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# Test vectors for trilinear forms, when two representations are unramified

Louise Nyssen

October 25, 2007

## 1 Introduction

Let  $F$  be a finite extension of  $\mathbb{Q}_p$ , with ring of integers  $\mathcal{O}_F$ , and uniformizing parameter  $\pi_F$ , whose residual field has  $q$  elements. For  $G = \mathrm{GL}_2(F)$ , let  $(\pi_1, V_1)$ ,  $(\pi_2, V_2)$  and  $(\pi_3, V_3)$  be three irreducible, admissible, infinite dimensional representations of  $G$ . Using the theory of Gelfand pairs, Diprenda Prasad proves in [P] that the space of  $G$ -invariant linear forms on  $V_1 \otimes V_2 \otimes V_3$  has dimension at most one. He gives a precise criterion for this dimension to be one, that we will explain now.

Let  $D_F^*$  be the group of invertible elements of the quaternion division algebra  $D_F$  over  $F$ . When  $(\pi_i, V_i)$  is a discrete series representation of  $G$ , denote by  $(\pi'_i, V'_i)$  the irreducible representation of  $D_F^*$  associated to  $(\pi_i, V_i)$  by the Jacquet-Langlands correspondance. Again, by the theory of Gelfand pairs, the space of  $D_F^*$ -invariant linear forms on  $V'_1 \otimes V'_2 \otimes V'_3$  has dimension at most one.

Let  $\sigma_i$  be the two dimensional representations of the Weil-Deligne group of  $F$  associated to the irreducible representations  $\pi_i$ . The triple tensor product  $\sigma_1 \otimes \sigma_2 \otimes \sigma_3$  is an eight dimensional symplectic representation of the Weil-Deligne group, and has local root number  $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = \pm 1$ . When  $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = -1$ , one can prove that the representations  $\pi_i$ 's are all discrete series representations of  $G$ .

**Theorem 1.** (*Prasad, theorem 1.4 of [P]*) *Let  $(\pi_1, V_1)$ ,  $(\pi_2, V_2)$ ,  $(\pi_3, V_3)$  be three irreducible, admissible, infinite dimensional representations of  $G$  such that the product of their central characters is trivial. If all the representations  $V_i$ 's are cuspidal, assume that the residue characteristic of  $F$  is not 2. Then*

- $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = 1$  if and only if there exists a non zero  $G$ -invariant linear form on  $V_1 \otimes V_2 \otimes V_3$
- $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = -1$  if and only if there exists a non zero  $D_k^*$  invariant linear form on  $V'_1 \otimes V'_2 \otimes V'_3$ .

Once you got a non zero  $G$ -invariant linear form  $\ell$  on  $V_1 \otimes V_2 \otimes V_3$ , or a non zero  $D_k^*$ -invariant linear form  $\ell'$  on  $V'_1 \otimes V'_2 \otimes V'_3$ , you want to find a vector in  $V_1 \otimes V_2 \otimes V_3$  which is not in the kernel of  $\ell$ , or a vector in  $V'_1 \otimes V'_2 \otimes V'_3$  which is not in the kernel of  $\ell'$ . Such a vector is called a test vector. At first sight, it appears to have strong connections with the new vectors  $v_1$ ,  $v_2$  and  $v_3$  of the representations  $\pi_1$ ,  $\pi_2$  et  $\pi_3$ .

**Theorem 2.** (*Prasad, theorem 1.3 of [P]*) *When all the  $\pi_i$ 's are unramified principal series representations of  $G$ ,  $v_1 \otimes v_2 \otimes v_3$  is a test vector for  $\ell$ .*

**Theorem 3.** (Gross and Prasad, proposition 6.3 of [G-P]) When all the  $\pi_i$ 's are unramified twists of the special representation of  $G$  :

- if  $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = 1$ , then  $v_1 \otimes v_2 \otimes v_3$  is a test vector for  $\ell$ ,
- if  $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = -1$ , let  $R'$  be the unique maximal order in  $D_F$ . Then the open compact subgroup  $R'^* \times R'^* \times R'^*$  fixes a unique line in  $V_1' \otimes V_2' \otimes V_3'$ . Any vector on this line is a test vector for  $\ell'$ .

The proof by Gross and Prasad of the first statement of this theorem, actually contains another result:

**Theorem 4.** When two of the  $\pi_i$ 's are unramified twists of the special representation of  $G$  and the third one belongs to the unramified principal series of  $G$ ,  $v_1 \otimes v_2 \otimes v_3$  is a test vector for  $\ell$ .

But the paper [G-P] ends up with an evidence that  $v_1 \otimes v_2 \otimes v_3$  is not always a test vector for  $\ell$ . Let  $K = \text{GL}(\mathcal{O}_F)$  be the maximal compact subgroup of  $G$ . If  $\pi_1$  and  $\pi_2$  are unramified and if  $\pi_3$  has conductor  $n \geq 1$ ,  $\ell$  being  $G$ -invariant,  $v_1$  and  $v_2$  being  $K$ -invariant, one gets a  $K$ -invariant linear form

$$\begin{cases} V_3 & \longrightarrow \mathbb{C} \\ v & \longmapsto \ell(v_1 \otimes v_2 \otimes v) \end{cases}$$

which must be 0 since  $\pi_3$  is ramified. Then  $\ell(v_1 \otimes v_2 \otimes v_3) = 0$ .

Now Gross and Prasad make the following suggestion. Let  $\Gamma_0(\pi_F^n)$  be the congruence subgroup

$$\Gamma_0(\pi_F^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \quad c \equiv 0 \pmod{\pi_F^n} \right\}$$

and  $R$  be a maximal order  $M_2(F)$  such that  $R^* \cap K = \Gamma_0(\pi_F^n)$ . If  $v_2^*$  is a  $R^*$ -invariant vector in  $V_2$ , the linear form

$$\begin{cases} V_3 & \longrightarrow \mathbb{C} \\ v & \longmapsto \ell(v_1 \otimes v_2^* \otimes v) \end{cases}$$

is invariant under the action of  $R^* \cap K = \Gamma_0(\pi_F^n)$ , and one can still hope that  $v_1 \otimes v_2^* \otimes v_3$  is a test vector for  $\ell$ . In theorem 5 we will prove that  $v_1 \otimes v_2^* \otimes v_3$  is a test vector for  $\ell$ , up to a condition on  $\pi_1$  and  $\pi_2$ . The case  $n = 1$  will almost complete the study of test vectors when the  $\pi_i$ 's have ramification 0 or 1.

In the long term, the search for test vectors is motivated by the subconvexity problem for  $L$ -functions. Roughly speaking, one wants to bound some  $L$ -functions along the critical line  $\Re(z) = \frac{1}{2}$ . A recent and successful idea in this direction has been to relate triple products of automorphic forms to special values of  $L$ -functions on the critical line. In [B-R 1] and [B-R 2] Joseph Bernstein and Andre Reznikov did this in the *eigenvalue* aspect, and in [V] Akshay Venkatesh did it in the level aspect. More details about subconvexity and those related techniques will be found in [M-V]. Test vectors are key ingredients. Bernstein and Reznikov use an explicit test vector. Venkatesh uses a theoretical one, but explains that the bounds would be better with an explicit one (see paragraph 5 of [V]). Unfortunately, the difficulty of finding them increases with the ramification of the representations involved.

There is an extension of Prasad's result in [H-S], where Harris and Scholl prove that the dimension of the space of  $G$ -invariant linear forms on  $V_1 \otimes V_2 \otimes V_3$  is one when  $\pi_1, \pi_2$  and  $\pi_3$  are principal series representations, either irreducible or reducible with their unique irreducible

subspace, infinite dimensional. They apply the global setting of this to the construction of elements in the motivic cohomology of the product of two modular curves constructed by Beilinson.

