int_M h \langle X, \nu \rangle dv_g \\ &= A_1 \varepsilon + h \text{Vol}(M) - \frac{s}{n(n-1)h} \int_M (H(x) - \varepsilon f_1(x)) \langle X, \nu \rangle dv_g \\ &= A_2 \varepsilon + h \text{Vol}(M) - \frac{s}{n(n-1)h} \int_M H \langle X, \nu \rangle dv_g \end{aligned}$$

But, we know that $\int_M H \langle X, \nu \rangle dv_g = \text{Vol}(M)$, which yields

$$A_2 \varepsilon + h - \frac{s}{n(n-1)h} = 0.$$

So, we deduce that $s = n(n-1)h^2 + A\varepsilon$, where A is a constant depending only on n and h . \square

Now, we use Corollary 1 to conclude. For this, we need to estimate the umbilicity tensor. By the Gauss formula, we have

$$\begin{aligned} |\tau|^2 &\leq n(n-1)H^2 - \text{Scal} \\ &\leq n(n-1)(h + \varepsilon f_1(x))^2 - (s + \varepsilon f_2(x)) \\ &\leq n(n-1)h^2 - s + \varepsilon h(x) \\ &\leq A'\varepsilon. \end{aligned}$$

Then, from the definition of $k_{p,r}$ and the assumptions, we can see easily that $|H^2 - k_{p,2}| \leq A''\varepsilon$. So the assumptions of Corollary 1 are satisfied and we can conclude that M is diffeomorphic and quasi-isometric to $\mathbb{S}^n\left(\frac{1}{k}\right)$ if we choose ε small enough depending only on n , θ and h . \square

Now, we can deduce Theorem 1 from this result.

Proof of Theorem 1. For $\varepsilon(n, h)$ small enough, we know, from Theorem 5, that M is diffeomorphic to \mathbb{S}^n (we take $\theta = \frac{1}{2}$ in Theorem 5). So we deduce that the immersion of M into \mathbb{R}^{n+1} is in fact an embedding. Since M has constant positive mean curvature, by the Alexandrov theorem, M is the sphere of corresponding radius, that is, $\mathbb{S}^n\left(\frac{1}{h}\right)$. \square

From Theorem 5, and using the Alexandrov theorem for constant scalar curvature due to Ros [11, 9], we get the following corollary.

Corollary 2. *Let (M^n, g) be a compact, connected and oriented Riemannian manifold without boundary isometrically immersed into \mathbb{R}^{n+1} . Let s be a positive constant. Then, there exists $\varepsilon > 0$ such that if*

$$(1) \text{ Scal} = s$$

$$(2) |H - h| \leq \varepsilon,$$

for some constant h , then M is the sphere $\mathbb{S}^n\left(\sqrt{\frac{n(n-1)}{s}}\right)$.

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References

- [1] A.D. Alexandrov, *Uniqueness theorems for the surfaces in the large I*, Vesnik Leningrad Univ. **11** (1956), 5–17.
- [2] E. Aubry, *Diameter pinching in almost positive Ricci curvature*, To appear.
- [3] B. Colbois and J.F. Grosjean, *A pinching theorem for the first eigenvalue of the Laplacian on hypersurfaces of the Euclidean space*, Comment. Math. Helv. **82** (2007), 175–195.
- [4] A. Fialkow, *Hypersurfaces of a space of constant curvature*, Ann. of Math. **39** (1938), 762–783.
- [5] J.F. Grosjean, *Eigenvalue pinching and application to the stability and the almost umbilicity of hypersurfaces*, math. DG/0709.0831.
- [6] ———, *Upper bounds for the first eigenvalue of the Laplacian on compact manifolds*, Pac. J. Math. **206** (2002), no. 1, 93–111.
- [7] H. Hopf, *Differential geometry in the large*, Lecture Notes in Mathematics, vol. 1000, Springer-Verlag, Berlin, 1983.
- [8] M. Jleli and F. Pacard, *An end-to-end construction for compact constant mean curvature surfaces*, Pac. J. Math. **221** (2005), no. 1, 81–108.
- [9] S. Montiel and A. Ros, *Compact hypersurfaces: the Alexandrov theorem for higher order mean curvature*, Pitman Monographs Surveys Pure Appl. Math. **52** (1991), 279–296, in honor of M.P. Do Carmo; edited by B. Lawson and K. Tenenblat.
- [10] R.C. Reilly, *On the first eigenvalue of the Laplacian for compact submanifolds of Euclidean space*, Comment. Math. Helv. **52** (1977), 525–533.
- [11] A. Ros, *Compact hypersurfaces with constant higher order mean curvatures*, Revista Mat. Iberoamer. **3** (1987), 477–483.

- [12] J. Roth, *Pinching of the first eigenvalue of the Laplacian and almost-Einstein hypersurfaces of Euclidean space*, Ann. Glob. Anal. Geom. (to appear).
- [13] T.Y. Thomas, *On closed space of constant mean curvature*, Amer. J. Math. **58** (1936), 702–704.
- [14] H. Wente, *Counterexample to a conjecture of H. Hopf*, Pac. J. Math. **121** (1986), no. 1, 193–243.