I would like to thank Philippe Michel for suggesting this problem, Wen-Ching Winnie Li who invited me to spend one semester at PennState University where I could write the first draft of this paper, and of course Benedict Gross and Diprenda Prasad for the inspiration. I would also like to thank Paul Broussous and Nicolas Templier for interesting discussions.

## 2 Strategy

### 2.1 About induced and contragredient representations

Let  $(\rho, W)$  be a smooth representation of a closed subgroup  $H$  of  $G$ . Let  $\Delta_H$  be the modular function on  $H$ . The induction of  $\rho$  from  $H$  to  $G$  is a representation  $\pi$  whose space is the space  $\text{Ind}_H^G(\rho)$  of functions  $f$  from  $G$  to  $W$  satisfying the two following conditions :

- (1)  $\forall h \in H \quad \forall g \in G \quad f(hg) = \Delta_H^{-\frac{1}{2}}(h)\rho(h)f(g),$
- (2) there exists an open compact subgroup  $K_f$  of  $G$  such that

$$\forall k \in K_f, \quad \forall g \in G, \quad f(gk) = f(g)$$

where  $G$  acts by right translation. The resulting function will be denoted  $\langle \pi(g), f \rangle$  that is

$$\forall g, g_0 \in G \quad \langle \pi(g), f \rangle(g_0) = f(g_0g).$$

With the additional condition that  $f$  must be compactly supported modulo  $H$ , one gets the *compact* induction denoted by  $\text{ind}_H^G$ . When  $G/H$  is compact, there is no difference between  $\text{Ind}_H^G$  and  $\text{ind}_H^G$ .

Let  $B$  the Borel subgroup of upper triangular matrices in  $G$  and  $T$  be the diagonal torus. Then we will use  $\delta = \Delta_B^{-1}$  with  $\delta\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = \left|\frac{a}{d}\right|$  and  $\Delta_T$  is trivial. The quotient  $B \backslash G$  is compact and can be identified with  $\mathbb{P}^1(F)$ .

For a smooth representation  $V$  of  $G$ ,  $V^*$  is the space of linear forms on  $V$ . The contragredient representation  $\tilde{\pi}$  is given by the action of  $G$  on  $\tilde{V}$ , the subspace of smooth vectors in  $V^*$ . If  $H$  is a subgroup of  $G$ ,  $\tilde{V} \subset \widetilde{V|_H} \subset V^*$ .

More information about induced and contragredient representations will be found in [B-Z].

### 2.2 The main result and the (\*)-condition

Let  $(\pi_1, V_1)$ ,  $(\pi_2, V_2)$  and  $(\pi_3, V_3)$  be three irreducible, admissible, infinite dimensional representations of  $G$  such that the product of their central characters is trivial. Assume that  $\pi_1$  and  $\pi_2$  are unramified principal series, and that  $\pi_3$  has conductor  $n \geq 1$ . Then, according to theorem 1, there exists a non-zero,  $G$ -invariant linear form  $\ell$  on  $V_1 \otimes V_2 \otimes V_3$ , and we are looking for a vector  $v$  in  $V_1 \otimes V_2 \otimes V_3$  which is not in the kernel of  $\ell$ . In order to follow Gross and Prasad suggestion, we will consider

$$\gamma = \begin{pmatrix} \pi_F & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R = \gamma^{-n} \text{M}_2(\mathcal{O}_F) \gamma^n.$$

One can easily check that

$$R^* = \gamma^{-n} K \gamma^n \quad \text{and} \quad R^* \cap K = \Gamma_0(\pi_F^n).$$

If  $v_1, v_2$  and  $v_3$  denote the new vectors of  $\pi_1, \pi_2$  and  $\pi_3$ , the vector

$$v_2^* = \pi_2(\gamma^{-n}) \cdot v_2$$

is invariant under the action of  $R^*$ . Hence we can write

$$v_1 \in V_1^K \quad v_2^* \in V_1^{R^*} \quad v_3 \in V_3^{R^* \cap K}$$

According to Gross and Prasad  $v_1 \otimes v_2^* \otimes v_3$  should be a test vector for  $\ell$ , for any  $n \geq 1$ . There will be a technical condition, denoted (\*) regarding  $\pi_1$  and  $\pi_2$ . Since they are unramified principal series, for  $i = 1$  and  $i = 2$ , there are unramified characters  $\mu_i$  and  $\eta_i$ , such that

$$\pi_i = \text{Ind}_B^G \chi_i \quad \text{with} \quad \chi_i \left( \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right) = \eta_i(ad) \mu_i \left( \frac{a}{d} \right)$$

Then

$$\chi_1 \left( \begin{pmatrix} \pi_F & 0 \\ 0 & \pi_F^{-1} \end{pmatrix} \right) = \mu_1(\pi_F)^2$$

The (\*)-condition is the following : if  $n = 1$

$$\mu_1(\pi_F)^2 \neq -1 \quad \text{or} \quad \mu_2(\pi_F)^2 \neq -1$$

if  $n \geq 2$

$$\begin{aligned} & \mu_1(\pi_F)^2 = 1 \quad \text{or} \quad \forall k \in \{1, \dots, n\} \quad \mu_1(\pi_F)^{2k} \neq 1 \\ \text{or} \quad & \mu_2(\pi_F)^2 = 1 \quad \text{or} \quad \forall k \in \{1, \dots, n\} \quad \mu_2(\pi_F)^{2k} \neq 1. \end{aligned}$$

We will prove

**Theorem 5.** *If the (\*)-condition is satisfied,  $v_1 \otimes v_2^* \otimes v_3$  is a test vector for  $\ell$ .*

The proof will follow the same pattern as Prasad's proof of theorem 2 in [P], with the necessary changes.

### 2.3 Central characters

Let  $\omega_1, \omega_2$  and  $\omega_3$  be the central characters of  $\pi_1, \pi_2$  and  $\pi_3$ . Notice that the condition  $\omega_1 \omega_2 \omega_3 = 1$  derives from the  $G$ -invariance of  $\ell$ . Since  $\pi_1$  and  $\pi_2$  are unramified,  $\omega_1$  and  $\omega_2$  are unramified too, and so is  $\omega_3$  because  $\omega_1 \omega_2 \omega_3 = 1$ . Let  $\eta_i$ , for  $i \in \{1, 2, 3\}$  be unramified quasi-characters of  $F^*$  with  $\eta_i^2 = \omega_i$  and  $\eta_1 \eta_2 \eta_3 = 1$ . Then

$$V_1 \otimes V_2 \otimes V_3 \simeq (V_1 \otimes \eta_1^{-1}) \otimes (V_2 \otimes \eta_2^{-1}) \otimes (V_3 \otimes \eta_3^{-1})$$

as a representation of  $G$ . Hence it is enough to prove theorem 5 when the central characters of the representations are trivial.

When  $n = 1$ , it is also enough to prove theorem 5 when  $V_3$  is the special representation  $\text{Sp}$  of  $G$  : take  $\eta_3$  to be the unramified character such that  $V_3 = \eta_3 \otimes \text{Sp}$ .

## 2.4 Prasad's exact sequences

Let us now explain how Prasad finds  $\ell$ . It is equivalent to search  $\ell$  or to search a non zero element in  $\text{Hom}_G(V_1 \otimes V_2, \widetilde{V}_3)$ . Since the central characters of  $\pi_1$  and  $\pi_2$  are trivial, there are unramified characters  $\mu_1$  and  $\mu_2$  such that for  $i = 1$  and  $i = 2$

$$\pi_i = \text{Ind}_B^G \chi_i \quad \text{with} \quad \chi_i \left( \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right) = \mu_i \left( \frac{a}{d} \right)$$

Hence

$$V_1 \otimes V_2 = \text{Res}_G \text{Ind}_{B \times B}^{G \times G} (\chi_1 \times \chi_2)$$

where  $G$  is diagonally embedded in  $G \times G$  for the restriction. The action of  $G$  on  $B \times B \backslash G \times G = \mathbb{P}^1(F) \times \mathbb{P}^1(F)$  has precisely two orbits : the first one is  $\{(u, v) \in \mathbb{P}^1(F) \times \mathbb{P}^1(F) \mid u \neq v\}$ , it is open and can be identified with  $T \backslash G$ , the second one is the diagonal embedding of  $\mathbb{P}^1(F)$  in  $\mathbb{P}^1(F) \times \mathbb{P}^1(F)$ , it is closed and it can be identified with  $B \backslash G$ . Then, we have a short exact sequence of  $G$ -modules

$$0 \rightarrow \text{ind}_T^G \left( \frac{\chi_1}{\chi_2} \right) \xrightarrow{\mathbf{ext}} V_1 \otimes V_2 \xrightarrow{\mathbf{res}} \text{Ind}_B^G (\chi_1 \chi_2 \delta^{\frac{1}{2}}) \rightarrow 0 \quad (1)$$

The surjection  $\mathbf{res}$  is the restriction of functions from  $G \times G$  to the diagonal part of  $B \backslash G \times B \backslash G$ , that is

$$\Delta_{B \backslash G} = \left\{ (g, bg) \mid b \in B, g \in G \right\}.$$

The injection  $\mathbf{ext}$  takes a function  $f \in \text{ind}_T^G \left( \frac{\chi_1}{\chi_2} \right)$  to a function  $F \in \text{Ind}_{B \times B}^{G \times G} (\chi_1 \times \chi_2)$  vanishing on  $\Delta_{B \backslash G}$ , given by the relation

$$F \left( g, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g \right) = f(g).$$

on the other orbit. Applying the functor  $\text{Hom}_G(\cdot, \widetilde{V}_3)$ , one gets a long exact sequence

$$\begin{aligned} 0 \rightarrow \text{Hom}_G \left( \text{Ind}_B^G (\chi_1 \chi_2 \delta^{\frac{1}{2}}), \widetilde{V}_3 \right) &\rightarrow \text{Hom}_G (V_1 \otimes V_2, \widetilde{V}_3) \rightarrow \text{Hom}_G \left( \text{ind}_T^G \left( \frac{\chi_1}{\chi_2} \right), \widetilde{V}_3 \right) \\ &\downarrow \\ \dots \leftarrow \text{Ext}_G^1 \left( \text{Ind}_B^G (\chi_1 \chi_2 \delta^{\frac{1}{2}}), \widetilde{V}_3 \right) &\quad (2) \end{aligned}$$

## 2.5 The simple case

The situation is easier when  $n = 1$  and  $\mu_1 \mu_2 | \cdot |^{\frac{1}{2}} = | \cdot |^{-\frac{1}{2}}$ . Then  $\pi_3$  is special and there is a natural surjection

$$\text{Ind}_B^G (\chi_1 \chi_2 \delta^{\frac{1}{2}}) \longrightarrow \widetilde{V}_3$$

whose kernel is the one dimensional subspace of constant functions. Thanks to the exact sequence (1) one gets a surjection  $\Psi$

$$\begin{array}{ccc} V_1 \otimes V_2 & \xrightarrow{\mathbf{res}} & \text{Ind}_B^G (\chi_1 \chi_2 \delta^{\frac{1}{2}}) \\ \Psi \searrow & & \swarrow \\ & & \widetilde{V}_3 \end{array}$$

which corresponds to

$$\ell \begin{cases} V_1 \otimes V_2 \otimes V_3 & \longrightarrow \mathbb{C} \\ v \otimes v' \otimes v'' & \longmapsto \Psi(v \otimes v').v'' \end{cases}$$

The surjection  $\Psi$  vanishes on  $v_1 \otimes v_2^*$  if and only if  $\mathbf{res}(v_1 \otimes v_2^*)$  has constant value on  $\mathbb{P}^1(F) \simeq B \backslash G$ . Easy computation proves that it is not constant : the new vectors  $v_1$  and  $v_2$  are functions from  $G$  to  $\mathbb{C}$  such that

$$\forall i \in \{1, 2\}, \quad \forall b \in B, \quad \forall k \in K, \quad v_i(bk) = \chi_i(b) \cdot \delta(b)^{\frac{1}{2}}$$

and

$$\forall g \in G, \quad v_2^*(g) = v_2(g\gamma^{-1}).$$

Then

$$(v_1 \otimes v_2^*)\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = v_1\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)v_2(\gamma^{-1}) = v_2\left(\begin{pmatrix} \pi_F^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) = \mu_2(\pi_F)^{-1}|\pi_F|^{-\frac{1}{2}} = \frac{\sqrt{q}}{\mu_2(\pi_F)}$$

and

$$(v_1 \otimes v_2^*)\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = v_2\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)\left(\begin{pmatrix} \pi_F^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) = v_2\left(\begin{pmatrix} 1 & 0 \\ 0 & \pi_F^{-1} \end{pmatrix}\right)\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \frac{\mu_2(\pi_F)}{\sqrt{q}}.$$

The representation  $\pi_2$  is principal so  $\frac{\sqrt{q}}{\mu_2(\pi_F)} \neq \frac{\mu_2(\pi_F)}{\sqrt{q}}$  and

$$(v_1 \otimes v_2^*)\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \neq (v_1 \otimes v_2^*)\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right).$$

Hence,  $\Psi$  does not vanish on  $v_1 \otimes v_2^*$ . Then,  $v_1$  being  $K$ -invariant and  $v_2^*$  being  $R^*$ -invariant,  $\Psi(v_1 \otimes v_2^*)$  is a non zero  $\Gamma_0(\pi_F^n)$ -invariant element of  $\widetilde{V}_3$ , that is, a new vector for  $\widetilde{\pi}_3$ , and it does not vanish on  $v_3$  :

$$\ell(v_1 \otimes v_2^* \otimes v_3) = \Psi(v_1 \otimes v_2^*).v_3 \neq 0$$

Then  $v_1 \otimes v_2^* \otimes v_3$  is a test vector for  $\ell$ .

## 2.6 The other case

If  $n \geq 2$  or  $\mu_1\mu_2|\cdot|^{\frac{1}{2}} \neq |\cdot|^{-\frac{1}{2}}$  then  $\mathrm{Hom}_G\left(\mathrm{Ind}_B^G\left(\chi_1\chi_2\delta^{\frac{1}{2}}\right), \widetilde{V}_3\right) = 0$  and by corollary 5.9 of [P]

$$\mathrm{Ext}_G^1\left(\mathrm{Ind}_B^G\left(\chi_1\chi_2\delta^{\frac{1}{2}}\right), \widetilde{V}_3\right) = 0$$

Through the long exact sequence (2) we get an isomorphism

$$\mathrm{Hom}_G\left(V_1 \otimes V_2, \widetilde{V}_3\right) \simeq \mathrm{Hom}_G\left(\mathrm{ind}_T^G\left(\frac{\chi_1}{\chi_2}\right), \widetilde{V}_3\right)$$

and by Frobenius reciprocity

$$\mathrm{Hom}_G\left(\mathrm{ind}_T^G\left(\frac{\chi_1}{\chi_2}\right), \widetilde{V}_3\right) \simeq \mathrm{Hom}_T\left(\left(\frac{\chi_1}{\chi_2}\right), \widetilde{V}_3|_T\right)$$

By lemmas 8 and 9 of [W], this latter space is one dimensional. Thus, we have a chain of isomorphic one dimensional vector spaces

$$\begin{aligned}
\ell &\in \text{Hom}_G(V_1 \otimes V_2 \otimes V_3, \mathbb{C}) \\
&\quad \downarrow \wr \\
\Psi &\in \text{Hom}_G(V_1 \otimes V_2, \widetilde{V}_3) \\
&\quad \downarrow \wr \\
\Phi &\in \text{Hom}_G(\text{ind}_T^G\left(\frac{\chi_1}{\chi_2}\right), \widetilde{V}_3) \\
&\quad \downarrow \wr \\
\varphi &\in \text{Hom}_T\left(\left(\frac{\chi_1}{\chi_2}\right), \widetilde{V}_{3|T}\right)
\end{aligned}$$

with generators  $\ell$ ,  $\Psi$ ,  $\Phi$  and  $\varphi$  corresponding via the isomorphisms. Notice that  $\varphi$  is a linear form on  $V_3$  such that

$$\forall t \in T \quad \forall v \in V_3 \quad \varphi(\pi_3(t)v) = \frac{\chi_2(t)}{\chi_1(t)}\varphi(v) \quad (3)$$

**Lemma 1.**  $\varphi(v_3) \neq 0$ .

*Proof:* this is proposition 2.6 of [G-P] with the following translation :

- the local field  $F$  is the same,
- the quadratic extension  $K/F$  of Gross and Prasad is  $F \times F$  and their group  $K^*$  is our torus  $T$ ,
- the infinite dimensional representation  $V_1$  of Gross and Prasad is our  $\pi_3$ ,
- the one dimensional, unramified representation  $V_2$  of Gross and Prasad is  $\frac{\chi_1}{\chi_2}$ .

Then the representation that Gross and Prasad call  $V$  is  $\frac{\chi_1}{\chi_2} \otimes \pi_3$  and their condition (1.3) is exactly our condition (3). In order to apply Gross and Prasad's proposition, we need to check the equality

$$\varepsilon(\sigma \otimes \sigma_3) = \alpha_{K/F}(-1)\omega(-1)$$

Basically, it is true because  $K$  is not a field. Let us give some details.

- In [G-P],  $\omega$  is the central character of the representation  $V_1$  which is trivial for us.
- The character  $\alpha_{K/F}$  is the quadratic character of  $F^*$  associated to the extension  $K/F$  by local class-field theory. Here, it is trivial because  $K$  is  $F \times F$ .
- To compute  $\varepsilon(\sigma \otimes \sigma_3)$  we will use the first pages of [T].

$$\forall \begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix} \in T \quad \frac{\chi_1}{\chi_2}\left(\begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix}\right) = \frac{\mu_1}{\mu_2}(x)\frac{\mu_2}{\mu_1}(z) \Rightarrow \varepsilon(\sigma \otimes \sigma_3) = \varepsilon\left(\frac{\mu_1}{\mu_2} \otimes \sigma_3\right)\varepsilon\left(\frac{\mu_2}{\mu_1} \otimes \sigma_3\right)$$

Since the determinant of  $\sigma_3$  is the central character of  $\pi_3$  which is trivial,  $\sigma_3$  is isomorphic to its own contragredient, and the contragredient representation of  $\frac{\mu_1}{\mu_2} \otimes \sigma_3$  is  $\frac{\mu_2}{\mu_1} \otimes \sigma_3$ . Then, by formula (1.1.6) of [T],

$$\varepsilon(\sigma \otimes \sigma_3) = \det(\sigma_3(-1)) = 1 = \alpha_{K/F}(-1)\omega(-1)$$

Then, according to [G-P], the restriction of  $\frac{\chi_1}{\chi_2} \otimes \pi_3$  to the group

$$M = \left\{ \begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix} \mid x, y \in \mathcal{O}_F^* \right\} \times \Gamma_0(\pi_F^n)$$



fixes a unique line in  $V_3$  : it is the line generated by the new vector  $v_3$ . According to Gross and Prasad, a non-zero linear form on  $V_3$  which satisfies (3) cannot vanish on  $v_3$ .  $\square$

We still need to prove that  $\ell(v_1 \otimes v_2^* \otimes v_3) \neq 0$ . For the reason described at the end of section 2.5, it is enough to prove that

$$\begin{cases} V_3 & \longrightarrow \mathbb{C} \\ v & \longmapsto \ell(v_1 \otimes v_2^* \otimes v) \end{cases}$$

is non zero in  $\widetilde{V}_3$ . In order to do that we want to build a function  $F$  in  $V_1 \otimes V_2$ , of the form

$$F = \sum_{i \in I} c_i \langle (\pi_1 \otimes \pi_2)(g_i), v_1 \otimes v_2^* \rangle \quad (4)$$

which vanishes on the closed orbit of  $G$  in  $\mathbb{P}^1(F) \times \mathbb{P}^1(F)$ . Then,  $F$  is in the kernel of **res** so it is the image by **ext** of a function  $f \in \text{ind}_T^G \left( \frac{\chi_1}{\chi_2} \right)$ . The point is that  $f$  must be the characteristic function of the orbit of the unit in the decomposition of  $T \backslash G$  under the action of  $\Gamma_0(\pi_F^n)$ , which means :

$$f(g) = \begin{cases} \frac{\chi_1(t)}{\chi_2(t)} & \text{if } g = tk \text{ with } t \in T \text{ and } k \in \Gamma_0(\pi_F^n) \\ 0 & \text{else} \end{cases} \quad (5)$$

Then, the function

$$\begin{cases} G & \longrightarrow \mathbb{C} \\ g & \longmapsto f(g) \varphi(\pi_3(g)v_3) \end{cases}$$

is invariant by the action of  $T$  by left translation and we can do the following computation: on the one hand

$$\begin{aligned} (\Psi(F))(v_3) &= (\Phi(f))(v_3) \\ &= \int_{T \backslash G} f(g) \varphi(\pi_3(g)v_3) dg \\ &= \int_{(T \cap K) \backslash \Gamma_0(\pi_F^n)} \varphi(\pi_3(k)v_3) dk \\ &= \lambda \cdot \varphi(v_3). \end{aligned}$$

where  $\lambda$  is a non zero constant. Thanks to lemma 1 we know that  $\varphi(v_3) \neq 0$  then

$$(\Psi(F))(v_3) \neq 0.$$

On the other hand, it comes from (4) that

$$\begin{aligned}
(\Psi(F))(v_3) &= \sum_{i \in I} c_i \ell \left( \pi_1(g_i)v_1 \otimes \pi_2(g_i)v_2^* \otimes v_3 \right) \\
&= \sum_{i \in I} c_i \ell \left( v_1 \otimes v_2^* \otimes \pi_3(g_i^{-1})v_3 \right) \\
&= \Psi(v_1 \otimes v_2^*) \left( \sum_{i \in I} c_i \pi_3(g_i^{-1})v_3 \right)
\end{aligned}$$

then  $\Psi(v_1 \otimes v_2^*) \neq 0$  and  $v_1 \otimes v_2^* \otimes v_3$  is a test vector for  $\ell$ .

### 3 Calculations

#### 3.1 The function $F$ as an element of $V_1 \otimes V_2$

Let  $a$  and  $b$  be the numbers

$$a = \frac{\mu_1(\pi_F)}{\sqrt{q}} \quad b = \frac{\mu_2(\pi_F)}{\sqrt{q}}$$

They verify

$$(a^2 - 1)(b^2 - 1) \neq 0$$

because  $\pi_1$  and  $\pi_2$  are principal series representations. For the sake of simplicity, we shall use the following notations : for any  $g$  in  $G$

$$gv_1 = \langle \pi_1(g), v_1 \rangle \quad \text{and} \quad gv_2^* = \langle \pi_2(g), v_2^* \rangle$$

and for a family  $(g_i)$  of elements of  $G$ , and  $(c_i)$  some complex numbers, denote

$$\left( \sum_i c_i \cdot g_i \right) (v_1 \otimes v_2^*) = \sum_i c_i \cdot \langle (\pi_1 \times \pi_2)(g_i), v_1 \otimes v_2^* \rangle.$$

**Lemma 2.** *As an element of  $V_1 \otimes V_2$ , the function  $F$  is given by the formula*

$$\begin{aligned}
F = \frac{a^n}{(a^2 - 1)(b^2 - 1)} \left\{ a \cdot b \cdot \gamma^{-(n-1)}v_1 \otimes \gamma^{-1}v_2 - a \cdot \gamma^{-(n-1)}v_1 \otimes v_2 - b \cdot \gamma^{-n}v_1 \otimes \gamma^{-1}v_2 \right. \\
\left. + \gamma^{-n}v_1 \otimes v_2 \right\}
\end{aligned}$$

*Proof :* The function  $F$  has to be  $\mathbf{ext}(f)$ , where  $f$  is the function described by formula (5), and  $\mathbf{ext}(f)$  is described by the short exact sequence (1) using the orbits of the action of  $G$  on  $B \times B \backslash G \times G$ . The function  $F$  must vanish on the closed orbit

$$\Delta_{B \backslash G} = \left\{ (g, bg) \mid b \in B, \quad g \in G \right\}$$

The open orbit can be identified with  $T \backslash G$  via the bijection

$$\begin{cases} T \backslash G & \longrightarrow & (B \backslash G \times B \backslash G) \setminus \Delta_{B \backslash G} \\ Tg & \longmapsto & (Bg, B \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g) \end{cases}$$

through which, the orbit of the unit in  $T \backslash G$  under the action of  $\Gamma_0(\pi_F^n)$  corresponds to

$$\left\{ \left( Bk, B \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} k \right) \mid k \in \Gamma_0(\pi_F^n) \right\}$$

Now, pick any  $(k, k') \in K \times K$ . If  $k' \in Bk$ , then

$$k \in \Gamma_0(\pi_F^n) \iff k' \in \Gamma_0(\pi_F^n) \quad \text{and} \quad k \in \Gamma_0(\pi_F) \iff k' \in \Gamma_0(\pi_F).$$

When  $k' \notin Bk$ , write  $k = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$  and  $k' = \begin{pmatrix} x' & y' \\ z' & t' \end{pmatrix}$ . There exists  $(b_1, b_2) \in B \times B$  such that

$$k = b_1 k_0 \quad \text{and} \quad k' = b_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} k_0 \quad \text{with} \quad k_0 = \begin{pmatrix} z' & t' \\ z & t \end{pmatrix} \in M_2(\mathcal{O}_F).$$

Then  $(k, k')$  is in the orbit of the unit in  $T \backslash G$  under the action of  $\Gamma_0(\pi_F^n)$  if and only if  $k_0$  is in  $T\Gamma_0(\pi_F^n)$ . Because  $k$  and  $k'$  are in  $K$ , one can see that

$$k_0 \in T\Gamma_0(\pi_F^n) \iff k_0 \in \Gamma_0(\pi_F^n)$$

and

$$\begin{aligned} k_0 \in \Gamma_0(\pi_F^n) &\iff z \equiv 0 \pmod{\pi_F^n} \quad \text{and} \quad z't \in \mathcal{O}_F^* \\ &\iff z \equiv 0 \pmod{\pi_F^n} \quad \text{and} \quad z' \in \mathcal{O}_F^* \\ &\iff k \in \Gamma_0(\pi_F^n) \quad \text{and} \quad k' \notin \Gamma_0(\pi_F). \end{aligned}$$

It follows that  $(k, k')$  corresponds to an element of the orbit of the unit in the decomposition of  $T \backslash G$  under the action of  $\Gamma_0(\pi_F^n)$  if and only if  $k \in \Gamma_0(\pi_F^n)$  and  $k' \notin \Gamma_0(\pi_F)$ . Then, it will be enough to check that

$$F(k, k') = \begin{cases} 1 & \text{if } k \in \Gamma_0(\pi_F^n) \quad \text{and} \quad k' \notin \Gamma_0(\pi_F) \\ 0 & \text{else} \end{cases} \quad (6)$$

This is mere calculation. We need, with  $a = \frac{\mu_1(\pi_F)}{\sqrt{q}}$  as in section 3.1

**Lemma 3.**

$$\forall i \in \mathbb{N}, \quad \forall k = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in K, \quad \gamma^{-i} v_1(k) = \begin{cases} a^i & \text{if } \text{val}(\frac{z}{t}) \leq 0 \\ a^{i-2 \text{val}(\frac{z}{t})} & \text{if } 1 \leq \text{val}(\frac{z}{t}) \leq i-1 \\ a^{-i} & \text{if } i \leq \text{val}(\frac{z}{t}) \end{cases} \quad (7)$$

*Proof*: since  $k$  is in  $K$ , either  $z$  or  $t$  is in  $\mathcal{O}_F^*$ . The other one is in  $\mathcal{O}_F$ . Write

$$k\gamma^{-i} = \begin{pmatrix} \frac{x}{\pi_F^i} & y \\ \frac{z}{\pi_F^i} & t \end{pmatrix}.$$

If  $\text{val} \frac{z}{t} \leq 0$ , then  $\text{val} z = 0$ ,  $\text{val}(\frac{\pi_F^i t}{z}) \geq 0$  and

$$k\gamma^{-i} = \begin{pmatrix} \frac{xt-yz}{z} & \frac{x}{\pi_F^i} \\ 0 & \frac{z}{\pi_F^i} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & \frac{\pi_F^i t}{z} \end{pmatrix}$$

with

$$\begin{pmatrix} 0 & -1 \\ 1 & \frac{\pi_F^i t}{z} \end{pmatrix} \in K \quad \text{and} \quad (\chi_1 \cdot \delta^{\frac{1}{2}}) \begin{pmatrix} \frac{xt-yz}{z} & \frac{x}{\pi_F^i} \\ 0 & \frac{z}{\pi_F^i} \end{pmatrix} = \left( \frac{\mu_1(\pi_F)}{\sqrt{q}} \right)^{i-2\text{val} z} = a^{i-2\text{val} z} = a^i$$

If  $1 \leq \text{val} \frac{z}{t} \leq i-1$ , then  $\text{val} t = 0$ ,  $\text{val}(\frac{\pi_F^i t}{z}) \geq 0$  and the computation is quite the same, except that

$$(\chi_1 \cdot \delta^{\frac{1}{2}}) \begin{pmatrix} \frac{xt-yz}{z} & \frac{x}{\pi_F^i} \\ 0 & \frac{z}{\pi_F^i} \end{pmatrix} = \left( \frac{\mu_1(\pi_F)}{\sqrt{q}} \right)^{i-2\text{val} z} = a^{i-2\text{val} \frac{z}{t}}$$

If  $i \leq \text{val} \frac{z}{t}$ , then  $\text{val} t = 0$ ,  $\text{val}(\frac{\pi_F^i t}{z}) \leq 0$  and

$$k\gamma^{-i} = \begin{pmatrix} \frac{xt-yz}{t \pi_F^i} & y \\ 0 & t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{z}{t \pi_F^i} & 1 \end{pmatrix}$$

with

$$\begin{pmatrix} 1 & 0 \\ \frac{z}{t \pi_F^i} & 1 \end{pmatrix} \in K \quad \text{and} \quad (\chi_1 \cdot \delta^{\frac{1}{2}}) \begin{pmatrix} \frac{xt-yz}{t \pi_F^i} & y \\ 0 & t \end{pmatrix} = \left( \frac{\mu_1(\pi_F)}{\sqrt{q}} \right)^{-i-2\text{val} t} = a^{-i}$$

□

**NB** : the case  $t = 0$  is included in  $\text{val} \frac{z}{t} \leq 0$ .

We can now finish the proof of lemma 2. The same computation, with  $v_2$  and  $b$  instead of  $v_1$  and  $a$ , gives the values of  $\gamma^{-i}v_2$  for any  $i \in \mathbb{N}$ . It is then easy to compute  $F(k, k')$  as given in lemma 2, for  $k$  and  $k'$  in  $K$ , and to check formula (6).

### 3.2 The nice modulus $\mathcal{M}$

We still need to express  $F$  obtained from  $v_1 \otimes v_2^*$  as in formula (4). Let  $\mathcal{M}$  denote the  $\mathbb{C}[G]$ -modulus generated by  $v_1 \otimes v_2^*$  in  $V_1 \otimes V_2$ . We want to prove that  $F$  is in  $\mathcal{M}$ . Thanks to formula (2), it is enough to prove that the three functions

$$\gamma^{-(n-2)}v_1 \otimes v_2 \quad \gamma^{-(n-1)}v_1 \otimes v_2 \quad \gamma^{-n}v_1 \otimes v_2$$

are in  $\mathcal{M}$ . We will do that by making Hecke operators act on  $v_1 \otimes v_2^*$ .

## 4 Hecke operators

### 4.1 Notations

Let  $k$  be an integer greater or equal to 1. The  $k^{\text{th}}$  Hecke operator is

$$T_{\pi_F^k} = K \begin{pmatrix} \pi_F^k & 0 \\ 0 & 1 \end{pmatrix} K$$

The new vector  $v_1$  is an eigenvector for  $T_{\pi_F^k}$ , with eigen value  $A_k$ . If we decide to take  $T_{\pi_F^0}$  to be the identity operator, then we know that

$$\begin{cases} A_0 = 1 \\ A_1 = \frac{\sqrt{q}}{\mu_1(\pi_F)} (1 + \mu_1^2(\pi_F)) = \frac{1}{a} (1 + \mu_1^2(\pi_F)) \\ A_2 = A_1^2 - (q + 1) \\ \forall k \geq 2 \quad A_{k+1} = A_1 A_k - q A_{k-1} \end{cases}$$

The same relations hold for the new vector  $v_2$  of  $V_2$ , and we call the eigenvalues  $B_k$ .

Let  $\{\tau_{k,i} \mid 1 \leq i \leq q^k\}$  be a set of representatives of  $\mathcal{O}_F / \pi_F^k \mathcal{O}_F$  in  $\mathcal{O}_F$ , such that  $\tau_{k,1} = 0$ . The subset of elements which are not invertible modulo  $\pi_F^k \mathcal{O}_F$  is  $\{\pi_F \tau_{k-1,j} \mid 1 \leq j \leq q^{k-1}\}$ . If we choose

$$\forall i, \quad 1 \leq i \leq q^k \quad g_{k,i} = \begin{pmatrix} \pi_F^k & \tau_{k,i} \\ 0 & 1 \end{pmatrix}$$

then

$$\forall i, \quad 1 \leq i \leq q^{k-1} \quad \begin{pmatrix} 0 & 1 \\ \pi_F & 0 \end{pmatrix} g_{k-1,i} = \begin{pmatrix} 0 & 1 \\ \pi_F^k & \pi_F \tau_{k-1,i} \end{pmatrix}$$

and we can write

$$T_{\pi_F^k} = \bigsqcup_{i=1}^{q^k} g_{k,i} K \bigsqcup \bigsqcup_{i=1}^{q^{k-1}} \begin{pmatrix} 0 & 1 \\ \pi_F & 0 \end{pmatrix} g_{k-1,i} K$$

Then we define an operator  $T_k$  on  $V_1 \otimes V_2$

$$\forall v \in V_1 \quad \forall v' \in V_2, \quad T_k(v \otimes v') = \sum_{i=1}^{q^k} g_{k,i}(v \otimes v') + \sum_{i=1}^{q^{k-1}} \begin{pmatrix} 0 & 1 \\ \pi_F & 0 \end{pmatrix} g_{k-1,i}(v \otimes v')$$

We will also use the truncated operator  $T_k^*$

$$\forall v \in V_1 \quad \forall v' \in V_2, \quad T_k^*(v \otimes v') = \sum_{i=1}^{q^k} g_{k,i}(v \otimes v')$$

### 4.2 Trying to put F in $\mathcal{M}$

**Lemma 4.** Let  $v_\bullet$  denote  $v_1$  or  $v_2$

$$\mathbf{a)} \quad \forall j, k \in \mathbb{N} \quad \begin{pmatrix} 0 & 1 \\ \pi_F^k & 0 \end{pmatrix} (\gamma^{-j} v_\bullet) = \gamma^{j-k} v_\bullet.$$

$$\mathbf{b)} \quad \forall i \in \mathbb{N}, \quad 1 \leq i \leq q \quad \forall j \in \mathbb{N}, \quad j \geq 1 \quad g_{1,i} \gamma^{-j} v_\bullet = \gamma^{-(j-1)} v_\bullet.$$

*Proof*: a) remembering that  $v_\bullet$  is invariant by  $K$  and by the center of  $G$ , we write

$$\begin{pmatrix} 0 & 1 \\ \pi_F^k & 0 \end{pmatrix} \gamma^{-j} = \begin{pmatrix} \pi_F^{k-j} & 0 \\ 0 & \pi_F^{k-j} \end{pmatrix} \begin{pmatrix} \pi_F^{j-k} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The first matrix belongs to the center of  $G$ , the second one is precisely  $\gamma^{j-k}$  and the third one is in  $K$ .

b) write

$$g_{1,i} \gamma^{-j} = \begin{pmatrix} \pi_F^{1-j} & \tau_{1,i} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \pi_F^{1-j} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \pi_F^{j-1} \tau_{1,i} \\ 0 & 1 \end{pmatrix}$$

The first matrix is  $\gamma^{-(j-1)}$  and the second one is in  $K$  because  $j-1 \geq 0$ .  $\square$

**Lemma 5.**  $T_n(v_1 \otimes \gamma^{-n} v_2) = A_n \cdot v_1 \otimes v_2$

*Proof*: it is enough to observe that

$$\forall i, \quad 1 \leq i \leq q^n \quad g_{n,i} \gamma^{-n} \in K \quad \text{and} \quad \forall i, \quad 1 \leq i \leq q^{n-1} \quad \begin{pmatrix} 0 & 1 \\ \pi_F & 0 \end{pmatrix} g_{n-1,i} \gamma^{-n} \in K$$

Then

$$\begin{aligned} T_n(v_1 \otimes \gamma^{-n} v_2) &= \sum_{i=1}^{q^n} g_{n,i} (v_1 \otimes \gamma^{-n} v_2) + \sum_{i=1}^{q^{n-1}} \begin{pmatrix} 0 & 1 \\ \pi_F & 0 \end{pmatrix} g_{n-1,i} (v_1 \otimes \gamma^{-n} v_2) \\ &= \sum_{i=1}^{q^n} g_{n,i} v_1 \otimes (g_{n,i} \gamma^{-n}) v_2 + \sum_{i=1}^{q^{n-1}} \begin{pmatrix} 0 & 1 \\ \pi_F & 0 \end{pmatrix} g_{n-1,i} v_1 \otimes \left( \begin{pmatrix} 0 & 1 \\ \pi_F & 0 \end{pmatrix} g_{n-1,i} \gamma^{-n} \right) v_2 \\ &= \sum_{i=1}^{q^n} g_{n,i} v_1 \otimes v_2 + \sum_{i=1}^{q^{n-1}} \begin{pmatrix} 0 & 1 \\ \pi_F & 0 \end{pmatrix} g_{n-1,i} v_1 \otimes v_2 \end{aligned}$$

because  $v_2$  is  $K$  invariant. Then

$$T_n(v_1 \otimes \gamma^{-n} v_2) = T_{\pi_F^n}(v_1) \otimes v_2 = A_n \cdot v_1 \otimes v_2$$

**Lemma 6.** For any integer  $\ell$  such that  $1 \leq \ell \leq n$

$$T_1^*(v_1 \otimes \gamma^{-\ell} v_2) = A_1 \cdot v_1 \otimes \gamma^{-(\ell-1)} v_2 - \gamma^{-1} v_1 \otimes \gamma^{-(\ell-1)} v_2$$

*Proof*: by lemma 4.b, we get

$$\forall i, \quad 1 \leq i \leq q \quad (g_{1,i} \gamma^{-\ell}) v_2 = \gamma^{-(\ell-1)} v_2$$

Then

$$\begin{aligned} T_1^*(v_1 \otimes \gamma^{-\ell} v_2) &= \sum_{i=1}^q g_{1,i} v_1 \otimes (g_{1,i} \gamma^{-\ell}) v_2 \\ &= \sum_{i=1}^q g_{1,i} v_1 \otimes \gamma^{-(\ell-1)} v_2 \\ &= \left( T_{\pi_F} v_1 \right) \otimes \gamma^{-(\ell-1)} v_2 - \left( \begin{pmatrix} 0 & 1 \\ \pi_F & 0 \end{pmatrix} v_1 \right) \otimes \gamma^{-(\ell-1)} v_2 \end{aligned}$$

Using lemma 4.a with  $k = 1$  and  $j = 0$ , we get  $\begin{pmatrix} 0 & 1 \\ \pi_F & 0 \end{pmatrix} v_1 = \gamma^{-1}v_1$  and

$$T_1^*(v_1 \otimes \gamma^{-\ell}v_2) = A_1 \cdot v_1 \otimes \gamma^{-(\ell-1)}v_2 - \gamma^{-1}v_1 \otimes \gamma^{-(\ell-1)}v_2$$

□

We will now define a sequence  $(C_k)_{k \in \mathbb{N}}$  of complex numbers with the same inductive relation as  $(A_k)_{k \in \mathbb{N}}$

$$C_0 = 0 \quad C_1 = 1 \quad \forall k \in \mathbb{N}, \quad C_{k+2} = A_1 \cdot C_{k+1} - q \cdot C_k$$

Then

**Lemma 7.** *For any integer  $k$  such that  $0 \leq k \leq n$*

$$\left(T_1^*\right)^k (v_1 \otimes \gamma^{-n}v_2) = C_{k+1} \cdot v_1 \otimes \gamma^{-(n-k)}v_2 - C_k \cdot \gamma^{-1}v_1 \otimes \gamma^{-(n-k)}v_2$$

*Proof:* by induction over  $k$ . The relation is obvious for  $k = 0$  and it is lemma 6 for  $k = 1$ . We assume it to be true for some  $1 \leq k \leq n - 1$  and we apply to it the operator  $T_1^*$ . First, thanks to lemma 6 with  $\ell = n - k$  we get

$$T_1^*(v_1 \otimes \gamma^{-(n-k)}v_2) = A_1 \cdot v_1 \otimes \gamma^{-(n-k-1)}v_2 - \gamma^{-1}v_1 \otimes \gamma^{-(n-k-1)}v_2.$$

Then compute

$$T_1^*(v_1 \otimes \gamma^{-(n-k)}v_2) = \sum_{i=1}^q (g_{1,i}\gamma^{-1})v_1 \otimes (g_{1,i}\gamma^{-(n-k)})v_2$$

We know that  $g_{1,i}\gamma^{-1} \in K$  and by lemma 4.b that  $(g_{1,i}\gamma^{-(n-k)})v_2 = \gamma^{-(n-k-1)}v_2$  since  $n - k \geq 1$ . Then

$$T_1^*(v_1 \otimes \gamma^{-(n-k)}v_2) = \sum_{i=1}^q v_1 \otimes \gamma^{-(n-k-1)}v_2 = q \cdot v_1 \otimes \gamma^{-(n-k-1)}v_2$$

and

$$\begin{aligned} & \left(T_1^*\right)^{k+1} (v_1 \otimes \gamma^{-n}v_2) \\ &= C_{k+1} \cdot \left( A_1 \cdot v_1 \otimes \gamma^{-(n-k-1)}v_2 - \gamma^{-1}v_1 \otimes \gamma^{-(n-k-1)}v_2 \right) - C_k \cdot q \cdot v_1 \otimes \gamma^{-(n-k-1)}v_2 \\ &= \left( A_1 \cdot C_{k+1} - q \cdot C_k \right) v_1 \otimes \gamma^{-(n-k-1)}v_2 - C_{k+1} \cdot \gamma^{-1}v_1 \otimes \gamma^{-(n-k-1)}v_2 \\ &= C_{k+2} \cdot v_1 \otimes \gamma^{-(n-k-1)}v_2 - C_{k+1} \cdot \gamma^{-1}v_1 \otimes \gamma^{-(n-k-1)}v_2 \end{aligned}$$

□

We need another technical lemma

**Lemma 8.** *Let  $n$  be an integer,  $n \geq 2$ . If  $C_{n-1} \neq 0$ ,  $A_n$  and  $(C_{n+1} - C_{n-1})$  cannot vanish together.*

*Proof* : Let us compute the coefficients  $C_k$  and the eigenvalues  $A_k$ . The second degree equation attached to the sequences  $(C_k)_{k \in \mathbb{N}}$  and  $(A_k)_{k \in \mathbb{N}}$  is  $X^2 - A_1 \cdot X + q$  and its discriminant is

$$\frac{q}{\mu_1(\pi_F)^2} \cdot \left(1 - \mu_1(\pi_F)^2\right)^2.$$

If  $\mu_1(\pi_F)^2 = 1$  then

$$\forall k \in \mathbb{N} \quad C_k = k \left( \frac{\sqrt{q}}{\mu_1(\pi_F)} \right)^{k-1} \quad \text{and} \quad \forall k \in \mathbb{N}^* \quad A_k = \left( \left(1 - \frac{1}{q}\right)k + \left(1 + \frac{1}{q}\right) \right) \cdot \left( \frac{\sqrt{q}}{\mu_1(\pi_F)} \right)^k$$

In this case,  $A_n \neq 0$ .

If  $\mu_1(\pi_F)^2 \neq 1$  then  $\forall k \in \mathbb{N}$

$$C_k = \frac{1}{1 - \mu_1(\pi_F)^2} \cdot \frac{\mu_1(\pi_F)}{\sqrt{q}} \cdot \left( \left( \frac{\sqrt{q}}{\mu_1(\pi_F)} \right)^k - (\sqrt{q} \cdot \mu_1(\pi_F))^k \right)$$

and  $\forall k \in \mathbb{N}^*$

$$A_k = \frac{1 - q \cdot \mu_1(\pi_F)^2}{1 - \mu_1(\pi_F)^2} \cdot \left( \frac{\sqrt{q}}{\mu_1(\pi_F)} \right)^k + \frac{q - \mu_1(\pi_F)^2}{1 - \mu_1(\pi_F)^2} \cdot \left( \sqrt{q} \cdot \mu_1(\pi_F) \right)^k.$$

Then

$$\begin{aligned} C_k = 0 &\iff \mu_1(\pi_F)^{2k} = 1 \\ C_{n+1} - C_{n-1} = 0 &\iff q \cdot \mu_1(\pi_F)^{2n+2} - \mu_1(\pi_F)^{2n} + \mu_1(\pi_F)^2 - q = 0 \\ A_n = 0 &\iff \mu_1(\pi_F)^{2n+2} - q \cdot \mu_1(\pi_F)^{2n} + q \cdot \mu_1(\pi_F)^2 - 1 = 0 \end{aligned}$$

If both  $A_n$  and  $C_{n+1} - C_{n-1}$  vanish, it follows that  $\mu_1(\pi_F)^{2n-2} = 1$ , that is  $C_{n-1} = 0$ .

Now comes the key point

**Lemma 9.** *Let  $n$  be a positive integer. If  $n = 1$  assume  $C_2 \neq 0$  and if  $n \geq 2$  assume that  $\forall k \in \{2, \dots, n\} \quad C_k \neq 0$ . Then  $\forall k \in \{0, \dots, n\} \quad \gamma^{-k}v_1 \otimes v_2 \in \mathcal{M}$ .*

*Proof* : recall that  $\mathcal{M}$  is the  $\mathbb{C}[G]$ -modulus generated by  $v_1 \otimes v_2^* = v_1 \otimes \gamma^{-n}v_2$ . Then

$$v_1 \otimes \gamma^{-n}v_2 \in \mathcal{M}$$

and by lemma 4.a

$$\gamma^{-n}v_1 \otimes v_2 = \begin{pmatrix} 0 & 1 \\ \pi_F^n & 0 \end{pmatrix} (v_1 \otimes \gamma^{-n}v_2) \in \mathcal{M}$$

First, assume that  $A_n \neq 0$ . When  $n = 1$ ,  $A_1 \neq 0$  because  $A_1 = C_2$ . Then by lemma 5

$$v_1 \otimes v_2 = \frac{1}{A_n} \cdot T_n(v_1 \otimes \gamma^{-n}v_2) \in \mathcal{M}$$

Use lemma 7 for  $k = n - 1$  to get

$$\left( T_1^* \right)^{(n-1)} (v_1 \otimes \gamma^{-n}v_2) = C_n \cdot v_1 \otimes \gamma^{-1}v_2 - C_{n-1} \cdot \gamma^{-1}(v_1 \otimes v_2)$$



Since  $v_1 \otimes v_2$  is in  $\mathcal{M}$  and  $C_n \neq 0$ ,  $v_1 \otimes \gamma^{-1}v_2$  is in  $\mathcal{M}$ . Then using lemma 7 for  $k = n - 2$  and the fact that  $C_{n-1} \neq 0$  you prove that  $v_1 \otimes \gamma^{-2}v_2$  is in  $\mathcal{M}$ . By induction, you put  $v_1 \otimes \gamma^{-k}v_2$  in  $\mathcal{M}$  for any  $k$  in  $\{1, \dots, n - 1\}$ . Then, use lemma 4.a for each  $k$  to get

$$\begin{pmatrix} 0 & 1 \\ \pi_F^k & 0 \end{pmatrix} (v_1 \otimes \gamma^{-k}v_2) = \gamma^{-k}v_1 \otimes v_2 \in \mathcal{M}$$

Now, if  $A_n = 0$  and  $n \geq 2$ , we know by lemma 8 that the assumption  $C_{n-1} \neq 0$  implies  $C_{n+1} - C_{n-1} \neq 0$ . Let us apply  $\begin{pmatrix} 0 & 1 \\ \pi_F & 0 \end{pmatrix}$  to the formula given by lemma 7 for  $k = n - 1$

$$\begin{pmatrix} 0 & 1 \\ \pi_F & 0 \end{pmatrix} \left(T_1^*\right)^{(n-1)} (v_1 \otimes \gamma^{-n}v_2) = C_n \cdot (\gamma^{-1}v_1 \otimes v_2) - C_{n-1} \cdot (v_1 \otimes v_2) \in \mathcal{M}$$

Then compare to the formula given by lemma 7 for  $k = n$

$$\left(T_1^*\right)^{(n)} (v_1 \otimes \gamma^{-n}v_2) = C_{n+1} \cdot (v_1 \otimes v_2) - C_n \cdot (\gamma^{-1}v_1 \otimes v_2) \in \mathcal{M}$$

Since  $C_{n+1} - C_{n-1} \neq 0$ , we can deduce that  $v_1 \otimes v_2$  is in  $\mathcal{M}$ , and finish the proof as in the case  $A_n = 0$ . □

Now, we know that the function  $F$  of lemma 2 is in  $\mathcal{M}$  which was precisely what we needed to complete the proof of theorem 5. We still have to explain the link between the fact that the coefficients  $C_k$  do not vanish and the (\*)-condition.

### 4.3 The (\*)-condition again

In theorem 5, it is not assumed that  $\pi_1$  has trivial central character. There might be a twist by some unramified character, but it is still true that

$$\chi_1 \begin{pmatrix} \pi_F & 0 \\ 0 & \pi_F^{-1} \end{pmatrix} = \mu_1(\pi_F)^2$$

From the proof of lemma 8, we know that either  $\mu_1(\pi_F)^2 = 1$  in which case none of the coefficients  $C_k$  vanishes, or  $\mu_1(\pi_F)^2 \neq 1$  and then

$$\forall k \in \mathbb{N}, \quad C_k = 0 \quad \iff \quad \mu_1(\pi_F)^{2k} = 1.$$

This explain the part of the (\*)-condition which lies on  $\pi_1$ . Now, if this condition is not fulfilled, it is possible to switch everything on  $\pi_2$  thanks to the following observation

Let  $(D_k)_{k \in \mathbb{N}}$  be the sequence of complex numbers defined by

$$D_0 = 0 \quad D_1 = 1 \quad \forall k \in \mathbb{N}, \quad D_{k+2} = B_1 \cdot D_{k+1} - q \cdot D_k.$$

This is the analogous for the eigenvalues  $(B_k)_{k \in \mathbb{N}}$  of the sequence  $(C_k)_{k \in \mathbb{N}}$  for the eigenvalues  $(A_k)_{k \in \mathbb{N}}$ . Then

**Lemma 10.**  $\forall k \in \{0, \dots, n-1\}$

$$(T_1^*)^k \begin{pmatrix} 0 & 1 \\ \pi_F^n & 0 \end{pmatrix} (v_1 \otimes \gamma^{-n} v_2) = D_{k+1} \cdot \gamma^{-(n-k)} v_1 \otimes v_2 - D_k \cdot \gamma^{-(n-k)} v_1 \otimes \gamma^{-1} v_2$$

*Proof:* Thanks to lemma 4.a for  $k = n$

$$\begin{pmatrix} 0 & 1 \\ \pi_F^n & 0 \end{pmatrix} (v_1 \otimes \gamma^{-n} v_2) = \gamma^{-n} v_1 \otimes v_2$$

Then the calculations are the same as in lemma 7, where  $v_1$  is replaced by  $v_2$ ,  $\mu_1$  by  $\mu_2$  and each eigenvalue  $A_k$  by  $B_k$ .  $\square$

This leads to the part of the (\*)-condition which lies on  $\pi_2$ . Theorem 5 is now proved.

It might be interesting to observe that the (\*)-condition for  $n = 1$  can be formulated in terms of Hecke eigenvalues : it means that  $A_1$  and  $B_1$  must not vanish together.

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